PLSC 503 – Spring 2018
Cases and Variables

March 1, 2018

Under the Hood of **X**

OLS (and regression methods more generally) requires:

- **X** is full column rank.
- $N > K$.
- "Sufficient" variability in **X**.

"Perfect" Multicollinearity

Formally: There cannot be any set of λs such that:

$$\lambda_0 \mathbf{1} + \lambda_1 \mathbf{X}_1 + ... + \lambda_K \mathbf{X}_K = \mathbf{0}$$

If there was, it would imply

$$\mathbf{X}_j = \frac{-\lambda_0}{\lambda_j}\mathbf{1} + \frac{-\lambda_1}{\lambda_j}\mathbf{X}_1 + ... + \frac{-\lambda_K}{\lambda_j}\mathbf{X}_K$$

which means

$$\begin{aligned}
Y &= \beta_0 \mathbf{1} + \beta_1 \mathbf{X}_1 + ... + \beta_j \mathbf{X}_j + ... + \beta_K \mathbf{X}_K + \mathbf{u} \\
&= \beta_0 \mathbf{1} + \beta_1 \mathbf{X}_1 + ... + \beta_j \left(\frac{-\lambda_0}{\lambda_j}\mathbf{1} + \frac{-\lambda_1}{\lambda_j}\mathbf{X}_1 + ... + \frac{-\lambda_K}{\lambda_j}\mathbf{X}_K \right) + ... + \beta_K \mathbf{X}_K + \mathbf{u} \\
&= \left[\beta_0 + \beta_j \left(\frac{-\lambda_0}{\lambda_j} \right) \right] \mathbf{1} + \left[\beta_1 + \beta_j \left(\frac{-\lambda_1}{\lambda_j} \right) \right] \mathbf{X}_1 + ... + \left[\beta_K + \beta_j \left(\frac{-\lambda_K}{\lambda_j} \right) \right] \mathbf{X}_K + \mathbf{u} \\
&= \left(\beta_0 + \frac{\gamma_0}{\lambda_j} \right) \mathbf{1} + \left(\beta_1 + \frac{\gamma_1}{\lambda_j} \right) \mathbf{X}_1 + ... + \left(\beta_K + \frac{\gamma_K}{\lambda_j} \right) \mathbf{X}_K + \mathbf{u}
\end{aligned}$$

In Practice

```
> Africa$newgdp<-(Africa$gdppppd-mean(Africa$gdppppd))*1000

> fit<-with(Africa, lm(adrate~gdppppd+newgdp+healthexp+subsaharan+
+                         muslperc+literacy))
> summary(fit)

Call:
lm(formula = adrate ~ gdppppd + newgdp + healthexp + subsaharan +
    muslperc + literacy)

Residuals:
    Min      1Q  Median      3Q     Max
-15.291  -4.329  -1.412   2.723  20.682

Coefficients: (1 not defined because of singularities)
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)           -7.78020   10.33872  -0.753   0.4565
gdppppd                0.36142    0.58214   0.621   0.5385
newgdp                      NA         NA      NA       NA
healthexp              1.87001    0.75667   2.471   0.0182 *
subsaharanSub-Saharan  3.64354    4.54163   0.802   0.4275
muslperc              -0.07908    0.05967  -1.325   0.1932
literacy               0.12445    0.09867   1.261   0.2151
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 7.665 on 37 degrees of freedom
Multiple R-squared:  0.4782,Adjusted R-squared:  0.4077
F-statistic: 6.782 on 5 and 37 DF,  p-value: 0.0001407
```

So...

- Perfect multicollinearity is terrible, but

- Perfect multicollinearity not a problem at all.

$N > K$...

Statistically,

- we lack sufficient degrees of freedom to identify $\hat{\boldsymbol{\beta}}$.
- $\hat{\boldsymbol{\beta}}$ is "overdetermined."

Conceptually:

- Variables $>$ Cases means
- ...no unique conclusion about explanatory / causal factors.

