Chapter 3.5-3.6 Definition and the Multiplication Rule

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Chapter 3 Conditional Probability

Introduction

– There is a formal definition of conditional probability .

- Suppose one has two events A and B where the probability of event B is positive, that is $P(B) > 0$.
- Then the probability of A given B is defined as the quotient

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

How many boys?

- Suppose a couple has four children. One is told that this couple has at least one boy. What is the chance that they have exactly two boys?
- If one lets L be the event "at least one boy" and B be the event "have two boys", one wishes to find $P(B \mid L)$.

The sample space

- Represent the genders of the four children (from youngest to oldest) as a sequence of four letters.
- There are 16 possible outcomes of four births

BBBB	*BGBB*	*GBBB*	*GGBB*
BBBG	*BGBG*	*GBBG*	*GGBG*
BBGB	*BGGB*	*GBGB*	*GGGB*
BBGG	*BGGG*	*GBGG*	*GGGG*

- If one assumes that boys and girls are equally likely, each outcome is assigned a probability of $1/16$.

Computation

- Applying the definition of conditional probability, one has

$$P(B \mid L) = \frac{P(B \cap L)}{P(L)}.$$

- There are 15 outcomes in the set L, and 6 outcomes where both events B and L occur.
- So using the definition

$$P(B \mid L) = \frac{6/16}{15/16} = \frac{6}{15}.$$

The multiplication rule

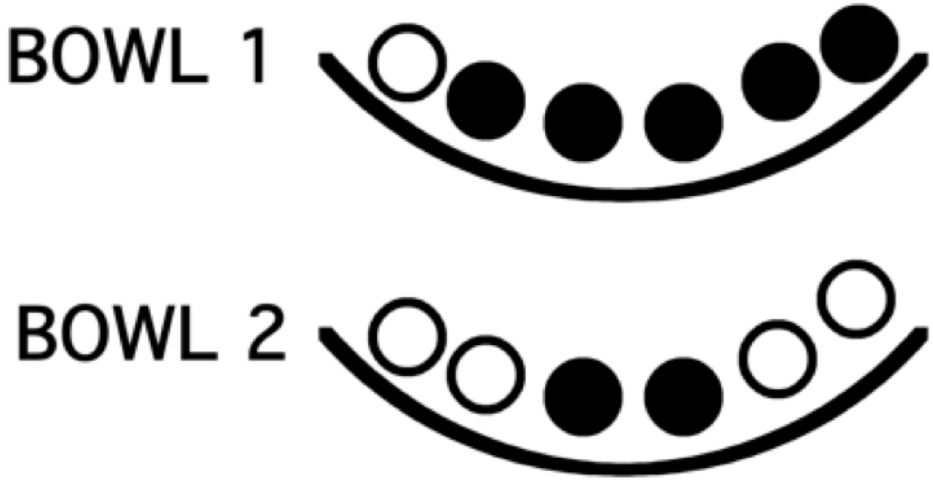
- If one takes the conditional probability definition and multiplies both sides of the equation by $P(B)$, one obtains the multiplication rule

