

PLSC 503 – Spring 2018

Ordinal Outcomes

April 19, 2018

Ordinal data are:

- Discrete: $Y \in \{1, 2, \dots\}$
- *Grouped Continuous Data*
- *Assessed Ordered Data*

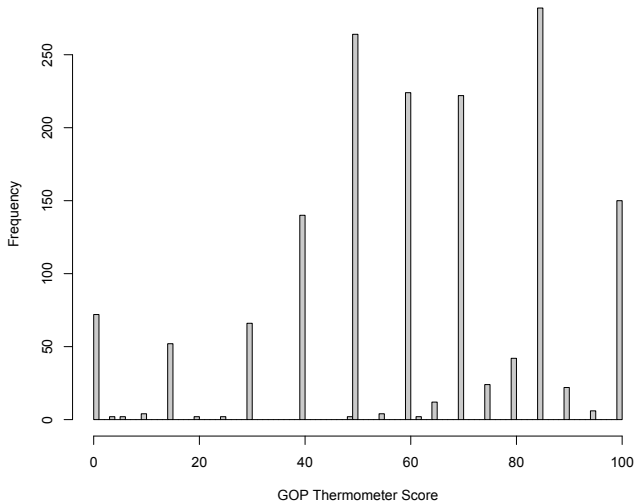
In general:

- Some things can be ordered, but shouldn't be
- Some things are ordered in some circumstances but not others
- Orderings can differ across applications

Ordinal vs. Continuous Response Models

"I'd like to get your feelings toward some of our political leaders and other people who are in the news these days. I'll read the name of a person and I'd like you to rate that person using something we call the feeling thermometer. Ratings between 50 and 100 degrees mean that you feel favorably and warm toward the person; ratings between 0 and 50 degrees mean that you don't feel favorably toward the person and that you don't care too much for that person. You would rate the person at the 50 degree mark if you don't feel particularly warm or cold toward the person."

GOP Thermometer Scores (1988)



A Fake-Data Example

$$Y_i^* = 0 + 1.0X_i + u_i,$$

$$X_i \sim U[0, 10]$$

$$u_i \sim N(0, 1)$$

$$\begin{aligned} Y_{1i} &= 1 \quad \text{if } Y_i^* < 2.5 \\ &= 2 \quad \text{if } 2.5 \leq Y_i^* < 5 \\ &= 3 \quad \text{if } 5 \leq Y_i^* < 7.5 \\ &= 4 \quad \text{if } Y_i^* \geq 7.5 \end{aligned}$$

$$\begin{aligned} Y_{2i} &= 1 \quad \text{if } Y_i^* < 2 \\ &= 2 \quad \text{if } 2 \leq Y_i^* < 8 \\ &= 3 \quad \text{if } 8 \leq Y_i^* < 9 \\ &= 4 \quad \text{if } Y_i^* \geq 9 \end{aligned}$$

World's Best Regression

```
> summary(lm(Ystar~X))
```

Call:

```
lm(formula = Ystar ~ X)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-3.006	-0.654	-0.049	0.643	3.298

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-0.0830	0.0609	-1.36	0.17
X	1.0110	0.0106	95.48	<0.0000000000000002 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 0.988 on 998 degrees of freedom

Multiple R-squared: 0.901, Adjusted R-squared: 0.901

F-statistic: 9.12e+03 on 1 and 998 DF, p-value: <0.0000000000000002

Also A Pretty Good Regression

```
> summary(lm(Y1~X))
```

Call:

```
lm(formula = Y1 ~ X)
```

Residuals:

Min	1Q	Median	3Q	Max
-1.2889	-0.2439	0.0158	0.2592	1.3968

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.69979	0.02639	26.5	<0.0000000000000002 ***
X	0.35825	0.00459	78.0	<0.0000000000000002 ***

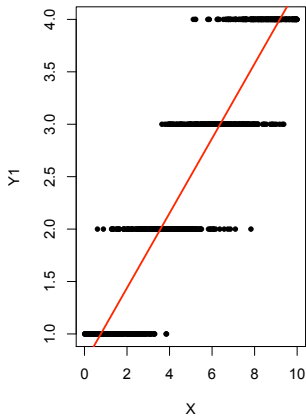
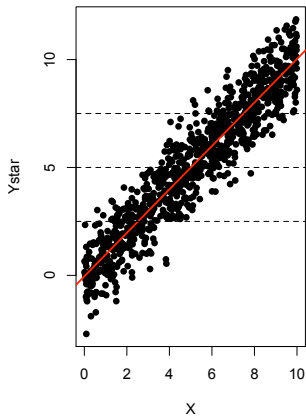
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 0.428 on 998 degrees of freedom

Multiple R-squared: 0.859, Adjusted R-squared: 0.859

F-statistic: 6.09e+03 on 1 and 998 DF, p-value: <0.0000000000000002

What That Looks Like



A Not-So-Good Regression

