Linear Regression Models
P8111

Lecture 25

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Today's Lecture

- Logistic regression / GLMs
  - Model framework ✓
  - Interpretation ✓
  - Estimation ✓

Linear regression

Course started with the model

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where

$$\epsilon_i \sim (0, \sigma_\epsilon^2)$$

In particular, y_i has been continuous throughout the course

$$y_i | x_i \sim N(x\beta, \sigma_\epsilon^2)$$

Binary responses

Binary outcomes are common in practice; usually indicate some event

- Yes vs no
- Transplant vs no transplant
- Death vs no death

Count

PROP [0, 1]

Binary responses

$y|x \sim N(x\beta, \sigma^2)$

$E(y|x)$

How should we deal with binary (0/1) y's?

Linear

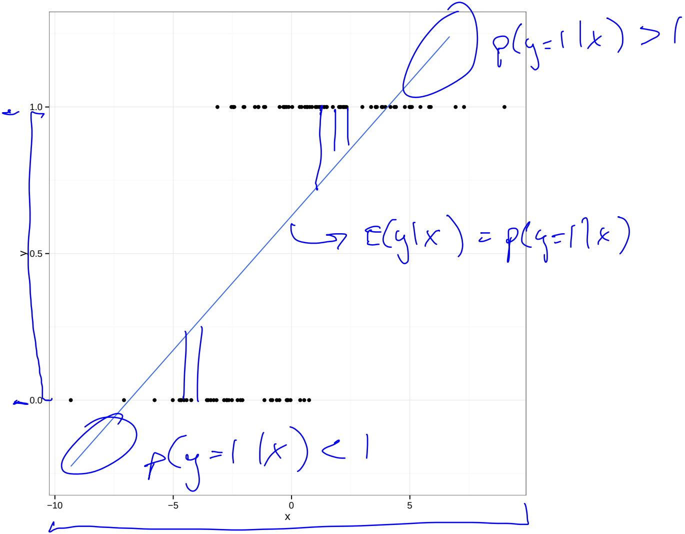
- Regression focuses on $E(y|x) = x\beta$ ~~$+ \epsilon$~~
- For binary outcomes, we want $E(y|x) = p(y = 1|x)$
- Does $p_i = p(y = 1|x) = \beta_0 + \beta_1 x_i$ work?

$\boxed{p_i = x\beta}$ X

$p_i = x\beta$? ?

$y_i = x\beta + \epsilon$

Linear regression for binary outcome



What we need for binary outcomes

- Fitted probabilities should be between 0 and 1
- Use a invertible function $g : (0, 1) \to \underline{(-\infty, \infty)}$ to *link* probabilities to the real line
- Build a model for $\underline{g(p_i)} = \underline{\beta_0 + \beta_1 x_i}$

$$g^{-1}(x\beta) = p_i$$

Link functions

$\Phi^{-1}(p_i)$

- Lots of possible link functions: logit, probit, complimentary log-log
- By far, most common is the logit link:

$$g(p_i) = logit(p_i) = \log \frac{p_i}{1-p_i} = z_i$$

- The inverse link function is also useful:

$$g^{-1}(z_i) = \frac{\exp(z_i)}{1+\exp(z_i)} = p_i$$

Logistic regression

$\{0, 1\}$

Model is now

$$
\begin{aligned}
E(y_i|x_i) &= p_i \\
g(p_i) &= \log \frac{p_i}{1-p_i} = \beta_0 + \beta_1 x_i
\end{aligned}
$$

Using the logit link, we have

$$\mu(y; x, \beta) = p_i = g^{-1}(\beta_0 + \beta_1 x_i) = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}$$

Parameter interpretation

Suppose we can estimate β_0, β_1; what do they mean?

For a binary predictor ...

$\beta_0 = E(y \mid x = 0)$ . . .

Log odds

β_1 = Log OR

Parameter interpretation

For a continuous predictor ...

Parameter estimation

$\min (y - X\beta)^T (y - X\beta)$

$\max \; Like(\beta; y, X)$

- For linear regression, we used least squares and found that this corresponded to ML
- Try using maximum likelihood for logistic regression; need a likelihood ...

