Week 2: Linear Regression

Slava Mikhaylov

PUBLG088 Advanced Quantitative Methods

11 October 2016

Week 2 Outline

- Simple linear regression
  - Estimation of the parameters
  - Confidence intervals
  - Hypothesis testing
  - Assessing overall accuracy of the model
  - Multiple Linear Regression
  - Interpretation
  - Model fit
- Qualitative predictors
  - Qualitative predictors in regression models
  - Interactions
  - Non-linear effects
- Additional models
  - Panel data

Simple linear regression

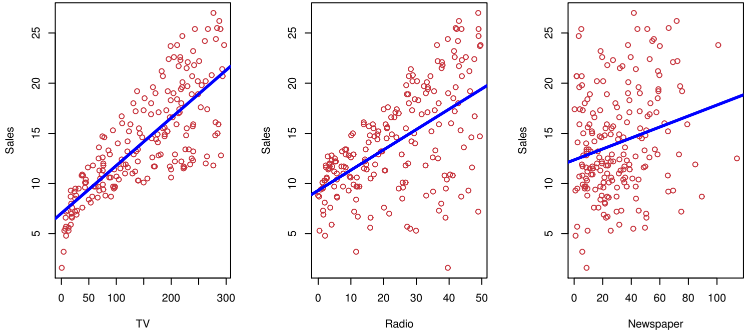
- Linear regression is a simple approach to supervised learning. It assumes that the dependence of Y on $X_1, X_2, \ldots, X_p$ is linear.
- True regression functions are never linear!
- Although it may seem overly simplistic, linear regression is extremely useful both conceptually and practically.

Linear regression for the advertising data

Consider the advertising data. Questions we might ask:

- Is there a relationship between advertising budget and sales?
- How strong is the relationship between advertising budget and sales?
- Which media contribute to sales?
- How accurately can we predict future sales?
- Is the relationship linear?
- Is there synergy among the advertising media?

Advertising data



Simple linear regression using a single predictor X

- We assume a model

$$Y = \beta_0 + \beta_1 X + \epsilon,$$

  where β_0 and β_1 are two unknown constants that represent the intercept and slope, also known as coefficients or parameters, and ϵ is the error term.
- Given some estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ for the model coefficients, we predict future sales using

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x,$$

  where $\hat{y}$ indicates a prediction of Y on the basis of $X = x$. The hat symbol denotes an estimated value.

Estimation of the parameters by least squares

- Let $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ be the prediction for Y based on the ith value of X. Then $e_i = y_i - \hat{y}_i$ represents the ith residual.
- We define the residual sum of squares (RSS) as

$$\mathrm{RSS} = e_1^2 + e_2^2 + \cdots + e_n^2,$$

or equivalently as

$$\mathrm{RSS} = (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + (y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2)^2 + \cdots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2.$$

Estimation of the parameters by least squares

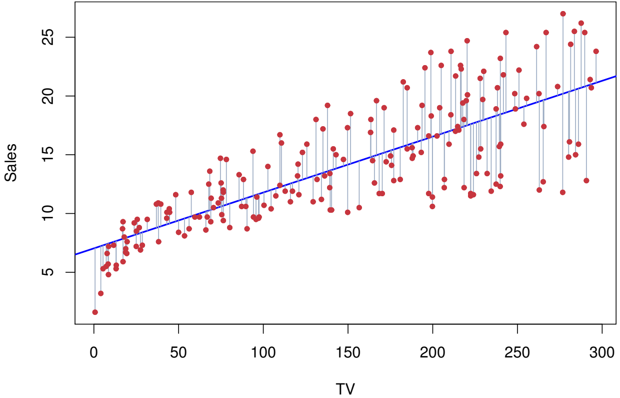
- The least squares approach chooses $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize the RSS. The minimizing values can be shown to be

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$

where $\bar{y} \equiv \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i$ are the sample means.

Example: advertising data



The least squares fit for the regression of sales on TV. The fit is found by minimizing the sum of squared residuals. In this case a linear fit captures the essence of the relationship, although it is somewhat deficient in the left of the plot.

