Chapter 7.2 Bayesian Inference with Discrete Priors

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Chapter 7 Learning About a Binomial Probability

Example: students' dining preference

- A popular restaurant in a college town has been in business for about 5 years
- The restaurant owner wishes to learn more about his customers
- Interested in learning about the dining preferences of the students
- The owner plans to conduct a survey by asking students "what is your favorite day for eating out?"
- He wants to find out what percentage of students prefer to dine on Friday
- Let p denote the proportion of all students whose answer is Friday

Discrete prior distributions for proportion p

- A set of plausible values of p:

$$p = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$$

- A laymen's prior distribution for p:

$$\pi_l(p) = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$$

- An expert's prior distribution for p:

$$\pi_e(p) = (0.125, 0.125, 0.250, 0.250, 0.125, 0.125)$$

R for $\pi_e(p)$

- The ProbBayes R package

```
bayes_table <- data.frame(p = seq(.3, .8, by=.1),
                          Prior = c(1, 1, 2, 2, 1, 1))
bayes_table
```

```
##     p Prior
## 1 0.3     1
## 2 0.4     1
## 3 0.5     2
## 4 0.6     2
## 5 0.7     1
## 6 0.8     1
```

R for $\pi_e(p)$ cont'd

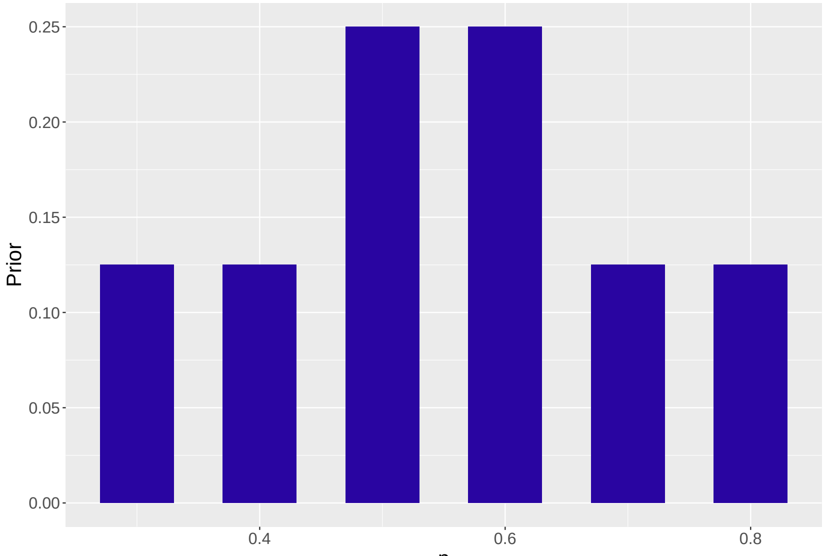
- Use the function `mutate()` to normalize these weights to obtain the prior probabilities in the Prior column

```
bayes_table %>%
  mutate(Prior = Prior / sum(Prior)) -> bayes_table
bayes_table
```

```
##     p Prior
## 1 0.3 0.125
## 2 0.4 0.125
## 3 0.5 0.250
## 4 0.6 0.250
## 5 0.7 0.125
## 6 0.8 0.125
```

R for $\pi_e(p)$ cont'd

- Plot the restaurant owner's prior distribution by use of `ggplot2` functions



Likelihood

- The restaurant owner gives a survey to 20 student diners at the restaurant
- Out of the 20 student respondents, 12 say that their favorite day for eating out is Friday
- The likelihood is a function of the quantity of interest, the proportion p
- The owner has conducted an experiment 20 times
  - each experiment involves a “yes” or “no” answer from the respondent to the rephrased question “whether Friday is your preferred day to dine out”
  - the proportion p is the probability a student answers “yes”

Review: binomial experiment

Four conditions for a binomial experiment:

- One is repeating the same basic task or trial many times – let the number of trials be denoted by n
- On each trial, there are two possible outcomes called "success" or "failure"
- The probability of a success, denoted by p, is the same for each trial
- The results of outcomes from different trials are independent

The binomial likelihood function

- The probability of y successes in a Binomial experiment is given by

$$Prob(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, y = 0, \cdots, n$$

- The likelihood is the chance of 12 successes in 20 trials viewed as a function of the probability of success p:

$$Likelihood = L(p) = \binom{20}{12} p^{12} (1-p)^8$$

  - generally use L to denote a likelihood function
  - L is a function of p

R for the likelihood

- The likelihood function $L(p)$ is efficiently computed using the `dbinom()` function in R
  - the sample size n: 20 in the dining survey
  - the number of successes y: 12 in the dining survey
  - p: the list of 6 plausible values
    $p = \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$

R for the likelihood cont'd

- The values are placed in the `Likelihood` column of the `bayes_table` data frame

```
bayes_table$Likelihood <- dbinom(12, size = 20,
                                  prob = bayes_table$p)
bayes_table
```

```
##     p Prior  Likelihood
## 1 0.3 0.125 0.003859282
## 2 0.4 0.125 0.035497440
## 3 0.5 0.250 0.120134354
## 4 0.6 0.250 0.179705788
## 5 0.7 0.125 0.114396740
## 6 0.8 0.125 0.022160877
```

