PLSC 503 – Spring 2018
MLE: Testing and Inference

April 5, 2018

Testing: The Plan

- “The Trinity”
- An example
- Practical advice

Inference, In General

1. Pick some $\mathbf{H}_A : \boldsymbol{\Theta} = \boldsymbol{\Theta}_A$
2. Estimate $\hat{\boldsymbol{\Theta}}$
3. Determine distribution of $\hat{\boldsymbol{\Theta}}$ under $\mathbf{H}_A$
4. Use (2) and (3) $\rightarrow \hat{\mathbf{S}} \sim h(\boldsymbol{\Theta}, \hat{\boldsymbol{\Theta}})$ (*test statistic*)
5. Assess $\Pr(\hat{\mathbf{S}}|\mathbf{H}_A)$

MLEs and Inference

$$\hat{\boldsymbol{\Theta}} \overset{a}{\sim} \mathbf{N}[\boldsymbol{\Theta}, \mathbf{I}(\hat{\boldsymbol{\Theta}})]$$

Means that

$$\frac{\hat{\theta}_k - \theta_k}{\sqrt{\hat{\sigma}_k^2}} \sim N(0, 1)$$

Single Coefficients: Significance Testing

- Choose θ_A
- Estimate $\hat{\theta}_k$, $\hat{\sigma}_k^2$
- Compare $z_k = \frac{\hat{\theta}_k - \theta_A}{\sqrt{\hat{\sigma}_k^2}}$ to a z-table
- (Or, just look at your output...)

Single Coefficients: Confidence Intervals

- $\alpha \in (0,1) =$ desired level of "significance"
- $(1-\alpha) \times 100$-percent confidence intervals for $\hat{\theta}_k$ are:

$$\hat{\theta}_k \pm \left(z_\alpha \sqrt{\hat{\sigma}_k^2} \right)$$

- (Or just look at your output...)

More General Tests: "The Trinity"

- Likelihood-Ratio ("LR")
- Wald
- Lagrangian Multiplier ("Score")

Linear Restrictions

$$\mathbf{R}\boldsymbol{\Theta} = \mathbf{r}$$

$$\theta_2 = -2 \iff (0\ 1\ 0)\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = -2$$

Linear Restrictions

$$\mathbf{\Theta}_A : \theta_2 = 1, \theta_1 = 2\theta_3$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$r = \text{rows}(\mathbf{R}) \in [0, K]$$

LR Test

$$L(\hat{\mathbf{\Theta}}) \geq L(\mathbf{\Theta}_{\mathbf{A}}), \text{ but}$$

By how much?

Odds of one thing vs. another:

$$\frac{\Pr(\text{Something})}{\Pr(\text{Something Else})}$$

LR Test

$$\frac{L(\boldsymbol{\Theta}_{\mathbf{A}})}{L(\hat{\boldsymbol{\Theta}})} \ \ (\leq 1)$$

Suggests:

$$\ln L(\boldsymbol{\Theta}_{\mathbf{A}}) - \ln L(\hat{\boldsymbol{\Theta}}) \ \ (\leq 0)$$

$$-2[\ln L(\boldsymbol{\Theta}_{\mathbf{A}}) - \ln L(\hat{\boldsymbol{\Theta}})] \ \overset{a}{\sim} \ \chi^2_r$$

LR Test

- Intuition: Difference in $\ln L$ under constraint(s)
- Asymptotic
- Unreliable if $r > 100$ (or so)
- Easy to compute, but
- Requires that we have $\ln L(\boldsymbol{\Theta}_{\mathbf{A}})$ *and* $\ln L(\hat{\boldsymbol{\Theta}})$

Wald Tests

Idea: If $\boldsymbol{\Theta}_{\mathbf{A}}$, then

$$\mathbf{R}\boldsymbol{\Theta} = \mathbf{r}$$

$$\mathbf{R}\boldsymbol{\Theta} - \mathbf{r} = \mathbf{0}$$

Wald Tests (continued)

But...

- We have only $\hat{\boldsymbol{\Theta}}$ (from sample data)
- Possible that $\mathbf{R}\hat{\boldsymbol{\Theta}} - \mathbf{r} = \mathbf{0}$ *due to chance* (sampling variability).
- Solution: Account for *variability* in $\hat{\boldsymbol{\Theta}}$.

