Linear Regression Models
P8111

Lecture 10

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Today's Lecture

- Review of tests
- Two new tests
- Confidence intervals
- Foreshadowing

Resampling

Some review notes

$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i$

$\epsilon \sim (0, \sigma^2)$

$\hat{\beta}$

- Do we have a test for the null $H_0 : \beta_3 = -14$ ✓
- Do we have a test for the null $H_0 : \underline{\beta_2 + \beta_3} = \pi$ ✓
- Do we have a test for the null $H_0 : \underline{\beta_2}\,\underline{\beta_3} = \pi$ ✗

$\dfrac{\hat{\beta}_3 - (-14)}{\hat{se}(\hat{\beta}_3)}$

$c = [0 \;\; 0 \;\; 1 \;\; 1]$

$\dfrac{c\hat{\beta} - \pi}{\hat{se}(c\hat{\beta})}$

Individual coefficients

For individual coefficients $H_0: \beta_j = 0$

- We can use the test statistic

$$T = \frac{\hat{\beta}_j - \beta_j}{\widehat{se}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j}{\sqrt{\hat{\sigma}^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}_{jj}}} \sim t_{n-p-1}$$

- For a two-sided test of size α, we reject if

$$|T| > t_{1-\alpha/2, n-p-1}$$

- The p-value gives $P(|t_{n-p-1}| > |T_{obs}| | H_0)$

Note that t is a symmetric distribution that converges to a Normal as $n - p - 1$ increases.

Inference for linear combinations

$H_0: \beta_2 = \beta_3$ (maybe categorical)

$H_0: \beta_2 - \beta_3 = 0$

$c = [0\ 0\ 1\ -1]$

$c: x_1 - x_2$

Sometimes we are interested in making claims about $c^T\boldsymbol{\beta}$ for some c.

- Define $H_0 : c^T\boldsymbol{\beta} = c^T\boldsymbol{\beta}_0$ or $H_0 : c^T\boldsymbol{\beta} = 0$
- We can use the test statistic

$c = [0\ 1\ 1]$

$$T = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}}{\widehat{se}(c^T\hat{\boldsymbol{\beta}})} = \frac{c^T\hat{\boldsymbol{\beta}} - c^T\boldsymbol{\beta}}{\sqrt{\underbrace{\hat{\sigma}^2 c^T(\boldsymbol{X}^T\boldsymbol{X})^{-1}c}_{c\,\widehat{Var}(\hat{\beta})\,c^T}}}$$

$c \underbrace{\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}_{\widehat{Var}(\hat{\beta})} c^T$

- This test statistic is asymptotically Normally distributed
- For a two-sided test of size α, we reject if

$$|T| > z_{1-\alpha/2}$$

Global F tests

$H_0: \beta_2 = \beta_3 = 0$ / useful for categorical

- Compute the test statistic

$$F_{obs} = \frac{(RSS_S - RSS_L)/(df_S - df_L)}{RSS_L/df_L}$$

- If H_0 (the null model) is true, then $F_{obs} \sim F_{df_S - df_L, df_L}$
- Note $df_s = n - p_S - 1$ and $df_L = n - p_L - 1$
- We reject the null hypothesis if the p-value is above α, where

$$\text{p-value} = P(F_{df_S - df_L, df_L} > F_{obs})$$

Alternative global tests: the Wald test

$$\left(\frac{\hat{\beta}_j - \beta_0}{\hat{se}(\hat{\beta}_j)}\right)^2 = t^2$$

For a vector of coefficients, we can test $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$:

- Use the test statistic

$$W = (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T [\hat{Var}(\hat{\boldsymbol{\beta}})]^{-1} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$$

- Under the null, this test statistic has an asymptotic χ^2_p distribution
- In practice, we replace $Var(\hat{\boldsymbol{\beta}})$ with $\widehat{Var}(\hat{\boldsymbol{\beta}})$ and use an F distribution

Alternative global tests: the Wald test

$9 \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \\ & & & & 0 \\ & & & & & 0 \end{bmatrix}$ $R\beta = \begin{bmatrix} 0 \\ 0 \\ \beta_2 \\ \beta_3 \\ 0 \\ 0 \end{bmatrix}$

The previous test is special case of $H_0 : R\boldsymbol{\beta} = R\boldsymbol{\beta}_0$ for a $d \times p$ matrix R, using $R = I_{p\times p}$:

- Use the test statistic

  $R\, Var(\hat{\beta})\, R^T$

$$W = (R\hat{\boldsymbol{\beta}} - R\boldsymbol{\beta}_0)^T [Var(R\hat{\boldsymbol{\beta}})]^{-1} (R\hat{\boldsymbol{\beta}} - R\boldsymbol{\beta}_0)$$

