PLSC 502 – Autumn 2016
Probability Distributions, I

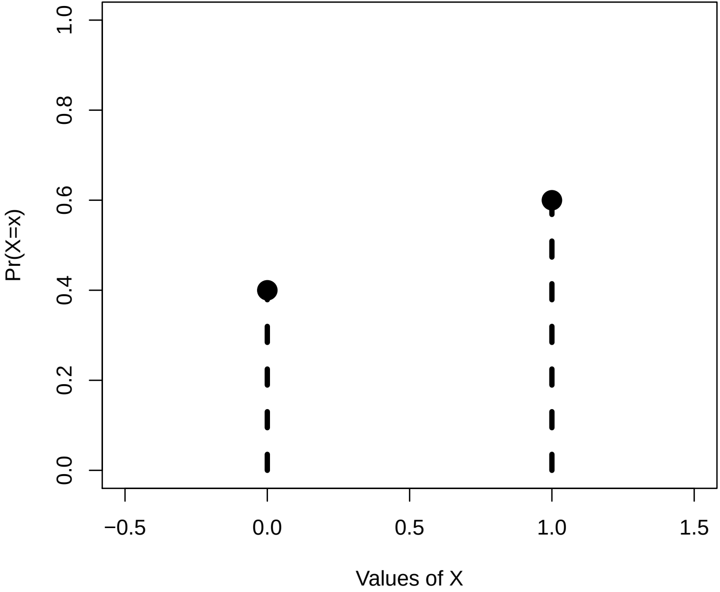
September 27, 2016

Bernoulli

$$\begin{aligned} X &= 0 \text{ with probability } 1-\pi \\ &= 1 \text{ with probability } \pi. \end{aligned}$$

$$\begin{aligned} f(x) &= \begin{cases} 1-\pi & \text{for } X=0 \\ \pi & \text{for } X=1 \end{cases} \\ &= \pi^x(1-\pi)^{1-x},\ x \in \{0,1\} \end{aligned}$$

$$X \sim \text{ Bernoulli}(\pi)$$


Pr(X=x)
0.0
0.2
0.4
0.6
0.8
1.0
−0.5
0.0
0.5
1.0
1.5
Values of X

Bernoulli, continued

CDF:

$$
\begin{aligned}
F(x) &= \sum_x f(x) \\
&= \begin{cases} 1-\pi & \text{for } X = 0 \\ 1 & \text{for } X = 1 \end{cases}
\end{aligned}
$$

Expectation:

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_x x f(x) \\
&= (0)(1-\pi) + (1)(\pi) \\
&= \pi
\end{aligned}
$$

Variance:

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum_x [X - \mathrm{E}(X)]^2 f(x) \\
&= \sum_x [X - \pi]^2 f(x) \\
&= (0-\pi)^2(1-\pi) + (1-\pi)^2\pi \\
&= \pi^2 - \pi^3 + \pi - 2\pi^2 + \pi^3 \\
&= \pi^2 - \pi \\
&= \pi(1-\pi)
\end{aligned}
$$

Skewness:

$$
\text{Skewness} = \frac{(1-\pi) - \pi}{\sqrt{(1-\pi)\pi}}
$$

Even More Bernoulli

MGF:

$$\begin{aligned}\psi(t) &= \int_{-\infty}^{\infty} \exp(tx)\, dF(x) \\ &= \sum_{n=0}^{1} \exp(tn)\pi^{n}(1-\pi)^{1-n} \\ &= \exp(0)(1-\pi) + \exp(t)\pi \\ &= (1-\pi) + \pi \exp(t)\end{aligned}$$

Implying:

$$\frac{\partial^k \psi(t)}{\partial^k t} = \pi \exp(t)\ \forall\ k$$

and *raw moments*:

$$\mathrm{E}(X^k) = \pi\ \forall\ k > 0$$

Feel the Bern...

Central moments:

$$M_1 = \pi,$$

$$M_2 = \pi(1 - \pi),$$

$$M_3 = \pi(1 - \pi)(1 - 2\pi),$$

etc.

