Chapter 6.2 Multinomial Distribution

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Chapter 6 Joint Probability Distributions

Introduction

- Suppose one rolls the usual six-sided die where one side shows 1, two sides show 2, and three sides show 3.
- One rolls this die ten times – what is the chance that one will observe three 1's and five 2's?
- This situation resembles the coin-tossing experiment described in Chapter 4.
- The difference is that the coin-tossing experiment had only two possible outcomes on a single trial, and here there are three outcomes on a single die roll, 1, 2, and 3.

A Multinomial Experiment

- Suppose a random experiment consists of a sequence of n independent trials where there are k possible outcomes on a single trial where $k \geq 2$.
- Denote the possible outcomes as 1, 2, . . . , k, and let $p_1, p_2, ..., p_k$ denote the associated probabilities.
- Let X_1, X_2, . . . , X_k denote the number of 1s, 2s, . . . , ks observed in the n trials
- The vector of outcomes $X = (X_1, X_2, ..., X_n)$ has a Multinomial distribution with sample size n and vector of probabilities $p = (p_1, p_2, ..., p_k)$.

Die Example

- Each die roll has $k = 3$ possible outcomes and the associated vector of probabilities is $p = (1/6, 2/6, 3/6)$.
- The number of observed 1s, 2s, 3s in $n = 10$ trials, $X = (X_1, X_2, X_3)$ has a Multinomial distribution with parameters n and p.
- The probability that $X_1 = x_1, ..., X_k = x_k$ has the general form

$$f(x_1, ..., x_k) = \left(\frac{n!}{n_1!...n_k!}\right) \prod_{j=1}^{k} p_j^{x_j},$$

where $x_j = 0, 1, 2, ..., j = 1, ...k$ and $\sum_{j=1}^{n} x_j = n$.

Computing a probability

- One has $n = 10$ trials and the outcome "three 1's and five 2's" is equivalent to the outcome $X_1 = 3, X_2 = 5$.
- Note that the number of 3s X_3 is not random since we know that $X_1 + X_2 + X_3 = 10$. The probability vector is $p = (1/6, 2/6, 3/6)$.
- By substitution, we get

$$P(X_1 = 3, X_2 = 5, X_3 = 2) = \left(\frac{10!}{3!\,5!\,2!}\right)\left(\frac{1}{6}\right)^3\left(\frac{2}{6}\right)^5\left(\frac{3}{6}\right)^2 = 0.012$$

Another probability

- Other probabilities can be found by summing the joint Multinomial pmf over sets of interest.
- Suppose one is interested in computing the probability that the number of 1s exceeds the number of 2's in our ten dice rolls.
- One is interested in the probability $P(X_1 > X_2)$ which is given by

$$P(X_1 > X_2) = \sum_{x_1 > x_2} \left(\frac{10!}{3!\,5!\,2!} \right) \left(\frac{1}{6} \right)^{x_1} \left(\frac{2}{6} \right)^{x_2} \left(\frac{3}{6} \right)^{10 - x_1 - x_2},$$

where one is summing over all of the outcomes (x_1, x_2) where $x_1 > x_2$.

Marginal distributions

- One attractive feature of the Multinomial distribution is that the marginal distributions have familiar functional forms.
- In the dice roll example, suppose one is interested only in X_1, the number of 1s in ten rolls of our die.
- One obtains the marginal probability distribution of X_1 directly by summing out the other variables from the joint pmf of X_1 and X_2.
- For example, one finds, say $P(X_1 = 2)$, by summing the joint probability values over all (x_1, x_2) pairs where $x_1 = 2$:

$$P(X_1 = 2) = \sum_{x_2, x_1 + x_2 \leq 10} f(x_1, x_2).$$

Another way of thinking about this

- In each die roll, suppose one records if one gets a one or not.
- Then X_1, the number of ones in n trials, will be Binomial distributed with parameters n and $p = 1/6$.
- Using a similar argument, X_2, the number of twos in n trials, will be Binomial with n trials and $p = 2/6$.

Conditional distributions

- One applies the knowledge about marginal distributions to compute conditional distributions in the Multinomial situation.
- Suppose that one is given that $X_2 = 3$ in $n = 10$ trials. What can one say about the probabilities of X_1?
- One uses the conditional pmf definition to compute the conditional probability $P(X_1 = x \mid X_2 = 3)$.

$$P(X_1 = x \mid X_2 = 3) = \frac{P(X_1 = x, X_2 = 3)}{P(X_2 = 3)}.$$

Conditional distributions

- Think about possible values for X_1.
- Since one has $n = 10$ rolls of the die and we know that we observe $X_2 = 3$ (three twos), the possible values of X_1 can be 0, 1, . . . , 7.
- For these values, we have

$$P(X_1 = x \mid X_2 = 3) = \frac{P(X_1 = x, X_2 = 3)}{P(X_2 = 3)}.$$

- The numerator is the Multinomial probability and since X_2 and the denominator is a Binomial probability.

Calculation

- Making the substitutions, one has

$$P(X_1 = x \mid X_2 = 3) = \frac{\left(\frac{10!}{x!\,3!\,(10-x-3)!}\right)\left(\frac{1}{6}\right)^x \left(\frac{2}{6}\right)^3 \left(\frac{3}{6}\right)^{10-x-3}}{\binom{10}{3}\left(\frac{2}{6}\right)^3\left(1-\frac{2}{6}\right)^{10-3}}.$$

- After some simplification, one obtains

$$P(X_1 = x \mid X_2 = 3) = \binom{7}{x}\left(\frac{1}{4}\right)^x \left(1-\frac{1}{4}\right)^{7-x},\ x = 0, \ldots, 7.$$

  which is a Binomial distribution with 7 trials and probability of success $1/4$.