PLSC 502 – Autumn 2016
Probability

September 20, 2016

Terminology

- **X**: A random variable
- **Outcome**: A possible event / result of a process
- **Realization**: One observation of the process (x)
- **Sample Space (S)**: The set of all possible outcomes

Sample Spaces

So for a variable X:

$$X \in S = \{x_1, x_2, ...x_J\}$$

E.g., for the NFL (week one):

$$X \in S = \{0, 1, 2, ...\}$$

and

$$X_{\text{Rams}} = 0.$$

Probability (*Frequentist*)

Probability = *Long-run relative frequency*.

$$\Pr(\text{Event}) = \frac{\text{The number of times the } \textit{event of interest} \text{ can or could occur}}{\text{The number of times } \textit{any event} \text{ can or could occur}}.$$

More formally:

$$\Pr(X = x) = \lim_{N \to \infty} \left(\frac{\sum_N I\{X_i = x\}}{N} \right)$$

where $I\{\cdot\}$ is an *indicator function* for $X_i = x$.

Probability: Characteristics

- Probabilities necessarily **range between zero and one**:

$$\Pr(X = x) \in [0, 1].$$

- The **sum of probabilities for all outcomes always equals one**:

$$\sum_{j=1}^{J} \Pr(X = x_j) \equiv \Pr(S) = 1.0$$

The Multiplication Rule

The probability of obtaining a *combination* of independent, mutually exclusive outcomes is equal to the *product* of their separate probabilities.

Formally:

$$\Pr(X = x_j \cap X = x_\ell) = \Pr(X = x_j) \times \Pr(X = x_\ell),\ j \neq \ell$$

The Addition Rule

The probability of obtaining *any one* (or more) of several independent, mutually exclusive outcomes is equal to the *sum* of the probabilities for those events.

Formally:

$$\Pr(X = x_j \cup X = x_\ell) = \Pr(X = x_j) + \Pr(X = x_\ell),\ j \neq \ell$$

Addition Rule (continued)

If events are not mutually exclusive:

$$\begin{aligned} \Pr(X = x_j \cup X = x_\ell) &= \Pr(X = x_j) + \Pr(X = x_\ell) \\ &\quad - \Pr(X = x_j \cap X = x_\ell) \end{aligned}$$

So, for example, Pr(Diamond *or* face-card):

$$\begin{aligned} \Pr(Z) &= \Pr(\text{Diamond}) + \Pr(\text{Face-Card}) \\ &\quad - \Pr(\text{Diamond-Suited Face Card}) \\ &= \frac{1}{4} + \frac{12}{52} - \frac{3}{52} \\ &= 0.25 + 0.23 - 0.06 \\ &= \mathbf{0.42} \end{aligned}$$

Independence

Consider $\Pr(X = x_j, X = x_\ell) = \Pr(X = x_j \;\cap\; X = x_\ell)$ ("joint PDF")...

If x_j and x_ℓ are *independent*:

- $\Pr(X = x_j, X = x_\ell) = \Pr(X = x_j) \times \Pr(X = x_\ell)$.
- *The joint PDF is equal to the product of the marginal PDFs.*
- We write $X_j \perp X_\ell$.

Conditional Probability

If x_j and x_ℓ are *not independent*...

Conditional probabilities:

- I.e., $\Pr(X = x_j | X = x_\ell)$ and/or $\Pr(X = x_\ell | X = x_j)$
- Say "The probability of x_j given x_ℓ," etc.

Implies:

$$\begin{aligned} \Pr(X = x_j | X = x_\ell) &= \frac{\Pr(X = x_j, X = x_\ell)}{\Pr(X = x_\ell)}, \text{ and} \\ \Pr(X = x_\ell | X = x_j) &= \frac{\Pr(X = x_j, X = x_\ell)}{\Pr(X = x_j)} \end{aligned}$$

Independence, Defined

If two variables are *independent*, then:

$$
\begin{aligned}
\Pr(X = x_j | X = x_\ell) &= \frac{\Pr(X = x_j, X = x_\ell)}{\Pr(X = x_\ell)} \\
&= \frac{\Pr(X = x_j) \times \Pr(X = x_\ell)}{\Pr(X = x_\ell)} \\
&= \Pr(X = x_j)
\end{aligned}
$$

Any number of variables; e.g. for x_j, x_ℓ, and x_k:

$$
\begin{aligned}
\Pr(X = x_j, X = x_\ell, X = x_k) &= \Pr(X = x_j | X = x_\ell, X = x_k) \\
&\quad \times \Pr(X = x_\ell | X = x_k) \\
&\quad \times \Pr(X = x_k)
\end{aligned}
$$

Bayes' Rule

Because $\Pr(X = x_j, X = x_\ell) = \Pr(X = x_\ell, X = x_j)$, we can write:

$$
\begin{aligned}
\Pr(X = x_j | X = x_\ell) \times \Pr(X = x_\ell) &= \Pr(X = x_\ell | X = x_j) \\
&\quad \times \Pr(X = x_j)
\end{aligned}
$$

and so:

$$
\Pr(X = x_j | X = x_\ell) = \frac{\Pr(X = x_\ell | X = x_j) \times \Pr(X = x_j)}{\Pr(X = x_\ell)}.
$$