Wald Tests (continued)

Test:

$$\mathbf{W} = (\mathbf{R}\hat{\boldsymbol{\Theta}} - \mathbf{r})' \left[\mathbf{R}\,\mathrm{Var}(\hat{\boldsymbol{\Theta}})\,\mathbf{R}'\right]^{-1} (\mathbf{R}\hat{\boldsymbol{\Theta}} - \mathbf{r})$$

$$\mathbf{W} \overset{a}{\sim} \chi^2_r$$

Two-Handed Wald Tests

- (+) Easy, fast
- (+) No need for $\ln L(\boldsymbol{\Theta}_{\mathbf{A}})$
- (-) Uses $\text{Var}(\hat{\boldsymbol{\Theta}})$, not $\text{Var}(\boldsymbol{\Theta}_{\mathbf{A}})$ (potentially poor coverage)
- (-) Can yield nonsensical results

Lagrange Multiplier (LM) Tests

Idea: If $\boldsymbol{\Theta}_{\mathbf{A}}$, then

$$\left.\frac{\partial \ln L}{\partial \theta}\right|_{\boldsymbol{\Theta}_{\mathbf{A}}} \approx \mathbf{0}$$

Consider a new problem:

$$\max_{\boldsymbol{\Theta}} \left[L(\boldsymbol{\Theta}) - \boldsymbol{\lambda}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_{\mathbf{A}}) \right]$$

LM Tests

Yields:

$$\tilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta}_{\mathsf{A}}$$

$$\tilde{\boldsymbol{\lambda}} = \mathbf{g}(\tilde{\boldsymbol{\Theta}})$$

Suggests

$$LM = \mathbf{g}(\tilde{\boldsymbol{\Theta}})' \, \mathbf{I}(\tilde{\boldsymbol{\Theta}})^{-1} \mathbf{g}(\tilde{\boldsymbol{\Theta}})$$

LM Tests

$$LM \overset{a}{\sim} \chi^2_r$$

Note: No need for $\hat{\boldsymbol{\Theta}}$!

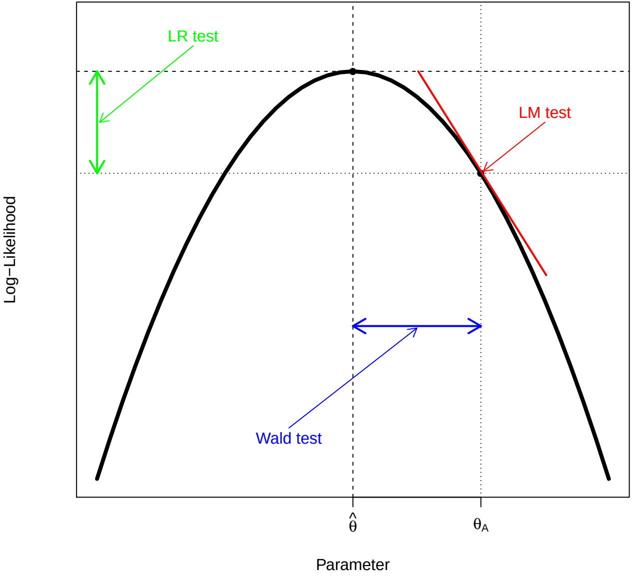
Tests, Conceptually (C. Franklin remix)

- The LR asks, "Did the likelihood change much under the null hypotheses versus the alternative?"
- The Wald test asks, "Are the estimated parameters very far away from what they would be under the null hypothesis?"
- The LM test asks, "If I had a less restrictive likelihood function, would its derivative be close to zero here at the restricted ML estimate?"

Tests, Conceptually (h.t.: Buse 1982)

- LR test $\approx$ manic mountaineer
- Wald test $\approx$ tired mountaineer
- LM test $\approx$ lazy mountaineer

Tests, Conceptually (A Picture)



Tests, Practically

- All are asymptotically identical...
- Require different estimates, but similar information
- Generally, LR $>$ Wald $>$ LM