Binomial

Assume n independent binary "trials," each with identical probability of "success" π. Then the number of "successes" in n trials follows a *binomial* distribution:

$$f(x) = \binom{n}{x} \pi^x (1-\pi)^{n-x}$$

where recall that

$$\binom{n}{x} \equiv \frac{n!}{x!(n-x)!}.$$

$$X \sim \text{binomial}(n, \pi).$$

Why "binomial"?

Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

e.g.

$$\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2, \\
(a+b)^3 &= a^3 + 3a^2 b + 3ab^2 + y^3, \\
(a+b)^4 &= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4,
\end{aligned}$$

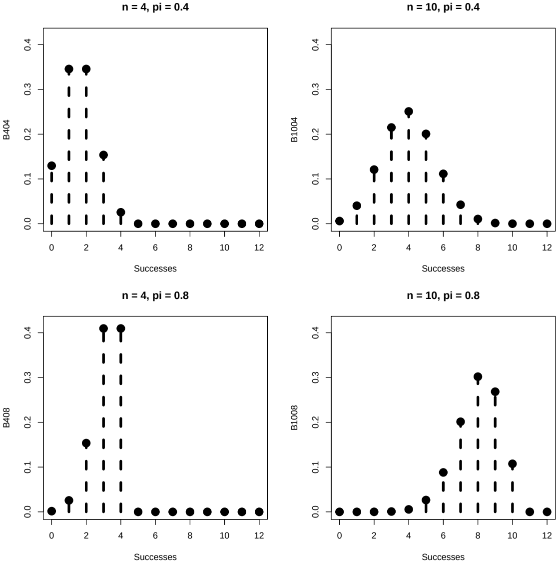
Binomial: Derivation

For $n = 2$:

$$\begin{aligned}\Pr(X=0) &= \Pr(X_1=0, X_2=0)\\ &= \Pr(X_1=0)\times\Pr(X_2=0)\\ &= (1-\pi)^2\end{aligned}$$

$$\begin{aligned}\Pr(X=1) &= \Pr(X_1=1, X_2=0 \textit{ or } X_1=0, X_2=1)\\ &= \Pr(X_1=1)\times\Pr(X_2=0)+\Pr(X_1=0)\times\Pr(X_2=1)\\ &= \pi(1-\pi)+(1-\pi)\pi\end{aligned}$$

$$\begin{aligned}\Pr(X=2) &= \Pr(X_1=1, X_2=1)\\ &= \Pr(X_1=1)\times\Pr(X_2=1)\\ &= \pi^2\end{aligned}$$


n = 4, pi = 0.4
n = 10, pi = 0.4
n = 4, pi = 0.8
n = 10, pi = 0.8
B404
B1004
B408
B1008
Successes
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10
12

More Binomial

CDF:

$$
\begin{aligned}
F(x) &= \sum_{x} f(x) \\
&= \sum_{j=0}^{x} \binom{n}{j} \pi^{j} (1-\pi)^{n-j}
\end{aligned}
$$

Expectation:

$$E(X) = n\pi,$$

More Binomial

Variance:

$$\begin{aligned}\mathrm{Var}(X) &= \sum_x [X - \mathrm{E}(X)]^2 f(x) \\ &= \sum_x (X - \pi n)^2 \binom{n}{x} \pi^x (1-\pi)^{n-x} \\ &= n\pi(1-\pi).\end{aligned}$$

Skewness:

$$\text{Skewness} = \frac{1 - 2\pi}{\sqrt{n\pi(1-\pi)}}$$

The Binomial...

- Is *unimodal* (except in certain cases),
- has median $\lceil n\pi \rceil$ or $\lfloor n\pi \rfloor$,
- has mode $\lceil (n+1)\pi \rceil$ or $\lfloor (n+1)\pi \rfloor$,
- has skewness that is:
  - increasing in n, and
  - is largest when $\pi = 0.5$ for a fixed value of n.