ML for logistic regression

$y|x \sim N(\dots)$

- Assume that $[y_i|x_i] \sim Bern(p_i)$
- Density function is $p(y_i) = p_i^{y_i}(1-p_i)^{1-y_i}$
- As before, use that $logit(p_i) = \beta_0 + \beta_1 x_i$
- Likelihood is

$$L(\beta_0, \beta_1; \boldsymbol{y}) = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$$

$$p_i = \frac{e^{x\beta}}{1+e^{x\beta}} \qquad \prod \left(\frac{e^{x\beta}}{1+e^{x\beta}}\right)^{y_i}\left(1 - \underline{\quad}\right)^{1-y_i}$$

ML for logistic regression

Lin $X^T y - X^T X \beta = 0$

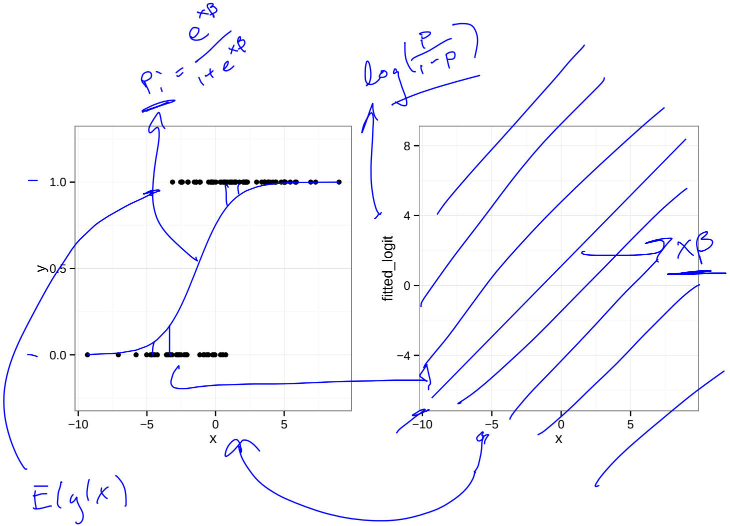
$\hat{\beta} = (X^T X)^{-1} X^T y$

$$\frac{\partial}{\partial \beta} \ell(\beta; y, x) = 0$$

$$\Rightarrow X^T(y - \mu(x, \beta)) = 0$$

- Log likelihood is easier to work with, but it is typically not possible to find a closed-form solution
- Iterative algorithms are used instead (Newton-Raphson, Iteratively Reweighted Least Squares)
- These are implemented for a variety of link functions in *R*

Example



$p_i = \frac{e^{x\beta}}{1+e^{x\beta}}$
$\log\left(\frac{p}{1-p}\right)$
1.0
y
0.5
0.0
−10
−5
0
5
x
8
4
fitted_logit
0
−4
−10
−5
0
5
x
$x\beta$
$E(y|x)$

Code

```
> model = glm(y~x, family = binomial(link = "logit"), data = data)
> summary(model)

Call:
glm(formula = y ~ x, family = binomial(link = "logit"), data = data)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.9360  -0.4631   0.1561   0.5564   1.8131

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   1.1072     0.3357   3.298 0.000974 ***
x             0.8097     0.1664   4.865 1.15e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 129.49  on 99  degrees of freedom
Residual deviance:  73.24  on 98  degrees of freedom
AIC: 77.24

Number of Fisher Scoring iterations: 6
```

Multiple predictors

$\beta_0 + \beta_1 x_i$ / $X\beta$

- Essentially everything that worked for linear models works for logistic models:
  - Multiple predictors of various types
  - Interactions
  - Polynomials
  - Piecewise, splines
  - (Penalization, random effects, Bayesian models)

Testing in Logistic

- In linear models, many of our inferential procedures (ANOVA, F tests, ...) were based on RSS
- For logistic regression (and GLMs), we'll use the asymptotic Normality of MLEs:

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \to \mathrm{N}\left[0, V\right]$$

with $V = (\boldsymbol{X}^T\boldsymbol{W}\boldsymbol{X})^{-1}$ and weight matrix W to construct Wald tests

- Likelihood ratio tests can be used to compare nested models

Wald tests

For individual coefficients

- We can use the test statistic

$$T = \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)}$$

- This is compared to a Normal distribution, trusting that the asymptotics have kicked in
- Recall that coefficients are on the logit scale ...