- Under the null, this test statistic has an asymptotic χ^2_d distribution, where

$$d = \text{rank}(Var(R\hat{\boldsymbol{\beta}}))$$

- This formulation is useful for testing subsets (e.g. $H_0 : \beta_1 = \beta_2 = 0$)

Alternative global tests: the likelihood ratio test

$H_0: \beta_2 = \beta_3 = 0$

If we are using maximum likelihood estimation (we'll cover this soon – turns out to be least squares in MLR), we can use a LRT:

- Use the test statistics

Likhood for null

for alt

$$\Delta = -2 \log \frac{L_0}{L_1} = -2(l_0 - l_1)$$

- This test statistic has an asymptotic χ^2_d distribution where d is the difference in the number of parameters between the two models.
- Must compare nest models

Example: LRT

$H_0: \beta_1 = \beta_4 = 0$

```
> linmod = lm(runs ~ at_bats + hits + homeruns + stolen_bases, data = mlb11)
> linmod.null1 = lm(runs ~ hits + homeruns, data = mlb11)
> anova(linmod.null1, linmod)
Model 1: runs ~ hits + homeruns
Model 2: runs ~ at_bats + hits + homeruns + stolen_bases
  Res.Df   RSS Df Sum of Sq      F   Pr(>F)
1     27 27128
2     25 18020  2    9107.8 6.3178 0.006015 **
>
> ## LRT
> Delta = -2*(logLik(linmod.null1) - logLik(linmod))
> 1-pchisq(Delta, 2)
'log Lik.' 0.002163305 (df=4)
```

Confidence intervals: individual parameters

$H_0: \beta_j = \beta_0 \Rightarrow \frac{\hat{\beta}_j - \beta_0}{\hat{se}(\hat{\beta}_j)}$

- A confidence interval with coverage $(1 - \alpha)$ is given by

$$\hat{\beta}_j \pm t_{1-\alpha/2,n-p-1}\widehat{se}(\hat{\beta}_j)$$

$\left(\hat{\beta}_j \pm 2\hat{se}(\hat{\beta}) \right)$

- Assuming all the standard assumptions hold,

$$(1 - \alpha) \text{ ''='' } P(LB < \beta_j < UB)$$

$.95$ $0 / 1$

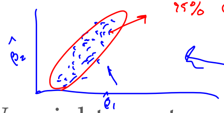
Note there is a one-to-one correspondence between this confidence interval and the hypothesis test.

$\frac{\hat{\beta}_j - \beta_0}{\hat{se}(\hat{\beta})} < t_{\alpha,df} \Rightarrow \beta_0 \in \left(\hat{\beta}_j \pm t_{\alpha,df}\,\hat{se}(\hat{\beta})\right)$

Example revisited

```
> confint(linmod)
                    2.5 %       97.5 %
(Intercept)  -502.9429878 1.665365e+03
at_bats        -0.4440385 3.938287e-02
hits            0.4643923 9.304364e-01
homeruns        0.9253836 1.581629e+00
stolen_bases    0.1756711 8.702772e-01
```

Confidence intervals: multiple parameters



95% CR for (β_1, β_2)

$\forall \beta_0 \in CI$, fail to reject

$\left(\hat{\beta}_j \pm 2\,\hat{se}(\hat{\beta}_j) \right)$ inverting [test]

We might want a confidence region for multiple coefficients simultaneously

CR(β_1, β_2)

- Invert Wald test for multiple coefficients – find region containing all values β_0 for which p-value from global Wald test is $> \alpha$

$$\forall \beta_0, \quad (\hat{\beta} - \beta_0)^T \left(\text{Var}(\hat{\beta})\right)^{-1} (\hat{\beta} - \beta_0) \le \text{Critical Value}$$

- Then

$$(1-\alpha) \text{ "=" } P[\boldsymbol{\beta} \in \text{region}]$$

- This region is an ellipsoid in higher dimensions; we can visualize in 2D most easily and 3D pretty well.