Assessing the Accuracy of the Coefficient Estimates

- The standard error of an estimator reflects how it varies under repeated sampling. We have

$$\mathrm{SE}(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2},$$

$$\mathrm{SE}(\hat{\beta}_0)^2 = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right],$$

  where $\sigma^2 = \mathrm{Var}(\epsilon)$

- These standard errors can be used to compute confidence intervals. A 95% confidence interval is defined as a range of values such that with 95% probability, the range will contain the true unknown value of the parameter. It has the form

$$\hat{\beta}_1 \pm 2 \times \mathrm{SE}(\hat{\beta}_1).$$

Confidence Intervals

That is, there is approximately a 95% chance that the interval

$$\left[\hat{\beta}_1 - 2 \times \mathrm{SE}(\hat{\beta}_1), \hat{\beta}_1 + 2 \times \mathrm{SE}(\hat{\beta}_1)\right]$$

will contain the true value of β_1 (under a scenario where we got repeated samples like the present sample).

Hypothesis testing

- Standard errors can also be used to perform hypothesis tests on the coefficients. The most common hypothesis test involves testing the null hypothesis of

  H_0: There is no relationship between X and Y versus the alternative hypothesis.

  H_A: There is some relationship between X and Y.

- Mathematically, this corresponds to testing versus

$$H_0 : \beta_1 = 0$$

versus

$$H_A : \beta_1 \neq 0,$$

since if $\beta_1 = 0$ then the model reduces to $Y = \beta_0 + \epsilon$, and X is not associated with Y.

Hypothesis testing

- To test the null hypothesis, we compute a t-statistic, given by

$$t = \frac{\hat{\beta}_1 - 0}{\mathrm{SE}(\hat{\beta}_1)},$$

- This will have a t-distribution with $n-2$ degrees of freedom, assuming $\beta_1 = 0$.
- Using statistical software, it is easy to compute the probability of observing any value equal to $\mid t \mid$ or larger. We call this probability the p-value.

Assessing the Overall Accuracy of the Model

- We compute the Residual Standard Error

$$\mathrm{RSE} = \sqrt{\frac{1}{n-2}\mathrm{RSS}} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2},$$

  where the residual sum-of-squares is $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$.

- R-squared or fraction of variance explained is

$$R^2 = \frac{\mathrm{TSS} - \mathrm{RSS}}{\mathrm{TSS}} = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}}$$

  where $\mathrm{TSS} = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the total sum of squares.

- It can be shown that in this simple linear regression setting that $R^2 = r^2$, where r is the correlation between X and Y:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}.$$

Results for the advertising data

```r
advertising <- read.csv("http://www-bcf.usc.edu/~gareth/ISL/Advertising.csv")
names(advertising)

## [1] "X"         "TV"        "Radio"     "Newspaper" "Sales"

simple.regression <- lm(advertising$Sales ~ advertising$TV)
```

Results for the advertising data

```
summary(simple.regression)
```

```
##
## Call:
## lm(formula = advertising$Sales ~ advertising$TV)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.3860 -1.9545 -0.1913  2.0671  7.2124
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    7.032594   0.457843   15.36   <2e-16 ***
## advertising$TV 0.047537   0.002691   17.67   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.259 on 198 degrees of freedom
## Multiple R-squared:  0.6119,Adjusted R-squared:  0.6099
## F-statistic: 312.1 on 1 and 198 DF,  p-value: < 2.2e-16
```

Multiple Linear Regression

- Here our model is

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon,$$

- We interpret β_j as the average effect on Y of a one unit increase in X_j, holding all other predictors fixed. In the advertising example, the model becomes

$$\text{sales} = \beta_0 + \beta_1 \times \text{TV} + \beta_2 \times \text{radio} + \beta_p \times \text{newspaper} + \epsilon.$$

Interpreting regression coefficients

- The ideal scenario is when the predictors are uncorrelated – a balanced design:
  - Each coefficient can be estimated and tested separately.
  - Interpretations such as "a unit change in X_j is associated with a β_j change in Y, while all the other variables stay fixed", are possible.
- Correlations amongst predictors cause problems:
  - The variance of all coefficients tends to increase, sometimes dramatically
  - Interpretations become hazardous – when X_j changes, everything else changes.
- Claims of causality should be avoided for observational data.