```
> summary(lm(Y2~X))
```

```
Call:
```

```
lm(formula = Y2 ~ X)
```

```
Residuals:
```

Min	1Q	Median	3Q	Max
-1.3115	-0.3205	-0.0405	0.2914	1.4876

```
Coefficients:
```

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.88919	0.03069	29.0	<0.00000000000000002 ***
X	0.24383	0.00534	45.7	<0.00000000000000002 ***

```
---
```

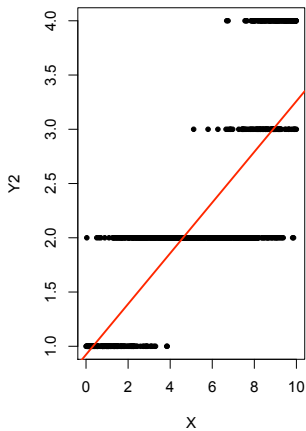
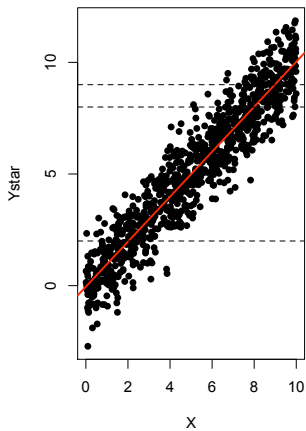
```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Residual standard error: 0.498 on 998 degrees of freedom
```

