PLSC 503 – Spring 2018
Multivariate Regression I

February 13, 2018

The Model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

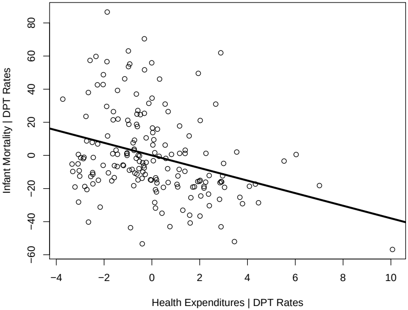
$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

Diversion: "Added Variable Plots"

- Regress Y on X_1 and save the residuals $\hat{u}_i$,
- Regress X_2 on X_1 and save the residuals (call these $\hat{e}_i$),
- Plot $\hat{u}_i$ (conventionally on the y-axis) vs. $\hat{e}_i$ (conventionally on the x-axis).

Example: Infant Mortality and Health Expenditures Given DPT Immunization Rates



Estimating $\boldsymbol{\beta}$

Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\boldsymbol{\beta}$$

The inner product of **u**:

$$\begin{aligned}
\mathbf{u}'\mathbf{u} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

Estimating $\boldsymbol{\beta}$

$$
\begin{aligned}
\mathbf{u}'\mathbf{u} &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\
&= \mathbf{Y}'\mathbf{Y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{Y}' + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}
\end{aligned}
$$

Now get:

$$
\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}
$$

Solve:

$$
\begin{aligned}
-2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\
-\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\
\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{X}'\mathbf{Y} \\
(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}
\end{aligned}
$$

OLS Assumptions

1. Zero Expectation Disturbances

$$\mathsf{E}(\mathbf{u}) = \mathbf{0}$$

OLS Assumptions

2. Homoscedasticity / No Error Correlation

$$\mathbf{u}\mathbf{u}' = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_N \\ u_2 u_1 & u_2^2 & \cdots & u_2 u_N \\ \vdots & \vdots & \ddots & \vdots \\ u_N u_1 & u_N u_2 & \cdots & u_N^2 \end{bmatrix}$$

Expectation must be:

$$\mathrm{E}(\mathbf{u}\mathbf{u}') = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

OLS Assumptions

3. "Fixed" $\mathbf{X}$...

- No *measurement error* in the $\mathbf{X}$s, and
- $\text{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$.

4. $\mathbf{X}$ is full column rank.

Means:

- no exact linear relationship among $\mathbf{X}$, and
- $K < N$.

5. Normal Disturbances

$$\mathbf{u} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

OLS: Unbaisedness

Unbiasedness:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

Substitute OLS $\hat{\boldsymbol{\beta}}$:

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\ &= \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\end{aligned}$$

and so:

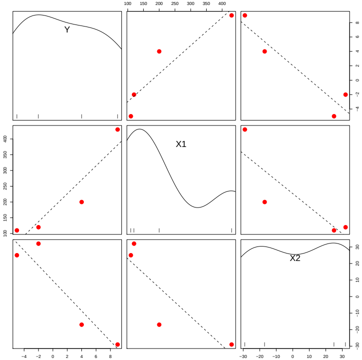
$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}.$$

By $\mathrm{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$, we have $\mathrm{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$.

An Example

$$\mathbf{Y} = \begin{bmatrix} 4 \\ -2 \\ 9 \\ -5 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 200 & -17 \\ 1 & 120 & 32 \\ 1 & 430 & -29 \\ 1 & 110 & 25 \end{bmatrix}$$



Example, continued

So:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 4 & 860 & 11 \\ 860 & 251400 & -9280 \\ 11 & -9280 & 2779 \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 3.2453 & -0.0132 & -0.05694 \\ -0.0132 & 0.000058 & 0.0002468 \\ -0.0569 & 0.000247 & 0.001409 \end{bmatrix}$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 6 \\ 3880 \\ 518 \end{bmatrix}$$

So:

$$\begin{aligned} \hat{\boldsymbol{\beta}} &= \begin{bmatrix} 3.2453 & -0.0132 & -0.05694 \\ -0.0132 & 0.000058 & 0.0002468 \\ -0.0569 & 0.000247 & 0.001409 \end{bmatrix} \begin{bmatrix} 6 \\ 3880 \\ 518 \end{bmatrix} \\ &= \begin{bmatrix} -2.264 \\ 0.0190 \\ -0.1141 \end{bmatrix} \end{aligned}$$

R Example: Correlation

```
Y<-c(4,-2,9,-5)
X1<-c(200,120,430,110)
X2<-c(-17,32,-29,25)
data<-cbind(Y,X1,X2)
```

```
cor(data)
```

```
        Y      X1      X2
Y   1.0000  0.9285 -0.9425
X1  0.9285  1.0000 -0.8613
X2 -0.9425 -0.8613  1.0000
```

Regression

```
fit<-lm(Y~X1+X2)
summary(fit)

Call:
lm(formula = Y ~ X1 + X2)

Residuals:
     1      2      3      4
 0.531  1.639 -0.201 -1.970

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.2643     4.7284   -0.48     0.72
X1            0.0190     0.0200    0.95     0.52
X2           -0.1141     0.0985   -1.16     0.45

Residual standard error: 2.62 on 1 degrees of freedom
Multiple R-Squared: 0.941,Adjusted R-squared: 0.823
F-statistic: 7.99 on 2 and 1 DF,  p-value: 0.243
```

Estimation Issues

"Do not compute the least squares estimates using $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$*!"*

– Sanford Weisberg (p. 61)

Most software uses:

$$\mathbf{X} = \mathbf{QR}$$

where $\mathbf{Q}$ is orthogonal ($\mathbf{Q}'\mathbf{Q} = \mathbf{I}$) and $\mathbf{R}$ is upper-triangular.

Why???

Estimation Example

```
options(digits=16)
options(scipen=99)

z<-c(-1000000000000,0.000000000000001,1000000000000)
x<-c(-50000,0.000007,5000000)
fit<-lm(z~x)
fit

Call:
lm(formula = z ~ x)

Coefficients:
          (Intercept)                         x
-494950994952.3740845         299970.2999707
```

Estimation Example (continued)

```
X<-as.matrix(x)
Z<-as.matrix(z)
beta.hat <- solve(t(X) %*% X) %*% t(X) %*% Z
beta.hat
                 [,1]
[1,] 201979.802019798
```

```
(fit$coefficients[2] / beta.hat) * 100
                  [,1]
[1,] 148.5149985152023
```