The woes of (interpreting) regression coefficients

"Data Analysis and Regression" Mosteller and Tukey 1977

- a regression coefficient β_j estimates the expected change in Y per unit change in X_j, with all other predictors held fixed. But predictors usually change together!
- Example: Y total amount of change in your pocket; $X_1 =$ number of coins; $X_2 =$ number of pennies, nickels and dimes. By itself, regression coefficient of Y on X_2 will be > 0. But how about with X_1 in model?
- $Y =$ number of tackles by a rugby player in a season; W and H are his weight and height. Fitted regression model is $\hat{Y} = \beta_0 + .50W - .10H$. How do we interpret $\hat{\beta}_2 < 0$?

Two quotes by famous Statisticians

- "Essentially, all models are wrong, but some are useful" George Box
- "The only way to find out what will happen when a complex system is disturbed is to disturb the system, not merely to observe it passively" Fred Mosteller and John Tukey, paraphrasing George Box

Estimation and Prediction for Multiple Regression

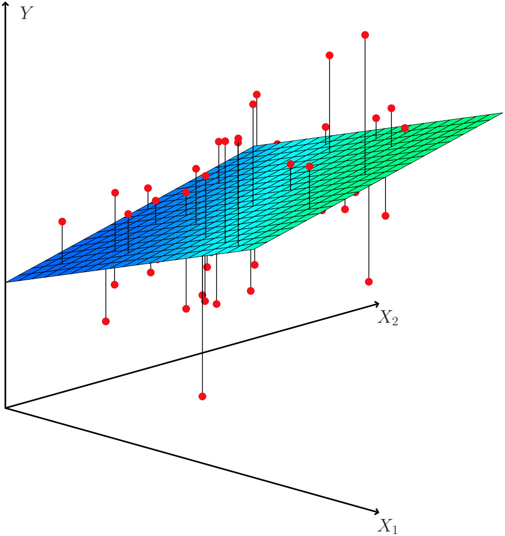
- Given estimates $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$, we can make predictions using the formula

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \cdots + \hat{\beta}_p x_p.$$

- We estimate $\beta_0, \beta_1, \ldots, \beta_p$ as the values that minimize the sum of squared residuals

$$\mathrm{RSS} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i1} - \hat{\beta}_2 x_{i2} - \cdots - \hat{\beta}_p x_{ip})^2.$$

This is done using standard statistical software. The values $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$ that minimize RSS are the multiple least squares regression coefficient estimates.


Y
X_2
X_1

Results for the advertising data

```
multiple.regression <- lm(advertising$Sales ~ advertising$TV + advertising$Radi
cor(advertising[,-1])

##                   TV      Radio  Newspaper     Sales
## TV        1.00000000 0.05480866 0.05664787 0.7822244
## Radio     0.05480866 1.00000000 0.35410375 0.5762226
## Newspaper 0.05664787 0.35410375 1.00000000 0.2282990
## Sales     0.78222442 0.57622257 0.22829903 1.0000000
```