```
Multiple R-squared:  0.676, Adjusted R-squared:  0.676
```

```
F-statistic: 2.09e+03 on 1 and 998 DF, p-value: <0.00000000000000002
```

What That Looks Like



Models for Ordinal Responses

$$Y_i^* = \mu + u_i$$

$$Y_i = j \text{ if } \tau_{j-1} \leq Y_i^* < \tau_j, j \in \{1, \dots, J\}$$

$$\begin{aligned} Y_i &= 1 \text{ if } -\infty \leq Y_i^* < \tau_1 \\ &= 2 \text{ if } \tau_1 \leq Y_i^* < \tau_2 \\ &= 3 \text{ if } \tau_2 \leq Y_i^* < \tau_3 \\ &= 4 \text{ if } \tau_3 \leq Y_i^* < \infty \end{aligned}$$

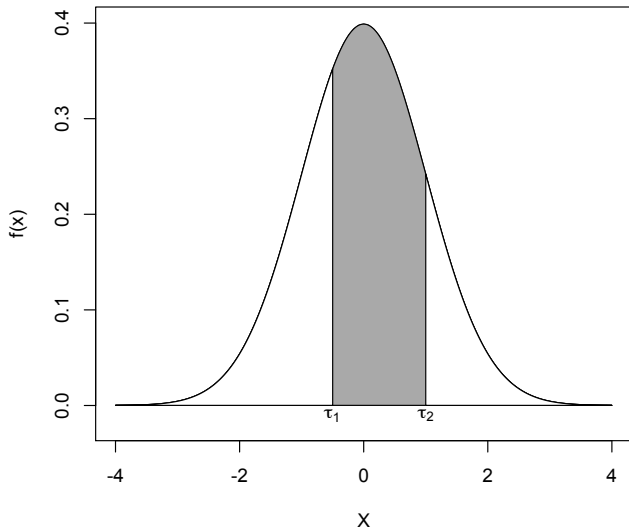
Ordinal Response Models: Probabilities

$$\begin{aligned}\Pr(Y_i = j) &= \Pr(\tau_{j-1} \leq Y^* < \tau_j) \\ &= \Pr(\tau_{j-1} \leq \mu_i + u_i < \tau_j)\end{aligned}\quad (1)$$

$$\mu_i = \mathbf{X}_i\boldsymbol{\beta}$$

$$\begin{aligned}\Pr(Y_i = j|\mathbf{X}, \boldsymbol{\beta}) &= \Pr(\tau_{j-1} \leq Y_i^* < \tau_j|\mathbf{X}) \\ &= \Pr(\tau_{j-1} \leq \mathbf{X}_i\boldsymbol{\beta} + u_i < \tau_j) \\ &= \Pr(\tau_{j-1} - \mathbf{X}_i\boldsymbol{\beta} \leq u_i < \tau_j - \mathbf{X}_i\boldsymbol{\beta}) \\ &= \int_{-\infty}^{\tau_j - \mathbf{X}_i\boldsymbol{\beta}} f(u_i) du - \int_{-\infty}^{\tau_{j-1} - \mathbf{X}_i\boldsymbol{\beta}} f(u_i) du \\ &= F(\tau_j - \mathbf{X}_i\boldsymbol{\beta}) - F(\tau_{j-1} - \mathbf{X}_i\boldsymbol{\beta})\end{aligned}$$

What That Looks Like



Probabilities, etc.

$$\Pr(Y_i = 1) = \Phi(\tau_1 - \mathbf{X}_i\boldsymbol{\beta}) - 0$$

$$\Pr(Y_i = 2) = \Phi(\tau_2 - \mathbf{X}_i\boldsymbol{\beta}) - \Phi(\tau_1 - \mathbf{X}_i\boldsymbol{\beta})$$

$$\Pr(Y_i = 3) = \Phi(\tau_3 - \mathbf{X}_i\boldsymbol{\beta}) - \Phi(\tau_2 - \mathbf{X}_i\boldsymbol{\beta})$$

$$\Pr(Y_i = 4) = 1 - \Phi(\tau_3 - \mathbf{X}_i\boldsymbol{\beta})$$

Define:

$$\begin{aligned}\delta_{ij} &= 1 \text{ if } Y_i = j \\ &= 0 \text{ otherwise.}\end{aligned}$$

Likelihood:

$$L(Y|\mathbf{X}, \beta, \tau) = \prod_{i=1}^N \prod_{j=1}^J [F(\tau_j - \mathbf{X}_i\beta) - F(\tau_{j-1} - \mathbf{X}_i\beta)]^{\delta_{ij}}$$

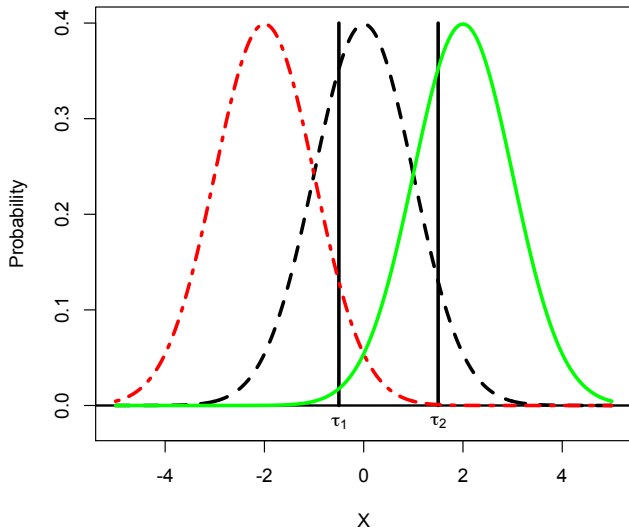
Log-Likelihood, probit:

$$\ln L(Y|\mathbf{X}, \beta, \tau) = \sum_{i=1}^N \sum_{j=1}^J \delta_{ij} \ln[\Phi(\tau_j - \mathbf{X}_i\beta) - \Phi(\tau_{j-1} - \mathbf{X}_i\beta)]$$

Log-Likelihood, logit:

$$\ln L(Y|\mathbf{X}, \beta, \tau) = \sum_{i=1}^N \sum_{j=1}^J \delta_{ij} \ln[\Lambda(\tau_j - \mathbf{X}_i\beta) - \Lambda(\tau_{j-1} - \mathbf{X}_i\beta)]$$

The Intuition



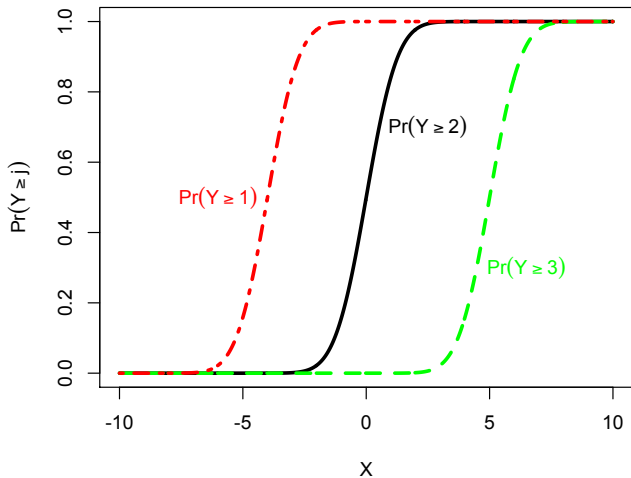
- (Usual) Assumption about σ_{Y*}^2
- β_0 vs. the τ s...
- Must either omit β_0 or drop one of the $J - 1$ τ s
- In practice: Stata & R omit β_0

Parallel Regressions

$$\frac{\partial \Pr(Y_i \geq j)}{\partial X} = \frac{\partial \Pr(Y_i \geq j')}{\partial X} \quad \forall j \neq j'$$

(aka “proportional odds” ...)

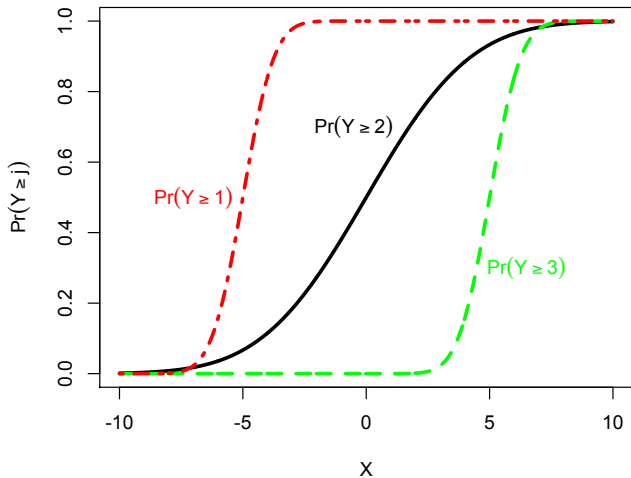
Parallel Regressions Envisioned



Relaxing Parallel Regressions

$$\frac{\partial \Pr(Y_i \geq j)}{\partial X} \neq \frac{\partial \Pr(Y_i \geq j')}{\partial X} \quad \forall j \neq j'$$

Nonparallel Regressions Envisioned



- R :
 - polr (in MASS)
 - ologit/oprobit (in Zelig; calls polr)
 - vglm (in VGAM)
- Stata : ologit, oprobit

Best Example Ever