Confidence intervals: multiple parameters

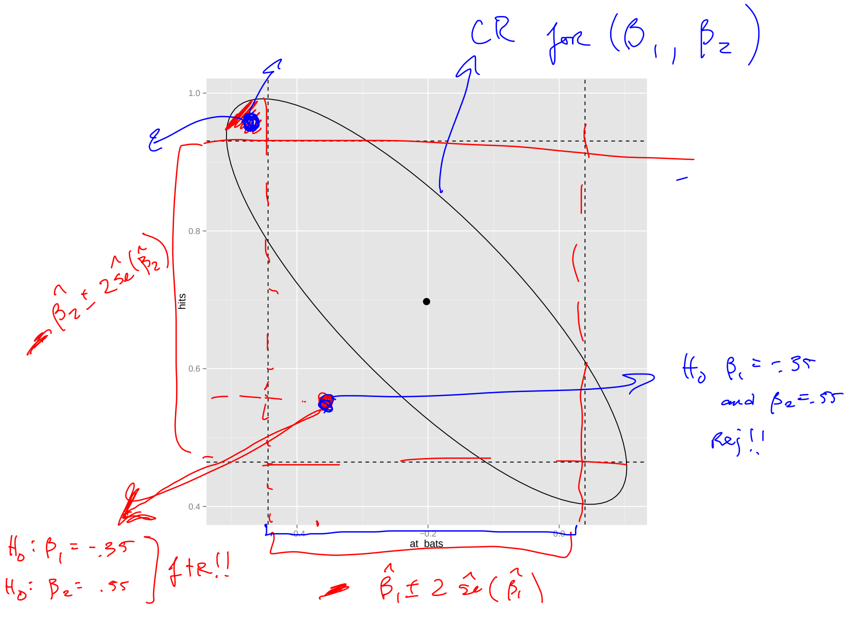
X, y

```
library(ellipse)

CI.ellipse = as.data.frame(ellipse(linmod,c(2,3)))
est = as.data.frame(t(as.matrix(coef(linmod)[2:3])))

## plot the joint confidence region
ggplot(CI.ellipse, aes(x = at_bats, y = hits)) + geom_path() +
  geom_hline(yintercept = confint(linmod)[3,], linetype = 2) +
  geom_vline(xintercept = confint(linmod)[2,], linetype = 2) +
  geom_point(data = est, size = 4)
```

Confidence intervals: multiple parameters



Expected value and it's variance

σ^2

$Var(y|x) + Var(\text{fitted})$

σ^2

- What is $\hat{E}(y|\boldsymbol{x} = \boldsymbol{x}_0)$? $\quad \hat{x}_0 \hat{\beta}$
- How can we estimate the variance of $\hat{E}(y|\boldsymbol{x} = \boldsymbol{x}_0)$?

$Var(x_0 \hat{\beta}) = \sigma^2 x_0^T (X^T X)^{-1} x$

In particular, a confidence interval for $E(y|\boldsymbol{x} = \boldsymbol{x}_0)$ is given by

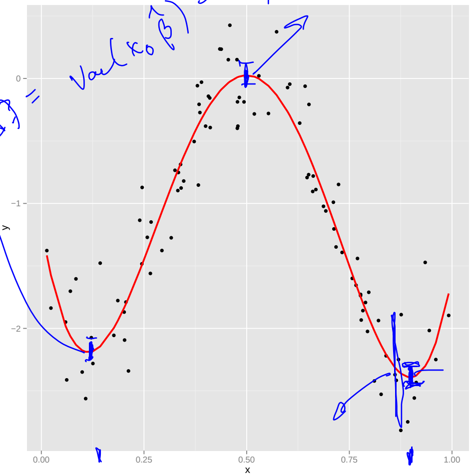
$$(\hat{y}|\boldsymbol{x} = \boldsymbol{x}_0) \pm t_{1-\alpha/2, n-p-1} \widehat{se}_{fit}(\hat{y}|\boldsymbol{x}_0)$$

This can be estimated for any $\boldsymbol{x}_0$.

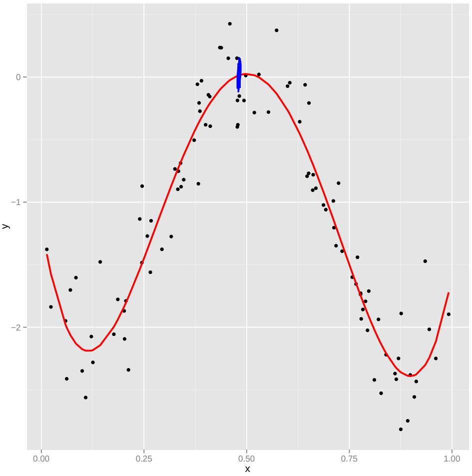
$\left[\sigma^2 \left(1 + \underline{x_0^T (X^T X)^{-1} x_0}\right)\right.$

Estimated mean

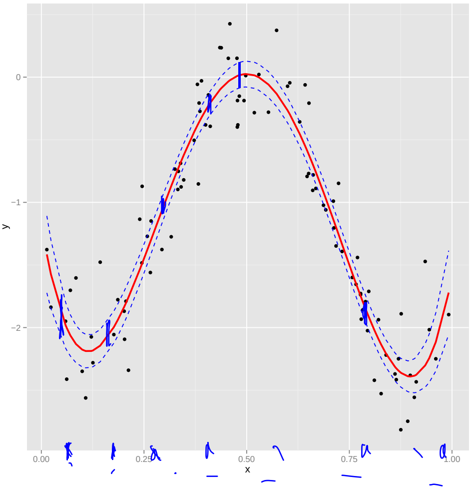
$\text{Var}(\text{fitted}) = \text{Var}(x_0 \hat{\beta}) = x_0^T \left(\sigma^2 (X^T X)^{-1}\right) x_0$