Software: R

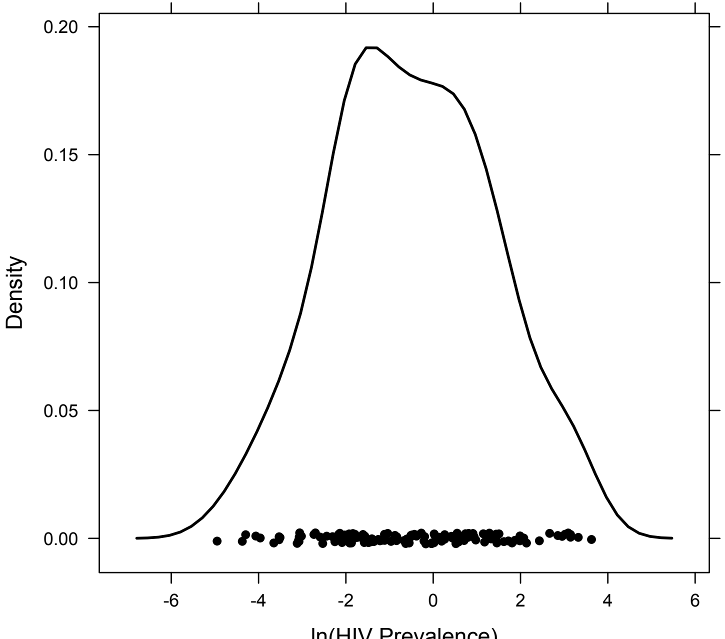
- Wald tests: `waldtest` (in `lmtest`), `wald.test` (in `aod`), etc.
- LR tests: `lrtest` (in `lmtest`), `RLRsim`, many others
- "by-hand" straightforward...

Software: Stata

- `test`, `testnl` $\rightarrow$ Wald tests
- `lrtest` $\rightarrow$ LR tests
- `waldtest` in `ml`
- LM tests require `enumopt`, `testomit` (see the example here)

Example: HIV Rates, 2005

- HIV prevalence rates, 144 countries
- Source: UNAIDS
- (Badly) Skewed $\rightarrow$ logged
- We're guessing $\sim N(\mu, \sigma^2)$...


Density
0.20
0.15
0.10
0.05
0.00
-6
-4
-2
0
2
4
6
ln(HIV Prevalence)

Preliminaries

```
> library(maxLik)
> library(aod)
> library(lmtest)
> HIV<-read.dta("HIV2005.dta")
> attach(HIV)

> HIVll <- function(param) {
+   mu <- param[1]
+   sigma <- param[2]
+   ll <- -0.5*log(sigma^2) - (0.5*((x-mu)^2/sigma^2))
+   ll
+  }

> x<-logHIV
```

Estimation

```
> hats <- maxLik(HIVll, start=c(0,1))
> summary(hats)
--------------------------------------------
Maximum Likelihood estimation
Newton-Raphson maximisation, 7 iterations
Return code 1: gradient close to zero. May be a solution
Log-Likelihood: -159.5
2  free parameters
Estimates:
     Estimate Std. error t value Pr(> t)
[1,]   -0.500      0.153   -3.27  0.0011 **
[2,]    1.836      0.108   16.97  <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
--------------------------------------------
```

≡ Mean-Only Linear Model

```
> HIVLM<-lm(logHIV~1)
> summary(HIVLM)

Call:
lm(formula = logHIV ~ 1)

Residuals:
    Min      1Q  Median      3Q     Max
-4.4493 -1.3474 -0.0622  1.3012  4.1264

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   -0.500      0.154   -3.26   0.0014 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
Residual standard error: 1.84 on 143 degrees of freedom
```

Moving parts...

```
> hats$estimate
[1] -0.5002  1.8357
```

```
> hats$gradient
[1] -1.645e-05  1.749e-04
```

```
> hats$hessian
       [,1]   [,2]
[1,] -42.73  -2.09
[2,]  -2.09 -63.63
```

More moving parts...

```
> -(solve(hats$hessian))
            [,1]       [,2]
[1,]  2.340e-02 -2.432e-07
[2,] -2.432e-07  1.170e-02
```

```
> sqrt(-(solve(hats$hessian)))
       [,1]   [,2]
[1,] 0.1530    NaN
[2,]    NaN 0.1082
```

Wald tests

```
> wald.test(Sigma=vcov(hats),b=coef(hats),Terms=1:2,verbose=TRUE)

Wald test:
----------
Coefficients:
[1] -0.5  1.8

Var-cov matrix of the coefficients:
      [,1]  [,2]
[1,] 0.023 0.000
[2,] 0.000 0.012

Test-design matrix:
   [,1] [,2]
L1    1    0
L2    0    1
```