Results for the advertising data

```
summary(multiple.regression)
```

```
##
## Call:
## lm(formula = advertising$Sales ~ advertising$TV + advertising$Radio +
##     advertising$Newspaper)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -8.8277 -0.8908  0.2418  1.1893  2.8292
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)            2.938889   0.311908   9.422   <2e-16 ***
## advertising$TV         0.045765   0.001395  32.809   <2e-16 ***
## advertising$Radio      0.188530   0.008611  21.893   <2e-16 ***
## advertising$Newspaper -0.001037   0.005871  -0.177     0.86
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.686 on 196 degrees of freedom
## Multiple R-squared:  0.8972,Adjusted R-squared:  0.8956
## F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

Some important questions

1. Is at least one of the predictors $X_1, X_2, \ldots, X_p$ useful in predicting the response?
2. Do all the predictors help to explain Y, or is only a subset of the predictors useful?
3. How well does the model fit the data?
4. Given a set of predictor values, what response value should we predict, and how accurate is our prediction?

Is at least one predictor useful?

- For the first question, we can use the F-statistic

$$F = \frac{(TSS - RSS)/p}{RSS/(n-p-1)} \sim F_{p,n-p-1}$$

Deciding on the important variables

- The most direct approach is called all subsets or best subsets regression: we compute the least squares fit for all possible subsets and then choose between them based on some criterion that balances training error with model size.
- However we often can't examine all possible models, since there are 2^p of them; for example when $p = 40$ there are over a billion models!
- Instead we need an automated approach that searches through a subset of them. We discuss two commonly use approaches next.

Forward selection

- Begin with the null model - a model that contains an intercept but no predictors.
- Fit p simple linear regressions and add to the null model the variable that results in the lowest RSS.
- Add to that model the variable that results in the lowest RSS amongst all two-variable models.
- Continue until some stopping rule is satisfied, for example when all remaining variables have a p-value above some threshold.

Backward selection

- Start with all variables in the model.
- Remove the variable with the largest p-value – that is, the variable that is the least statistically significant.
- The new (p - 1) - variable model is fit, and the variable with the largest p-value is removed.
- Continue until a stopping rule is reached. For instance, we may stop when all remaining variables have a significant p-value defined by some significance threshold.

Model selection - continued

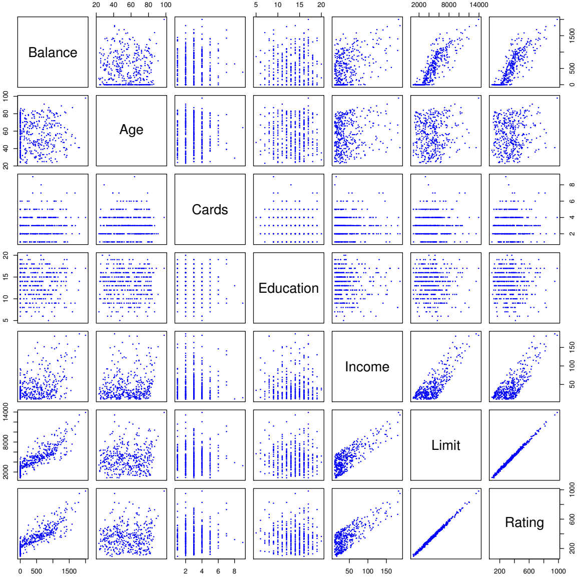
- Later we discuss more systematic criteria for choosing an "optimal" member in the path of models produced by forward or backward stepwise selection.
- These include Mallow's C_p, Akaike information criterion (AIC), Bayesian information criterion (BIC), adjusted R^2 and Cross-validation (CV).

Qualitative predictors

Other Considerations in the Regression Model

Qualitative Predictors

- Some predictors are not quantitative but are qualitative, taking a discrete set of values.
- These are also called categorical predictors or factor variables.
- See for example the scatterplot matrix of the credit card data in the next slide.
- In addition to the 7 quantitative variables shown, there are four qualitative variables: `gender`, `student` (student status), `status` (marital status), and `ethnicity` (Caucasian, African American (AA) or Asian).


Balance
Age
Cards
Education
Income
Limit
Rating
20 40 60 80 100
5 10 15 20
2000 8000 14000
0 500 1500
20 40 60 80 100
2 4 6 8
5 10 15 20
50 100 150
2000 8000 14000
200 600 1000
0 500 1500
2 4 6 8
50 100 150
200 600 1000

Qualitative Predictors – continued

- Example: investigate differences in credit card balance between males and females, ignoring the other variables. We create a new variable

$$x_i = \begin{cases} 1 & \text{if } i\text{th person is female} \\ 0 & \text{if } i\text{th person is male} \end{cases}$$

- Resulting model:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is female} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is male} \end{cases}$$

- Interpretation?

Credit card data

```
credit <- read.csv("http://www-bcf.usc.edu/~gareth/ISL/Credit.csv")
names(credit)

##  [1] "X"         "Income"    "Limit"     "Rating"    "Cards"
##  [6] "Age"       "Education" "Gender"    "Student"   "Married"
## [11] "Ethnicity" "Balance"

gender.regression <- lm(credit$Balance ~ credit$Gender)
```

Results for gender model

```
summary(gender.regression)

