

PLSC 503 – Spring 2017

Bivariate Regression, I

January 19, 2017

$$Y_i = \mu + u_i \quad (1)$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

so:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \quad (2)$$

Goals:

- Estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
- Estimate the *variability* $\hat{\beta}_0$ and $\hat{\beta}_1$

If we have $\hat{\beta}_0$ and $\hat{\beta}_1$, then:

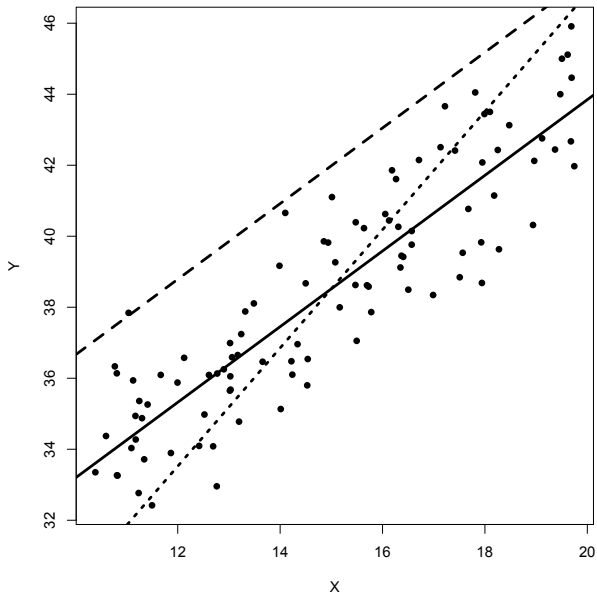
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \quad (3)$$

and

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \end{aligned} \quad (4)$$

Q: How to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$?

Scatterplot: X and Y (with regression lines)



Ordinary Least Squares

Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize $\hat{S} = \sum_{i=1}^N \hat{u}_i^2$.

$$\begin{aligned}\hat{S} &= \sum_{i=1}^N (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^N (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\ &= \sum_{i=1}^N (Y_i^2 - 2Y_i\hat{\beta}_0 - 2Y_i\hat{\beta}_1 X_i + \hat{\beta}_0^2 + 2\hat{\beta}_0\hat{\beta}_1 X_i + \hat{\beta}_1^2 X_i^2)\end{aligned}$$

Differentiate:

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= \sum_{i=1}^N (-2Y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 X_i) \\ &= -2 \sum_{i=1}^N (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) \\ &= -2 \sum_{i=1}^N \hat{u}_i\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= \sum_{i=1}^N (-2Y_i X_i + 2\hat{\beta}_0 X_i + 2\hat{\beta}_1 X_i^2) \\ &= -2 \sum_{i=1}^N (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) X_i \\ &= -2 \sum_{i=1}^N \hat{u}_i X_i\end{aligned}$$

Yields:

$$\sum_{i=1}^N Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^N X_i$$

and

$$\sum_{i=1}^N Y_i X_i = \hat{\beta}_0 \sum_{i=1}^N X_i + \hat{\beta}_1 \sum_{i=1}^N X_i^2$$

Solving yields:

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^N (X_i - \bar{X})^2} \\ &= \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X}\end{aligned}\tag{5}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}\tag{6}$$

Infant Mortality Data

```
> url <- getURL("https://raw.githubusercontent.com/PrisonRodeo/
  PLSC503-Spring-2016-git/master/Data/CountryData2000.csv")
> Data <- read.csv(text = url) # read the "votes" data
> rm(url)
>
> # Summary statistics
>
> # install.packages("psych") <- Install psych package, if necessary
> library(psych)

> with(Data, describe(infantmortalityperK))
  vars    n  mean    sd median trimmed  mad min max range skew kurtosis   se
1     1 179 43.83 40.39     29   38.38 34.26 2.9 167 164.1    1    0.06 3.02

> with(Data, describe(DPTpct))
  vars    n  mean    sd median trimmed  mad min max range skew kurtosis   se
1     1 181 81.71 19.77     90   85.23 11.86 24  99   75 -1.31    0.57 1.47
```

OLS Regression

```
> IMDPT<-lm(infantmortalityperK~DTPpct,data=Data,na.action=na.exclude)
> summary.lm(IMDPT)
```

Call:

```
lm(formula = infantmortalityperK ~ DTPpct, data = Data)
```

Residuals:

Min	1Q	Median	3Q	Max
-56.801	-16.328	-5.105	11.777	86.590

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	173.2771	8.4893	20.41	<2e-16 ***
DTPpct	-1.5763	0.1009	-15.62	<2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 26.19 on 175 degrees of freedom

(14 observations deleted due to missingness)

