# PLSC 503 – Spring 2018
# Binary Response Models

April 10, 2018

# Linear Probability Model (LPM)

$$\mathrm{E}(Y) = \mathbf{X}\boldsymbol{\beta}$$

$$Y \in \{0, 1\}$$

$$\begin{aligned} \mathrm{E}(Y) &= 1[\Pr(Y = 1)] + 0[\Pr(Y = 0)] \\ &= \Pr(Y = 1) \end{aligned}$$

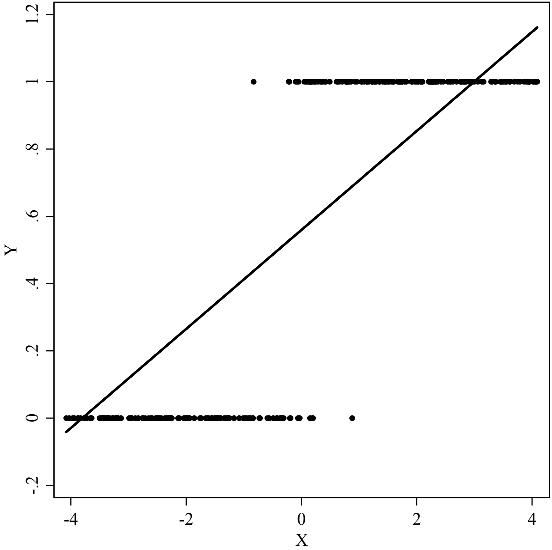
So:

$$\Pr(Y_i = 1) = \mathbf{X}_i\boldsymbol{\beta}$$

or:

$$Y_i = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

# LPM Illustrated



# LPM Issues

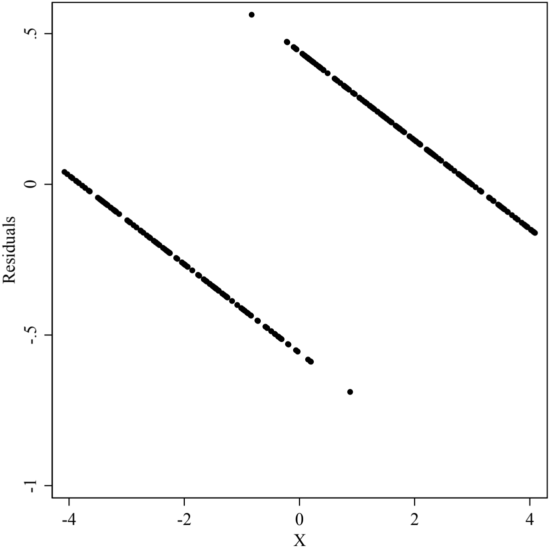
Variance:

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}(Y)[1 - \mathrm{E}(Y)] \\ &= \mathbf{X}_i\boldsymbol{\beta}(1 - \mathbf{X}_i\boldsymbol{\beta}) \end{aligned}$$

Residuals:

$$\hat{u}_i \in \{1 - \mathbf{X}_i\hat{\boldsymbol{\beta}}, -\mathbf{X}_i\hat{\boldsymbol{\beta}}\}$$

# LPM Residuals



# LPM Issues (continued)

- Predictions $\notin [0, 1]$
- Functional form $\rightarrow \frac{\partial \mathsf{E}(Y)}{\partial X} = \beta$ (reasonable?)