Estimated mean and variance



Estimated mean and variance



Pointwise and simultaneous CIs

- Pointwise confidence intervals construct CI's at each point independently of all other points
- Implicit multiple comparisons problem
- Simultaneous intervals can be constructed so that

$$(1-\alpha) \text{ "="} \; P(f(x) \in SCI)$$

Which is wider?

Predictions and prediction intervals

$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$

$\epsilon_i \sim (0, \sigma^2)$

$\mathrm{Var}(\hat{y}) = x_0^T (\sigma^2 (X^T X)^{-1}) x_0$

- What is the prediction value y for a given $\boldsymbol{x}_0$
- What range would you give for the value of a new outcome?
- Two sources of variance to consider: variance in estimates and variance in outcome

$\sigma^2 x_0^T (X^T X)^{-1} x_0$ $\qquad \sigma^2$

$\mathrm{Var}(y) = \mathrm{Var}(E(y|x)) + E(\mathrm{Var}(y|x))$

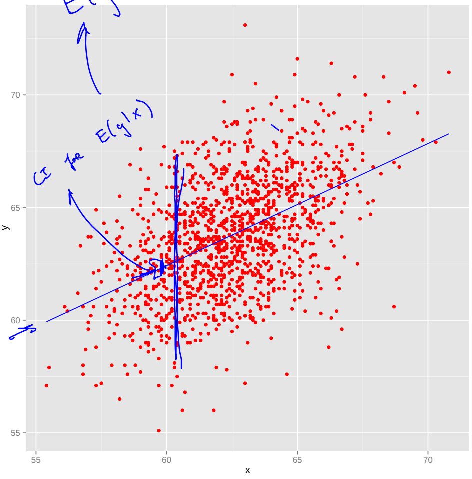
A prediction interval is given by

$$(\hat{y}|\boldsymbol{x} = \boldsymbol{x}_0) \pm t_{1-\alpha/2,n-p-1}\widehat{se}_{pred}(\hat{y}|\boldsymbol{x}_0)$$

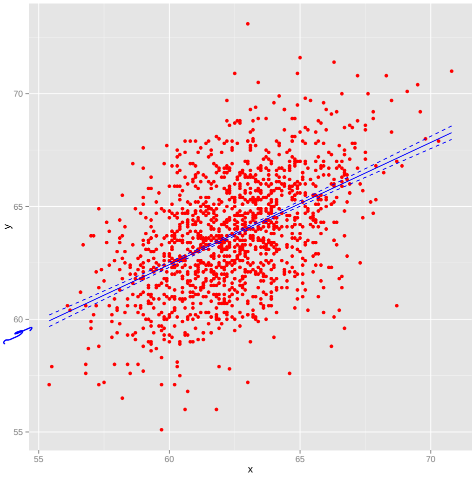
This can be estimated for any $\boldsymbol{x}_0$.

Predictions - mother/daughter height

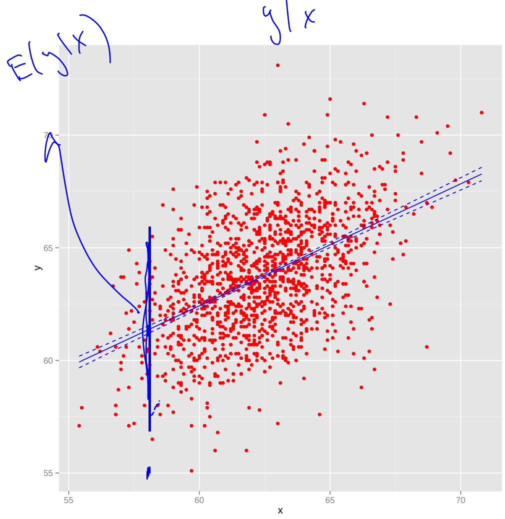
$\hat{E}(y|x) \pm 2\,se$



Variance of fitted values



Prediction interval



Some things to think on

- understand Association
- prediction

- Why are we building models?
- How should we assess models?
- What kinds of predictors should we included, and how should we decide to include them?

Some things to think on

Three general goals are

- Prediction
- Estimation of association
- Testing of associations

These goals will often not lead to the same final model.

Today's big ideas

- Inference for MLRs: global tests; confidence intervals for coefficients, predictions, functions

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- Suggested reading: Faraway Ch 3.6 - 3.9