##
## Call:
## lm(formula = credit$Balance ~ credit$Gender)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -529.54 -455.35  -60.17  334.71 1489.20
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)           509.80      33.13  15.389   <2e-16 ***
## credit$GenderFemale    19.73      46.05   0.429    0.669
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 460.2 on 398 degrees of freedom
## Multiple R-squared:  0.0004611,Adjusted R-squared:  -0.00205
## F-statistic: 0.1836 on 1 and 398 DF,  p-value: 0.6685
```

Qualitative predictors with more than two levels

- With more than two levels, we create additional dummy variables. For example, for the `ethnicity` variable we create two dummy variables. The first could be

$$x_{i1} = \begin{cases} 1 & \text{if } i\text{th person is Asian} \\ 0 & \text{if } i\text{th person is not Asian,} \end{cases}$$

- and the second could be

$$x_{i2} = \begin{cases} 1 & \text{if } i\text{th person is Caucasian} \\ 0 & \text{if } i\text{th person is not Caucasian.} \end{cases}$$

Qualitative predictors with more than two levels

- Then both of these variables can be used in the regression equation, in order to obtain the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{th person is Asian} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{th person is Caucasian} \\ \beta_0 + \epsilon_i & \text{if } i\text{th person is AA} \end{cases}$$

- There will always be one fewer dummy variable than the number of levels. The level with no dummy variable – African American in this example – is known as the baseline.

Credit card data

```
ethnicity.regression <- lm(credit$Balance ~ credit$Ethnicity)
summary(ethnicity.regression)

##
## Call:
## lm(formula = credit$Balance ~ credit$Ethnicity)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -531.00 -457.08  -63.25  339.25 1480.50
##
## Coefficients:
##                           Estimate Std. Error t value Pr(>|t|)
## (Intercept)                 531.00      46.32  11.464   <2e-16 ***
## credit$EthnicityAsian       -18.69      65.02  -0.287    0.774
## credit$EthnicityCaucasian   -12.50      56.68  -0.221    0.826
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 460.9 on 397 degrees of freedom
## Multiple R-squared:  0.0002188,Adjusted R-squared:  -0.004818
## F-statistic: 0.04344 on 2 and 397 DF,  p-value: 0.9575
```

Extensions of the Linear Model

Removing the additive assumption: interactions and nonlinearity

Interactions:

- In our previous analysis of the `Advertising` data, we assumed that the effect on `sales` of increasing one advertising medium is independent of the amount spent on the other media.
- For example, the linear model

$$\widehat{sales} = \beta_0 + \beta_1 \times TV + \beta_2 \times radio + \beta_3 \times newspaper$$

states that the average effect on `sales` of a one-unit increase in `TV` is always β_1, regardless of the amount spent on `radio`.

Interactions – continued

- But suppose that spending money on radio advertising actually increases the effectiveness of TV advertising, so that the slope term for TV should increase as `radio` increases.
- In this situation, given a fixed budget of $100,000, spending half on `radio` and half on `TV` may increase sales more than allocating the entire amount to either `TV` or to `radio`.
- In marketing, this is known as a synergy effect, and in statistics it is referred to as an interaction effect.

Modelling interactions – Advertising data

Model takes the form

$$sales = \beta_0 + \beta_1 \times TV + \beta_2 \times radio + \beta_3 \times (radio \times TV) + \epsilon$$

$$= \beta_0 + (\beta_1 + \beta_3 \times radio) \times TV + \beta_2 \times radio + \epsilon$$

Modelling interactions – Advertising data

```
interaction.model <- lm(advertising$Sales ~ advertising$TV*advertising$Radio)
summary(interaction.model)