Multiple R-squared: 0.5824, Adjusted R-squared: 0.58

F-statistic: 244.1 on 1 and 175 DF, p-value: < 2.2e-16

Analysis of Variance

```
> anova(IMDPT)
```

```
Analysis of Variance Table
```

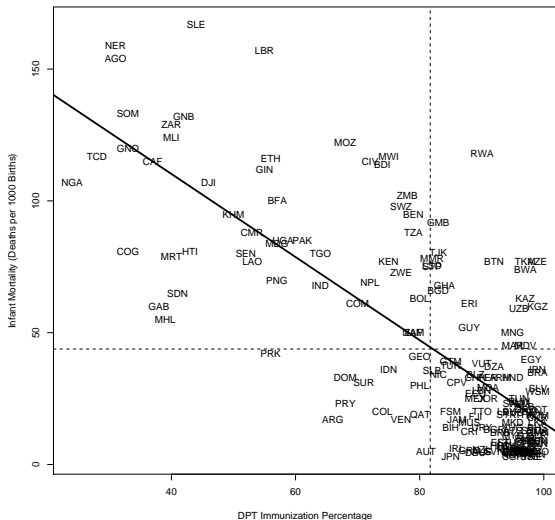
```
Response: infantmortalityperK
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
DPTpct	1	167423	167423	244.09	< 2.2e-16 ***
Residuals	175	120033	686		

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

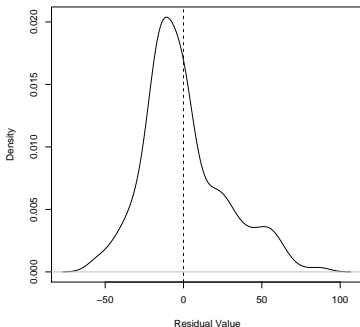
Regression of Infant Mortality on DPT Immunization Rates



Fitted Values, Residuals, etc.

```
> # Residuals (u):  
> Data$IMDPTres <- with(Data, residuals(IMDPT))  
> describe(Data$IMDPTres)
```

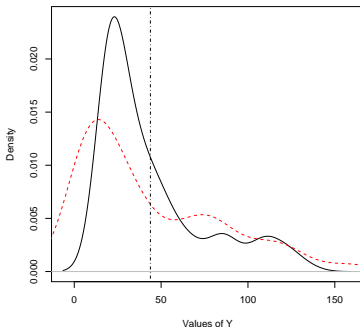
	var	n	mean	sd	median	mad	min	max	range	skew	kurtosis	se
1	1	177	0	26.12	-5.1	19.42	-56.8	86.59	143.4	0.75	0.44	1.96



```
> # Fitted Values:  
> Data$IMDPThat<-fitted.values(IMDPT)  
> describe(Data$IMDPThat)
```

	var	n	mean	sd	median	mad	min	max	range	skew	kurtosis	se
1	1	177	44.26	30.84	31.41	18.7	17.22	135.4	118.2	1.3	0.59	2.32

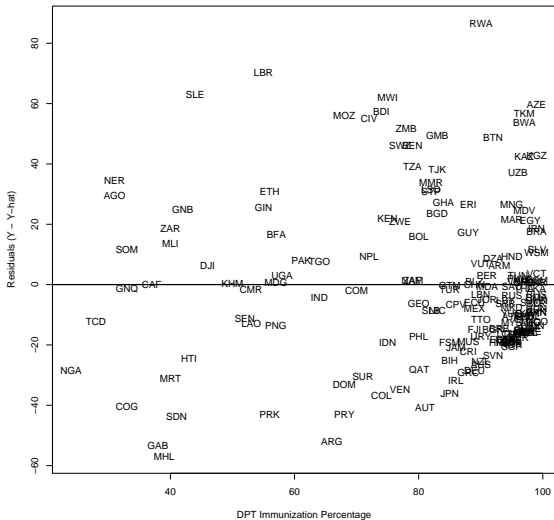
Figure: Density Plot: Actual (Y) and Fitted Values (\hat{Y})



Some Correlations

```
> with(Data, cor(infantmortalityperK,DPTpct,use="complete.obs"))  
[1] -0.7632  
  
> with(Data, cor(IMDPTres,infantmortalityperK,use="complete.obs"))  
[1] 0.6462  
  
> with(Data, cor(IMDPTres,DPTpct,use="complete.obs"))  
[1] 9.573e-17  
  
> with(Data, cor(IMDPTthat,infantmortalityperK,use="complete.obs"))  
[1] 0.7632  
  
> with(Data, cor(IMDPTthat,DPTpct,use="complete.obs"))  
[1] -1  
  
> with(Data, cor(IMDPTres,IMDPTthat,use="complete.obs"))  
[1] 5.302e-17
```

Regression Residuals (\hat{u}) vs. DPT Percentage



Squared Residuals vs. DPT Percentage

