Linear Regression Models
P8111

Lecture 22

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Today's Lecture

- Random intercept models
- Random slope
- Example (pig data!)
- Example (CD4 data!)

Recall the setting

- We observe data $y_{ij}, \boldsymbol{x}_{ij}$ for subjects $i = 1, \ldots I$ at visits $j = 1, \ldots, J_i$
- Overall, we pose the model

marginal

$$\boxed{\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}}$$

where $\text{Var}(\boldsymbol{\epsilon}) = \sigma^2 V$ and

$$V = \begin{bmatrix} V_1 & 0 & \ldots & 0 \\ 0 & V_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & V_I \end{bmatrix}$$

Recall the setting

$Cov(y_{ij}, y_{ij'}) \neq 0$

- We've focused on random intercept models and (equivalently) uniform correlation marginal models
- Today we'll review random intercept approaches and introduce random slope models

Random intercept model

A random intercept model with one covariate is given by

$$y_{ij} = \beta_0 + b_i + \beta_1 x_{ij} + \epsilon_{ij}$$

where

- $b_i \sim \mathrm{N}\left[0, \tau^2\right]$
- $\epsilon_{ij} \sim \mathrm{N}\left[0, \nu^2\right]$

$$E_b\left(E(y_{ij} \mid b_i)\right) = E_b\left(\beta_0 + b_i + \beta_1 x_{ij}\right)$$

$$= \beta_0 + \beta_1 x_{ij}$$

$$= E(y_{ij} \mid b_i = 0)$$

Random intercept model

More compactly, we write

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{b} + \boldsymbol{\epsilon}$$

where

- $\boldsymbol{b} \sim \mathrm{N}\left[0, \tau^2 I_I\right]$
- $\boldsymbol{\epsilon} \sim \mathrm{N}\left[0, \nu^2 I_n\right]$

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{IJ_I} \end{bmatrix} = X\beta + \underbrace{\begin{bmatrix} \vdots & 0 & 0 \\ 0 & \vdots & 0 \\ & \ddots & 0 \\ & & \vdots \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_I \end{bmatrix}}_{b} + \epsilon$$

Random intercept model

In the model

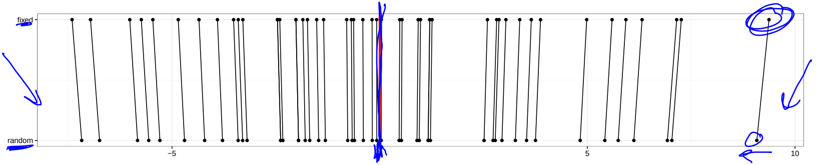
$$y = X\beta + Zb + \epsilon$$

we've discussed why we use random effects rather than fixed effects:

- Random effects induce correlation; fixed effects don't
- This reduces the number of parameters we estimate ✓
- Random effect modeling is similar mathematically to introducing penalization

$b_i | y_{ij}$

BLUP "=" $\hat{\beta}_{Ridge}$ $\quad b_i \sim N(0, \sigma^2)$



Estimation – random intercept model

Estimation is done using MLE with model and distributional assumptions

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zb} + \boldsymbol{\epsilon}$$

where

- $\boldsymbol{b} \sim \mathrm{N}\left[0, \tau^2 I_I\right]$
- $\boldsymbol{\epsilon} \sim \mathrm{N}\left[0, \nu^2 I_n\right]$

Remember that BLUPs from this model can be derived without distributional assumptions (similarly to OLS and BLUEs).

Estimation – BLUPs

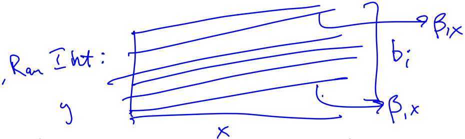
Our estimate for fixed and random effects are

$$\hat{\gamma} = \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{b}} \end{bmatrix} = \left(\boldsymbol{C}^T \boldsymbol{C} + \frac{\nu^2}{\tau^2} R \right)^{-1} \boldsymbol{C}^T \boldsymbol{y}$$

where $\boldsymbol{C} = [\boldsymbol{X}\ \boldsymbol{Z}]$ and

$$R = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \dots & 0 & 0 & & 1 \end{bmatrix}$$