# A Different Model

$$Y_i^* = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

$$\begin{aligned} Y_i &= 0 \text{ if } Y_i^* < 0 \\ Y_i &= 1 \text{ if } Y_i^* \geq 0 \end{aligned}$$

So:

$$\begin{aligned} \Pr(Y_i = 1) &= \Pr(Y_i^* \geq 0) \\ &= \Pr(\mathbf{X}_i\boldsymbol{\beta} + u_i \geq 0) \\ &= \Pr(u_i \geq -\mathbf{X}_i\beta) \\ &= \Pr(u_i \leq \mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}} f(u)du \end{aligned}$$

# Logit

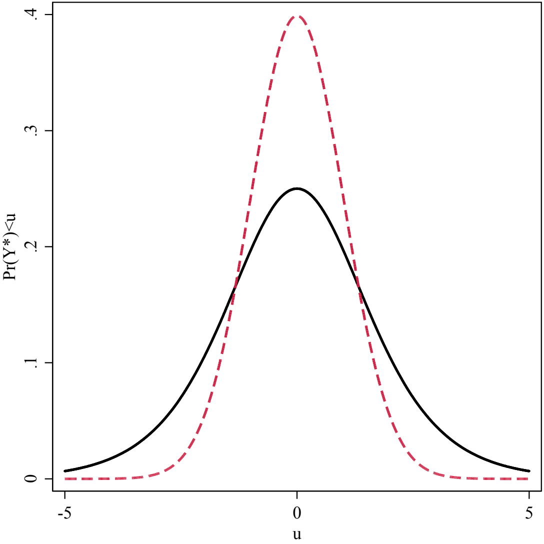
"Standard logistic" PDF:

$$\Pr(u) \equiv \lambda(u) = \frac{\exp(u)}{[1 + \exp(u)]^2}$$

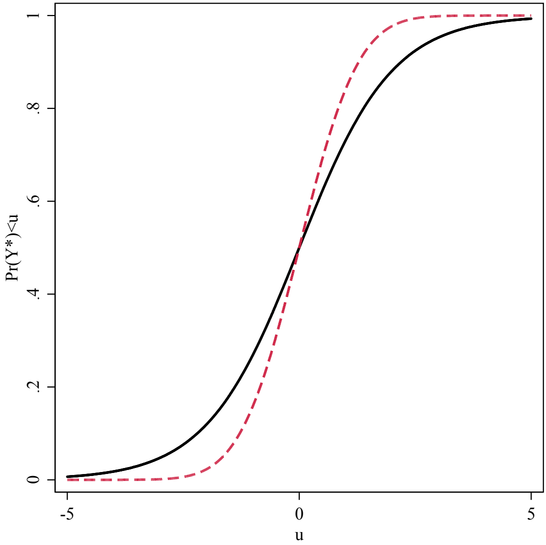
CDF:

$$\begin{aligned} \Lambda(u) &= \int \lambda(u) du \\ &= \frac{\exp(u)}{1 + \exp(u)} \\ &= \frac{1}{1 + \exp(-u)} \end{aligned}$$

# Standard Normal and Logistic PDFs



# Standard Normal and Logistic CDFs



- $\lambda(u) = 1 - \lambda(-u)$
- $\Lambda(u) = 1 - \Lambda(-u)$
- $\mathrm{Var}(u) = \frac{\pi^2}{3} \approx 3.29$

# Logistic → "Logit"

$$\begin{aligned}
\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) \\
&= \Pr(u_i \leq \mathbf{X}_i\boldsymbol{\beta}) \\
&= \Lambda(\mathbf{X}_i\boldsymbol{\beta}) \\
&= \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i\boldsymbol{\beta})}
\end{aligned}$$

$$\text{(equivalently)} = \frac{1}{1 + \exp(-\mathbf{X}_i\boldsymbol{\beta})}$$

$$L_i = \left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)^{Y_i}\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]^{1-Y_i}$$

$$L = \prod_{i=1}^{N}\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)^{Y_i}\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right]^{1-Y_i}$$

$$\begin{aligned} \ln L \quad = \quad & \sum_{i=1}^{N} Y_i \ln\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right) + \\ & (1-Y_i)\ln\left[1-\left(\frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}\right)\right] \end{aligned}$$

