Chapter 2.6 Arrangements of Nondistinct Objects

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Chapter 2 Counting Methods

Creating silly words

- Suppose one is making up silly words from the letters "a", "b", "c", "d", "e", "f", like

bacedf, decabf, eabcfd

- How many silly words can one make up?
- Using the permutation rule, the number of possible permutations is

$$6! = 6 \times 5 \times 4 \times ... \times 1.$$

Second counting rule

- Suppose one has six letters "a", "b", "c", "d", "e", "f", and one is going to choose three of the letters to construct a three-letter word.
- One cannot choose the same letter twice and the order in which one chooses the letters is not important.
- Here one is interested in the number of combinations – the number of ways of choosing three letters from six is equal to

$$\binom{6}{3} = \frac{6!}{3!\,3!}.$$

Different arrangement problem

- One randomly arranges the four triangles and five squares as shown below.

- What is the chance that the first and last locations are occupied by triangles?

What is the difference?

- This is an arrangement problem with one difference – the objects are not all distinct
- So one cannot use the earlier permutations rule that assumes the objects are distinguishable.
- How can one count the number of possible arrangements?

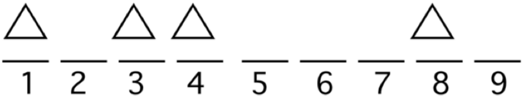
Possible arrangements

- Suppose one writes down a list of nine slots and an arrangement is constructed by placing the triangles and the squares in the nine slots.

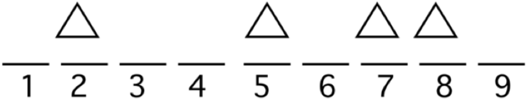
___	___	___	___	___	___	___	___	___
1	2	3	4	5	6	7	8	9

Constructing an arrangement

- First, place the four triangles in four slots, and then place the squares in the remaining slots.
- How many ways can one put the triangles in the slots? For example, one could place the triangles in slots 1, 3, 4, and 8.



- Or one could place the four triangles in slots 2, 5, 7, and 8.



Constructing an arrangement

- One specifies an arrangement by choosing four locations from the slot locations $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- One knows that the number of ways of selecting four objects from a group of nine objects is

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = 126.$$

- One finishes the arrangement by putting in the squares. But there is only one way of doing this.
- So applying the multiplication rule, the number of ways of arranging four triangles and five squares is $126 \times 1 = 126$.

Permutations Rule for Non-Distinct Objects

- The number of permutations of n non-distinct objects where r are of one type and $n - r$ are of a second type is

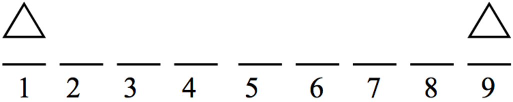
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Back to example

- Suppose four triangles and five squares are randomly arranged. What is the chance that the first and last locations are occupied by triangles?
- We know there are 126 ways of mixing up four triangles and five squares.
- Each possible arrangement is equally likely and has a chance of $1/126$ of occurring.

Finding the probability

- One needs to count the number of ways of arranging the triangles and squares so that the first and last positions are filled with triangles.



- If one places triangles in slots 1 and 9, then one is free to arrange the remaining two triangles and five squares in the remaining 7 slots. By use of the new arrangements formula, the number of ways is

$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = 21$$

  and so the probability the first and last slots are filled with triangles is equal to $21/126$.