Random slope model



A random slope model with one covariate is given by

$$y_{ij} = \beta_0 + b_{i,0} + \beta_1 x_{ij} + b_{i,1} x_{ij} + \epsilon_{ij}$$

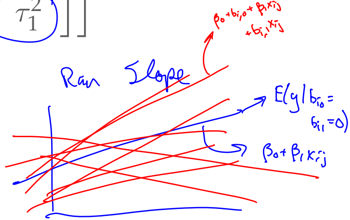
where

$$\begin{bmatrix} b_{i,0} \\ b_{i,1} \end{bmatrix} \sim \mathrm{N} \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{10} & \tau_1^2 \end{bmatrix} \right]$$

and

$$\epsilon_{ij} \sim \mathrm{N}\left[0, \nu^2\right]$$

$$E(y_{ij} \mid b_{i0}, b_{i1}) = \underbrace{(\beta_0 + b_{i,0})} + \underbrace{(\beta_1 + b_{i,1})} x_{ij}$$



Random slope model

Using vectors, we can write

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}_0\boldsymbol{b}_0 + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{\epsilon}$$

$$y = X\beta + \overbrace{Z_0 b_0}^{\text{Int}} + \begin{bmatrix} x_{11} & 0 \\ x_{12} & 0 \\ x_{13} & 0 \\ 0 & x_{21} \\ 0 & x_{22} \\ & \vdots \end{bmatrix} \begin{bmatrix} b_{11} \\ \vdots \\ b_{I1} \end{bmatrix}$$

Estimation – random slope model

Omitting the details –

- Again use MLE to set up approach and derive BLUPs (which don't depend on distributional assumptions)
- This is easier if one assumes $\tau_{01} = 0$, and usually the results aren't affected much
- The estimates look similar to the BLUPs for one random intercept, although there are more "R"s to deal with
- Results again resemble ridge regression estimates, although with more than one penalty

Random effect models

Our random slope model with one covariate given by

$$y_{ij} = \beta_0 + b_{i,0} + \beta_1 x_{ij} + b_{i,1} x_{ij} + \epsilon_{ij}$$

$E_b(E_{y|b}(y|b))$

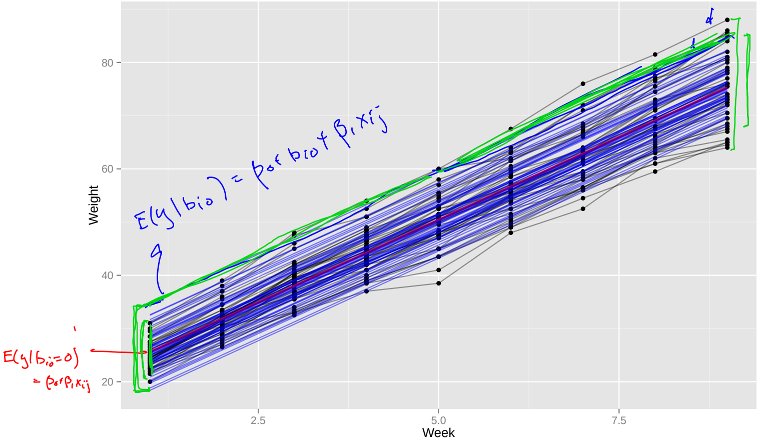
has the following properties

- $E(\boldsymbol{y}) = \underline{\beta_0 + \beta_1 x_{ij}}$ $= E(y \mid \underline{b_{i0} = 0, b_{i1} = 0})$
- $E(\boldsymbol{y}|b_{i,0}, b_{i,1}) = \underline{(\beta_0 + b_{i,0})} + \underline{(\beta_1 + b_{i,1})}x_{ij}$

So main effect parameters are interpreted as the effect for *an average* subject; the interpretation for *a particular* subject is conditional on the random effects.

Pig data

Random intercept model fit for pig data



Pig data

Random intercept model code

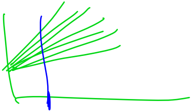
E(y_ij | b_i) = ...

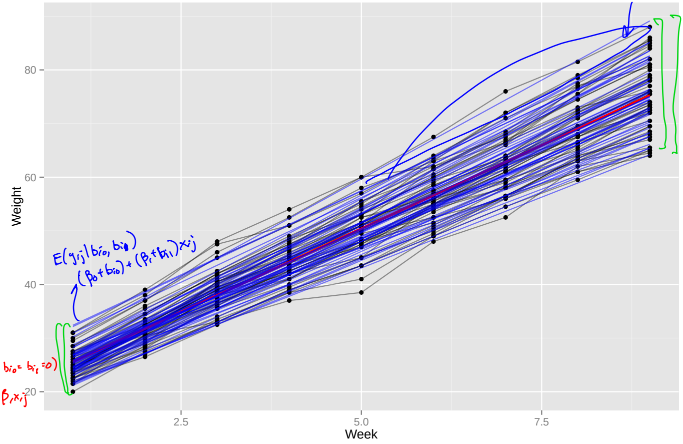
```
> library(lme4)
> ranef.mod = lmer(weight ~ (1 | id.num) + num.weeks, data = pig.weights)
> summary(ranef.mod)
Linear mixed model fit by REML
Formula: weight ~ (1 | id.num) + num.weeks
   Data: pig.weights
  AIC  BIC logLik deviance REMLdev
 2042 2058  -1017     2030    2034
Random effects:
 Groups   Name        Variance Std.Dev.
 id.num   (Intercept) 15.1418  3.8913
 Residual              4.3947  2.0964
Number of obs: 432, groups: id.num, 48

Fixed effects:
            Estimate Std. Error t value
(Intercept) 19.35561    0.60311   32.09
num.weeks    6.20990    0.03906  158.97

>   (15.1418) / (15.1418 + 4.3947)
[1] 0.7750518
```

