Chapter 6.1c Conditional PMFs

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Chapter 6 Joint Probability Distributions

Introduction

- In Chapter 3, the conditional probability of an event A was defined given knowledge of another event B.
- Back to the sampling balls from a box example, suppose one is told that exactly two red balls are sampled, that is $X = 2$.
- How does that information change the probabilities about the number of white balls Y?

A Conditional Calculation

In this example, one is interested in finding $P(Y = y \mid X = 2)$.

- Using the definition of conditional probability, one has

$$P(Y = y \mid X = 2) = \frac{P(Y = y, X = 2)}{P(X = 2)}.$$

$$= \frac{f(2, y)}{f_X(2)}$$

A Conditional Calculation

- For example, the probability of observing two white balls given that we have two red balls is equal to

$$P(Y = 2 \mid X = 2) = \frac{P(Y = 2, X = 2)}{P(X = 2)}$$

$$= \frac{f(2, 2)}{f_X(2)}$$

$$= \frac{54/252}{105/252} = \frac{54}{105}.$$

A Conditional pmf

- Suppose this calculation is repeated for all possible values of Y – one obtains the values displayed in this table.

y	$f_{Y\|X}(y \mid X = 2)$
0	$3/105$
1	$36/105$
2	$54/105$
3	$12/105$

- These probabilities represent the conditional pmf for Y conditional on $X = 2$.

A Conditional pmf is a Probability Distribution

- This conditional pmf is just like any other probability distribution – the values are nonnegative and they sum to one.
- That is ...

1. $f_{Y|X}(y|x) \geq 0$ for all y
2. $\sum_y f_{Y|X}(y|x) = 1$

Conditional pmf Example

- To illustrate using this distribution, suppose one is told that two red balls are selected (that is, $X = 2$) and one wants to find the probability that more than one white ball is chosen.
- This probability is given by

$$P(Y > 1 \mid X = 2) = \Sigma_{y>1}\ f_{Y|X}(y \mid X = 2)$$

$$= f_{Y|X}(2 \mid X = 2) + f_{Y|X}(3 \mid X = 2)$$
$$= \frac{54}{105} + \frac{12}{105} = \frac{66}{105}.$$

General Formula

- In general, the conditional probability mass function of Y conditional on $X = x$, denoted by $f_{Y\,|X}(y \mid x)$, is defined to be

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)}, \text{ if } f_X(x) > 0.$$

- We can turn this around and find the conditional mass function of X conditional on the value $Y = y$.

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)}, \text{ if } f_Y(y) > 0.$$