##
## Call:
## lm(formula = advertising$Sales ~ advertising$TV * advertising$Radio)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.3366 -0.4028  0.1831  0.5948  1.5246
##
## Coefficients:
##                                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)                     6.750e+00  2.479e-01  27.233   <2e-16 ***
## advertising$TV                  1.910e-02  1.504e-03  12.699   <2e-16 ***
## advertising$Radio               2.886e-02  8.905e-03   3.241   0.0014 **
## advertising$TV:advertising$Radio 1.086e-03  5.242e-05  20.727   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9435 on 196 degrees of freedom
## Multiple R-squared:  0.9678,Adjusted R-squared:  0.9673
## F-statistic:  1963 on 3 and 196 DF,  p-value: < 2.2e-16
```

Interpretation

- The results in this estimation suggests that interactions are important.
- The p-value for the interaction term $TV \times radio$ is extremely low, indicating that there is strong evidence for $H_A : \beta_3 \neq 0$.
- The R^2 for the interaction model is 96.8%, compared to only 89.7% for the model that predicts *sales* using *TV* and *radio* without an interaction term.

Interpretation – continued

- This means that (96.8 - 89.7)/(100 - 89.7) = 69% of the variability in sales that remains after fitting the additive model has been explained by the interaction term.
- The coefficient estimates in the table suggest that an increase in TV advertising of $1,000 is associated with increased sales of

$$(\hat{\beta}_1 + \hat{\beta}_3 \times radio) \times 1000 = 19 + 1.1 \times radio\ \ units.$$

- An increase in radio advertising of $1,000 will be associated with an increase in sales of

$$(\hat{\beta}_2 + \hat{\beta}_3 \times TV) \times 1000 = 29 + 1.1 \times TV\ \ units.$$

Hierarchy

- Sometimes it is the case that an interaction term has a very small p-value, but the associated main effects (in this case, *TV* and *radio*) do not.
- The hierarchy principle: If we include an interaction in a model, we should also include the main effects, even if the p-values associated with their coefficients are not significant.

Hierarchy

- The rationale for this principle is that interactions are hard to interpret in a model without main effects – their meaning is changed.
- Specifically, the interaction terms also contain main effects, if the model has no main effect terms.

Interactions between qualitative and quantitative variables

- Consider the *Credit* dataset, and suppose that we wish to predict *balance* using *income* (quantitative) and *student* (qualitative).
- Without an interaction term, the model takes the form

$$balance_i \approx \beta_0 + \beta_1 \times income_i + \begin{cases} \beta_2 & \text{if } i\text{th person is a student} \\ 0 & \text{if } i\text{th person is not a student} \end{cases}$$

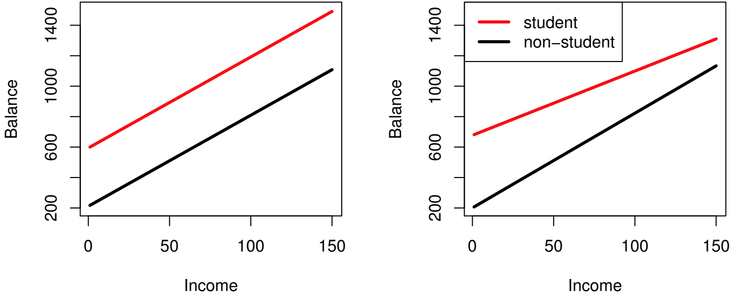
$$= \beta_1 \times income_i + \begin{cases} \beta_0 + \beta_2 & \text{if } i\text{th person is a student} \\ \beta_0 & \text{if } i\text{th person is not a student} \end{cases}$$

- With interactions, it takes the form

$$balance_i \approx \beta_0 + \beta_1 \times income_i + \begin{cases} \beta_2 + \beta_3 \times income_i & \text{if } i\text{th person is a student} \\ 0 & \text{if } i\text{th person is not a student} \end{cases}$$

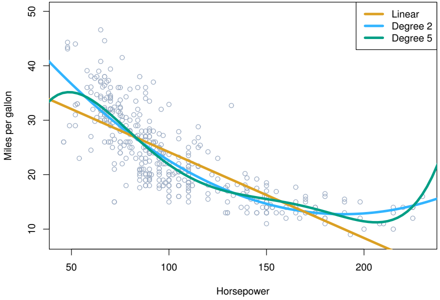
$$= \begin{cases} (\beta_0 + \beta_2) + (\beta_1 + \beta_3) & \text{if } i\text{th person is a student} \\ \beta_0 + \beta_1 \times income_i & \text{if } i\text{th person is not a student} \end{cases}$$

Credit data



- For the *Credit* data, the least squares lines are shown for prediction of balance from income for students and non-students.
- Left: no interaction between income and student.
- Right: with an interaction term between income and student.

Non-linear effects of predictors



- Polynomial regression on *Auto* data
- The figure suggests that

$$mpg = \beta_0 + \beta_1 \times horsepower + \beta_2 \times horsepower^2 + \epsilon$$

may provide a better fit.