Pig data

int +

Random slope model fit for pig data





$E(y_{ij} \mid b_{i0}, b_{i1})$
$(\beta_0 + b_{i0}) + (\beta_1 + b_{i1}) x_{ij}$

$E(y_{ij} \mid b_{i0} = b_{i1} = 0)$
$\beta_0 + \beta_1 x_{ij}$

Pig data

Random slope model code

```
>   ranef.mod = lmer(weight ~ (1 + num.weeks | id.num) + num.weeks, data = pig.weights)
>   summary(ranef.mod)
Linear mixed model fit by REML
Formula: weight ~ (1 + num.weeks | id.num) + num.weeks
   Data: pig.weights
  AIC  BIC logLik deviance REMLdev
 1753 1777 -870.4     1738    1741
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 id.num   (Intercept) 6.9865   2.64319
          num.weeks   0.3800   0.61644  -0.063
 Residual             1.5968   1.26366
Number of obs: 432, groups: id.num, 48

Fixed effects:
            Estimate Std. Error t value
(Intercept) 19.35561    0.40387   47.93
num.weeks    6.20990    0.09204   67.47
```

Hypothesis test

$$y_{ij} = \beta_0 + b_{i0} + \beta_1 x_{ij} + b_{i1} x_{ij} + \epsilon_{ij}$$

vs

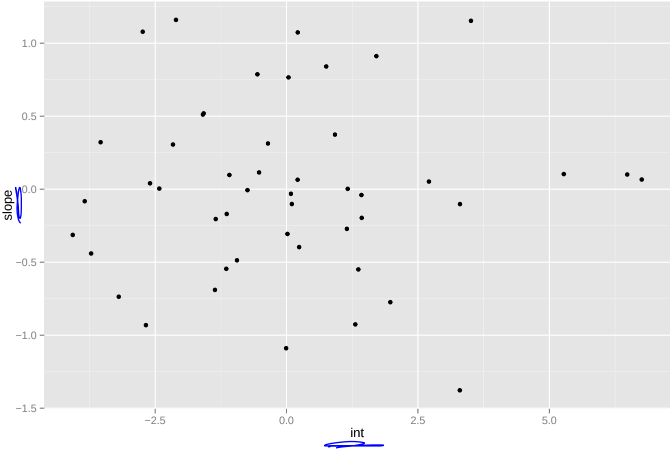
$$y_{ij} = \beta_0 + b_{i0} + \beta_1 x_{ij} + \epsilon_{ij}$$

~~$H_0: b_{11} = b_{21} = \dots = b_{I1} = 0$~~

$H_0: \tau_{11}^2 = 0$

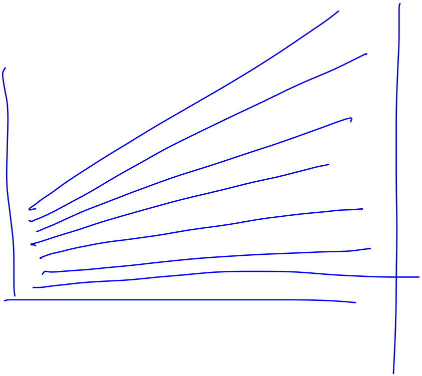
Pig data

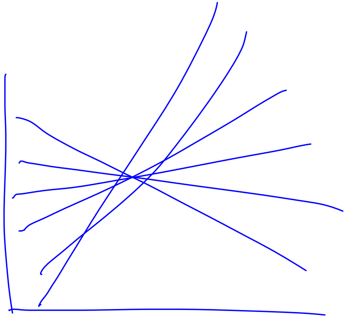
Random intercept against random slope



Quick questions

What data would lead to the random intercept and random slope having high positive correlation? High negative correlation?

