

# PLSC 502 – Autumn 2016

## Random Variables

September 22, 2016

# Continuous and Discrete Variables

## Discrete Variables

- $X \in S = \{s_0, s_1, \dots\}$
- $\Pr(s) \geq 0$  for each  $s \in S$
- $\sum \Pr(s) = 1$

## Continuous Variables

- $X \in S \in \Re$
- $\exists f(x)$  such that for any closed interval  $[a, b]$   
 $\Pr(a < x \leq b) = \int_a^b f(x)dx.$
- Requires:
  - $f(x) \geq 0$  for all  $x$
  - $\int_{-\infty}^{\infty} f(x)dx = 1$

# Probability Density Function

The PDF is the function  $f(x)$  that maps the possible values of  $X$  to some associated probability of their occurrence.

Discrete  $X$ :

$$f(x) = \Pr(X = x) \forall x$$

Continuous  $X$ :

$$\Pr(a < X \leq b) = \int_a^b f(x) dx$$

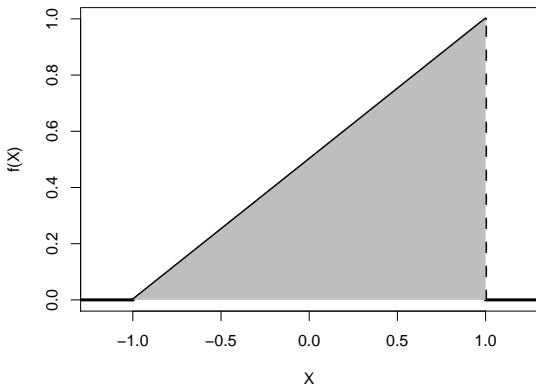
Again: Requires:

- $f(x) \geq 0$  for all  $x$
- $\int_{-\infty}^{\infty} f(x) dx = 1$

# An Example

Consider:

$$f(X) = \begin{cases} \frac{X+1}{2} & \text{for } -1 \leq y < 1, \\ 0 & \text{otherwise.} \end{cases}$$



1. Is  $f(x) \geq 0 \forall x$ ? – Yes.

2. Is  $\Pr(-\infty \leq x \leq \infty) \equiv \Pr(-1 \leq x \leq 1) = \int_{-1}^1 f(x)dx = 1$ ?

$$\begin{aligned}\Pr(-1 \leq Y \leq 1) &= \int_{-1}^1 \frac{1}{2}(x+1)dx \\&= \frac{1}{2} \left( \frac{x^2}{2} + x \right) \Big|_{-1}^1 \\&= \frac{1}{2} \left( \frac{1^2}{2} + 1 \right) - \frac{1}{2} \left( \frac{-1^2}{2} - 1 \right) \\&= 0.75 - (-0.25) \\&= 1.\end{aligned}$$

# Cumulative Distribution Function (CDF)

The CDF is the probability that  $X$  will take on a value less than or equal to than some value  $x$  in its range.

Discrete  $X$ :

$$\begin{aligned}\Pr(X \leq x) \equiv F(x) &= \sum_{X \leq x} \Pr(X = x) \\ &= 1 - \sum_{X > x} \Pr(X = x)\end{aligned}$$

## Continuous $X$ :

$$\Pr(X \leq x) \equiv F(x) = \int_{-\infty}^x f(t) dt$$

## Properties:

- $0 \leq F(x) \leq 1$ .
- Nondecreasing in  $X$ .
- $\Pr(x < k) = 1 - F(k)$ .
- $\Pr(a < x \leq b) = F(b) - F(a)$ .
- $F(-\infty) = 0$ .
- $F(\infty) = 1$ .