```r
library(ISLR)
auto.model <- lm(Auto$mpg ~ Auto$horsepower + I(Auto$horsepower^2))
summary(auto.model)
```

```
##
## Call:
## lm(formula = Auto$mpg ~ Auto$horsepower + I(Auto$horsepower^2))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -14.7135  -2.5943  -0.0859   2.2868  15.8961
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)          56.9000997  1.8004268   31.60   <2e-16 ***
## Auto$horsepower      -0.4661896  0.0311246  -14.98   <2e-16 ***
## I(Auto$horsepower^2)  0.0012305  0.0001221   10.08   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.374 on 389 degrees of freedom
## Multiple R-squared:  0.6876,Adjusted R-squared:  0.686
## F-statistic:   428 on 2 and 389 DF,  p-value: < 2.2e-16
```

What we did not cover

- Correlation of the error-terms.
- Non-constant variance of error terms.
- Outliers.
- High leverage points.
- Collinearity.

See text Section 3.3.3

The basic fixed effects model

- The fixed effects model is simply a variation on the linear regression model.
- Its key advantage is that it enables us to control for all variables that vary over the cross-sectional units but are constant over time.

Assumptions of the fixed effects model

- The fixed effects model assumes that the true relationship is:

$$y_{i,t} = \beta_0 + \beta_1 x_{i,t} + \beta_2 z_i + u_{i,t} \tag{1}$$

  where e.g. $y_{i,t}$ would be the number of traffic fatalities and $x_{i,t}$ the beer tax in state i in year t.
- Note that the variable z_i does not have a time index and is therefore assumed to be constant over time.
- In this example z_i could be the social attitude towards drunk driving in state i.

Assumptions of the fixed effects mode

- If we define $\alpha_i = \beta_0 + \beta_2 z_i$, then (1) simplifies to

$$y_{i,t} = \alpha_i + \beta_1 x_{i,t} + u_{i,t} \tag{2}$$

- The graphical interpretation of α_i is that it is the intercept of the relationship between alcohol taxes and traffic fatalities in state i.
- It is straightforward to allow for further variables which are constant over time in (1).
- In this case the intercepts α_i reflect the combined effect of several variables which are constant over time.

```
> library(plm)
> library(foreign)
> fatality <- read.dta("fatality.dta")
>
> #Rescale the outcome variable to make coefficients easier to read.
> #Note: substantive interpretation will need to take this into account.
>
> fatality$mrall <- fatality$mrall*100000
>
> #FE individual
> fatality.fe <- plm(mrall ~ beertax + mlda + jaild + vmiles + unrate + perinc,
+              data = fatality,
+              index = c("state","year"),
+              model = "within",
+              effect = "individual")
>
```

```
> summary(fatality.fe)
Oneway (individual) effect Within Model

Call:
plm(formula = mrall ~ beertax + mlda + jaild + vmiles + unrate +
    perinc, data = fatality, effect = "individual", model = "within",
    index = c("state", "year"))

Unbalanced Panel: n=48, T=6-7, N=335

Residuals :
   Min. 1st Qu.  Median 3rd Qu.    Max.
-5.3700 -0.7380 -0.0258  0.7790  8.4700

Coefficients :
           Estimate  Std. Error t-value Pr(>|t|)
beertax -4.1024e+00  1.9317e+00 -2.1237 0.034569 *
mlda    -3.2178e-01  1.9836e-01 -1.6222 0.105881
jaild   -2.9744e-01  6.9316e-01 -0.4291 0.668167
vmiles  -2.6827e-05  1.0165e-04 -0.2639 0.792031
unrate  -2.8645e-01  1.0705e-01 -2.6758 0.007892 **
perinc   2.0616e-04  2.2690e-04  0.9086 0.364341
---
Signif. codes:  0 `***' 0.001 `**' 0.01 `*' 0.05 `.' 0.1 ` ' 1

Total Sum of Squares:    1078.5
Residual Sum of Squares: 949.22
R-Squared      :  0.11986
      Adj. R-Squared :  0.10054
F-statistic: 6.37764 on 6 and 281 DF, p-value: 2.6041e-06
```

Advantages and disadvantages of the fixed effects model

- The key advantage of the fixed effects model is that it allows us to control for all time invariant omitted variables.
- This is particularly important in the case of variables which are difficult or impossible to observe.
- The key disadvantage is that we have to estimate a number of additional parameters.
- Furthermore, it will be impossible to estimate the effect of variables which do not (or hardly) vary over time.

Time and state fixed effects

- In most applications we use both state and time fixed effects at the same time.
- This model is sometimes referred to as the "twoway fixed effects" model.
- In the literature the cross-sectional fixed effects are referred to as "fixed effects", "state (fixed) effects", "firm (fixed) effects" or "person (fixed) effects".
- Similarly, time fixed effects are often referred to as "time effects".

