PLSC 502 – Fall 2016
Linear Regression II

November 15, 2016

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

Estimators:

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

and

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} \\ &= \frac{\sum (X_i - \bar{X}) Y_i}{\sum (X_i - \bar{X})^2}\end{aligned}$$

$$u_i \sim \text{i.i.d. } N(0, \sigma^2)$$

meaning:

$$\mathrm{Var}(Y|X, \beta) = \sigma^2$$

so:

$$\begin{aligned}
\mathrm{Var}(\hat{\beta}_1) &= \mathrm{Var}\left[\frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}\right] \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2\, \mathrm{Var}(Y) \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2\, \sigma^2 \\
&= \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.
\end{aligned}$$

$\text{Var}(\hat{\beta}_0)$ and $\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)$

Similarly:

$$\text{Var}(\hat{\beta}_0) = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2$$

and :

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2$$

Important Things

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto \sigma^2$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -\sum(X_i - \bar{X})$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -N$
- $\text{sign}[\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)] = \text{sign}(\bar{X})$

The Gauss-Markov Theorem

"Given the assumptions of the classical linear regression model, the least squares estimators are the minimum variance estimators among the class of unbiased linear estimators. (They are BLUE)."

Gauss-Markov, continued

Imagine:

$$\hat{\beta}_1 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2}$$

Rewrite:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}.$$

k are "weights":

$$\hat{\beta}_1 = \sum k_i Y_i$$

with $k_i = \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}$.

Gauss-Markov (continued)

Alternative (non-LS) estimator:

$$\tilde{\beta}_1 = \sum w_i Y_i$$

Unbiasedness requires:

$$\begin{aligned} E(\tilde{\beta}_1) &= \sum w_i E(Y_i) \\ &= \sum w_i(\beta_0 + \beta_1 X_i) \\ &= \beta_0 \sum w_i + \beta_1 \sum w_i X_i \end{aligned}$$

Gauss-Markov (continued)

Variance:

$$
\begin{aligned}
\mathrm{Var}(\tilde{\beta}_1) &= \mathrm{Var}\left(\sum w_i Y_i\right) \\
&= \sigma^2 \sum w_i^2 \\
&= \sigma^2 \sum \left[w_i - \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} + \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right]^2 \\
&= \sigma^2 \sum \left[w_i - \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right]^2 + \sigma^2 \left[\frac{1}{\sum (X_i - \bar{X})^2} \right]
\end{aligned}
$$

Gauss-Markov (continued)

Because $\sigma^2 \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]$ is a constant, $\min[\mathrm{Var}(\tilde{\beta}_1)]$ minimizes

$$\sum \left[w_i - \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2} \right]^2.$$

Minimized at:

$$w_i = \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2}.$$

implying:

$$\begin{aligned} \mathrm{Var}(\tilde{\beta}_1) &= \frac{\sigma^2}{\sum(X_i - \bar{X})^2} \\ &= \mathrm{Var}(\hat{\beta}_1) \end{aligned}$$

If $u_i \sim N(0, \sigma^2)$, then:

$$\hat{\beta}_0 \sim N[\beta_0, \text{Var}(\hat{\beta}_0)]$$

and

$$\hat{\beta}_1 \sim N[\beta_1, \text{Var}(\hat{\beta}_1)]$$

Means:

$$\begin{aligned} z_{\hat{\beta}_1} &= \frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)}{\text{s.e.}(\hat{\beta}_1)} \\ &= \sim N(0, 1) \end{aligned}$$

A Small Problem...

$\sigma^2 =$???

Solution: use

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{N - k}$$

Gives:

$$\widehat{\text{Var}(\hat{\beta}_1)} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2},$$

and

$$\widehat{\text{Var}(\hat{\beta}_0)} = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \hat{\sigma}^2$$

$$\begin{aligned}
\widehat{\text{s.e.}(\hat{\beta}_1)} &= \sqrt{\widehat{\text{Var}(\hat{\beta}_1)}} \\
&= \sqrt{\frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2}} \\
&= \frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}
\end{aligned}$$

implies:

$$\begin{aligned}
t_{\hat{\beta}_1} \equiv \frac{(\hat{\beta}_1 - \beta_1)}{\widehat{\text{s.e.}(\hat{\beta}_1)}} &= \frac{(\hat{\beta}_1 - \beta_1)}{\frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}} \\
&= \frac{(\hat{\beta}_1 - \beta_1)\sqrt{\sum(X_i - \bar{X})^2}}{\hat{\sigma}} \\
&\sim t_{N-k}
\end{aligned}$$