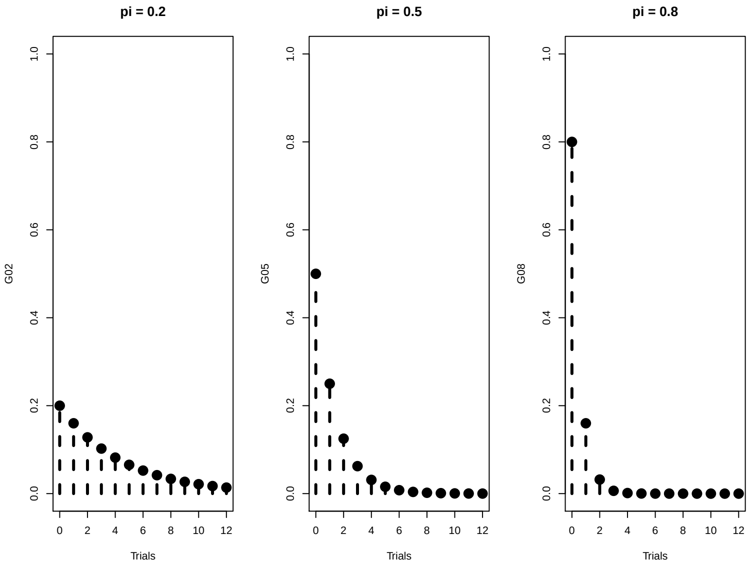
Geometric

The number of independent Bernoulli trials needed to achieve *one* success is a *geometric* random variable.

PDF:

$$f(x) = \pi(1-\pi)^{x-1}$$

$$X \sim \text{geometric}(\pi).$$


pi = 0.2
pi = 0.5
pi = 0.8
G02
G05
G08
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10
12
Trials

CDF:

$$\begin{aligned} F(x) &= \sum_{j=1}^{x} \pi(1-\pi)^{x-1} \\ &= 1-(1-\pi)^x \end{aligned}$$

Expectation:

$$\mathrm{E}(X) = \frac{1}{\pi}$$

Variance:

$$\mathrm{Var}(X) = \frac{1-\pi}{\pi^2}$$

Negative Binomial

The number of *failures we observe* (x) before achieving the rth success in n independent binomial trials (each with probability of success π) is distributed according to a *negative binomial* distribution.

PDF:

$$f(x) = \binom{r+x-1}{r-1} \pi^r (1-\pi)^x$$

More Negative Binomial

CDF:

$$
\begin{aligned}
F(x) &= \sum_{j=0}^{x} \binom{r+j-1}{r-1} \pi^r (1-\pi)^j \\
&= 1 - \mathrm{CDF}_{\text{binomial}}
\end{aligned}
$$

Expected value:

$$
\mathrm{E}(X) = \frac{(1-\pi)r}{\pi}
$$

Even More Negative Binomial

Variance:

$$\mathsf{Var}(X) = \frac{(1-\pi)r}{\pi^2}.$$

Skewness:

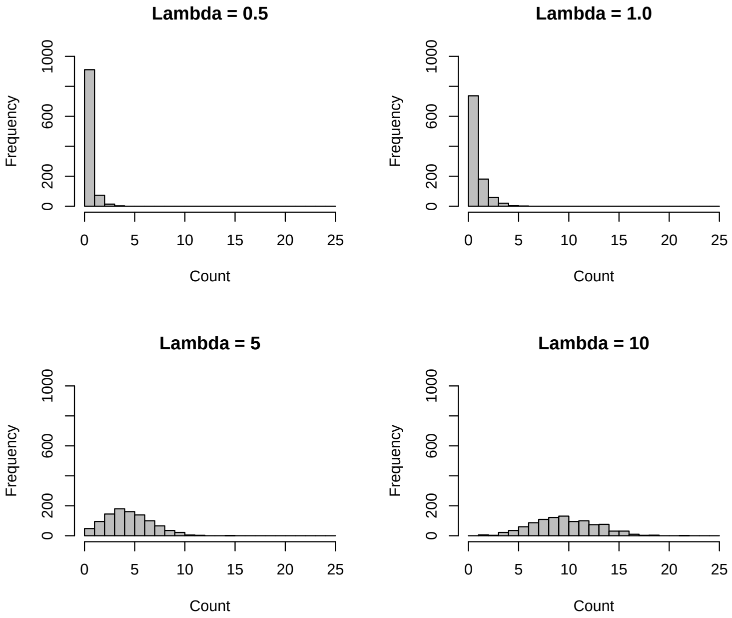
$$\mathsf{Skewness} = \frac{1+\pi}{\sqrt{\pi r}}$$

Poisson

For n independent Bernoulli trials with (sufficiently small) probability of success π and where $n\pi \equiv \lambda > 0$, the probability of observing exactly x total "successes" as the number of trials grows without limit is the *Poisson distribution*.

