Linear Regression Models
P8111

Lecture 06

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Today's lecture

- Multiple Linear Regression
  - Assumptions
  - Interpretation
  - Some models
    - categorical
    - interactions

Motivation

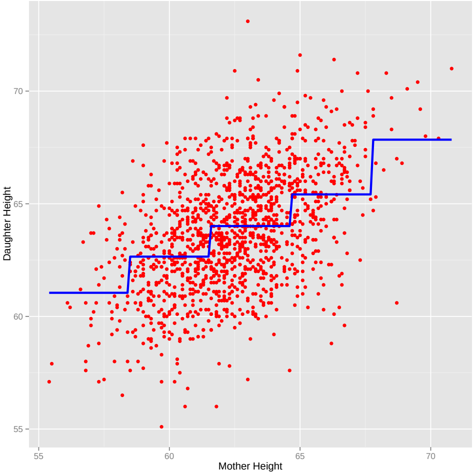
Most applications involve more that one covariate – if more than one thing can influence an outcome, you need multiple linear regression.

- Improved description of $y|\boldsymbol{x}$ $E(y|x)$
- More accurate estimates and predictions
- Allow testing of multiple effects
- Includes multiple predictor types

Why not bin all predictors?

- Divide x_i into k_i bins
- Stratify data based on inclusion in bins across x's
- Find mean of the y_i in each category
- Possibly a reasonable non-parametric model

Why not bin all predictors?



Why not bin all predictors?

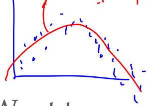
- More predictors = more bins
- If each x has 5 bins, you have 5^p overall categories
- May not have enough data to estimate distribution in each category
- Curse of dimensionality is a problem in a lot of non-parametric statistics

Multiple linear regression model

$E(y|x) = f(x; \beta)$

$y = X\beta + \epsilon$

$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \epsilon_i$



- Observe data $(y_i, x_{i1}, \ldots, x_{ip})$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1, \ldots, \beta_p$ in the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \epsilon_i; \ \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

- Assumptions (residuals have mean zero, constant variance, are independent) are as in SLR
- Impose linearity which (as in the SLR) is a big assumption
- Our primary interest will be $E(y|\boldsymbol{x})$
- Eventually estimate model parameters using least squares

$\hat{\beta}$??

Predictor types

- Continuous
- Categorical
- Ordinal

Interpretation of coefficients

$\beta_0 = E(y|x_1 = 0, \ldots, x = 0)$

$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + \epsilon_i$

- Centering some of the x's may make this more interpretable

Interpretation of coefficients

$$\boxed{\beta_1 =} \cancel{\underline{(\beta_0 + \beta_1)} - \underline{(\beta_0)}}$$

$$= (\beta_0 + \beta_1 + 17\beta_2 + 4.3\beta_3 \ldots) - (\beta_0 + \beta_2 17 + \beta_3 4.3 \ldots)$$

$$E(y \mid x_1 = 1, x_2 = 17 \ldots) - E(y \mid x_1 = 0, x_2 = 17)$$

β_1 = the diff in $E(y)$ for a 1-unit Δx_1

<u>Keeping everything else fixed!</u>

Example with two predictors

$E(y \mid x_1 = 10, \; x_2 = 0)$

$x_2 = 1)$

Suppose we want to regress weight on age and sex.

- Model is $y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \epsilon_i$
- Age is continuous starting with age 0; sex is binary, coded so that $x_{i,sex} = 0$ for men and $x_{i,sex} = 1$ for women
  - In your dataset, sex should be a `factor` variable ... ?

Example with two predictors

$\beta_1 =$ change in E(y) for a 1 unit Δ age,

keeping sex fixed.

$\beta_2 =$ " " "

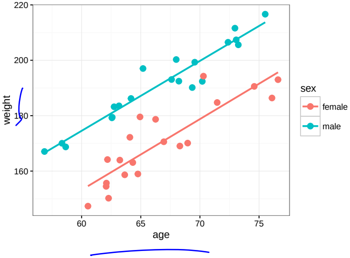
comparing females to males.

