Chapter 3.7 The Multiplication Rule Under Independence

Jim Albert and Monika Hu

Chapter 3 Conditional Probability

Definition

- When two events A and B are independent, then the multiplication rule takes the simple form

$$P(A \cap B) = P(A) \times P(B). \tag{1}$$

- f one has a sequence of independent events, say $A_1, A_2, \cdots, A_k$, then the probability that all events happen simultaneously is the product

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1) \times P(A_2) \times \cdots \times P(A_k). \tag{2}$$

Blood Types of Couples

- White Americans have the blood types O, A, B, and AB with respectively proportions 0.45, 0.40, 0.11, and 0.04.
- Suppose two people in this group are married.
- Look at probabilities of some interesting events.

What is the probability that the man has blood type O and the woman has blood type A?

- Let O_M denote the event that the man has O blood type and A_W the event that the woman has A blood type.
- Since these two people are not related, it is reasonable to assume that O_M and A_W are independent.
- Applying the multiplication rule, get

$$\begin{aligned} P(O_M \cap A_W) &= P(O_M) \times P(A_W) \\ &= (0.45) \times (0.40) = 0.18. \end{aligned}$$

What is the probability the couple have O and A blood types?

- Either the man has blood type O and the woman has blood type A, or the other way around.
- So the probability of interest is

$$
\begin{aligned}
P(\text{two have A, O types}) &= P((O_M \cap A_W) \cup (O_W \cap A_M)) \\
&= P(O_M \cap A_W) + P(O_W \cap A_M).
\end{aligned}
$$

- One adds the probabilities since $O_M \cap A_W$ and $O_W \cap A_M$ are different outcomes.

$$
\begin{aligned}
P(\text{two have A, O types}) &= P((O_M \cap A_W) \cup (O_W \cap A_M)) \\
&= P(O_M \cap A_W) + P(O_W \cap A_M) \\
&= (0.45) \times (0.40) + (0.45) \times (0.40) \\
&= 0.36.
\end{aligned}
$$

What is the probability the man and the woman have the same blood type?

- There are four possible ways for this to happen: they can both have type O, they both have type A, they have type B, or they have type AB.
- One first finds the probability of each possible outcome and then sum the outcome probabilities.

$$\begin{aligned} P(\text{same type}) &= P((O_M \cap O_W) \cup (A_M \cap A_W) \cup \\ &\quad (B_M \cap B_W) \cup (AB_W \cap AB_M)) \\ &= (0.45)^2 + (0.40)^2 + (0.11)^2 + (0.04)^2 \\ &= 0.3762. \end{aligned}$$

What is the probability the couple have different blood types?

- A simple way of doing this is to note that the event "having different blood types" is the complement of the event "have the same blood type".
- Then using the complement property of probability,

$$
\begin{aligned}
P(\text{different type}) &= 1 - P(\text{same type}) \\
&= 1 - 0.3762 \\
&= 0.6238.
\end{aligned}
$$

A Five-Game Playoff

- Suppose two baseball teams play in a "best of five" playoff series, where the first team to win three games wins the series.
- Suppose the Yankees play the Angels and one believes that the probability the Yankees will win a single game is 0.6.
- If the results of the games are assumed independent, what is the probability the Yankees win the series?

Comments

- There are numerous outcomes of this series of games.
- Note that the playoff can last three games, four games, or five games.
- In listing outcomes, one lets Y and A denote respectively the single-game outcomes "Yankees win" and "Angels win".
- Then a series result is represented by a sequence of letters. For example, $YYAY$ means that the Yankees won the first two games, the Angels won the third game, and the Yankees won the fourth game and the series.

Playoff outcomes

Three games	Four games	Five games
YYY	$YYAY$, $AAYA$	$YYAAY$, $AAYYA$
AAA	$YAYY$, $AYAA$	$YAYAY$, $AYAYA$
	$AYYY$, $YAAA$	$YAAYY$, $AYYAA$
		$AYYAY$, $YAAYA$
		$AYAYY$, $YAYAA$
		$AAYYY$, $YYAAA$

Probability Yankees win series

- Identify the outcomes above where the Yankees win are underlined.
- By the assumption of independence, one finds the probability of a specific outcome – for example, the probability of the outcome $YYAY$ as

$$
\begin{aligned}
P(YYAY) &= (0.6) \times (0.6) \times (0.4) \times (0.6) \\
&= 0.0864.
\end{aligned}
$$

- One finds the probability that the Yankees win the series by finding the probabilities of each type of Yankees win and adding the outcome probabilities.

Table of probabilities of Yankees winning outcomes

Three games	Four games	Five games
$P(YYY) = 0.216$	$P(YYAY) = 0.0864$	$P(YYAAY) = 0.0346$
	$P(YAYY) = 0.0864$	$P(YAYAY) = 0.0346$
	$P(AYYY) = .0864$	$P(YAAYY) = 0.0346$
		$P(AYYAY) = 0.0346$
		$P(AYAYY) = 0.0346$
		$P(AAYYY) = 0.0346$

- So the probability of interest is given by

$$\begin{aligned} P(\text{Yankees win series}) &= P(YYY, YYAY, YAYY, \ldots) \\ &= 0.216 + 3 \times 0.864 + 6 \times 0.0346 \\ &= 0.683. \end{aligned}$$