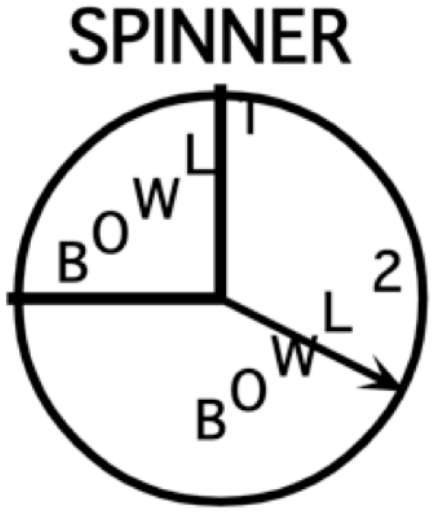
$$P(A \cap B) = P(B)P(A \mid B).$$

Choosing balls from a random bowl

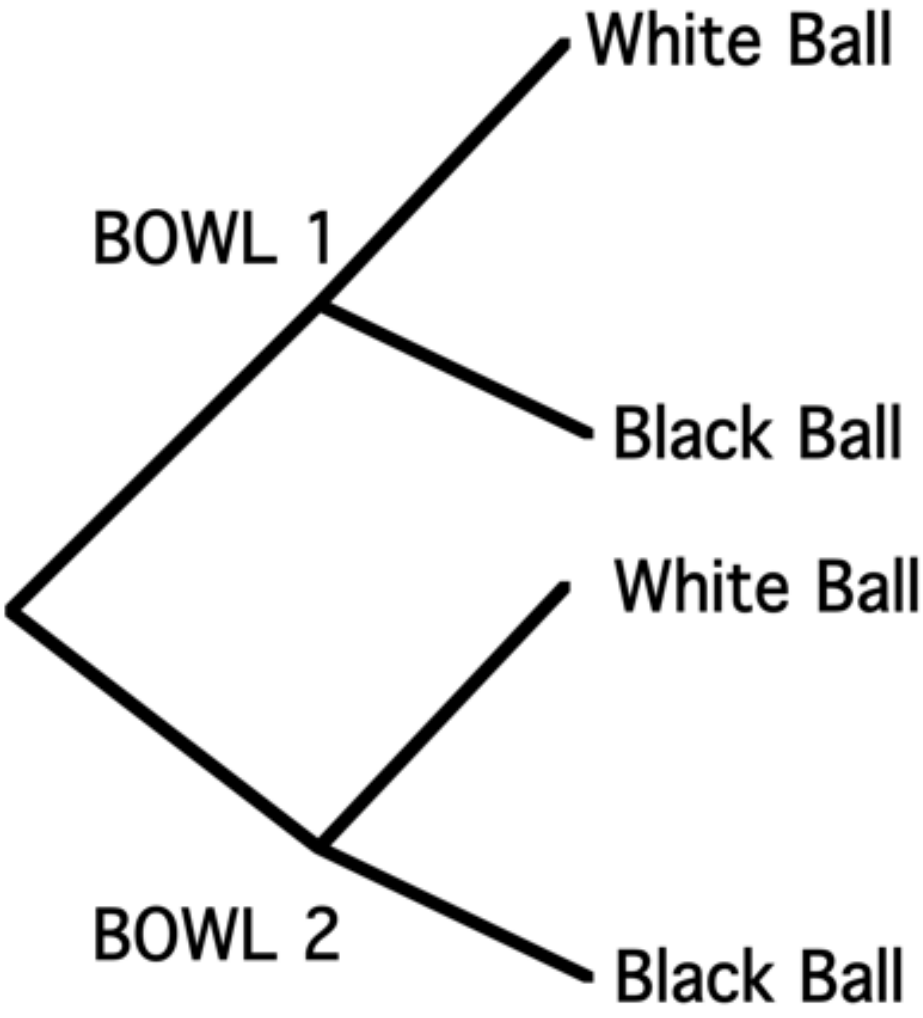
- Suppose one has two bowls – Bowl 1 is filled with one white and 5 black balls, and Bowl 2 has 4 white and 2 black balls.
- One first spins the spinner below that determines which bowl to select, and then selects one ball from the bowl.
- What the chance that the ball one selects is white?

