PLSC 502 – Autumn 2016
Random Variables

September 22, 2016

Continuous and Discrete Variables

<u>Discrete Variables</u>

- $X \in S = \{s_0, s_1, ...\}$
- $\Pr(s) \geq 0$ for each $s \in S$
- $\sum \Pr(s) = 1$

<u>Continuous Variables</u>

- $X \in S \in \Re$
- $\exists\ f(x)$ such that for any closed interval $[a, b]$ $\Pr(a < x \leq b) = \int_b^a f(x)dx$.
- Requires:
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f(x)dx = 1$

Probability Density Function

The PDF is the function $f(x)$ that maps the possible values of X to some associated probability of their occurrence.

Discrete X:

$$f(x) = \Pr(X = x) \, \forall \, x$$

Continuous X:

$$\Pr(a < X \leq b) = \int_a^b f(x) \, dx$$

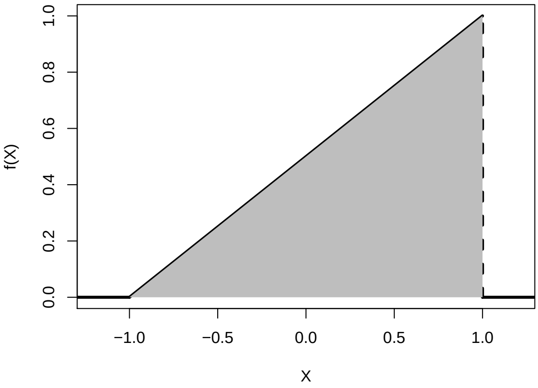
Again: Requires:

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x) dx = 1$

An Example

Consider:

$$f(X) = \begin{cases} \frac{X+1}{2} & \text{for} -1 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$



PDF?

1. Is $f(x) \geq 0 \;\forall\; x$? – Yes.
2. Is $\Pr(-\infty \leq x \leq \infty) \equiv \Pr(-1 \leq x \leq 1) = \int_{-1}^{1} f(x)dx = 1$?

$$
\begin{aligned}
\Pr(-1 \leq x \leq 1) &= \int_{-1}^{1} \frac{1}{2}(x+1)dx \\
&= \frac{1}{2}\left(\frac{X^2}{2} + x\right)\bigg|_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1^2}{2} + 1\right) - \frac{1}{2}\left(\frac{-1^2}{2} - 1\right) \\
&= 0.75 - (-0.25) \\
&= 1.
\end{aligned}
$$

Cumulative Distribution Function (CDF)

The CDF is the probability that X will take on a value less than or equal to than some value x in its range.

Discrete X:

$$\begin{aligned} \Pr(X \leq x) \equiv F(x) &= \sum_{X \leq x} \Pr(X = x) \\ &= 1 - \sum_{X > x} \Pr(X = x) \end{aligned}$$

CDF (continued)

Continuous X:

$$\Pr(X \leq x) \equiv F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Properties:

- $0 \leq F(x) \leq 1$.
- Nondecreasing in X.
- $\Pr(x < k) = 1$ - F(k).
- $\Pr(a < x \leq b) = F(b) - F(a)$.
- $F(-\infty) = 0$.
- $F(\infty) = 1$.

Example, Again

For

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for} -1 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

we already know that $\int_{-1}^{1} f(x)dx = 1$.

$$\begin{aligned} F(x) &= \int_{-1}^{1} f(t)dt \\ &= \int_{-1}^{1} \frac{1}{2}(t+1)dt \\ &= \frac{1}{2}\left(\frac{t^2}{2} + t\right)\Big|_{-1}^{1} \\ &= \frac{1}{2}\left(\frac{1}{2}t^2 + t\right) + c. \end{aligned}$$

Expected Value

For X with PDF $f(x)$ and CDF $F(x) = \int_{-\infty}^{x} f(t)\, dt$, the *expected value* of X ($\mathrm{E}(X)$ or μ) is the *probability-weighted mean of the potential values of that variable*.

