Chapter 7.3 Continuous Priors

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Chapter 7 Learning About a Binomial Probability

Introduction

- p is continuous on [0, 1]
- One possible choice of a continuous prior: the continuous uniform distribution
  - expresses the opinion that p is equally likely to take any value between 0 and 1
- The probability density function of the continuous uniform on the interval (a, b) is
  - $\pi(p) = \frac{1}{b-a}$for $a \leq p \leq b$
  - $\pi(p) = 0$for $p < a$ or $p > b$

Beta distribution

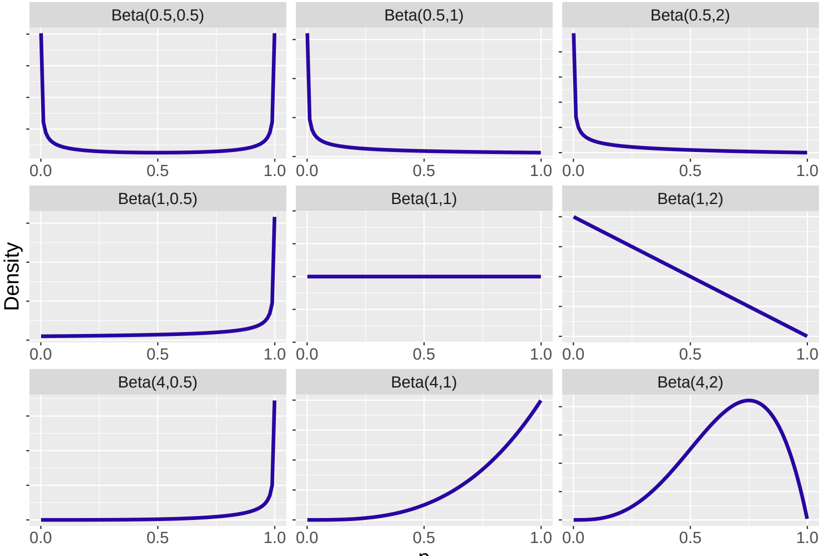
- A beta distribution, denoted by Beta(a, b), represents probabilities for a random variable falling between 0 and 1
- The beta distribution has two shape parameters, a and b, with probability density function given by

$$\pi(p) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1},\ 0 \leq p \leq 1,$$

  - $B(a, b)$ is the beta function: $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
  - mean $E[p] = \frac{a}{a+b}$
  - variance $V(p) = \frac{ab}{(a+b)^2(a+b+1)}$
- $\mathrm{Uniform}(0,1) = \mathrm{Beta}(1,1)$

Beta distribution cont'd

- Density curves of beta distributions for several choices of the shape parameters



R for beta distribution

- `dbeta()`: the probability density function for a Beta(a, b) which takes a value of the random variable as its input and outputs the probability density function at that value

```
dbeta(c(0.5, 0.8, 1.2), 1, 1)
```

```
## [1] 1 1 0
```

R for beta distribution cont'd

- `pbeta()`: the distribution function of a Beta(a, b) random variable, which takes a value x and gives the value of the random variable at that value, $F(x)$

```r
pbeta(c(0.5, 0.8), 1, 1)
```

```
## [1] 0.5 0.8
```

One calculates the probability of p between two values by taking the difference of the cdf at the two values

```r
pbeta(0.8, 1, 1) - pbeta(0.5, 1, 1)
```

```
## [1] 0.3
```

R for beta distribution cont'd

- `qbeta()`: the quantile function of a Beta(a, b), which inputs a probability value p and outputs the value of x such that $F(x) = p$

```
qbeta(c(0.5, 0.8), 1, 1)
```

```
## [1] 0.5 0.8
```

- `rbeta()`: the random number generator for Beta(a, b), which inputs the size of a random sample and gives a vector of the simulated random variates

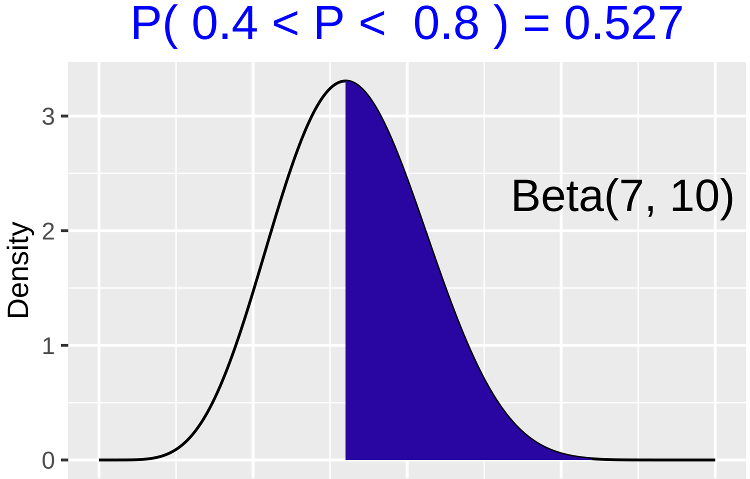
```
rbeta(3, 1, 1)
```

```
## [1] 0.4464770 0.9064875 0.3148758
```