The experiment





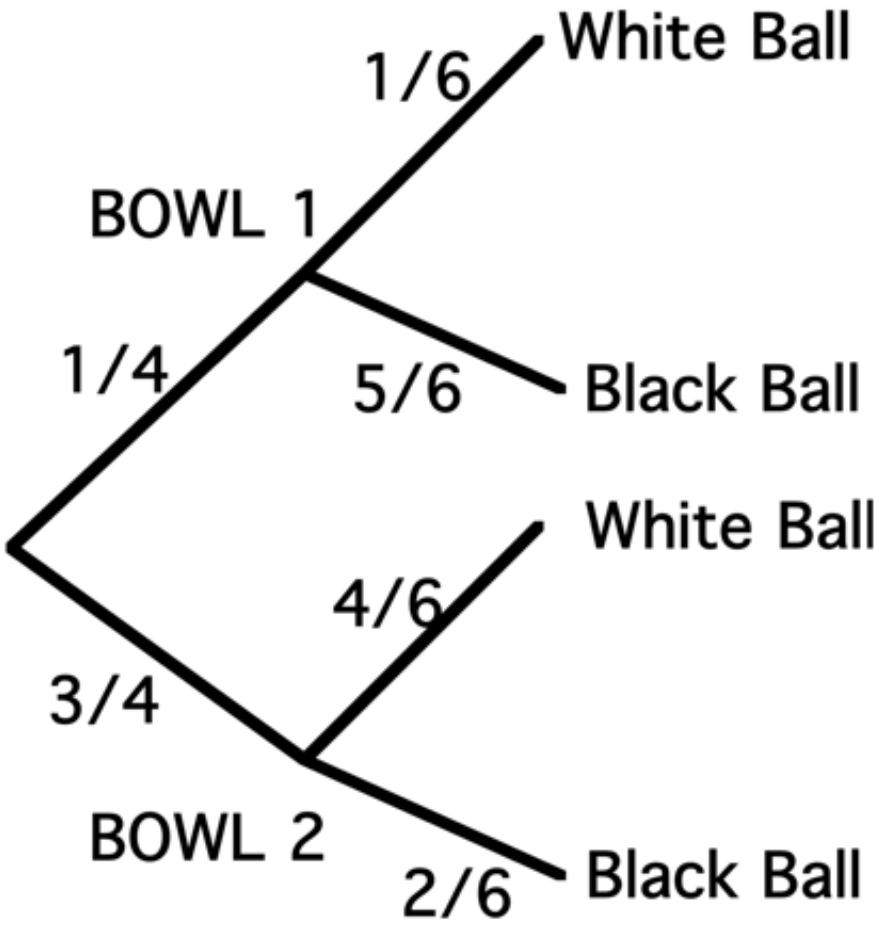
A tree diagram



Add probabilities to diagram

- Since one quarter of the spinner region is "Bowl 1", the chance of choosing Bowl 1 is $1/4$ and so the chance of choosing Bowl 2 is $3/4$
- Also one knows that if Bowl 1 is selected, the chances of choosing a white ball and a black ball are respectively $1/6$ and $5/6$
- Also, if one selects Bowl 2, the conditional probabilities of selecting a white ball and a black ball are given by $P(\text{white} \mid \text{Bowl 2}) = 4/6$ and $P(\text{black} \mid \text{Bowl 2}) = 2/6$

Labeled tree diagram



Computing probabilities

- What is the probability of selecting Bowl 1 and selecting a white ball?
- By the multiplication rule

$$P(\text{Bowl 1} \cap \text{white ball}) = P(\text{Bowl 1})P(\text{white ball} \mid \text{Bowl 1})$$
$$= \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}.$$

- One is just multiplying probabilities along the top branch of the tree.

Computing probabilities

- What is the probability of selecting a white ball?
- There are two ways of selecting a white depending on which bowl is selected. One can either (1) select Bowl 1 and choose a white ball or (2) select Bowl 2 and choose a white ball.
- One finds the probability of each of the two outcomes and add the probabilities

$$\begin{aligned} P(\text{white ball}) &= P(\text{Bowl } 1 \cap \text{white ball}) + P(\text{Bowl } 2 \cap \text{white ball}) \\ &= \frac{1}{4} \times \frac{1}{6} + \frac{3}{4} \times \frac{4}{6} = \frac{13}{24}. \end{aligned}$$