Posterior distribution for proportion p

- Bayes' rule for a discrete parameter:

$$\pi(p_i \mid y) = \frac{\pi(p_i) \times L(p_i)}{\sum_j \pi(p_j) \times L(p_j)}$$

  - $\pi(p_i)$ is the prior probability of $p = p_i$
  - $L(p_i)$ is the likelihood function evaluated at $p = p_i$
  - $\pi(p_i \mid y)$ is the posterior probability of $p = p_i$ given the number of successes y
  - by the **Law of Total Probability**, the denominator gives the marginal distribution of the observation y.

Bayes' rule

- Bayes' rule can also be expressed as "prior times likelihood":
$$\pi(p_i \mid y) \propto \pi(p_i) \times L(p_i)$$

Posterior probability calculation

- First, calculate the denominator and denote the value as D.

  $$D = \pi(0.3)\times L(0.3)+\pi(0.4)\times L(0.4)+\cdots+\pi(0.8)\times L(0.8)$$

- Then the posterior probability of $p = 0.3$ is given by

  $$\pi(p = 0.3 \mid 12) = \frac{\pi(0.3) \times L(0.3)}{D} \approx 0.005$$

- In a similar fashion, the posterior probability of $p = 0.5$ is calculated as

  $$\pi(p = 0.5 \mid 12) = \frac{\pi(0.5) \times L(0.5)}{D} \approx 0.310$$

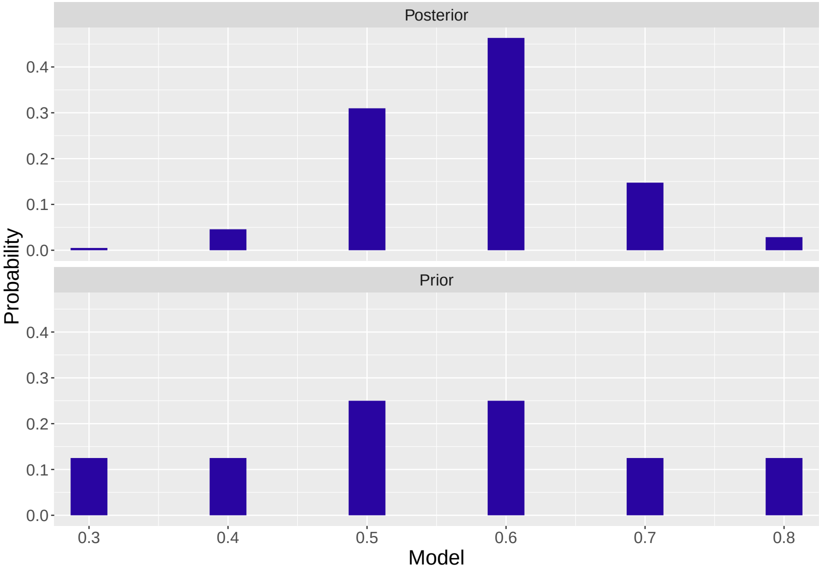
R for posterior probability calculation

- Use the `bayesian_crank()` function to compute the posterior probabilities

```
bayesian_crank(bayes_table) -> bayes_table
bayes_table
```

```
##     p Prior  Likelihood      Product   Posterior
## 1 0.3 0.125 0.003859282 0.0004824102 0.004975901
## 2 0.4 0.125 0.035497440 0.0044371799 0.045768032
## 3 0.5 0.250 0.120134354 0.0300335884 0.309786454
## 4 0.6 0.250 0.179705788 0.0449264469 0.463401326
## 5 0.7 0.125 0.114396740 0.0142995925 0.147495530
## 6 0.8 0.125 0.022160877 0.0027701096 0.028572757
```

Comparing prior and psoterior



Inference: students' dining preference

- What is the posterior probability that over half of the students prefer eating out on Friday?
- i.e. one is interested in the probability that $p > 0.5$, in the posterior
- Looking at the table, this posterior probability is equal to

$$Prob(p > 0.5) \approx 0.463 + 0.147 + 0.029 = 0.639$$

- This means the owner is reasonably confident (with probability 0.639) that over half of the college students prefer to eat out on Friday

Inference: students' dining preference cont'd

- Obtain the probability from the R output

```
sum(bayes_table$Posterior[bayes_table$p > 0.5])
```

```
## [1] 0.6394696
```

Discussion: issues with discrete priors

- If a plausible value is not specified in the prior distribution, it will be assigned a probability of zero in the posterior
- It generally is more desirable to have p to be any value in [0, 1] including less plausible values such as $p = 1.0$
- To make this happen, the proportion p should be allowed to take any value between 0 and 1, which means p will be a continuous variable
- i.e. it is necessary to construct a continuous prior distribution for p
- A popular class of continuous prior distributions for proportion is: the beta distribution