```
> summary(beer)
```

name	contqual	quality	price	calories
Length:69	Min. :24.00	Min. :1.000	Min. :2.360	Min. : 58.0
Class :character	1st Qu.:49.00	1st Qu.:2.000	1st Qu.:3.900	1st Qu.:142.0
Mode :character	Median :70.00	Median :3.000	Median :4.790	Median :148.0
	Mean :64.78	Mean :2.536	Mean :4.963	Mean :142.3
	3rd Qu.:80.00	3rd Qu.:4.000	3rd Qu.:6.240	3rd Qu.:160.0
	Max. :98.00	Max. :4.000	Max. :7.800	Max. :201.0

alcohol	craftbeer	bitter	malty	class
Min. :0.500	Min. :0.0000	Min. : 8.00	Min. : 5.00	Craft Lager :13
1st Qu.:4.400	1st Qu.:0.0000	1st Qu.:21.00	1st Qu.:12.00	Craft Ale :17
Median :4.900	Median :0.0000	Median :31.00	Median :23.00	Imported Lager :10
Mean :4.471	Mean :0.4348	Mean :35.44	Mean :33.13	Regular or Ice Beer:16
3rd Qu.:5.100	3rd Qu.:1.0000	3rd Qu.:52.50	3rd Qu.:50.50	Light Beer : 6
Max. :6.000	Max. :1.0000	Max. :80.50	Max. :86.00	Nonalcoholic : 7

Ordered Logit

```
> library(MASS)
> beer.logit<-polr(as.factor(quality)~price+calories+craftbeer+bitter
  +malty,data=beer)
> summary(beer.logit)
```

Call:

```
polr(formula = as.factor(quality) ~ price + calories + craftbeer +
      bitter + malt) )
```

Coefficients:

	Value	Std. Error	t value
price	-0.451	0.293	-1.5
calories	0.047	0.012	3.8
craftbeer	-1.705	0.942	-1.8
bitter	-0.030	0.042	-0.7
malty	0.051	0.025	2.1

Intercepts:

	Value	Std. Error	t value
1 2	2.771	1.674	1.655
2 3	4.270	1.725	2.475
3 4	5.578	1.760	3.170

Ordered Probit

```
> beer.probit<-polr(as.factor(quality)~price+calories+craftbeer+bitter+malty,  
+ data=beer,method="probit")  
> summary(beer.probit)
```

Call:

```
polr(formula = as.factor(quality) ~ price + calories + craftbeer +  
      bitter + malt, method = "probit")
```

Coefficients:

	Value	Std. Error	t value
price	-0.27914	0.172012	-1.6228
calories	0.02800	0.007184	3.8979
craftbeer	-0.98427	0.559020	-1.7607
bitter	-0.01737	0.024719	-0.7025
malty	0.02855	0.014321	1.9937

Intercepts:

	Value	Std. Error	t value
1 2	1.647	1.018	1.619
2 3	2.508	1.034	2.426
3 4	3.290	1.049	3.136

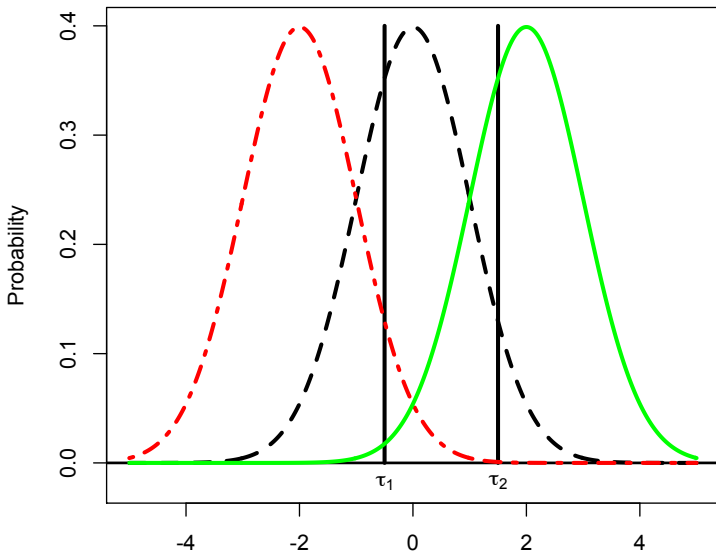
Interpretation: Marginal Effects

$$\begin{aligned}\frac{\partial \Pr(Y = j)}{\partial X_k} &= \frac{\partial F(\hat{\tau}_{j-1} - \bar{\mathbf{X}}\hat{\beta})}{\partial X_k} - \frac{\partial F(\hat{\tau}_j - \bar{\mathbf{X}}\hat{\beta})}{\partial X_k} \\ &= \hat{\beta}_k [f(\hat{\tau}_{j-1} - \bar{\mathbf{X}}\hat{\beta}) - f(\hat{\tau}_j - \bar{\mathbf{X}}\hat{\beta})]\end{aligned}$$

So:

- $\text{sign}\left(\frac{\partial \Pr(Y=1)}{\partial X_k}\right) = -\text{sign}(\hat{\beta}_k)$
- $\text{sign}\left(\frac{\partial \Pr(Y=J)}{\partial X_k}\right) = \text{sign}(\hat{\beta}_k)$
- $\frac{\partial \Pr(Y=\ell)}{\partial X_k}$, $\ell \in \{2, 3, \dots, J-1\}$ are non-monotonic

Marginal Effects, Illustrated



Interpretation: Odds Ratios

For a δ -unit change in X_k :

$$\begin{aligned}\text{OR}_{X_k} &= \frac{\frac{\Pr(Y > j | \mathbf{X}, X_k + \delta)}{\Pr(Y \leq j | \mathbf{X}, X_k + \delta)}}{\frac{\Pr(Y > j | \mathbf{X}, X_k)}{\Pr(Y \leq j | \mathbf{X}, X_k)}} \\ &= \exp(\delta \hat{\beta}_k)\end{aligned}$$

Calculating Odds Ratios