More Bayes' Rule

Generally:

$$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$$

Informally:

$$\text{Posterior} = \frac{\text{Likelihood} \;\times\; \text{Prior}}{\text{Marginal}}$$

Probability and Odds

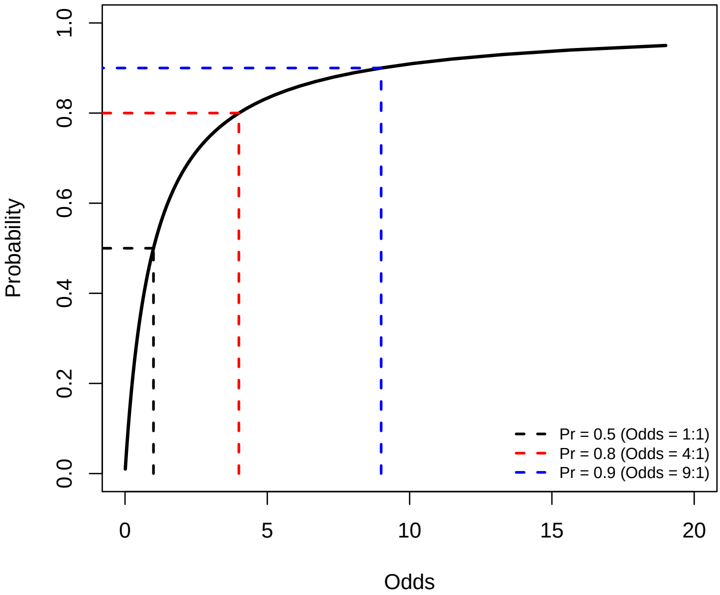
$$\begin{aligned}\text{Odds}(X = x_j) &= \frac{\Pr(X = x_j)}{\Pr(X \neq x_j)} \\ &= \frac{\Pr(X = x_j)}{1 - \Pr(X = x_j)}\end{aligned}$$

Often written as $\Pr(X = x_j) : [1 - \Pr(X = x_j)]$.

E.g., "The odds of x_j are 4:1 (in favor)":

- $\Pr(X = x_j) = \frac{4}{4+1} = 0.8$
- $\Pr(X \neq x_j) = \frac{1}{4+1} = 0.2$

Probability and Odds

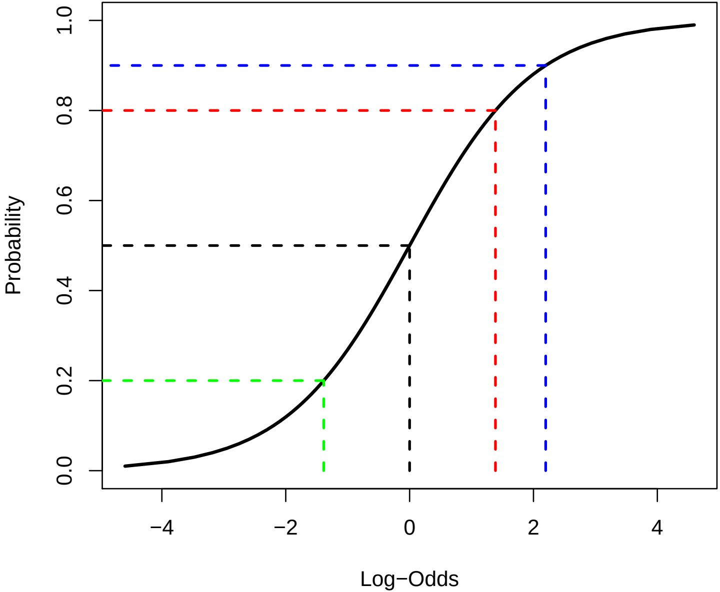


Log-Odds

$$
\begin{aligned}
\ln[\text{Odds}(X = x_j)] &= \ln\left[\frac{\Pr(X = x_j)}{\Pr(X \neq x_j)}\right] \\
&= \ln\left[\frac{\Pr(X = x_j)}{1 - \Pr(X = x_j)}\right]
\end{aligned}
$$

- Odds $\in\ [0, \infty)$
- Log-odds $\in\ (-\infty, \infty)$

Log-Odds and Probability



Likelihood

For N realizations of X:

$$\begin{array}{rcl} X_1 & = & x_1 \\ X_2 & = & x_2 \\ X_3 & = & x_3 \\ \vdots & & \vdots \\ X_N & = & x_N \end{array}$$

Likelihood:

$$L(X) = \Pr(X_1 = x_1, X_2 = x_2, \ldots X_N = x_N)$$

Likelihood (continued)

If $X_j \perp X_k \; \forall \, j, k$ then

$$\begin{aligned} L(X) &= \Pr(X_1 = x_1) \times \Pr(X_2 = x_2) \times \ldots \times \Pr(X_N = x_N) \\ &= \prod_{i=1}^{N} \Pr(X_i = x_i). \end{aligned}$$

Log-Likelihood:

$$\begin{aligned} \ln L(X) &= \ln \left[\prod_{i=1}^{N} \Pr(X_i = x_i) \right] \\ &= \sum_{i=1}^{N} \ln[\Pr(X_i = x_i)] \end{aligned}$$