Chapter 5.4-5.5 The Cumulative Distribution Function

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Chapter 5 Continuous Random Variables

Definition

- There is a basic function that can be computed that will simplify these probability computations for a continuous random variable.
- Choose an arbitrary point x – the cumulative distribution function at x, or cdf for short, is the probability that W is less than or equal to x:

$$F(x) = P(W \leq x) = \int_{-\infty}^{x} f(w)dw. \tag{1}$$

Our Example

- Here is the density function for our longest waiting time.

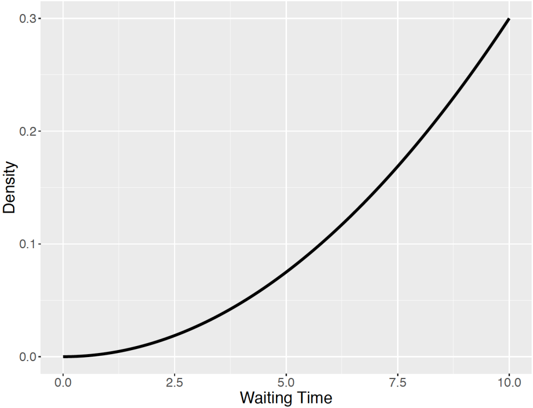

Figure 1: Density curve for the longest waiting time W.

Defining the cdf $F(x)$

- Suppose one chooses a value of x in the interval (0, 10). Then $F(x)$ would be the area under the density curve between 0 and x shown below.

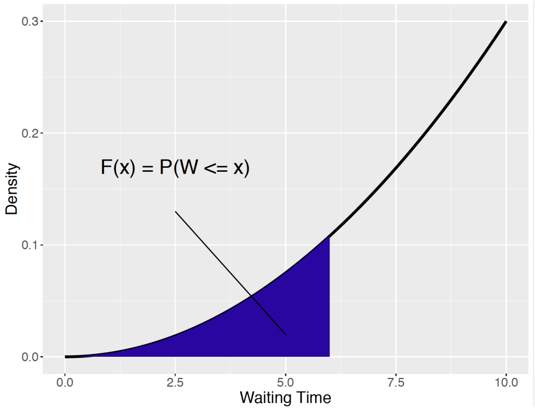

Figure 2: Illustration of the cumulative density function.

Some Calculus

Writing this area as an integral, one computes $F(x)$ as

$$F(x) = P(W \leq x) = \int_0^x \frac{3w^2}{1000} dw = \frac{w^3}{1000}\bigg|_0^x = \frac{x^3}{1000}.$$

This formula is valid for any value of x in the interval (0, 10).

Define $F(x)$ for all values of x

- If x is a value smaller or equal to 0, the probability that W is smaller than x is equal to 0.
- If x is greater or equal to 10, the probability that W is smaller than x is 1.

Putting all together, one sees that the cdf F is given by

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x^3/1000, & 0 < x < 10 \\ 1, & x \geq 10, \end{cases}$$

Here is the graph

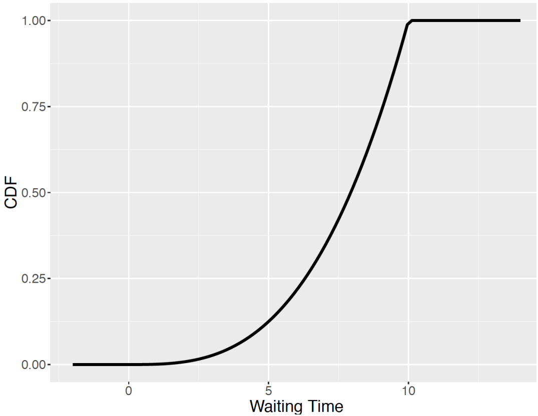

Figure 3: The cumulative density function, $F(x)$, of the bus waiting example.

Computing Probabilities Using the CDF

- Once the cdf function F is given, probabilities are found by evaluating F at different points.
- No additional integration is needed.

Waiting Time Example

- Recall that we found cdf to be equal to:

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x^3/1000, & 0 < x < 10 \\ 1, & x \geq 10, \end{cases}$$

Computing a Probability

- To find the probability that the maximum waiting time W is less than equal to 6 minutes, one just computes $F(6) = P(W \leq 6) = 6^3/1000 = 0.216$ which is shown in Figure 5.15.

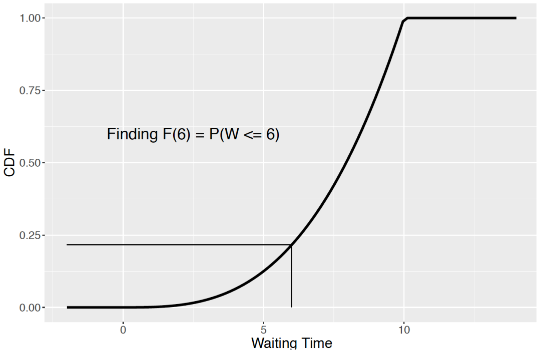

Figure 4: The cumulative density function $F(x)$ and evaluation of $F(6) = P(W <= 6)$.

Computing a Greater-Than Probability

To compute the probability that the maximum waiting time exceeds 8 minutes, first note that "exceeding 8 minutes" is the complement event to "less than or equal to 8 minutes", and so

$$P(W > 8) = 1 - P(W \leq 8)$$

$$= 1 - F(8)$$

$$= 1 - \frac{8^3}{1000} = 0.488.$$

Computing a Between Probability

Likewise, if one is interested in the chance that the waiting time W falls between 2 and 4, represent the probability as the difference of two "less-than" probabilities, and then subtract the two values of F.

$$\begin{aligned} P(2 < W < 4) &= P(W \leq 4) - P(W \leq 2) \\ &= F(4) - F(2) \\ &= \frac{4^3}{1000} - \frac{2^3}{1000} = 0.056. \end{aligned}$$