```
> olreg.or <- function(model)
+ {
+   coeffs <- coef(summary(model))
+   lci <- exp(coeffs[,1] - 1.96 * coeffs[,2])
+   or <- exp(coeffs[,1])
+   uci <- exp(coeffs[,1] + 1.96 * coeffs[,2])
+   lreg.or <- cbind(lci, or, uci)
+   lreg.or
+ }
```

```
> olreg.or(beer.logit)
```

	lci	or	uci
price	0.3586	0.6373	1.133
calories	1.0231	1.0479	1.073
craftbeer	0.0287	0.1818	1.152
bitter	0.8933	0.9707	1.055
malty	1.0023	1.0518	1.104
1 2	0.6003	15.9748	425.133
2 3	2.4319	71.4963	2101.961
3 4	8.4053	264.4357	8319.319

Odds Ratios: Explication

- craftbeer:
 - $\exp(-1.705) = 0.18$
 - “The odds of being rated “Good” or better (versus “Fair”) are more than 80 percent lower for a craft beer than for a regular beer.”
 - “The odds of being rated “Very Good” or better (versus “Fair” or “Good”) are more than 80 percent lower for a craft beer than for a regular beer.”
- calories:
 - $\exp(0.047) = 1.05$
 - “A one-calorie increase raises the odds of being in a higher set of categories (versus all lower ones) by about five percent.”
 - etc.

Predicted Probabilities: Basics

$$\Pr(\widehat{Y_i = j} | \mathbf{X}) = F(\hat{\tau}_j - \bar{\mathbf{X}}_i \hat{\beta}) - F(\hat{\tau}_{j-1} - \bar{\mathbf{X}}_i \hat{\beta})$$

Means:

- price = 4.96, calories = 142, craftbeer = 0, bitter = 35.4, malty = 33.1.
- Yields:

$$\begin{aligned} \sum_{k=1}^K \bar{\mathbf{X}}_k \hat{\beta}_k &= -0.45 \times 4.96 + 0.047 \times 142 - 1.70 \times 0 - \\ &\quad 0.03 \times 35.4 + 0.05 \times 33.1 \\ &= -2.23 + 6.67 - 0 - 1.06 + 1.66 \\ &= \mathbf{5.04}. \end{aligned}$$

Predicted Probabilities: “By Hand”

$$\begin{aligned}\Pr(Y = 1) &= \Lambda(2.77 - 5.04) - 0 \\ &= \frac{\exp(-2.27)}{1 + \exp(-2.27)} \\ &= \mathbf{0.09}.\end{aligned}$$

$$\begin{aligned}\Pr(Y = 2) &= \Lambda(4.27 - 5.04) - \Lambda(2.77 - 5.04) \\ &= \Lambda(-0.77) - \Lambda(-2.27) \\ &= 0.32 - 0.09 \\ &= \mathbf{0.23}.\end{aligned}$$

$$\begin{aligned}\Pr(Y = 3) &= \Lambda(5.58 - 5.04) - \Lambda(4.27 - 5.04) \\ &= \Lambda(0.54) - \Lambda(-0.77) \\ &= 0.63 - 0.32 \\ &= \mathbf{0.31}.\end{aligned}$$

$$\begin{aligned}\Pr(Y = 4) &= 1 - \Lambda(5.58 - 5.04) \\ &= 1 - \Lambda(0.54) \\ &= 1 - 0.63 \\ &= \mathbf{0.37}.\end{aligned}$$

Changes in Predicted Probabilities

For `craftbeer=1`:

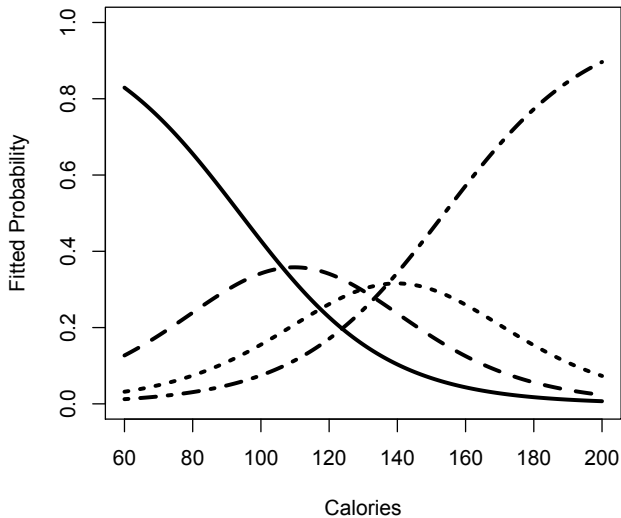
- $\Pr(Y = 1) = \Lambda(2.77 - 3.34) - 0 = \mathbf{0.36}$.
- $\Pr(Y = 2) = \Lambda(4.27 - 3.34) - \Lambda(2.77 - 3.34) = 0.72 - 0.36 = \mathbf{0.36}$.
- $\Pr(Y = 3) = \Lambda(5.58 - 3.34) - \Lambda(4.27 - 3.34) = 0.90 - 0.72 = \mathbf{0.18}$.
- $\Pr(Y = 4) = 1 - 0.90 = \mathbf{0.10}$.

Outcome	Change in Probability
$\Delta\Pr(\text{Fair})$	0.27
$\Delta\Pr(\text{Good})$	0.13
$\Delta\Pr(\text{Very Good})$	-0.13
$\Delta\Pr(\text{Excellent})$	-0.27

Predicted Probability Plots

- Can be category-specific or “cumulative”
- `polr`:
 - In-sample in `$fitted.values`
 - `polr` class supports `predict`, `confint`, etc.
- `ologit` / `oprobit`: using `predict`

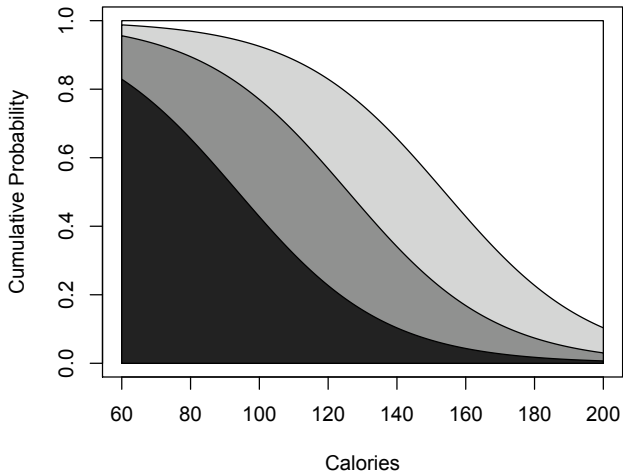
Plot by Outcome



(How'd He Do That?)