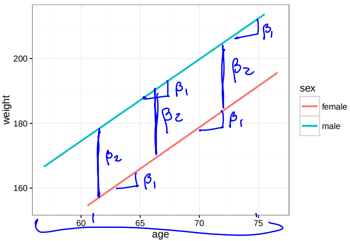
keeping age fixed

Example with two predictors

$$E(y|\underset{\sim}{x}) = \beta_0 + \beta_1 x_{age} + \beta_2 x_{sex}$$

$$\beta_0 = E(y \mid age = 0, sex = male)$$





Example: MLR

```
> summary(data.mlr)
      age            sex          weight
 Min.   :56.86   female:20   Min.   :147.3
 1st Qu.:62.71   male  :20   1st Qu.:168.3
 Median :65.72               Median :181.4
 Mean   :66.70               Mean   :180.9
 3rd Qu.:70.23               3rd Qu.:193.0
 Max.   :76.60               Max.   :216.6
```

Example: MLR

```
> linmod = lm(weight ~ age + sex, data = data.mlr)
> tidy(linmod)
         term  estimate  std.error   statistic      p.value
1 (Intercept)  1.060460 12.2596341  0.08650011 9.315353e-01
2         age  2.537768  0.1828033 13.88250888 3.022310e-16
3     sexmale 21.116049  1.8470731 11.43216717 1.056992e-13
```

$\hat{\beta}_0$

$\hat{\beta}_1$

$\hat{\beta}_2$

Example: MLR

```
> summary(linmod)

Call:
lm(formula = weight ~ age + sex, data = data.mlr)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.8987 -3.2152 -0.2969  2.3688 14.8074 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   1.0605    12.2596   0.087    0.932    
age           2.5378     0.1828  13.883 3.02e-16 ***
sexmale      21.1160     1.8471  11.432 1.06e-13 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5.841 on 37 degrees of freedom
Multiple R-squared:  0.8977,Adjusted R-squared:  0.8921
F-statistic: 162.3 on 2 and 37 DF,  p-value: < 2.2e-16
```

$n = 40$

Example: MLR

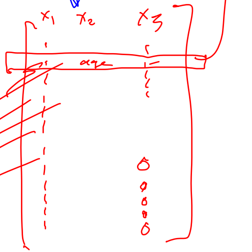
$x_0 = 1$

$$y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \epsilon_i$$

$$y = X\beta + \epsilon$$

```
> head(data.mlr)
Source: local data frame [6 x 3]

       age     sex   weight
     (dbl)  (fctr)    (dbl)
1 62.58799   male 179.4342
2 65.18893   male 197.0306
3 73.06852   male 207.3838
4 56.85860   male 167.0692
5 69.56368   male 199.3080
6 67.99770   male 200.2703
> model.matrix(linmod) %>% head
  (Intercept)      age sexmale
1           1 62.58799       1
2           1 65.18893       1
3           1 73.06852       1
4           1 56.85860       1
5           1 69.56368       1
6           1 67.99770       1
```



Example: MLR

```
> tail(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 64.75572 female 158.9645
2 63.64315 female 158.6567
3 64.08004 female 172.2003
4 64.32532 female 163.0857
5 68.96513 female 170.1063
6 64.93602 female 179.5558
> model.matrix(linmod) %>% tail
   (Intercept)      age sexmale
35           1 64.75572       0
36           1 63.64315       0
37           1 64.08004       0
38           1 64.32532       0
39           1 68.96513       0
40           1 64.93602       0
```

Omitted variable bias

$$y_i = \beta_0 + \beta_1 x_{1i} + \cancel{\beta_2 x_{i2}} + \epsilon_i$$

What happens if we ignore x_2 and fit the simple linear regression:

$$y_i = \beta_0^* + \beta_1^* x_{i,1} + \epsilon_i^*$$

Does $\beta_1^* = \beta_1$? Does "total" association equal "partial" association?