# Be Normal?

$$\Pr(u) \equiv \phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

$$\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

# Normal → "Probit"

$$\begin{aligned}
\Pr(Y_i = 1) &= \Phi(\mathbf{X}_i\boldsymbol{\beta}) \\
&= \int_{-\infty}^{\mathbf{X}_i\boldsymbol{\beta}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mathbf{X}_i\boldsymbol{\beta})^2}{2}\right) d\mathbf{X}_i\boldsymbol{\beta}
\end{aligned}$$

$$L = \prod_{i=1}^{N} \left[\Phi(\mathbf{X}_i\boldsymbol{\beta})\right]^{Y_i} \left[1 - \Phi(\mathbf{X}_i\boldsymbol{\beta})\right]^{(1-Y_i)}$$

$$\ln L = \sum_{i=1}^{N} Y_i \ln\Phi(\mathbf{X}_i\boldsymbol{\beta}) + (1 - Y_i) \ln[1 - \Phi(\mathbf{X}_i\boldsymbol{\beta})]$$

# Digression I: Logit as an Odds Model

$$\text{Odds}(Z) \equiv \Omega(Z) = \frac{\Pr(Z)}{1-\Pr(Z)}.$$

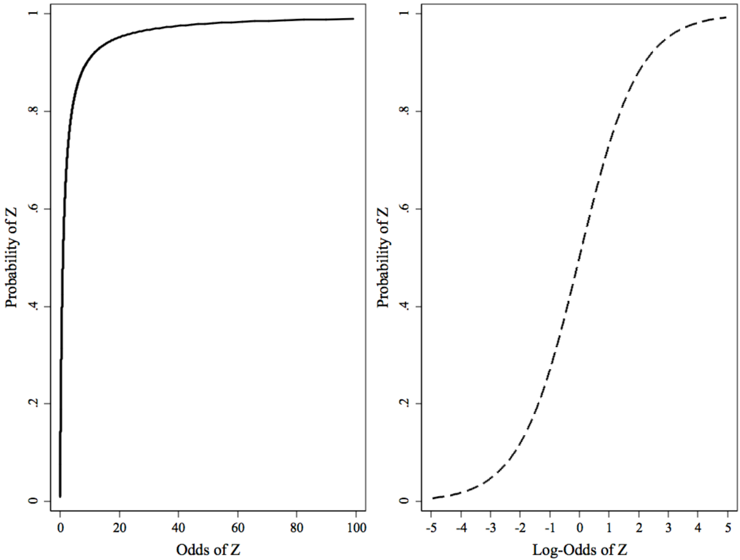
$$\ln[\Omega(Z)] = \ln\left[\frac{\Pr(Z)}{1-\Pr(Z)}\right]$$

$$\ln[\Omega(Z_i)] = \mathbf{X}_i\boldsymbol{\beta}$$

$$\begin{aligned}\Omega(Z_i) &= \frac{\Pr(Z)}{1-\Pr(Z)} \\ &= \exp(\mathbf{X}_i\boldsymbol{\beta})\end{aligned}$$

$$\Pr(Z_i) = \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1+\exp(\mathbf{X}_i\boldsymbol{\beta})}$$

# Visualizing Log-Odds



# Digression II: The Random Utility Model

$$Y \in \{SQ, A\}$$

$$\begin{aligned} Y_i &= A \quad \text{if} \quad \mathrm{E}[\mathrm{U}_i(A)] \geq \mathrm{E}[\mathrm{U}_i(SQ)] \\ &= SQ \quad \text{if} \quad \mathrm{E}[\mathrm{U}_i(A)] < \mathrm{E}[\mathrm{U}_i(SQ)] \end{aligned}$$

$$\mathrm{E}[\mathrm{U}_i(A)] = \mathbf{X}_{iA}\boldsymbol{\beta} + u_{iA}$$

So:

$$\begin{aligned} \Pr(Y = A) &= \Pr\{\mathrm{E}[\mathrm{U}_i(A)] \geq \mathrm{E}[\mathrm{U}_i(SQ)]\} \\ &= \Pr\{(\mathbf{X}_{iA}\boldsymbol{\beta} + u_{iA}) \geq \mathrm{E}[\mathrm{U}_i(SQ)]\} \end{aligned}$$