(continued)

```
Positions of tested coefficients in the vector of coefficients: 1, 2

H0:  -0.5002095 = 0; 1.8357192 = 0

Chi-squared test:
X2 = 298.7, df = 2, P(> X2) = 0.0
```

More Wald tests

```
> wald.test(Sigma=vcov(hats),b=coef(hats),Terms=1:2,H0=c(0,2))
Wald test:
----------

Chi-squared test:
X2 = 13.0, df = 2, P(> X2) = 0.0015
```

```
> wald.test(Sigma=vcov(hats),b=coef(hats),Terms=1:2,H0=c(-0.5,2))
Wald test:
----------

Chi-squared test:
X2 = 2.3, df = 2, P(> X2) = 0.32
```

Even More Wald tests (equivalence)

```
> wald.test(Sigma=vcov(hats),b=coef(hats),Terms=2:2,H0=2)
Wald test:
----------

Chi-squared test:
X2 = 2.3, df = 1, P(> X2) = 0.13
```

```
> ((1.836-2)/.108)^2
[1] 2.306

> pchisq(2.306,df=1,lower.tail=FALSE)
[1] 0.1289
```

A Nonsensical Wald Test

```
> wald.test(Sigma=vcov(hats),b=coef(hats),Terms=1:2,H0=c(1,-2))
Wald test:
----------

Chi-squared test:
X2 = 1353.6, df = 2, P(> X2) = 0.0
```

LR tests: Preliminaries

```
> HIVllOne <- function(param) {
+       mu <- param[1]
+    ll <- -0.5*log(4) - (0.5*((x - mu)^(2)/4))
+    ll
+  }
```

```
> hatsF <- maxLik(HIVll, start=c(0,1))
> hatsR <- maxLik(HIVllOne, start=c(0))
```

LR tests

```
> hatsF$maximum

[1] -159.5

> hatsR$maximum
[1] -160.5

> -2*(hatsR$maximum-hatsF$maximum)
[1] 1.999861

> pchisq(-2*(hatsR$maximum-hatsF$maximum),df=1,lower.tail=FALSE)
[1] 0.1573
```

Linear Model Redux

Linear Model: $\text{Var}(\hat{\beta})$ with $\mathbf{uu}' = \sigma^2\Omega$:

$$\begin{aligned} \text{Var}(\beta_{\text{Het.}}) &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{Q}\,(\mathbf{X}'\mathbf{X})^{-1} \end{aligned}$$

where $\mathbf{Q} = (\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})$ and $\mathbf{W} = \sigma^2\boldsymbol{\Omega}$.

Rewrite:

$$\begin{aligned} \mathbf{Q} &= \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}) \\ &= \sum_{i=1}^{N} \sigma_i^2 \mathbf{X}_i \mathbf{X}_i' \end{aligned}$$

White's Insight:

$$\widehat{\mathbf{Q}} = \sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'$$

$$\begin{aligned}
\widehat{\text{Var}(\boldsymbol{\beta})}_{\text{Robust}} &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\widehat{\mathbf{Q}}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\left[\mathbf{X}'\left(\sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'\right)^{-1} \mathbf{X}\right](\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

Recall:

$$\begin{aligned}\mathsf{Var}(\hat{\theta}) &= \mathsf{E}[(\hat{\theta}-\theta)(\hat{\theta}-\theta)'] \\ &= \mathsf{E}\left[\left(-\frac{\partial^2 \ln L}{\partial\theta^2}\right)^{-1}\frac{\partial \ln L}{\partial\theta}\frac{\partial \ln L}{\partial\theta}'\left(-\frac{\partial^2 \ln L}{\partial\theta^2}\right)^{-1}\right]\end{aligned}$$

We assumed:

$$\mathsf{E}\left[\frac{\partial \ln L}{\partial\theta}\frac{\partial \ln L}{\partial\theta}'\right] = \mathsf{E}\left[\frac{\partial^2 \ln L}{\partial\theta^2}\right]$$

So, "naive" is:

$$\begin{aligned}\mathsf{Var}(\hat{\theta}) &= \left[-\mathsf{E}\left(\frac{\partial^2 \ln L}{\partial\theta^2}\right)\right]^{-1} \\ &= [\mathbf{I}(\theta)]^{-1}\end{aligned}$$

"Robust" MLE

Alternatively:

$$\mathsf{Var}(\hat{\theta})_{\mathsf{Robust}} \quad = \quad [\mathbf{I}(\theta)]^{-1} \left(\frac{\partial \ln L}{\partial \hat{\theta}} \frac{\partial \ln L'}{\partial \hat{\theta}} \right) [\mathbf{I}(\theta)]^{-1}$$

Appendix: Optimization Using Stata

Software Issues: Stata

`ml` is `it`...