Confidence intervals

- A confidence interval with coverage $(1-\alpha)$ is given by

$$\beta_j \pm t_{1-\alpha/2,n-p-1}\widehat{se}(\hat{\beta}_j)$$

- To create a confidence interval for the $\exp(\hat{\beta}_j)$, the estimated odds ratio, exponentiate:

$$(\exp(\hat{\beta}_j - 2\widehat{se}(\hat{\beta}_j)), \exp(\hat{\beta}_j + 2\widehat{se}(\hat{\beta}_j)))$$

Wald tests for multiple coefficients

- Define $H_0 : c^T\boldsymbol{\beta} = c^T\boldsymbol{\beta}_0$ or $H_0 : c^T\boldsymbol{\beta} = 0$
- We can use the test statistic

$$T = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}_0}{\widehat{se}(c^T\hat{\boldsymbol{\beta}})} = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}_0}{\sqrt{c^T Var(\hat{\boldsymbol{\beta}})c}}$$

- Useful for some tests, looking at fitted values

Model building

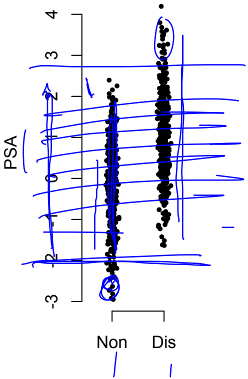
- Can define a model building strategy (at least for nested models) using these
- Other tools, like AIC and BIC, can compare non-nested models

CV??

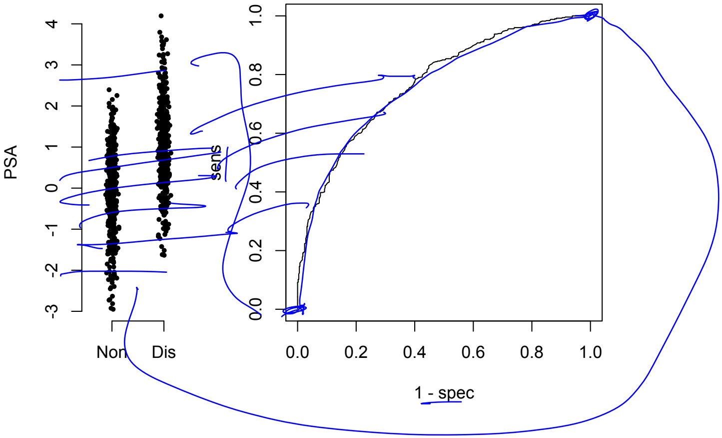
ROC curves

- Forget logistic for a minute
- Suppose you have some test to classifying subjects as diseased or non-diseased
- You can describe that test using sensitivity $P(+|D)$ and specificity $P(-|D')$
- These values depend on what threshold you use for your test

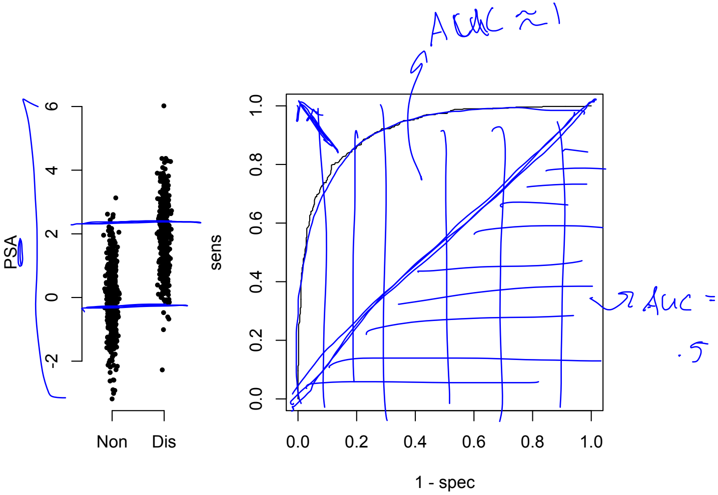
Threshold effect on sens, spec



Threshold effect on sens, spec



Better tests give better ROCs



Summarizing ROCs

- Area under the curve is a useful summary of an ROC
- AUC shouldn't be less than .5; can't be more than 1
- Bigger AUC indicates better classification
- Useful alternative to AIC, BIC, etc

Connection with logistic regression

- Your "test" might be $\hat{\mu}_i = \hat{p}(y_i = 1|\boldsymbol{x}_i)$
- You can model this probability using logistic regression
- Cross-validated ROCs are a way to compare the predictive performance of different models:
    - Based on fitted model (from training set) you construct fitted probabilities $\hat{\mu}_i = \frac{\exp(\boldsymbol{x}_i\boldsymbol{\beta})}{1+\exp(\boldsymbol{x}_i\boldsymbol{\beta})}$ for subjects in the validation set
    - Validation subjects test "positive" or "negative" based on their fitted value; compare to the observed value

Generalizing this approach

Lin / Logist / ......