Predictions and Variance

Point prediction:

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

Y_k is unbiased:

$$\begin{aligned}
E(\hat{Y}_k) &= E(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\
&= E(\hat{\beta}_0) + X_k E(\hat{\beta}_1) \\
&= \beta_0 + \beta_1 X_k \\
&= E(Y_k)
\end{aligned}$$

Variability:

$$\begin{aligned}
\text{Var}(\hat{Y}_k) &= \text{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\
&= \frac{\sum X_i^2}{N\sum(X_i - \bar{X})^2}\sigma^2 + \left[\frac{\sigma^2}{\sum(X_i - \bar{X})^2}\right] X_k^2 + 2\left[\frac{-\bar{X}}{\sum(X_i - \bar{X})^2}\sigma^2\right] X_k \\
&= \sigma^2\left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right]
\end{aligned}$$

Variability of Predictions

$$\text{Var}(\hat{Y}_k) = \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]$$

means that $\text{Var}(\hat{Y}_k)$:

- Decreases in N
- Decreases in $\text{Var}(X)$
- Increases in $|X - \bar{X}|$

Standard error of the prediction:

$$\widehat{\text{s.e.}(\hat{Y}_k)} = \sqrt{\sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]}$$

$\rightarrow$ (e.g.) confidence intervals:

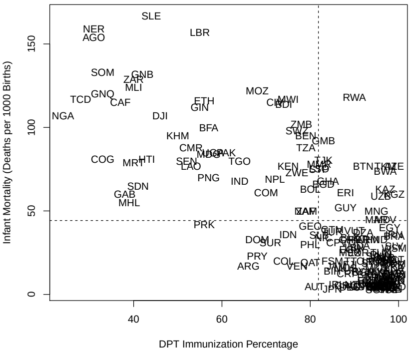
$$95\%\,\text{c.i.}(\hat{Y}_k) = \hat{Y}_k \pm [1.96 \times \widehat{\text{s.e.}(\hat{Y}_k)}]$$

Example: Infant Mortality and DPT Immunizations

```
> with(IMdata, describe(infantmortalityperK))
   vars   n mean   sd median trimmed  mad min max range skew kurtosis   se
X1    1 177 44.3 40.4   29.3    38.9 34.7 2.9 167   164 0.99     0.03 3.04

> with(IMdata, describe(DPTpct))
   vars   n mean   sd median trimmed  mad min max range skew kurtosis   se
X1    1 177 81.8 19.6     90    85.2 11.9  24  99    75 -1.3     0.59 1.47
```

Scatterplot of Infant Mortality and DPT Immunizations (2000)



Example, Continued

```
> IMDPT<-with(IMdata, lm(infantmortalityperK~DPTpct))
> summary(IMDPT)   # regression

Call:
lm(formula = infantmortalityperK ~ DPTpct, data = IMdata)

Residuals:
   Min     1Q Median     3Q    Max
 -56.8  -16.3   -5.1   11.8   86.6

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  173.277      8.489    20.4   <2e-16 ***
DPTpct        -1.576      0.101   -15.6   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.2 on 175 degrees of freedom
Multiple R-Squared: 0.582,Adjusted R-squared: 0.58
F-statistic:  244 on 1 and 175 DF,  p-value: <2e-16
```

Example, Continued

```
> anova(IMDPT)

Analysis of Variance Table

Response: infantmortalityperK
           Df Sum Sq Mean Sq F value Pr(>F)
DPTpct      1 167423  167423     244 <2e-16 ***
Residuals 175 120033     686
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Var($\hat{\beta}$):

```
> vcov(IMDPT)

            (Intercept)   DPTpct
(Intercept)     72.0677 -0.83317
DPTpct          -0.8332  0.01018
```

95 percent c.i.s:

```
> confint(IMDPT)

              2.5 %  97.5 %
(Intercept) 156.523 190.032
DPTpct       -1.775  -1.377
```

```
> SEs<-predict(IMDPT,interval="confidence")
> SEs
       fit    lwr    upr
1    25.10  20.53  29.68
3    17.22  12.05  22.40
4    23.53  18.84  28.21
.
.
<rows omitted>
.
.
189  21.95  17.15  26.75
190  39.29  35.36  43.23
191  17.22  12.05  22.40
```

A Plot, With CIs

Scatterplot of Infant Mortality and DPT Immunizations, along with Least-Squares Line and 95% Prediction Confidence Intervals