```
> fatality.twfe <- plm(mrall ~ beertax + mlda + jaild + vmiles + unrate + perinc,
+                      data = fatality, index = c("state","year"), model = "within", effect="twoways")
> summary(fatality.twfe)
Twoways effects Within Model

Call:
plm(formula = mrall ~ beertax + mlda + jaild + vmiles + unrate +
    perinc, data = fatality, effect = "twoways", model = "within",
    index = c("state", "year"))

Unbalanced Panel: n=48, T=6-7, N=335

Residuals :
   Min. 1st Qu.  Median 3rd Qu.    Max.
-4.5600 -0.8440  0.0766  0.8280  5.0700

Coefficients :
           Estimate  Std. Error t-value  Pr(>|t|)
beertax -5.2835e+00  1.6922e+00 -3.1223  0.001986 **
mlda     8.7754e-03  1.8192e-01  0.0482  0.961562
jaild    4.1344e-01  6.1279e-01  0.6747  0.500440
vmiles   9.3576e-05  8.9517e-05  1.0453  0.296783
unrate  -7.6731e-01  1.0778e-01 -7.1189 9.491e-12 ***
perinc   6.9770e-04  2.1886e-04  3.1878  0.001599 **
---
Signif. codes:  0 `***' 0.001 `**' 0.01 `*' 0.05 `.' 0.1 ` ' 1

Total Sum of Squares:    1028.9
Residual Sum of Squares: 694.58
R-Squared      :  0.32495
      Adj. R-Squared :  0.26675
F-statistic: 22.063 on 6 and 275 DF, p-value: < 2.22e-16
```

Serial correlation in the error term

- If there is substantial autocorrelation (serial correlation) in the error term, even heteroskedasticity-robust standard errors will be inconsistent.
- In panel data as in any other time series data, autocorrelation can be a very serious concern.

- We can test for serial correlation after our fixed effects estimation using the Breusch-Godfrey test.
- The null hypothesis in this test is that the autocorrelation of the error term is 0.

```
> pbgtest(fatality.twfe)

Breusch-Godfrey/Wooldridge test for serial correlation in panel models

data:  mrall ~ beertax + mlda + jaild + vmiles + unrate + perinc
chisq = 28.394, df = 6, p-value = 7.918e-05
alternative hypothesis: serial correlation in idiosyncratic errors
```

Clustered standard error

- A popular solution to the problem of autocorrelation in the error term are clustered standard errors, aka heteroskedasticity and autocorrelation consistent (HAC) standard errors.
- Such standard errors assume no correlation between errors of different groups while allowing for heteroskedasticity across groups.
- For the *intragroup* error structure the assumption is that there is a general form of heteroskedasticity and autocorrelation.
- In panel data models with fixed effects HAC standard errors are calculated using the so called "arellano" method, suggested in Arellano (1987).

```
> library(sandwich)
> library(lmtest)
> fatality.hac <- coeftest(fatality.twfe, vcov = vcovHC(fatality.twfe, method= "arellano",  type="HC3"))
> fatality.hac

t test of coefficients:

           Estimate  Std. Error t value  Pr(>|t|)
beertax -5.2835e+00  3.4487e+00 -1.5320   0.12666
mlda     8.7754e-03  2.2491e-01  0.0390   0.96890
jaild    4.1344e-01  1.1267e+00  0.3670   0.71394
vmiles   9.3576e-05  2.8130e-04  0.3327   0.73965
unrate  -7.6731e-01  1.3190e-01 -5.8175 1.657e-08 ***
perinc   6.9770e-04  3.7941e-04  1.8389   0.06701 .
---
Signif. codes:  0 `***' 0.001 `**' 0.01 `*' 0.05 `.' 0.1 ` ' 1
```

Generalizations of the Linear Model

In much of the rest of this course, we discuss methods that expand the scope of linear models and how they are fit:

- Classification problems: logistic regression, support vector machines
- Non-linearity: kernel smoothing, splines and generalized additive models; nearest neighbor methods.
- Interactions: Tree-based methods, bagging, random forests and boosting (these also capture non-linearities)
- Regularized fitting: Ridge regression and lasso