Suppose instead of binary data, we have

$$y_i \sim \underline{EF(\mu_i, \theta)}$$

where

$$\underline{E(y_i|x_i) = \mu_i}$$

and

$$Var(y_i|x_i) = a(\phi)V(x_i)$$

with known variance function $V(\cdot)$ and dispersion parameter ϕ

Generalized Linear Model

Norm, Bern, ...

Model components are the

- Probability distribution ✓
- Link function
- Linear predictor

$E(y|x)$

$X\beta$

Linear regression as a GLM

$$y_i | x_i \sim \underline{N}(\mu_i, \sigma_\epsilon^2)$$

$$E(y_i | x_i) = \underline{\mu_i}$$

$$g(\mu_i) = \mu_i = \boxed{X\beta}$$

$$y_i | x_i \sim \underline{\text{Bern}}(p_i)$$

$$E(y_i | x_i) = p_i$$

$$g(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \boxed{X\beta}$$

Comparing linear and logistic

- Comparing linear, logistic, and Poisson regression models:

	Linear	Logistic	Poisson
Outcome	Continuous	Binary	Count
Distribution	Normal	Binomial	Poisson
Parameter	$E(Y) = \mu$	$E(Y) = p$	$E(Y) = \lambda$
Range of mean	$-\infty < \mu < \infty$	$0 < p < 1$	$0 < \lambda < \infty$
Variance	σ^2	$p(1-p)$	λ
"Natural" Link	identity	logit	log

$g(\lambda) = \log(\lambda) = X\beta$

Other link functions?

Logistic

$E(y_i|x_i) = p_i$

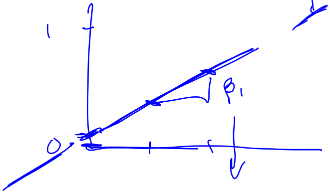
$g(p_i) = p_i = X\beta$

$g(p_i) = \log(p_i)$

$\Rightarrow \beta_1$ is a Rel Risk

$E(y|x) = P(y=1|x) = \beta_0 + \beta_1 x_i$

$E(y|x+1) \quad P(y=1|x) = \beta_0 + \beta_1(x_i+1)$



$\beta_1 = (\beta_0 + \beta_1) - \beta_0$

Risk Differences

Other GLMs

Framework holds for any member of the exponential family

- Probability distribution
- Link function
- Linear predictor

$x\beta$

Exponential family distribution

Any distribution whose density can be expressed as

$$f(y|\theta,\phi) = \exp\left(\frac{y\theta + b(\theta)}{a(\phi)} + c(y,\phi)\right)$$

where $b'(\theta) = \mu$ and $b''(\theta) = V$

- Can take some effort to convert usual density to this form
- Includes Normal, Bern, Poisson, Gamma, Multinomial, ...

Exponential family examples

Normal:

$$
\begin{aligned}
f(y;\mu,\sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{1}{2\sigma^2}(y-\mu)^2\right) \\
&= \exp\left((y\mu - \underbrace{\mu^2/2}_{})/\sigma^2 - \underbrace{\frac{1}{2}(y^2/\sigma^2 + log(2\pi\sigma^2))}_{}\right)
\end{aligned}
$$

$\underbrace{\theta = \mu}_{}$ $\qquad b(\theta) = -\mu^2/2$

$b'(\theta) = \mu$

$g(\mu) = \theta = x\beta$

Exponential family examples

Bernoulli:

$p^y (1-p)^{1-y}$

$$
\begin{aligned}
f(y;p) &= \exp\left(y\log(p) + (1-y)\log(1-p)\right) \\
&= \exp\left(y\underbrace{\log\frac{p}{1-p}}_{\theta} + \underbrace{(-\log(1-p))}_{}\right)
\end{aligned}
$$

$b(\theta)$ $b\left(\log\frac{p}{1-p}\right)$

$b(\theta) = \log(1+e^{\theta})$

$g(p) = \theta = x\beta$

$e^{(yx\beta + \ldots)}$

Today's big ideas

- Logistic regression and GLMs

- Suggested reading: ISLR Ch 4.2 and 4.3