- Syntax is

```
.ml model <method> <progname> <eq>...
.ml maximize
```

- Optimizers are Newton, BHHH, BFGS, and DFP
- Many, many options...

Stata Example

The Rayleigh again...

```
. set obs 100
. gen rayleigh = 3*sqrt(-2*ln(1-(uniform())))
. program define loglik
     args lnf beta
     qui replace ‘lnf’ = (ln($ML_y1)-ln(‘beta’^2))
        + ((-$ML_y1^2)/(2*‘beta’^2))
  end
```

Stata : Example

```
. ml model lf loglik (rayleigh = one, noconstant)
. ml search
. ml maximize
                                                  Number of obs   =        100
                                                  Wald chi2(1)    =     400.00
Log likelihood = -198.07937                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
 rayleigh |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
----------+-------------------------------------------------------------------
      one |   2.886389   .1443195    20.00   0.000     2.603528     3.16925
------------------------------------------------------------------------------
```

Tests Using Stata

HIV Example: Stata Remix

```
. program define HIV
   args lnf beta theta
   qui replace ‘lnf’ = ln(normalden(($ML_y1-‘beta’) / ‘theta’))
   - ln(‘theta’)
end

. gen one=1
. ml model lf HIV (logHIV = one) /sigma
. ml search
```

HIV Example Redux: Results

```
. ml maximize
initial:       log likelihood = -302.50517
rescale:       log likelihood = -302.50517
rescale eq:    log likelihood = -302.50517
Iteration 0:   log likelihood = -302.50517
Iteration 1:   log likelihood = -294.30557
Iteration 2:   log likelihood = -291.79938
Iteration 3:   log likelihood = -291.79798
Iteration 4:   log likelihood = -291.79798
                                                  Number of obs   =        144
                                                  Wald chi2(0)    =          .
Log likelihood = -291.79798                       Prob > chi2     =          .
------------------------------------------------------------------------------
      logHIV |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
eq1          |
       _cons |  -.5002095   .1529766    -3.27   0.001    -.8000381   -.2003808
-------------+----------------------------------------------------------------
sigma        |
       _cons |   1.835719   .1081708    16.97   0.000     1.623708     2.04773
------------------------------------------------------------------------------
```

HIV Example Redux: Tests

```
. test [eq1]_cons [sigma]_cons

 ( 1)  [eq1]_cons = 0
 ( 2)  [sigma]_cons = 0

           chi2(  2) =  298.69
         Prob > chi2 =    0.0000
```

```
. test ([eq1]_cons=0) ([sigma]_cons=2)

 ( 1)  [eq1]_cons = 0
 ( 2)  [sigma]_cons = 2

           chi2(  2) =   13.00
         Prob > chi2 =    0.0015
```

HIV Example Redux: More Tests

```
. test ([eq1]_cons=-0.5) ([sigma]_cons=2)

 ( 1)  [eq1]_cons = -.5
 ( 2)  [sigma]_cons = 2

           chi2(  2) =    2.31
         Prob > chi2 =    0.3156
```

```
. test [sigma]_cons=2

 ( 1)  [sigma]_cons = 2

           chi2(  1) =    2.31
         Prob > chi2 =    0.1288
```

HIV Example Redux: Even More Tests

```
. testnl ([eq1]_cons=0) ([sigma]_cons=-2)

  (1)  [eq1]_cons = 0
  (2)  [sigma]_cons = -2

               chi2(2) =     1268.09
           Prob > chi2 =        0.0000
```

```
. test [sigma]_cons=2

 ( 1)  [sigma]_cons = 2

           chi2(  1) =    2.31
         Prob > chi2 =    0.1288
```