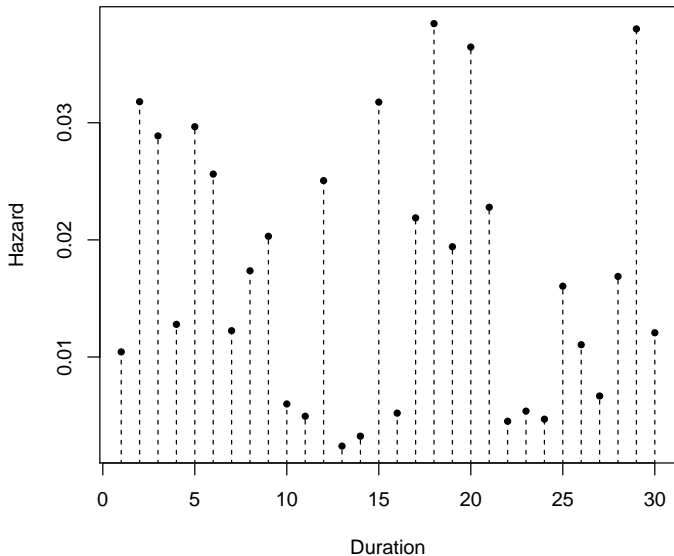


PLSC 504 - Fall 2017

Discrete-Time Survival Models

September 28, 2017

The Discrete-Time Idea



A General Discrete-Time Model

Process:

$$t \in \{1, 2, \dots, t_{\max}\}$$

Density:

$$f(t) = \Pr(T = t)$$

CDF:

$$\begin{aligned} F(t) &= \Pr(T \leq t) \\ &= \sum_{j=1}^t f(t_j) \end{aligned}$$

A General Discrete-Time Model

Survival function:

$$\begin{aligned} S(t) &\equiv \Pr(T \geq t) \\ &= 1 - F(t) \\ &= \sum_{j=t}^{t_{\max}} f(t_j) \end{aligned}$$

Hazard function:

$$\begin{aligned} h(t) &\equiv \Pr(T = t | T \geq t) \\ &= \frac{f(t)}{S(t)} \end{aligned}$$

A General Discrete-Time Model

Conditional Pr(Survival):

$$\Pr(T > t | T \geq t) = 1 - h(t)$$

Implies:

$$\begin{aligned} S(t) &= \Pr(T > t | T \geq t) \times \Pr(T > t-1 | T \geq t-1) \times \Pr(T > t-2 | T \geq t-2) \times \dots \\ &\quad \times \Pr(T > 2 | T \geq 2) \times \Pr(T > 1 | T \geq 1) \\ &= [1 - h(t)] \times [1 - h(t-1)] \times [1 - h(t-2)] \times \dots \times [1 - h(2)] \times [1 - h(1)] \\ &= \prod_{j=0}^t [1 - h(t-j)] \end{aligned}$$

A General Discrete-Time Model

which means:

$$\begin{aligned} f(t) &= h(t)S(t) \\ &= h(t) \times [1 - h(t-1)] \times [1 - h(t-2)] \times \dots \\ &\quad \times [1 - h(2)] \times [1 - h(1)] \\ &= h(t) \prod_{j=1}^{t-1} [1 - h(t-j)] \end{aligned}$$

General Discrete-Time Model: Likelihood

$$L = \prod_{i=1}^N \left\{ h(t) \prod_{j=1}^{t-1} [1 - h(t-j)] \right\}^{Y_{it}} \left\{ \prod_{j=0}^t [1 - h(t-j)] \right\}^{1-Y_{it}}$$

Ordered-Categorical Models

For K small:

$$\Pr(T_i \leq k) = \frac{\exp(\tau_k - \mathbf{X}_i\beta)}{1 + \exp(\tau_k - \mathbf{X}_i\beta)}$$

$$\ln \left[\frac{\Pr(T_i \leq \kappa)}{\Pr(T_i > \kappa)} \right] = \tau_\kappa - \mathbf{X}_i\beta$$

Grouped-Data (“BTSCS”) Approaches

$$\Pr(Y_{it} = 1) = f(\mathbf{X}_{it}\beta)$$

- logit
- probit
- c-log-log
- etc.