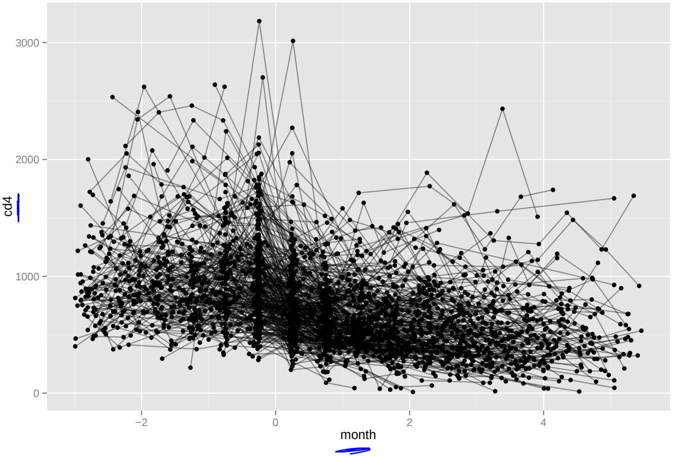




Pig data summary

- Overall, the random slope model provides a pretty good fit
- Lowest AIC of all models considered (linear model not shown, but trust me)
- Visual inspection of data indicates a good fit
- Easy to interpret

CD4 data



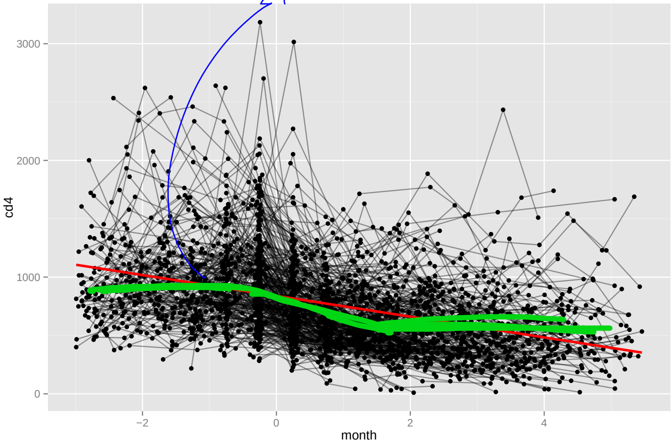
CD4 data

SLR model code

```
>   lin.mod = lm(cd4 ~ month, data = cd4)
>   summary(lin.mod)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  839.398      8.147  103.03   <2e-16 ***
month        -89.027      3.965  -22.45   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 362.9 on 2369 degrees of freedom
Multiple R-squared: 0.1754,Adjusted R-squared: 0.1751
>   AIC(lin.mod)
[1] 34682.64
```

CD4 data

SLR model fit for CD4 data PWL, Poly, Bsplines



CD4 data

B spline model code

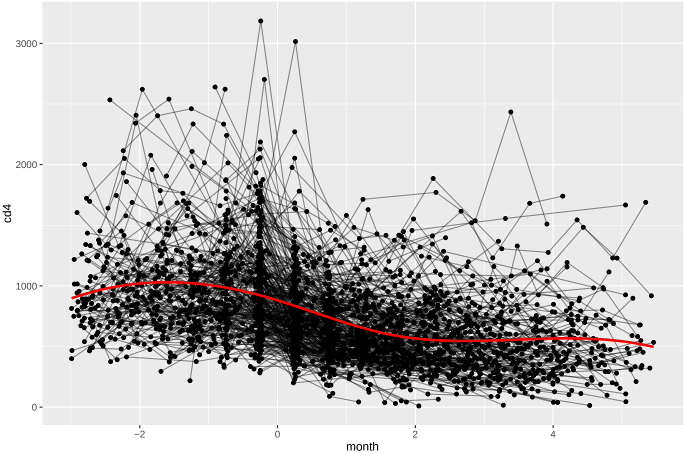
```
> bs.mod = lm(cd4 ~ bs(month, 5), data = cd4)
> summary(bs.mod)

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)      898.19      57.65  15.580  < 2e-16 ***
bs(month, 5)1    181.75     101.47   1.791  0.07340 .
bs(month, 5)2    154.21      57.27   2.693  0.00713 **
bs(month, 5)3   -544.31      84.28  -6.459 1.28e-10 ***
bs(month, 5)4   -230.25      80.16  -2.873  0.00411 **
bs(month, 5)5   -400.92      98.69  -4.063 5.01e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 356.2 on 2365 degrees of freedom
Multiple R-squared:  0.2068,Adjusted R-squared:  0.2051
> AIC(bs.mod)
[1] 34598.69
```

not LDA

CD4 data

Spline!