# Digression II: The Random Utility Model

Normalize:

$$\mathrm{E}[\mathrm{U}_i(SQ)] = 0$$

Then:

$$\begin{aligned}
\Pr(Y = A) &= \Pr\{(\mathbf{X}_{iA}\boldsymbol{\beta} + u_{iA}) \geq 0\} \\
&= \Pr\{u_{iA} \geq -\mathbf{X}_{iA}\boldsymbol{\beta}\} \\
&= F(\mathbf{X}_{iA}\boldsymbol{\beta})
\end{aligned}$$

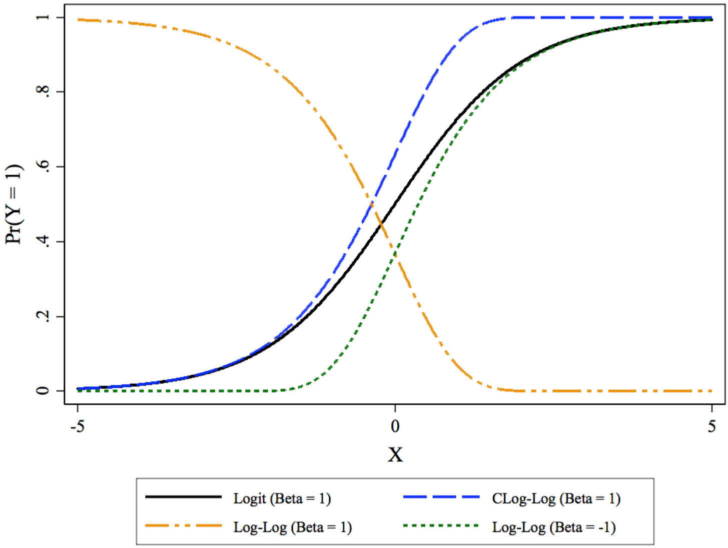
# Other Models: Complementary Log-Log

$$\Pr(Y_i = 1) = 1 - \exp[-\exp(\mathbf{X}_i\boldsymbol{\beta})]$$

or

$$\ln\{-\ln[1 - \Pr(Y_i = 1)]\} = \mathbf{X}_i\boldsymbol{\beta}$$

# Logit and C-log-log CDFs

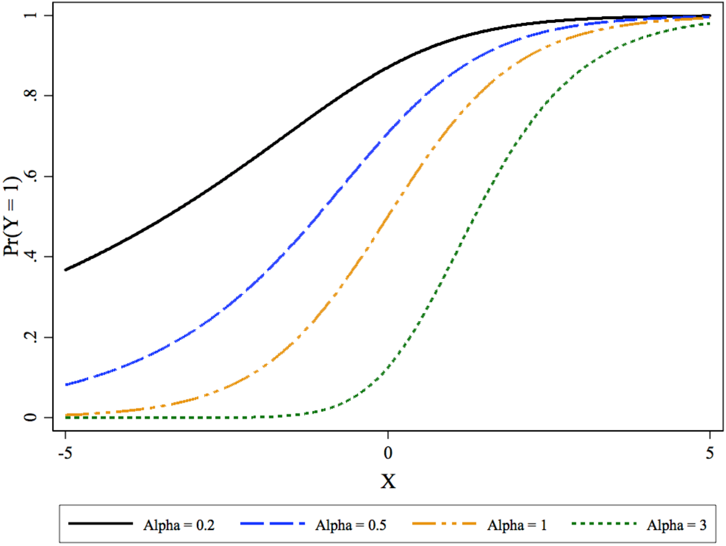


# "Scobit"

$$\Pr(Y_i = 1) = \frac{1}{[1 + \exp(-\mathbf{X}_i \boldsymbol{\beta})]^{\alpha}}, \quad \alpha > 0$$

$$\begin{aligned} \alpha = 1 \; \rightarrow \; \frac{1}{[1 + \exp(-\mathbf{X}_i \boldsymbol{\beta})]^{1}} \quad &= \quad \frac{1}{1 + \exp(-\mathbf{X}_i \boldsymbol{\beta})} \\ &= \quad \frac{\exp(\mathbf{X}_i \boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i \boldsymbol{\beta})} \end{aligned}$$