BTSCS: Advantages

- Easily estimated, interpreted and understood
- Natural interpretations:
 - $\hat{\beta}_0 \approx$ “baseline hazard”
 - Covariates shift this up or down.
- Can incorporate data in time-varying covariates
- Lots of software

(Potential) Disadvantages

- Requires time-varying data
- *Must deal with time dependence explicitly*

Temporal Issues in Grouped-Data Models

(Implicit) “Baseline” hazard:

$$h_0(t) = \frac{\exp(\beta_0)}{1 + \exp(\beta_0)}$$

→ No temporal dependence / “flat” hazard

Temporal Issues in Grouped-Data Models

Time trend:

$$\Pr(Y_{it} = 1) = f(\mathbf{X}_{it}\beta + \gamma T_{it})$$

- $\hat{\gamma} > 0 \rightarrow$ rising hazard
- $\hat{\gamma} < 0 \rightarrow$ declining hazard
- $\hat{\gamma} = 0 \rightarrow$ “flat” (exponential) hazard

Variants/extensions: Polynomials...

$$\Pr(Y_{it} = 1) = f(\mathbf{X}_{it}\beta + \gamma_1 T_{it} + \gamma_2 T_{it}^2 + \gamma_3 T_{it}^3 + \dots)$$

Temporal Issues in Grouped-Data Models

“Time dummies”:

$$\Pr(Y_{it} = 1) = f[\mathbf{X}_{it}\beta + \alpha_1 I(T_{i1}) + \alpha_2 I(T_{i2}) + \dots + \alpha_{t_{\max}} I(T_{it_{\max}})]$$

→ BKT's cubic splines; might also use:

- Fractional polynomials
- Smoothed duration
- Loess/lowess fits
- Other splines (B-splines, P-splines, natural splines, etc.)

Discrete-Time Model Selection

- Theory
- Formal tests
- Fitted values

Equivalency One: Cox \equiv Conditional Logit

$$\Pr(Y_i = j) = \frac{\exp(\mathbf{X}_{ij}\beta + \mathbf{Z}_j\gamma)}{\sum_{\ell=1}^J \exp(\mathbf{X}_{i\ell}\beta + \mathbf{Z}_\ell\gamma)}$$

$$\Pr(Y_i = j) = \frac{\exp(\mathbf{X}_{ij}\beta)}{\sum_{\ell=1}^J \exp(\mathbf{X}_{i\ell}\beta)}$$

$$L_k = \frac{\exp(\mathbf{X}_k\beta)}{\sum_{\ell \in R_j} \exp(\mathbf{X}_\ell\beta)}.$$

The point: Cox \equiv Conditional logit

Cox-Poisson Equivalence

**Grouped-data duration models
and the continuous-time Cox
model are equivalent.**

Cox-Poisson Equivalence

Cox:

$$S_i(t) = \exp \left[-\exp(\mathbf{X}_i\beta) \int_0^t h_0(t) dt \right]$$

Poisson:

$$\Pr(Y = y) = \frac{\exp(-\lambda)\lambda^y}{y!}$$

$$\begin{aligned} \Pr(Y_{it} = 0) &= \exp(-\lambda) \\ &= \exp[-\exp(\mathbf{X}_i\beta)] \end{aligned}$$

Example: Oneal & Russett (1950-1985)

No time variable / "flat" hazard:

```
> OR.logit<-glm(dispute~allies+contig+capratio+growth+democracy+trade,  
                data=OR,na.action=na.exclude,family="binomial")  
> summary(OR.logit)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	-4.32668	0.11451	-37.785	< 2e-16 ***
allies	-0.47969	0.11275	-4.255	2.09e-05 ***
contig	1.35358	0.12091	11.195	< 2e-16 ***
capratio	-0.19620	0.05011	-3.916	9.01e-05 ***
growth	-3.42753	1.25181	-2.738	0.00618 **
democracy	-0.40120	0.10063	-3.987	6.70e-05 ***
trade	-21.07611	11.30396	-1.864	0.06225 .

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Example, Continued

Linear trend:

```
> OR$duration<-OR$stop
> OR.trend<-glm(dispute~allies+contig+capratio+growth+democracy+trade
                +duration,data=OR,na.action=na.exclude,family="binomial")
> summary(OR.trend)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	-3.271136	0.134709	-24.283	< 2e-16	***
allies	-0.362966	0.114140	-3.180	0.001473	**
contig	0.996908	0.123978	8.041	8.91e-16	***
capratio	-0.235655	0.052763	-4.466	7.96e-06	***
growth	-3.957428	1.225716	-3.229	0.001244	**
democracy	-0.361150	0.099515	-3.629	0.000284	***
trade	-2.870981	9.861298	-0.291	0.770947	
duration	-0.091189	0.008098	-11.260	< 2e-16	***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Example, Continued

Fourth-Order polynomial trend:

```
OR$d2<-OR$duration^2*0.1
OR$d3<-OR$duration^3*0.01
OR$d4<-OR$duration^4*0.001

OR.P4<-glm(dispute~allies+contig+capratio+growth+democracy+trade
           +duration+d2+d3+d4,data=OR,na.action=na.exclude,
           family="binomial")

> summary(OR.P4)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	-3.401363	0.206815	-16.446	< 2e-16	***
allies	-0.364127	0.114201	-3.188	0.00143	**
contig	0.995584	0.124074	8.024	1.02e-15	***
capratio	-0.228355	0.052257	-4.370	1.24e-05	***
growth	-3.864329	1.245617	-3.102	0.00192	**
democracy	-0.392457	0.100693	-3.898	9.72e-05	***
trade	-4.032292	9.631171	-0.419	0.67546	
duration	0.058036	0.091465	0.635	0.52574	
d2	-0.274958	0.128454	-2.141	0.03231	*
d3	0.136086	0.063230	2.152	0.03138	*
d4	-0.018863	0.009914	-1.903	0.05709	.

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Polynomial Improvement?