PDF:

$$
\begin{aligned}
f(x) &= \lim_{n\to\infty}\left[\binom{n}{x}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}\right] \\
&= \frac{\lambda^{x}\exp(-\lambda)}{x!}.
\end{aligned}
$$


Lambda = 0.5
Lambda = 1.0
Lambda = 5
Lambda = 10
Frequency
Count
0
200
600
1000
0
5
10
15
20
25

More Poisson

CDF:

$$F(x) = \sum_{j=0}^{x} \frac{\lambda^j \exp(-\lambda)}{x!}.$$

Mean & Variance:

$$E(X) = \mathrm{Var}(X) = \lambda$$

All higher moments are zero...

Alternative Poisson

Independent, *constant-probability* events occurring in time...

$$\text{“arrival rate”} = \lambda$$

Implies:

$$\begin{aligned} \Pr(\text{Event in } (t, t+h]) &= \lambda \\ \Pr(\text{No event in } (t, t+h]) &= 1 - \lambda \end{aligned}$$

$$N_{\text{Events occurring in } (t, t+h]} = \frac{\exp(-\lambda h)\lambda h^x}{x!}$$

If $h = 1 \; \forall \; h$, then:

$$f(x) = \frac{\exp(-\lambda)\lambda^x}{x!}$$

Multinomial

Imagine K possible distinct *outcomes* for each "trial," where each possible outcome has π_k and $\sum_{k=1}^{K} \pi_k = 1$.

x_k = number of times we observe outcome k out of n trials.

Then for:

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{pmatrix}$$

the PDF is:

$$f(\mathbf{x}) = \frac{n!}{x_1!x_2!...x_K!} \pi_1^{x_1} \pi_2^{x_2} ... \pi_K^{x_K}$$

Multinomial, continued

Expected value:

$$\mathrm{E}(\mathbf{X}) \equiv \mathrm{E}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{pmatrix} = n\begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_K \end{pmatrix}$$

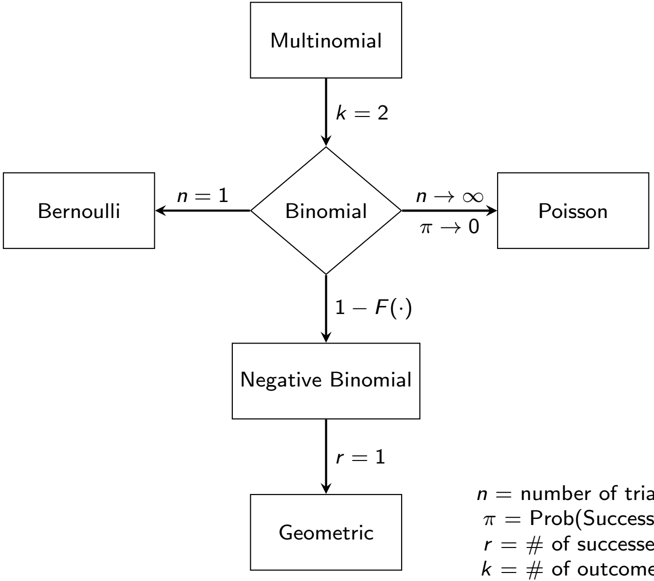
Variance:

$$\mathrm{Var}(\mathbf{X}) \equiv \mathrm{Var}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_K \end{pmatrix} = n\begin{bmatrix} \pi_1(1-\pi_1) \\ \pi_2(1-\pi_2) \\ \vdots \\ \pi_K(1-\pi_K) \end{bmatrix}$$

Covariance between X_s and X_t, $s \neq t$:

$$\mathrm{Cov}(X_s, X_t) = -n\pi_s\pi_t$$

Schematic



n = number of trials
π = Prob(Success)
r = # of successes
k = # of outcomes