~~Polynomial~~ model fit for CD4 data



CD4 data

Random intercept model code

```
>   ranint.mod = lmer(cd4 ~ (1 | ID) + month, data = cd4)
>   summary(ranint.mod)
Random effects:
 Groups   Name        Variance Std.Dev.
 ID       (Intercept) 64982    254.92
 Residual             66532    257.94
Number of obs: 2371, groups: ID, 364

Fixed effects:
            Estimate Std. Error t value
(Intercept)  838.925     14.724   56.98
month        -99.703      3.449  -28.91
>   AIC(ranint.mod)
[1] 33755.67
```

$ICC = \frac{\tau^2}{\tau^2 + \nu^2} \approx .5$

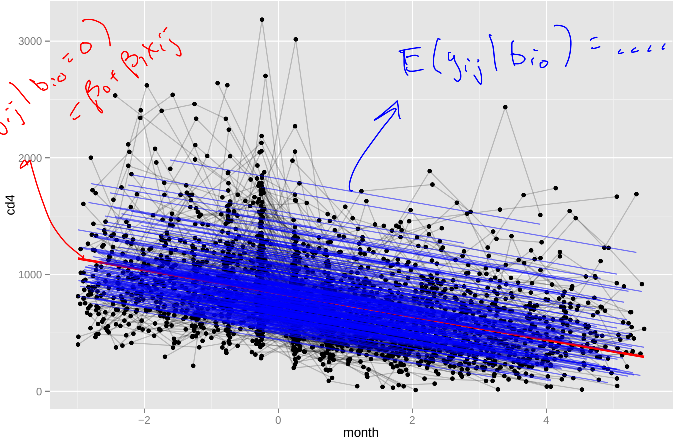
CD4 data

Random intercept fit for CD4 data

$$E(y_{ij} \mid b_{i0}, b_{i1}) = \beta_0 + b_{i0} + \beta_1 x_{ij} + b_{i1} x_{ij}$$

$$E(y_{ij} \mid b_{i0} = 0) = \beta_0 + \beta_1 x_{ij}$$

$$E(y_{ij} \mid b_{i0}) = \ldots$$



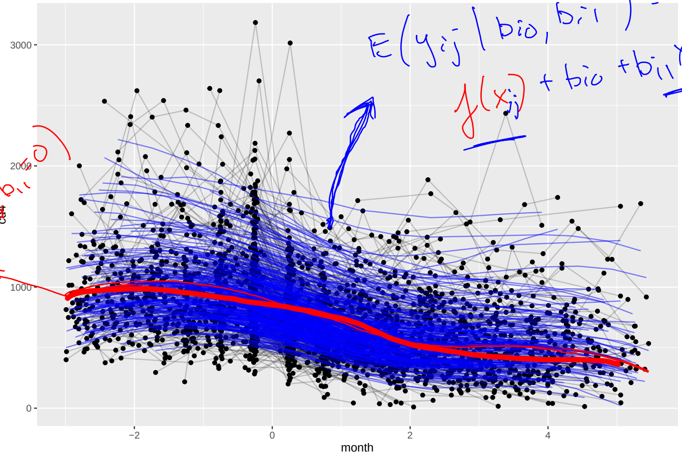
CD4 data

Random intercept, slope + B spline model code

```
> ranbs.mod = lmer(cd4 ~ (1 + month | ID) + bs(month, 5), data = cd4)
> summary(ranbs.mod)
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 ID       (Intercept) 71333    267.08
          month        4851     69.65   -0.43
 Residual             50484    224.69
Number of obs: 2371, groups:  ID, 364

Fixed effects:
              Estimate Std. Error t value
(Intercept)     910.11      50.09  18.169
bs(month, 5)1   157.54      73.43   2.145
bs(month, 5)2   164.87      47.42   3.477
bs(month, 5)3  -588.42      64.43  -9.133
bs(month, 5)4  -264.07      65.57  -4.027
bs(month, 5)5  -609.54      82.14  -7.421
> AIC(ranbs.mod)
[1] 33428.49
```

CD4 data

Random intercept, slope + B spline fit for CD4 data



$E(y_{ij} \mid b_{i0}, b_{i1}) = f(x_{ij}) + b_{i0} + b_{i1} x_{ij}$

$E(y_{ij} \mid b_{i0} = b_{i1} = 0) = f(x)$

CD4 data summary

Which model do you prefer?

Today's big ideas

- Random slope models
- Pig data analysis
- CD4 data analysis