```
> P4test
```

```
Analysis of Deviance Table
```

```
Model 1: dispute ~ allies + contig + capratio + growth + democracy + trade
```

```
Model 2: dispute ~ allies + contig + capratio + growth + democracy + trade +  
duration + d2 + d3 + d4
```

	Resid. Df	Resid. Dev	Df	Deviance	Pr(>Chi)
1	20441	3693.8			
2	20437	3510.0	4	183.76	< 2.2e-16 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example: "Time Dummies"

"Time dummies":

```
> OR.dummy<-glm(dispute~allies+contig+capratio+growth+democracy+trade
+as.factor(duration),data=OR,na.action=na.exclude,
family="binomial")
```

```
> summary(OR.dummy)
```

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)	
(Intercept)	-3.61115	0.18219	-19.820	< 2e-16	***
allies	-0.36922	0.11441	-3.227	0.001251	**
contig	0.99389	0.12417	8.005	1.20e-15	***
capratio	-0.22778	0.05219	-4.364	1.27e-05	***
growth	-3.97619	1.24940	-3.182	0.001460	**
democracy	-0.39559	0.10077	-3.926	8.65e-05	***
trade	-3.46727	9.62606	-0.360	0.718700	
as.factor(duration)2	0.45489	0.19606	2.320	0.020331	*
as.factor(duration)3	0.36020	0.20632	1.746	0.080843	.
as.factor(duration)4	0.14188	0.22175	0.640	0.522289	

<output omitted>

as.factor(duration)33	-1.64467	1.01715	-1.617	0.105891
as.factor(duration)34	-0.86966	0.73158	-1.189	0.234541
as.factor(duration)35	-1.38777	1.01857	-1.362	0.173049

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

"Time Dummies," continued

```
> Test.Dummies<-anova(OR.logit,OR.dummy,test="Chisq")
```

```
> Test.Dummies
```

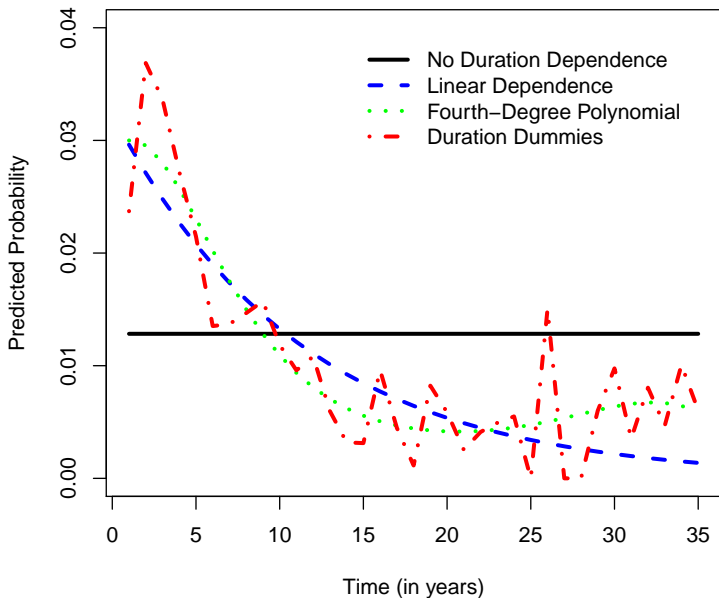
Analysis of Deviance Table

Model 1: dispute ~ allies + contig + capratio + growth + democracy + trade

Model 2: dispute ~ allies + contig + capratio + growth + democracy + trade +
as.factor(duration)

	Resid. Df	Resid. Dev	Df	Deviance	Pr(>Chi)
1	20441	3693.8			
2	20407	3464.4	34	229.38	< 2.2e-16 ***

Predicted “Hazards” (Probabilities)



Cox / Poisson Equivalence

Cox model:

```
OR.Cox<-coxph(Surv(OR$start,OR$stop,OR$dispute)~allies+contig+capratio+  
growth+democracy+trade,data=OR,method="breslow")
```

Poisson:

```
OR.Poisson<-glm(dispute~allies+contig+capratio+growth+democracy+trade  
+as.factor(duration),data=OR,na.action=na.exclude,  
family="poisson")
```