```
> calories<-seq(60,200,1)
> price<-mean(beer$price)
> craftbeer<-median(beer$craftbeer)
> bitter<-mean(beer$bitter)
> malty<-mean(beer$malty)
> beersim<-cbind(calories,price,craftbeer,bitter,malty)
> beer.hat<-predict(beer.logit,beersim,type='probs')
> plot(c(60,200), c(0,1), type='n', xlab="Calories", ylab='Fitted
  Probability')
> lines(60:200, beer.hat[1:141, 1], lty=1, lwd=3)
> lines(60:200, beer.hat[1:141, 2], lty=2, lwd=3)
> lines(60:200, beer.hat[1:141, 3], lty=3, lwd=3)
> lines(60:200, beer.hat[1:141, 4], lty=4, lwd=3)
```

Cumulative Predicted Probabilities



(code...)

```
> xaxis<-c(60,60:200,200)
> yaxis1<-c(0,beer.hat[,1],0)
> yaxis2<-c(0,beer.hat[,2]+beer.hat[,1],0)
> yaxis3<-c(0,beer.hat[,3]+beer.hat[,2]+beer.hat[,1],0)
> yaxis4<-c(0,beer.hat[,4]+beer.hat[,3]+beer.hat[,2]+beer.hat[,1],0)
>
> plot(c(60,200), c(0,1), type='n', xlab="Calories", ylab="Cumulative
  Probability")
> polygon(xaxis,yaxis4,col="white")
> polygon(xaxis,yaxis3,col="grey80")
> polygon(xaxis,yaxis2,col="grey50")
> polygon(xaxis,yaxis1,col="grey10")
```

- Generalized (relaxes parallel regressions; Brant (1990))
- Heteroscedastic
- Varying τ s (Maddala, Terza, Sanders)
- Models for “balanced” scales (Jones & Sobel)
- Compound Ordered Hierarchical Probit (“chopit”) (Wand & King)
- “Zero-Inflated” Ordered Models (Hill, Bagozzi, Moore & Mukherjee)
- Latent class/mixture models (Winkelmann, etc.)