Chapter 5.3 Probability Density

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Chapter 5 Continuous Random Variables

A Probability Density

- Have random experiment where a continuous random variable X is observed
- To describe probabilities about X, a density function denoted by $f(x)$ is defined.
- This density must satisfy two properties:

Property 1. The probability density f must be **nonnegative** which means that

$$f(x) \geq 0, \text{for all } x. \tag{1}$$

Property 2. The total area under the probability density curve f must be equal to 1. Mathematically,

$$\int_{-\infty}^{\infty} f(x) dx = 1. \tag{2}$$

Waiting for a Bus

- A professor walks to a bus stop and wait for a bus to go to school. Does this three times a week.
- Each waiting time between 0 and 10 minutes is equally likely.
- For a given week, what's the chance that her longest wait will be under 7 minutes?

Random Variable

- Let W denote her longest waiting time for the week. One can show that the density for W is given by

$$f(w) = \frac{3w^2}{1000}, 0 < w < 10.$$

Picture of the density for X

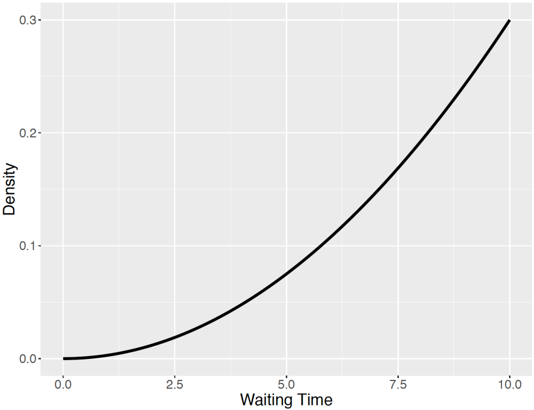

Figure 1: Density curve for the longest waiting time W.

Is this a legitimate density function?

1. Note from the graph that the density does not take on negative values, so the first property is satisfied.
2. Second, the entire area under the curve must be equal to 1. Let's check:

$$\int_0^{10} \frac{3w^2}{1000} dw = \frac{w^3}{1000}\Big|_0^{10} = \frac{10^3}{1000} - \frac{0^3}{1000} = 1.$$

Finding probabilities

To find the probability that this longest waiting time is less than 7 minutes, $P(W < 7)$, one wishes to compute the area under the density curve between 0 and 7.

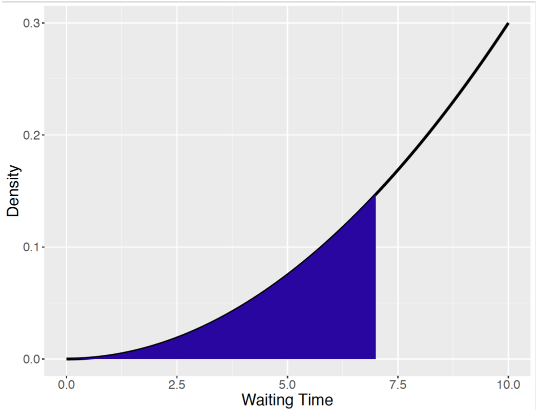

Figure 2: Density curve for the longest waiting time W, and $P(W < 7)$.

Some calculus

This is equivalent to the integral

$$\int_0^7 \frac{3w^2}{1000} dw$$

and, by evaluating this, one obtains the probability

$$\int_0^7 \frac{3w^2}{1000} dw = \frac{w^3}{1000}\Big|_0^7 = \frac{7^3}{1000} - \frac{0^3}{1000} = 0.343.$$

Another problem

Suppose one is interested in the probability that the longest waiting time is between 6 and 8 minutes. This is represented by the shaded area.

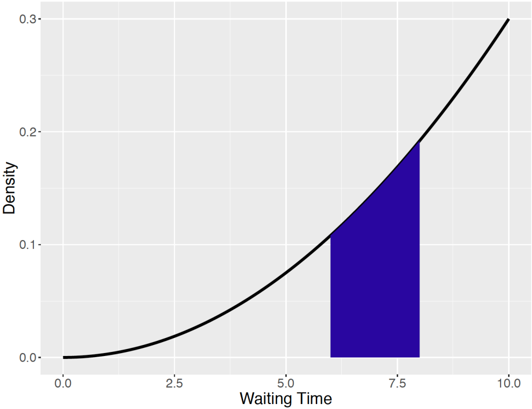

Figure 3: Density curve for the longest waiting time W, and $P(6 < W < 8)$.

Computation

To compute this area, one finds the integral of the density between 6 and 8:

$$\int_6^8 \frac{3w^2}{1000} dw = \frac{w^3}{1000}\Big|_6^8 = \frac{8^3}{1000} - \frac{6^3}{1000} = 0.296.$$