## Example, Again

For

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

we already know that  $\int_{-1}^1 f(x)dx = 1$ .

$$\begin{aligned} F(x) &= \int_{-1}^1 f(t)dt \\ &= \int_{-1}^1 \frac{1}{2}(t+1)dt \\ &= \frac{1}{2} \left( \frac{t^2}{2} + t \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left( \frac{1}{2}t^2 + t \right) + c. \end{aligned}$$



## Expected Value

For  $X$  with PDF  $f(x)$  and CDF  $F(x) = \int_{-\infty}^x f(t) dt$ , the *expected value* of  $X$  ( $E(X)$  or  $\mu$ ) is the *probability-weighted mean of the potential values of that variable*.

Discrete  $X$ :

$$E(X) = \sum_x [x \times f(x)]$$

E.g., number of heads in two coin flips:

0 Heads	Prob. = .25	Prob×Value = .25×0	=	0
1 Head	Prob. = .50	Prob×Value = .50×1	=	.50
2 Heads	Prob. = .25	Prob×Value = .25×2	=	.50
			$\Sigma$	= 1.0

# Expected Value (continued)

Continuous X:

$$E(X) = \int [x \times f(x)] dx$$

Properties:

- $E(c) = c$
- $E(x + y + z) = E(x) + E(y) + E(z)$
- If  $g(x)$  is some function of  $x$ , then

$$\begin{aligned} E[g(x)] &= \sum [g(x) \times \text{Prob}(X = x)] \forall x \text{ (discrete case)} \\ &= \int g(x)f(x) dx \text{ (continuous case)} \end{aligned}$$

- This includes a constant function:  $E(cx) = cE(x)$ .
- Implies that for  $g(x) = a + bx$ ,  $E(a + bx) = a + bE(x)$ .

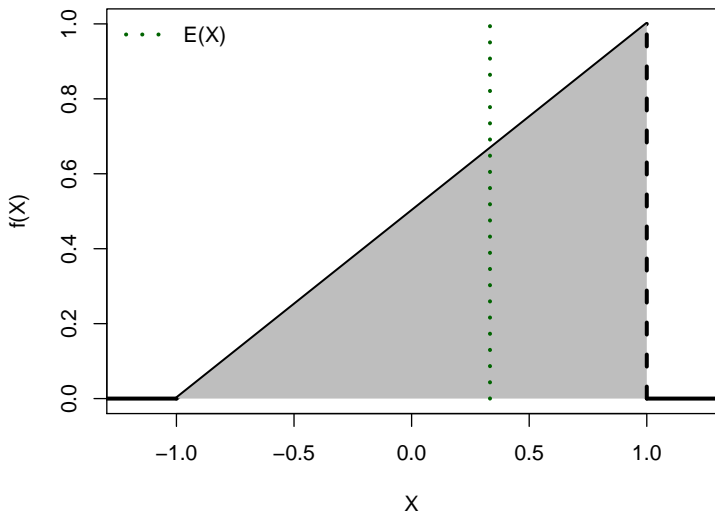
## Example Again

For

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is  $E(X)$ ?

$$\begin{aligned} E[f(X)] &= \int_{-1}^1 x \left( \frac{x+1}{2} \right) dx \\ &= \int_{-1}^1 \frac{1}{2}(x^2 + x) dx \\ &= \frac{1}{2} \int_{-1}^1 x^2 dx + \frac{1}{2} \int_{-1}^1 x dx \\ &= \frac{1}{2} \left( \frac{x^3}{3} + c_1 \right) \Big|_{-1}^1 + \frac{1}{2} \left( \frac{x^2}{2} + c_2 \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left( \frac{x^3}{3} + \frac{x^2}{2} + c_3 \right) \Big|_{-1}^1 \\ &= \frac{1}{2} \left[ \left( \frac{1^3}{3} + \frac{1^2}{2} + c_3 \right) - \left( \frac{-1^3}{3} + \frac{-1^2}{2} + c_3 \right) \right] \\ &= \frac{1}{3} \end{aligned}$$



