Chapter 6.3a Joint Density Functions

Jim Albert and Monika Hu

Chapter 6 Joint Probability Distributions

Introduction

- One can also describe probabilities when the two variables X and Y are continuous.
- As a simple example, suppose that one randomly chooses two points X and Y on the interval (0, 2) where $X < Y$. One defines the joint probability density function of X and Y to be the function

$$f(x, y) = \begin{cases} \frac{1}{2}, & 0 < x < y < 2; \\ 0, & \text{elsewhere.} \end{cases}$$

Picture

This joint pdf is viewed as a plane of constant height over the set of points (x, y) where $0 < x < y < 2$.

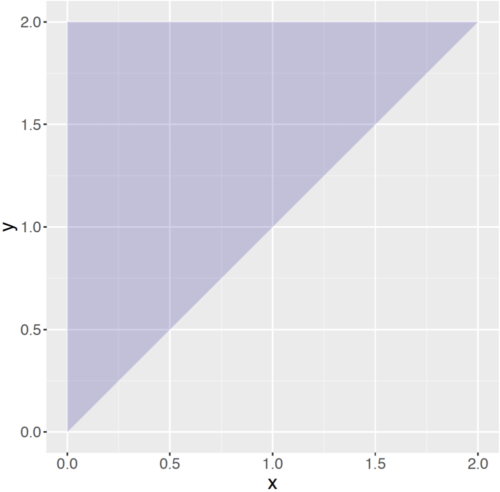


Figure 1: Region where the joint pdf $f(x, y)$ is positive in the

Definition of a Joint Density Function

- In the one variable situation in Chapter 5, a function f is a legitimate density function or pdf if it is nonnegative over the real line and the total area under the curve is equal t to one.
- Similarly for two variables, any function $f(x, y)$ is considered a pdf if it satisfies two properties:

1. Density is nonnegative over the whole plane:

$$f(x, y) \geq 0, \text{ for all } x, y.$$

2. The total volume under the density is equal to one:

$$\int \int f(x, y) dx dy = 1.$$

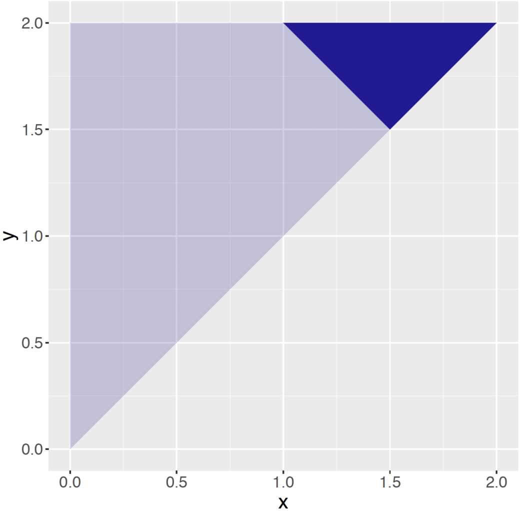
Back to Example

- One can check that the pdf in our example is indeed a legitimate pdf.
- It is pretty obvious that the density that was defined is nonnegative, but it is less clear that the integral of the density is equal to one.
- Since the density is a plane of constant height, one computes this double integral geometrically.
- Using the familiar "one half base times height" argument, the area of the triangle in the plane is $(1/2)(2)(2) = 2$ and since the pdf has constant height of $1/2$, the volume under the surface is equal to $2(1/2) = 1$.

Finding Probabilities

- Probabilities about X and Y are found by finding volumes under the pdf surface.
- For example, suppose one wants to find the probability $P(X + Y > 3)$.
- The region in the (x, y) plane of interest is first identified, and then one finds the volume under the joint pdf over this region.

Region of Interest $x + y > 3$



- The probability $P(X + Y > 3)$ is the volume under the pdf over this shaded region.

Calculation of Probability

- Applying a geometric argument, one notes that the area of the shaded region is $1/4$, and so the probability of interest is $(1/4)(1/2) = 1/8$.
- One also finds this probability by integrating the joint pdf over the region as follows:

$$P(X + Y < 3) = \int_{1.5}^{2} \int_{3-y}^{y} f(x, y) dxdy$$

$$= \int_{1.5}^{2} \int_{3-y}^{y} \frac{1}{2} dxdy$$

$$= \int_{1.5}^{2} \frac{2y - 3}{2} dy$$

$$= \frac{y^2 - 3y}{2} \Big|_{1.5}^{2} = \frac{1}{8}.$$