Omitted variable bias

HW2

Omitted variable bias

There are two conditions under which $E(\hat{\beta}_1^*) = \beta_1$:

- The omitted variable is unrelated to the outcome
- The omitted variable is uncorrelated with the retained variable

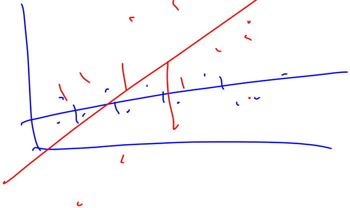
Still only two predictors

x_1 y_i

Suppose we think that the effect of age on weight is different for men and women. How might we approach this problem?

sex $= x_2$

- Separate models?
- Interactions?



$$y_i = \underline{\beta_0} + \underline{\beta_1} x_{i\,age} + \underline{\beta_2} x_{i\,sex} + \underline{\beta_3} x_{i\,age} \cdot x_{i\,sex} + \epsilon_i$$

Interpretation of coefficients

$x_{i\,sex} = 0$

$\downarrow$ $y_i = \beta_0 + \beta_1 x_{i\,age} + \epsilon_i$

$\beta_0 = E(y \mid age = 0, \text{men})$

$\beta_1 = \Delta E(y)$ for a 1 unit Δ age for men

$x_{i\,sex} = 1$

$* \ \beta_0 + \beta_1 x_{i\,age} + \beta_2 + \beta_3 x_{i\,age}$

$y_i = \underbrace{(\beta_0 + \beta_2)} + \underbrace{(\beta_1 + \beta_3)} x_{i\,age} + \epsilon_i$

$\beta_0 + \beta_2$ = Int for women

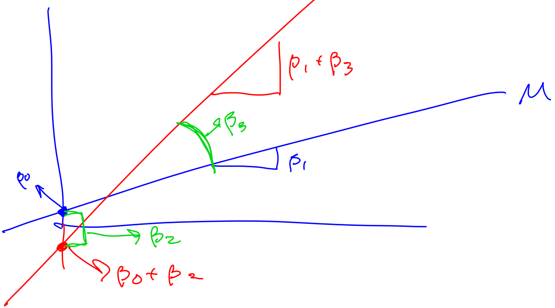
$\beta_1 + \beta_3$ = Age slope for women

$\beta_2 = (\beta_0 + \beta_2) - (\beta_0)$

$\beta_3 = (\beta_1 + \beta_3) - (\beta_1)$

Example: Interactions

```
> linmod = lm(weight ~ age * sex, data = data.mlr)
> tidy(linmod)
         term   estimate  std.error statistic      p.value
1 (Intercept) -33.529162 16.3317392 -2.053006 4.739377e-02
2         age   2.714362  0.2468607 10.995522 4.636445e-13
3     sexmale -63.357026 22.7687183 -2.782635 8.532471e-03
4 age:sexmale   2.049536  0.3412012  6.006825 6.805174e-07
```



Example: Interactions

```
> head(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 62.24128   male 199.0986
2 67.10186   male 220.4382
3 60.98623   male 198.6198
4 75.57168   male 263.4126
5 67.97705   male 221.7642
6 61.07719   male 190.6024
> model.matrix(linmod) %>% head
  (Intercept)      age sexmale age:sexmale
1           1 62.24128       1    62.24128
2           1 67.10186       1    67.10186
3           1 60.98623       1    60.98623
4           1 75.57168       1    75.57168
5           1 67.97705       1    67.97705
6           1 61.07719       1    61.07719
```

$$y_i = \beta_0 + \beta_1 x_{i,age} + \beta_2 x_{i,sex} + \beta_3 x_{i,age} x_{i,sex} + \epsilon_i$$

$$y = X\beta + \epsilon$$

$$\begin{bmatrix} 1 & 62 & 1 & 62 \\ 1 & 67 & 1 & 67 \\ 1 & 60 & 1 & 60 \\ 1 & & & \vdots \\ 1 & & 0 & 0 \\ 1 & & 0 & 0 \\ 1 & & 0 & 0 \\ 1 & & 0 & 0 \\ 1 & & 0 & 0 \\ & & & 0 \end{bmatrix}$$