Generally:

$$\text{Var}(X) = E[(x - \mu)^2]$$

Discrete  $X$ :

$$\text{Var}(X) = \sum (x - \mu)^2 f(x)$$

Continuous  $X$ :

$$\text{Var}(X) = \int (x - \mu)^2 f(x) dx$$

## Variance (continued)

$$\begin{aligned} E[(x - \mu)^2] \equiv \sigma^2 &= E[x^2 - 2x\mu + \mu^2] \\ &= E(x^2) - 2\mu E(x) + E(\mu^2) \\ &= E(x^2) - 2\mu^2 + \mu^2 \\ &= E(x^2) - \mu^2 \\ &\equiv \left( \int x^2 f(x) dx - \mu^2 \right) \end{aligned}$$

- We often write the variance as  $\sigma^2$ , and the positive square root of it (the standard deviation) as  $\sigma$ .
- This also implies that the expectation of the square of a variable  $X$  is  $E(x^2) = \sigma^2 + \mu^2$ .

# Variance Properties

- $\text{Var}(X) > 0$ , except
- $\text{Var}(c) = 0$
- $\text{Var}(a + bX) = b^2\text{Var}(X)$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$

## Example Again

What is the variance of  $f(X) = \frac{1}{2}(x+1)$  for the range  $-1 < x < 1$ ?

Recall that  $\mu = 1/3$ :

$$\begin{aligned} E[(x - \mu)^2] \equiv \sigma^2 &= \int_{-1}^1 X^2 f(x) dx - \mu^2 \\ &= \int_{-1}^1 \frac{1}{2} x^2 (x+1) dx - \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{2} \left(\frac{x^4}{4} + c_1\right) \Big|_{-1}^1 + \frac{1}{2} \left(\frac{x^3}{3} + c_2\right) \Big|_{-1}^1 - \frac{1}{9} \\ &= \frac{1}{2} \left(\frac{X^4}{4} + \frac{X^3}{3} + c_3\right) \Big|_{-1}^1 - \frac{1}{9} \\ &= \frac{1}{2} \left[ \left(\frac{1^4}{4} + \frac{1^3}{3} + c_3\right) - \left(\frac{-1^4}{4} + \frac{-1^3}{3} + c_3\right) \right] - \frac{1}{9} \\ &= \frac{19}{72} (\approx 0.2639). \end{aligned}$$



The  $k$ th moment of  $X$  is:

$$M_k = E(X^k)$$

The  $k$ th moment exists if:

$$\begin{aligned} E(|X|^k) &< \infty \\ &= \int_{-\infty}^{\infty} |x|^k f(x) dx < \infty \text{ (for continuous } X) \end{aligned}$$

“Central” moments:

$$\mu_k = E[(X - \mu)^k]$$

# Moment-Generating Functions

For  $t \in \Re$ ,

$$\psi(t) = E[\exp(tX)]$$

For continuous  $X$ :

$$\begin{aligned}\psi(t) &= \int_{-\infty}^{\infty} \exp(tx) f(x) dx \\ &= \int_{-\infty}^{\infty} \left( 1 + tx + \frac{t^2 x^2}{2!} + \dots \right) f(x) dx \\ &= 1 + tE(X) + \frac{t^2 E(X^2)}{2} + \dots \\ &= 1 + tM_1 + \frac{t^2 M_2}{2} + \dots\end{aligned}$$

Note that:

$$\begin{aligned}\psi(0) &= E[\exp(0)] \\ &= 1\end{aligned}$$

# MGFs Can Be Useful

First:

$$\psi(t) = \int_{-\infty}^{\infty} \exp(tx) dF(x).$$

Second:

$$\begin{aligned} \left. \frac{\partial^k \psi(t)}{\partial^k t} \right|_{t=0} &= \left. \frac{\partial^k E[\exp(tX)]}{\partial^k t} \right|_{t=0} \\ &= E \left[ \left. \frac{\partial^k \exp(tX)}{\partial^k t} \right|_{t=0} \right] \\ &= E\{[X^k \exp(tX)]|_{t=0}\} \\ &= E(X^k) \end{aligned}$$