<u>Discrete X</u>:

$$\mathrm{E}(X) = \sum_{x} [x \times f(x)]$$

E.g., number of heads in two coin flips:

0 Heads	Prob. = .25	Prob×Value = .25×0	=	0
1 Head	Prob. = .50	Prob×Value = .50×1	=	.50
2 Heads	Prob. = .25	Prob×Value = .25×2	=	.50
		$\sum$	=	1.0

Expected Value (continued)

Continuous X:

$$\mathrm{E}(X) = \int [x \times f(x)] dx$$

Properties:

- $\mathrm{E}(c) = c$
- $\mathrm{E}(x + y + z) = \mathrm{E}(x) + \mathrm{E}(y) + \mathrm{E}(z)$
- If $g(x)$ is some function of x, then

$$\begin{aligned} \mathrm{E}[g(x)] &= \sum [g(x) \times \mathrm{Prob}(X = x)] \,\forall x \text{ (discrete case)} \\ &= \int g(x) f(x)\, dx \text{ (continuous case)} \end{aligned}$$

- This includes a constant function:$\mathrm{E}(cx) = c\mathrm{E}(x)$.
- Implies that for $g(x) = a + bx$, $\mathrm{E}(a + bx) = a + b\mathrm{E}(x)$.

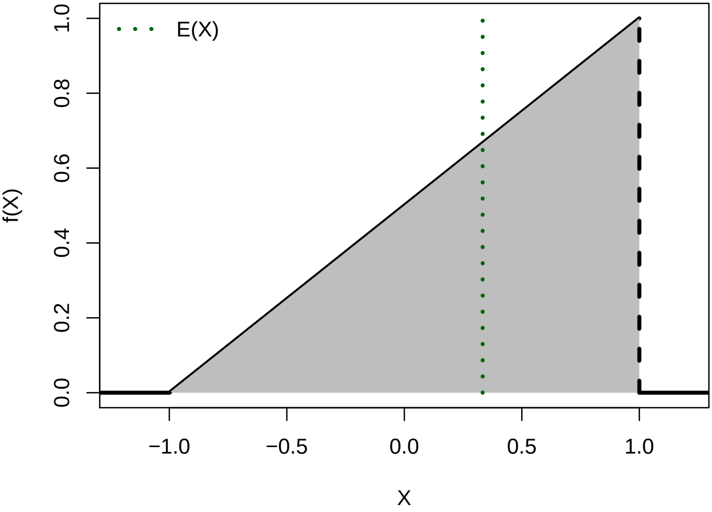
Example Again

For

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $E(X)$?

$$\begin{aligned}
\mathrm{E}[f(X)] &= \int_{-1}^{1} x \left(\frac{x+1}{2} \right) dx \\
&= \int_{-1}^{1} \frac{1}{2}(x^2 + x)\, dx \\
&= \frac{1}{2} \int_{-1}^{1} x^2\, dx + \frac{1}{2} \int_{-1}^{1} x\, dx \\
&= \frac{1}{2} \left(\frac{x^3}{3} + c_1 \right) \Bigg|_{-1}^{1} + \frac{1}{2} \left(\frac{x^2}{2} + c_2 \right) \Bigg|_{-1}^{1} \\
&= \frac{1}{2} \left(\frac{x^3}{3} + \frac{x^2}{2} + c_3 \right) \Bigg|_{-1}^{1} \\
&= \frac{1}{2} \left[\left(\frac{1^3}{3} + \frac{1^2}{2} + c_3 \right) - \left(\frac{-1^3}{3} + \frac{-1^2}{2} + c_3 \right) \right] \\
&= \frac{1}{3}
\end{aligned}$$



E(X)
f(X)
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0
X

Variance

Generally:

$$\mathrm{Var}(X) = \mathrm{E}[(x-\mu)^2]$$

Discrete X:

$$\mathrm{Var}(X) = \sum (x-\mu)^2 f(x)$$

Continuous X:

$$\mathrm{Var}(X) = \int (x-\mu)^2 f(x)\, dx$$

Variance (continued)

$$
\begin{aligned}
\mathrm{E}[(x-\mu)^2] \equiv \sigma^2 &= \mathrm{E}[x^2 - 2x\mu + \mu^2] \\
&= \mathrm{E}(x^2) - 2\mu\mathrm{E}(x) + \mathrm{E}(\mu^2) \\
&= \mathrm{E}(x^2) - 2\mu^2 + \mu^2 \\
&= \mathrm{E}(x^2) - \mu^2 \\
&\equiv \left(\int x^2 f(x)\, dx - \mu^2\right)
\end{aligned}
$$

