Chapter 6.1a Joint Probability Mass Function

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Chapter 6 Joint Probability Distributions

Sampling From a Box

- Suppose one has a box of ten balls – four are white, three are red, and three are black.
- Select five balls out of the box without replacement and count the number of white and red balls in the sample.
- What is the probability one observes two white and two red balls in the sample?
- We can find this probability can be found using ideas from previous chapters.

Probability of 2 white and 2 red balls in sample

- Assume the balls are distinct and order is not important, so the total number ways of choosing 5 balls out of 10 is $N = \binom{10}{5} = 252$.
- Count the number of ways of selecting two white and two red balls by selecting the white balls ($\binom{4}{2}$ ways), then selecting the red balls ($\binom{3}{2}$ ways), and then selecting the one remaining black ball ($\binom{3}{1}$ ways). So the total number of ways is

$$\binom{4}{2} \times \binom{3}{2} \times \binom{3}{1} = 6 \times 3 \times 3 = 54.$$

Probability of 2 white and 2 red balls in sample

- Each one of the $\binom{10}{5} = 252$ possible outcomes of five balls is equally likely
- So the probability of choosing two white and two red balls is

$$P(2\,\mathrm{white\,and}\,2\,\mathrm{red}) = \frac{54}{252}.$$

Generalize calculation

- Suppose we want probability of choosing a specific number of white and specific number of red balls.
- We define two random variables.

$X =$ number of red balls selected, $Y =$ number of white balls selected.

Based on what was found,

$$P(X = 2, Y = 2) = \frac{54}{252}.$$

Joint probability mass function

Suppose this calculation is done for every possible pair of values of X and Y. The table of probabilities is given in this table.

	$Y = \#ofWhite$				
$X = \#$ of Red	0	1	2	3	4
0	0	0	$6/252$	$12/252$	$3/252$
1	0	$12/252$	$54/252$	$36/252$	$3/252$
2	$3/252$	$36/252$	$54/252$	$12/252$	0
3	$3/252$	$12/252$	$6/252$	0	0

This table is called the joint probability mass function (pmf) $f(x, y)$ of (X, Y).

Joint probability mass function

- To be a proper joint probability mass function, or joint pmf, one requires that each of the probability values are nonnegative and the sum of the probabilities over all values of X and Y is one. That is,

1. $f(x, y) \geq 0$, for all x, y
2. $\Sigma_{x,y} f(x, y) = 1$

- One can check all of the probabilities in our example joint pmf are nonnegative
- You can confirm that the sum of the probabilities is equal to one.

Back to our example

- Most likely values of (x, y) are (1, 1) and (2, 1) - each has a probability of $54/252$.
- Some particular pairs (x, y) are not possible as $f(x, y) = 0$.
- For example, $f(0, 1) = 0$ which means that it is not possible to observe 0 red balls and 1 white ball in the sample.

Finding Probabilities

One finds probabilities of any event involving X and Y by summing probabilities over the joint pmf.

- **What is $P(X = Y)$, the probability that one samples the same number of red and white balls?**
- By the table, one sees that this is possible only when $X = 1, Y = 1$ or $X = 2, Y = 2$.
- So the probability

$$P(X = Y) = f(1,1) + f(2,2) = \frac{12}{252} + \frac{54}{252} = \frac{66}{252}.$$

Finding Probabilities

- **What is $P(X > Y)$, the probability one samples more red balls than white balls?**
- From the table, one identifies the outcomes where $X > Y$, and then sums the corresponding probabilities.

$$P(X > Y) = f(1,0) + f(2,0) + f(2,1) + f(3,0)$$

$$+f(3,1) + f(3,2)$$

$$= \frac{12}{252} + \frac{3}{252} + \frac{36}{252} + \frac{3}{252} + \frac{12}{252} + \frac{6}{252}$$

$$= \frac{72}{252}$$