# Scobit, Visualized



# Binary Response Models: Identification

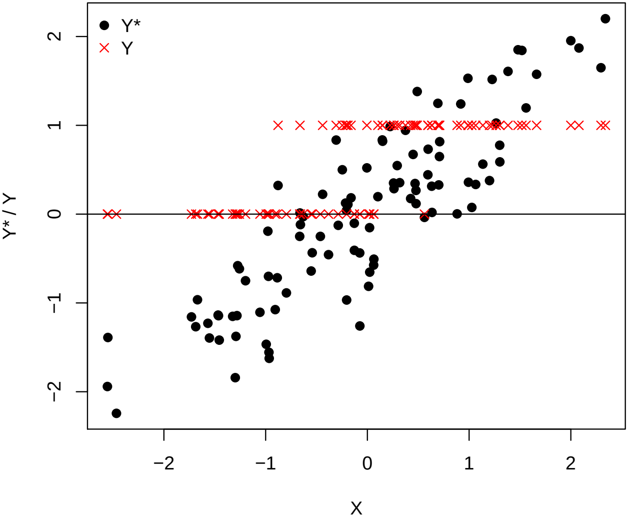
- "Threshold" $= Y^* > 0$
- $\mathrm{E}(u_i|\mathbf{X}, \beta) = 0$
- $\mathrm{Var}(u_i) = \frac{\pi^2}{3}$ or 1.0.

## Logit vs. Probit

- The Universe: Logit > Probit
- The (Social Science) Universe: Meh...
- $\hat{\boldsymbol{\beta}}_{\text{Logit}} \approx 1.8 \times \hat{\boldsymbol{\beta}}_{\text{Probit}}$
- Four reasons to prefer / use logit

```
> set.seed(7222009)
> ystar<-rnorm(100)
> y<-ifelse(ystar>0,1,0)
> x<-ystar+(0.5*rnorm(100))
> data<-data.frame(ystar,y,x)
> head(data)
        ystar y           x
1 -0.64045247 0 -0.55254581
2  0.58855848 1  1.30215029
3  0.64815988 1  0.70827789
4 -0.50684531 0  0.06377499
5  0.01932982 1  0.63521460
```

# A Toy Example



# Toy Example: Probit

```
> myprobit<-glm(y~x,family=binomial(link="probit"), data=data)
> summary(myprobit)

Deviance Residuals:
     Min        1Q    Median        3Q       Max
-2.28477  -0.32228   0.00975   0.38602   2.27744

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   0.3228     0.1923   1.679   0.0932 .
x             2.0090     0.3718   5.404 6.51e-08 ***
---
Signif. codes:
0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 137.989  on 99  degrees of freedom
Residual deviance:  57.908  on 98  degrees of freedom
AIC: 61.908

Number of Fisher Scoring iterations: 7
```

# Toy Example: Logit

```
> mylogit<-glm(y~x,family=binomial(link="logit"), data=data)
> summary(mylogit)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.2708  -0.3286   0.0456   0.3934   2.2899

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   0.5320     0.3390   1.569    0.117
x             3.5061     0.7261   4.828 1.38e-06 ***
---
Signif. codes:
0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 137.989  on 99  degrees of freedom
Residual deviance:  58.498  on 98  degrees of freedom
AIC: 62.498

Number of Fisher Scoring iterations: 6
```

# Toy Example (continued)

Note:

- $z$s, $P$s, $\ln L$s (via "residual deviance") nearly identical
- $\hat{\beta}_{\text{Logit}}$ is $\frac{3.5061}{2.0090} = 1.745 \times \hat{\beta}_{\text{Probit}}$

# Toy Example: Predicted Probabilities