- We often write the variance as σ^2, and the positive square root of it (the standard deviation) as σ.
- This also implies that the expectation of the square of a variable X is $\mathrm{E}(x^2) = \sigma^2 + \mu^2$.

Variance Properties

- $\mathrm{Var}(X) > 0$, except
- $\mathrm{Var}(c) = 0$
- $\mathrm{Var}(a + bX) = b^2\mathrm{Var}(X)$
- $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$

Example Again

What is the variance of $f(X) = \frac{1}{2}(x+1)$ for the range $-1 < x < 1$?

Recall that $\mu = 1/3$:

$$
\begin{aligned}
\mathrm{E}[(x-\mu)^2] \equiv \sigma^2 &= \int_{-1}^{1} X^2 f(x)\,dx - \mu^2 \\
&= \int_{-1}^{1} \frac{1}{2} x^2 (x+1)\,dx - \left(\frac{1}{3}\right)^2 \\
&= \frac{1}{2}\left(\frac{x^4}{4} + c_1\right)\Bigg|_{-1}^{1} + \frac{1}{2}\left(\frac{x^3}{3} + c_2\right)\Bigg|_{-1}^{1} - \frac{1}{9} \\
&= \frac{1}{2}\left(\frac{X^4}{4} + \frac{X^3}{3} + c_3\right)\Bigg|_{-1}^{1} - \frac{1}{9} \\
&= \frac{1}{2}\left[\left(\frac{1^4}{4} + \frac{1^3}{3} + c_3\right) - \left(\frac{-1^4}{4} + \frac{-1^3}{3} + c_3\right)\right] - \frac{1}{9} \\
&= \frac{19}{72}\ (\approx 0.2639).
\end{aligned}
$$

Moments, redux

The kth moment of X is:

$$M_k = \mathsf{E}(X^k)$$

The kth moment exists if:

$$\begin{aligned} \mathsf{E}(|X|^k) &< \infty \\ = \int_{-\infty}^{\infty} |x|^k f(x) dx &< \infty \text{ (for continuous } X\text{)} \end{aligned}$$

"Central" moments:

$$\mu_k = \mathsf{E}[(X - \mu)^k]$$

Moment-Generating Functions

For $t \in \mathfrak{R}$,

$$\psi(t) = \mathrm{E}[\exp(tX)]$$

For continuous X:

$$\begin{aligned}
\psi(t) &= \int_{-\infty}^{\infty} \exp(tx) f(x) dx \\
&= \int_{-\infty}^{\infty} \left(1 + tx + \frac{t^2 x^2}{2!} + \dots\right) f(x) dx \\
&= 1 + t\mathrm{E}(X) + \frac{t^2 \mathrm{E}(X^2)}{2} + \dots \\
&= 1 + tM_1 + \frac{t^2 M_2}{2} + \dots
\end{aligned}$$

Note that:

$$\begin{aligned}
\psi(0) &= \mathrm{E}[\exp(0)] \\
&= 1
\end{aligned}$$

MGFs Can Be Useful

First:

$$\psi(t) = \int_{-\infty}^{\infty} \exp(tx)\, d\, F(x).$$

Second:

$$\begin{aligned}
\left.\frac{\partial^k \psi(t)}{\partial^k t}\right|_{t=0} &= \left.\frac{\partial^k \mathsf{E}[\exp(tX)]}{\partial^k t}\right|_{t=0} \\
&= \mathsf{E}\left[\left.\frac{\partial^k \exp(tX)}{\partial^k t}\right|_{t=0}\right] \\
&= \mathsf{E}\{[X^k \exp(tX)]|_{t=0}\} \\
&= \mathsf{E}(X^k)
\end{aligned}$$