Chapter 6.5 The Beta-Binomial Distribution

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Chapter 6 Joint Probability Distributions

Flipping a Random Coin

- Suppose one has a box of coins where the coin probabilities vary.
- If one selects a coin from the box, p, the probability the coin lands heads follows the distribution

$$g(p) = \frac{1}{B(6,6)} p^5 (1-p)^5, \; 0 < p < 1,$$

where $B(6,6)$ is the Beta function.

Graph of varying coin probabilities

This density is plotted below.

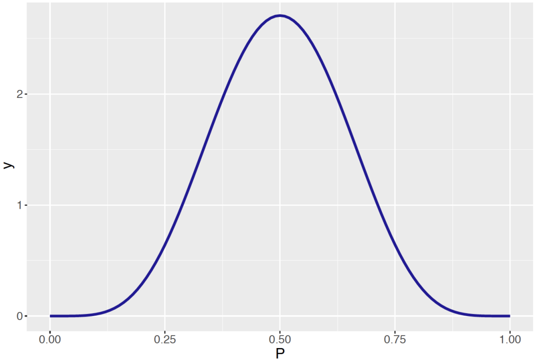

Figure 1: Beta(6, 6) density representing the distribution of probabilities of heads for a large collection of random coins.

Density of coin probabilities

- A couple of things to notice about this density.
- First, the density has a significant height over much of the plausible values of the probability – this reflects the idea that one are really unsure about the chance of observing a heads when flipped.
- Second, the density is symmetric about $p = 0.5$, which means that the coin is equally likely to be biased towards heads or biased towards tails.

Flipping the random coin

- One next flips this “random" coin 20 times.
- Denote the outcome of this experiment by the random variable Y which is equal to the count of heads.
- If we are given a value of the probability p, then Y has a Binomial distribution with $n = 20$ trials and success probability p.
- This probability function is actually the conditional probability of observing y heads given a value of the probability p:

$$f(y \mid p) = \binom{20}{y} p^y (1-p)^{20-y}, \ \ y = 0, 1, ..., 20.$$

The Beta-Binomial density

- Given the density of p and the conditional density of Y conditional on p, one computes the joint density by the product

$$f(y, p) = g(p) f(y \mid p) = \left[\frac{1}{B(6,6)} p^5 (1-p)^5 \right] \left[\binom{20}{y} p^y (1-p)^{20-y} \right]$$

$$= \frac{1}{B(6,6)} \binom{20}{y} p^{y+5} (1-p)^{25-y}, \ 0 < p < 1, y = 0, 1, ..., 20.$$

- This is a mixed density in the sense that one variable (p) is continuous and one (Y) is discrete.
- This density will be seen to be very important in our study of inference about a binomial proportion p in Chapter 7.