Example: Interactions

```
> tail(data.mlr)
Source: local data frame [6 x 3]

       age    sex   weight
     (dbl) (fctr)    (dbl)
1 57.73764 female 116.8223
2 63.51003 female 140.5238
3 63.63426 female 136.4259
4 65.64412 female 144.1169
5 72.60015 female 161.9464
6 70.57905 female 152.9105
> model.matrix(linmod) %>% tail
   (Intercept)      age sexmale age:sexmale
35           1 57.73764       0           0
36           1 63.51003       0           0
37           1 63.63426       0           0
38           1 65.64412       0           0
39           1 72.60015       0           0
40           1 70.57905       0           0
```

Categorical predictors

$$\cancel{y_i = \beta_0 + \beta_1 x_{i1}}$$

- Assume X is a categorical / nominal / factor variable with k levels
- With only one categorical X, we have the classic one-way ANOVA design
- Can't use a single predictor with levels $1, 2, \ldots, K$ – this has the wrong interpretation
- Need to create *indicator* or *dummy* variables

Indicator variables

- Let x be a categorical variable with k levels (e.g. with $k = 3$ "low", "med", "high").
- Choose one group as the baseline (e.g. "low")
- Create $(k-1)$ binary terms to include in the model:

$$\left.\begin{aligned} x_{\text{med},i} &= I(x_i = \text{"med"}) \\ x_{\text{high},i} &= I(x_i = \text{"high"}) \end{aligned}\right]$$

$$\left.\begin{aligned} x_{med} &= 0 \\ x_{high} &= 0 \end{aligned}\right\} \Rightarrow \text{low!}$$

- For a model with no additional predictors, pose the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_{k-1} x_{i,k-1} + \epsilon_i$$

and estimate parameters using least squares
- Note distinction between *predictors* and *terms*

Categorical predictor design matrix

X has 3 levels

$$
\begin{array}{ccc}
 & x_{i1} & x_{i2} \\
\end{array}
$$

$$
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\hline
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
\hline
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
$$

ANOVA model interpretation

Using the model $y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_{k-1} x_{i,k-1} + \epsilon_i$, interpret

$\beta_0 =$ $E(y \mid \text{Ref group} \; x_{i1} = x_{i2} = x_{i_{k-1}} = 0)$

$\beta_1 =$ $E(y \mid \underline{x_{i1} = 1}, \text{ all else} = 0) - E(y \mid \text{all } 0 \text{ Reference})$

$\beta_2 = E(y \mid x_{i2} = 1, \; 0) - E(y \mid \text{all } 0 / \text{Ref})$

Equivalent model

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Define the model $y_i = \beta_1 x_{i1} + \ldots + \beta_k x_{i,k} + \epsilon_i$ where there are indicators for each possible group

$\beta_1 =$ $E(y \mid x_{i1} = 1)$

$\beta_2 =$ $E(y \mid x_{i2} = 1)$

Example for categorical predictor

Suppose you want to compare the effect of placebo, exercise and a drug on blood pressure. You set up a trial to do this and gather data y_i, $treatment_i$ on n subjects.

Plac = Ref

- Analyze results using the model

exer drug

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$$

where x_i1 indicates that subject i exercised and x_i2 indicates that subject i received medication.

Example: categorical predictor

```
> ## load data
> load("BPDat.RDA")
>
> ## see what we've loaded
> head(BP)
  x1       x2
1  1 149.5939
2  1 155.5605
3  1 129.5920
4  1 149.3057
5  1 139.2455
6  1 120.3280
> summary(BP)
       x1          x2
 Min.   :1   Min.   :107.5
 1st Qu.:1   1st Qu.:128.0
 Median :2   Median :136.8
 Mean   :2   Mean   :169.9
 3rd Qu.:3   3rd Qu.:144.8
 Max.   :3   Max.   :999.0
```

