Chapter 7.4 Updating the Beta Prior

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Chapter 7 Learning About a Binomial Probability

Introduction

- Beta prior: Beta(3.06, 2.56)
- Binomial data: 12 yes's out of 20

$$Likelihood = L(p) = \binom{20}{12} p^{12}(1-p)^{8}$$

- We will see that a beta prior leads to a beta posterior: conjugacy

Bayes' rule calculation

- Bayes' rule
$$\pi(p \mid y) \propto \pi(p) \times L(p)$$
- A beta prior
$$p \sim \mathrm{Beta}(3.06, 2.56)$$
- The data / likelihood
$$Y \sim \mathrm{Binomial}(20, p)$$

Bayes' rule calculation cont'd

- The prior distribution:

$$\pi(p) = \frac{1}{B(3.06, 2.56)} p^{3.06-1}(1-p)^{2.56-1}$$

- The likelihood:

$$f(Y = 12 \mid p) = L(p) = \binom{20}{12} p^{12}(1-p)^{8}$$

- By Bayes' rule, the posterior density $\pi(p \mid y)$ is proportional to the product of the prior and the likelihood.

$$\pi(p \mid y) \propto \pi(p) \times L(p).$$

Bayes' rule calculation cont'd

- Substituting the current prior and likelihood, one can perform the algebra for the posterior density.

$$\pi(p \mid Y = 12) \propto p^{3.06-1}(1-p)^{2.56-1} \times p^{12}(1-p)^{8}$$

$$\pi(p \mid Y = 12) \propto p^{15.06-1}(1-p)^{10.56-1}$$

- What is the posterior?

A beta posterior

- Beta(a,b) density:

$$\propto p^{a-1}(1-p)^{b-1}$$

- The posterior density:

$$\pi(p \mid Y = 12) \propto p^{15.06-1}(1-p)^{10.56-1}$$

- The posterior distribution is another beta:

$$p \mid Y = 12 \sim \mathrm{Beta}(15.06, 10.56)$$

From beta prior to beta posterior

- The prior distribution:

$$p \sim \mathrm{Beta}(a, b)$$

- The sampling density:

$$Y \sim \mathrm{Binomial}(n, p)$$

- The posterior distribution:

$$p \mid Y = y \sim \mathrm{Beta}(a + y, b + n - y)$$

A summary table

Table 7.1. Updating the Beta prior.

Source	Successes	Failures
Prior	a	b
Data/Likelihood	y	$n - y$
Posterior	$a + y$	$b + n - y$

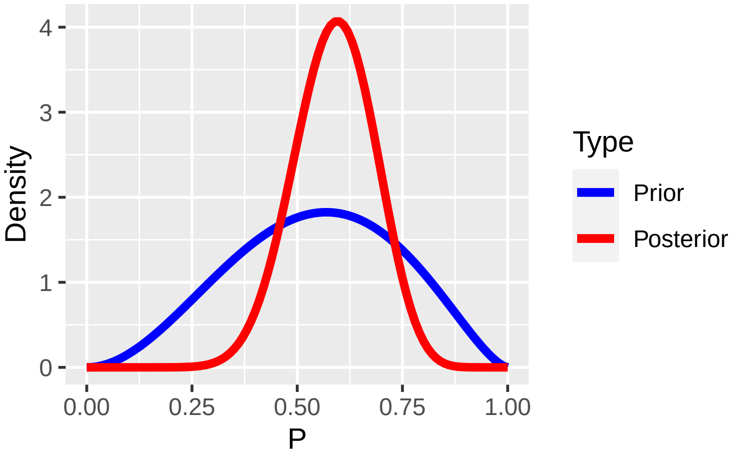
Example

```r
ab <- c(3.06, 2.56)
yny <- c(12, 8)
(ab_new <- ab + yny)
```

```
## [1] 15.06 10.56
```

Example cont'd

The function `beta_prior_post()` in the `ProbBayes` R package plots the prior and posterior Beta curves together on one graph.



Example cont'd

- Compare the prior and posterior Beta curves using the respective means
  - The mean of a Beta(a, b) distribution is a / (a + b)
  - Prior mean: 3.06 / (30.6 + 2.56) = 0.544
  - Sample mean: 12 / 20 = 0.6
  - Posterior mean: 15.06 / (15.06 + 10.56) = 0.588
- Compare the spreads of the two curves
  - Spread of prior is wider
  - The data helps sharpen the belief about the parameter of interest

Summary

- Conjugate prior: beta prior to beta posterior with binomial sampling model
- Choose a prior that matches one's belief, not one that is convenient to use