Chapter 6.3b Marginal and Conditional Density Functions

Jim Albert and Monika Hu

Chapter 6 Joint Probability Distributions

Marginal probability density functions

- Given a joint pdf $f(x, y)$, one summarizes probabilities about each variable individually by the computation of marginal pdfs.
- The marginal pdf of X, $f_X(x)$, is obtained by integrating out y from the joint pdf.

$$f_X(x) = \int f(x, y)dy.$$

- In a similar fashion, one defines the marginal pdf of Y by integrating out x from the joint pdf.

$$f_Y(x) = \int f(x, y)dx.$$

Example

- Let's illustrate the computation of marginal pdfs for our example.
- Looking back at the figure, one sees that if the value of x is fixed, then the limits for y go from x to 2.
- So the marginal density of X is given by

$$f_X(x) = \int f(x, y) dy$$

$$= \int_x^2 \frac{1}{2} dy$$

$$= \frac{2 - x}{2}, \; 0 < x < 2.$$

Example

- By a similar calculation, one can verify that the marginal density of Y is equal to

$$f_Y(y) = \frac{y}{2},\ 0 < y < 2.$$

Conditional probability density functions

- Once a joint pdf $f(x, y)$ has been defined, one can also define conditional pdfs.
- In our example, suppose one is told that the first random location is equal to $X = 1.5$. What has one learned about the value of the second random variable Y?
- One defines the notion of a conditional pdf. The conditional pdf of the random variable Y given the value $X = x$ is defined as the quotient

$$f_{Y|X}(y \mid X = x) = \frac{f(x, y)}{f_X(x)}, \text{ if } f_X(x) > 0.$$

Conditional density function for example

- In our example one is given that $X = 1.5$.
- Looking at the figure, one sees that when $X = 1.5$, the only possible values of Y are between 1.5 and 2.

Conditional density function for example

- By substituting the values of $f(x, y)$ and $f_X(x)$, one obtains

$$f_{Y|X}(y \mid X = 1.5) = \frac{f(1.5, y)}{f_X(1.5)}$$

$$= \frac{1/2}{(2 - 1.5)/2}$$

$$= 2, \; 1.5 < y < 2.$$

- In other words, the conditional density for Y when $X = 1.5$ is uniform from 1.5 to 2.

Working with a conditional density function

- A conditional pdf is a legitimate density function, so the integral of the pdf over all values y is equal to one.
- Use this density to compute conditional probabilities.
- For example, if $X = 1.5$, what is the probability that Y is greater than 1.7?

Working with a conditional density function

- This probability is the conditional probability

$$P(Y > 1.7 \mid X = 1.5)$$

that is given by:

$$P(Y > 1.7 \mid X = 1.5) = \int_{1.7}^{2} f_{Y|X}(y \mid 1.5) dy$$

$$= \int_{1.7}^{2} 2dy$$
$$= 0.6.$$

Turn the random variables around

- Above, we looked at the pdf of Y conditional on a value of X.
- One can also consider a pdf of X conditional on a value of Y.
- Returning to our example, suppose that one learns that Y, the larger random variable on the interval is equal to 0.8. In this case, what would one expect for the random variable X?
- This question is answered in two steps – one first finds the conditional pdf of X conditional on $Y = 0.8$. Then once this conditional pdf is found, one finds the mean of this distribution.

The conditional pdf of X given $Y = y$

- The conditional pdf of X given the value $Y = y$ is defined as the quotient

$$f_{X|Y}(x \mid Y = y) = \frac{f(x,y)}{f_Y(y)}, \text{ if } f_Y(y) > 0.$$

- Looking back the figure, one sees that if $Y = 0.8$, the possible values of X are from 0 to 0.8.
- Over these values the conditional pdf of X is given by

$$f_{X|Y}(x \mid 0.8) = \frac{f(x, 0.8)}{f_Y(0.8)}$$

$$= \frac{1/2}{0.8/2} = 1.25, \; 0 < x < 0.8.$$

Continuing

- So if one knows that $Y = 0.8$, then the conditional pdf for X is Uniform on (0, 0.8).
- To find the "expected" value of X knowing that $Y = 0.8$, one finds the mean of this distribution.

$$E(X \mid Y = 0.8) = \int_0^{0.8} x f_{X|Y}(x \mid 0.8) dx$$

$$= \int_0^{0.8} x \, 1.25 \, dx$$

$$= (0.8)^2/2 \times 1.25 = 0.4.$$