$N = K$ in Practice

```
> smallAfrica<-subset(Africa,subsaharan=="Not Sub-Saharan")
> fit2<-with(smallAfrica,lm(adrate~gdppppd+healthexp+muslperc+
+                             literacy+war))
> summary(fit2)

Call:
lm(formula = adrate ~ gdppppd + healthexp + muslperc + literacy +
    war)

Residuals:
ALL 6 residuals are 0: no residual degrees of freedom!

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.12430         NA      NA       NA
gdppppd     -0.97906         NA      NA       NA
healthexp   -0.45166         NA      NA       NA
muslperc     0.01413         NA      NA       NA
literacy     0.09512         NA      NA       NA
war         -0.96429         NA      NA       NA

Residual standard error: NaN on 0 degrees of freedom
Multiple R-squared:      1,Adjusted R-squared:    NaN
F-statistic:   NaN on 5 and 0 DF,  p-value: NA
```

High (Non-Perfect) Multicollinearity

Recall that

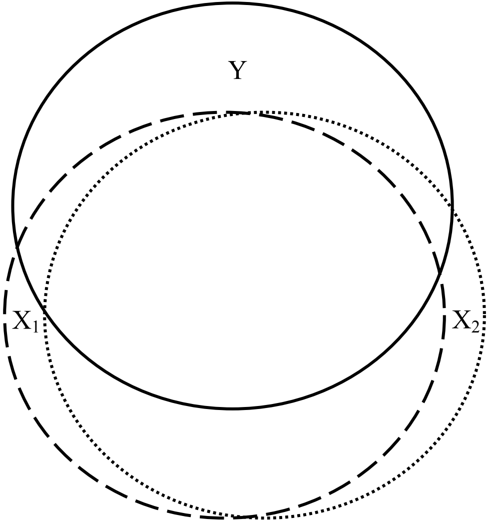
$$\widehat{\mathrm{Var}(\hat{\boldsymbol{\beta}})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

We can write the kth diagonal element of $(\mathbf{X}'\mathbf{X})^{-1}$ as:

$$\frac{1}{(\mathbf{X}_k'\mathbf{X}_k)(1 - \hat{R}_k^2)}$$

where $\hat{R}_k^2$ is the R^2 from the regression of $\mathbf{X}_k$ on all the other variables in $\mathbf{X}$.

The Obligatory Venn Diagram



High (Non-Perfect) Multicollinearity

Things to understand:

1. Multicollinearity is a *sample problem*.

2. Multicollinearity is a matter of *degree*.

Near-Perfect Collinearity: An Example

$$\text{HIV}_i = \beta_0 + \beta_1(\text{Civil War}_i) + \beta_2(\text{Intensity}_i) + u_i$$

```
> with(Africa, table(internalwar,intensity))

           intensity
internalwar  0  1  2  3
          0 30  0  0  0
          1  0  6  2  5
```

Table: Three Models

	Dependent variable:		
	adrate		
	(1)	(2)	(3)
internalwar	-4.459		-2.849
	(3.274)		(6.682)
intensity		-1.955	-0.837
		(1.481)	(3.018)
Constant	10.713^{***}	10.502^{***}	10.713^{***}
	(1.800)	(1.734)	(1.821)
Observations	43	43	43
R^2	0.043	0.041	0.045
Adjusted R^2	0.020	0.017	-0.003
Residual Std. Error	9.860 (df = 41)	9.873 (df = 41)	9.973 (df = 40)
F Statistic	1.855 (df = 1; 41)	1.743 (df = 1; 41)	0.945 (df = 2; 40)
Note:			*p<0.1; **p<0.05; ***p<0.01

(Near-Perfect) Multicollinearity: Detection

1. *High R^2, but nonsignificant coefficients.*
2. *High pairwise correlations among independent variables.*
3. *High partial correlations among the* $\mathbf{X}$*s.*
4. *VIF and Tolerance.*

VIF / Tolerance

If $\hat{R}_k^2 = 0$, then

$$\widehat{\text{Var}(\hat{\beta}_k)} = \frac{\hat{\sigma}^2}{\mathbf{X}_k'\mathbf{X}_k};$$

So:

$$\text{VIF}_k = \frac{1}{1 - \hat{R}_k^2}$$

$$\text{Tolerance} = \frac{1}{\text{VIF}_k}$$

Rule of Thumb: VIF > 10 is a problem...

What To Do?

Don't:

- **Drop Covariates!!!**
- Restrict βs...

Do:

- **Add data**.
- **Transform the covariates**
  - Data reduction
  - First differences
  - Orthogonalize
- **Shrinkage Methods** (e.g., "ridge regression")