ProbBayes for beta distribution

- Suppose one has a Beta(7, 10) curve and we want to find the chance that p is between 0.4 and 0.8

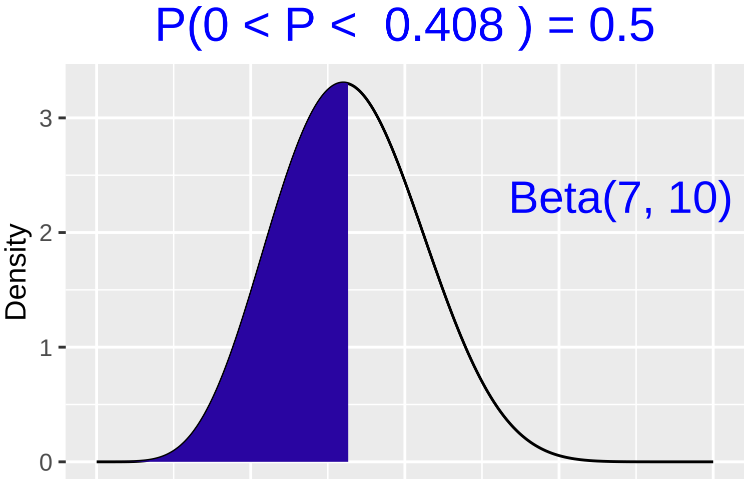
```
beta_area(0.4, 0.8, c(7, 10), Color = crcblue)
```



ProbBayes for beta distribution cont'd

- `beta_quantile()` automatically produces a plot with the shaded probability area

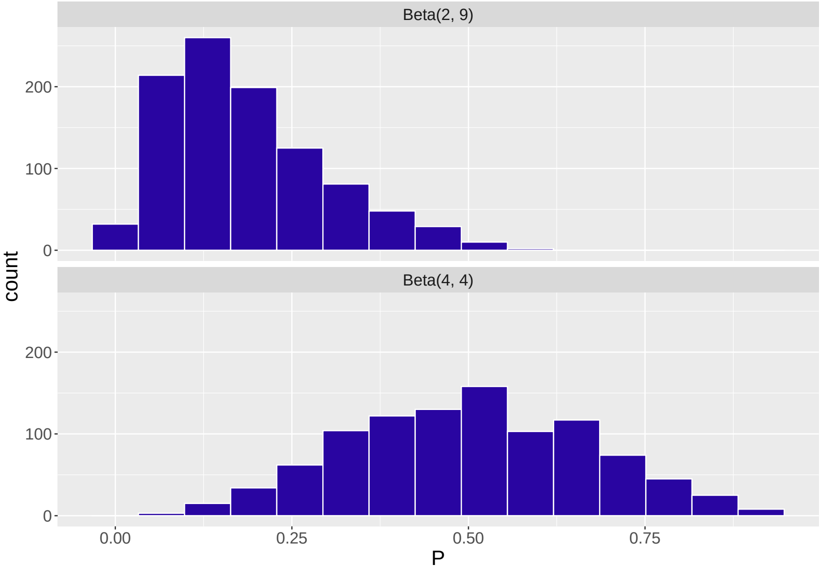
```
beta_quantile(0.5, c(7, 10), Color = crcblue)
```



Choosing a beta density as prior

- We cover two methods
- The first method is to consider:
  - the shape parameter a as the prior count of "successes"
  - the shape parameter b as the prior count of "failures"
- Subsequently, the values:
  - $a + b$ represents the **prior sample size**
  - n represents the **data sample size**

Examples: Beta(4, 4) and Beta(2, 9)



Examples: Beta(4, 4) and Beta(2, 9) cont'd

- To further check the quantiles of the prior, use the `quantile()` function

```
Beta44samples <- rbeta(1000, 4, 4)
quantile(Beta44samples, c(0.25, 0.75))
```

```
##       25%       75% 
## 0.3821453 0.6238341
```

- Discussion: what are the differences and similarities between Beta(4, 4) and Beta(40, 40)?

Choosing a beta density as prior

- The second indirect method is by specification of quantiles of the distribution
  - determine the shape parameters a and b by first specifying two quantiles of the beta density curve
  - find the beta density curve that matches these quantiles

Example: specifying the 0.5 and 0.9 quantiles

- The restaurant owner thinks of a value p_{50} such that the proportion p is equally likely to be smaller or larger than p_{50}: $p_{50} = 0.55$

The owner thinks of a value p_{90} that he is pretty sure (with probability 0.90) that the proportion p is smaller than p_{90}: $p_{90} = 0.80$

```
beta.select(list(x = 0.55, p = 0.5),
            list(x = 0.80, p = 0.9))
```

```
## [1] 3.06 2.56
```

Example: specifying the 0.5 and 0.9 quantiles cont'd

- Check the chosen beta prior