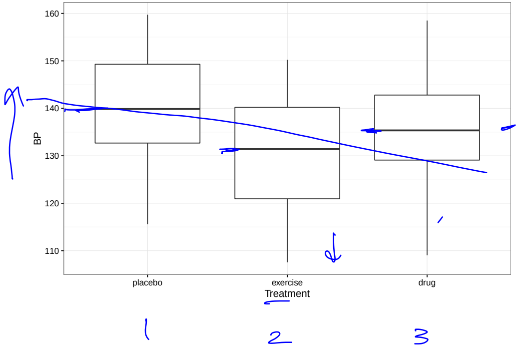
$y_i = \beta_0 + \beta_1 x_i$

Example: categorical predictor

```
> ## tidy data
> BP = BP %>% rename(Treatment = x1, BP = x2) %>%
+   mutate(Treatment = factor(Treatment, levels = 1:3,
+                             labels = c("placebo", "exercise", "drug"))) %>%
+   filter(BP != 999)
>
> summary(BP)
   Treatment         BP
 placebo :47   Min.   :107.5
 exercise:47   1st Qu.:127.0
 drug    :50   Median :136.7
               Mean   :135.3
               3rd Qu.:143.5
               Max.   :159.7
>
> BP %>% group_by(Treatment) %>% summarize(n = n(),
+                                          group_mean = mean(BP),
+                                          group_median = median(BP))
Source: local data frame [3 x 4]

  Treatment     n group_mean group_median
     (fctr) (int)      (dbl)        (dbl)
1   placebo    47   140.3368     139.8598
2  exercise    47   130.6135     131.4055
3      drug    50   135.0942     135.3504
```

Example: categorical predictor



Example: categorical predictor

$$bp_i = \beta_0 + \beta_1 tx_{\text{exer},i} + \beta_2 tx_{\text{drug},i} + \epsilon_i$$

```
> lm(BP ~ Treatment, data = BP) %>% tidy
               term   estimate std.error statistic       p.value
1       (Intercept) 140.336772  1.647753 85.168558 3.906601e-123
2 Treatmentexercise  -9.723234  2.330275 -4.172569  5.240892e-05
3     Treatmentdrug  -5.242587  2.295055 -2.284297  2.384739e-02
>
>
> lm(BP ~ Treatment, data = BP) %>% model.matrix %>% head
  (Intercept) Treatmentexercise Treatmentdrug
1           1                 0             0
2           1                 0             0
3           1                 0             0
4           1                 0             0
5           1                 0             0
6           1                 0             0
```

Example: releveling categorical predictor

$$bp_i = \beta_0 + \beta_1 tx_{\text{plac},i} + \beta_2 tx_{\text{drug},i} + \epsilon_i$$

```
> BP %>% mutate(Treatment = relevel(Treatment, ref = "exercise")) %>%
+   lm(BP ~ Treatment, data = .) %>%
+   tidy
             term   estimate std.error statistic       p.value
1     (Intercept) 130.613538  1.647753 79.267654 7.929548e-119
2 Treatmentplacebo   9.723234  2.330275  4.172569  5.240892e-05
3    Treatmentdrug   4.480647  2.295055  1.952305  5.288319e-02
```

Example: no intercept

$$bp_i = \beta_1 tx_{\text{exer},i} + \beta_2 tx_{\text{plac},i} + \beta_3 tx_{\text{drug},i} + \epsilon_i$$

-1 +

```
> lm(BP ~ 0 + Treatment, data = BP) %>% tidy
               term estimate std.error statistic       p.value
1  Treatmentplacebo 140.3368  1.647753  85.16856 3.906601e-123
2 Treatmentexercise 130.6135  1.647753  79.26765 7.929548e-119
3     Treatmentdrug 135.0942  1.597556  84.56303 1.048075e-122
>
>
> BP %>% group_by(Treatment) %>% summarize(n = n(),
+                                         group_mean = mean(BP),
+                                         group_median = median(BP))
Source: local data frame [3 x 4]

  Treatment     n group_mean group_median
     (fctr) (int)      (dbl)        (dbl)
1   placebo    47   140.3368     139.8598
2  exercise    47   130.6135     131.4055
3      drug    50   135.0942     135.3504
```

Today's big ideas

- Multiple linear regression models, interpretation, interactions, categorical predictors

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- Suggested reading: Faraway Ch 2.2 - 2.3; ISLR 3.2