MATH 369/650 Fall 2020 Homework #2

Professor Adam Kapelner

Due by email noon Tuesday, September 22, 2020

(this document last updated Tuesday 15$^{\text{th}}$ September, 2020 at 5:09pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning the normal distribution.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for the 600-level class and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 7 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

In lecture 3, we did a two-sided binomial exact test. Now we will now test the same theory (i.e. that the iPhone percentage in our class is different than the national average), but use the approximate one-proportion z test to do it. Recall that our data was as follows: for $n = 20$, the $\hat{\hat{\theta}} = 0.60$ where the estimator we chose was the sample proportion.

(a) [easy] Write the alternative and null hypotheses again (from lecture 3).

(b) [easy] Write the asymptotic distribution of our estimator under the null hypothesis which we denote $\hat{\theta} \mid H_0$. Answer in terms of standard error and round it to three decimal places. The answer is in lecture 4. Then illustrate the sampling distribution. Label the x-axis and provide tick marks on the x-axis.

(c) [easy] What theorem did you use to get the asymptotic distribution of the estimator? State the conditions of the theorem and the theorem's result.

(d) [easy] If we employ this asymptotic distribution of the estimator to test our theory, why is this no longer an *exact test* but instead an *approximate test*?

(e) [harder] I want to make an apples-apples comparison with the two-sided binomial exact test from lecture 3. There $\alpha = 7.06\%$ so I want to use that same Type I error setting here. Compute the retainment region and rejection region (remember $\Theta = [0, 1]$) and denote these two regions in your illustration in (b). To compute these regions, I'll provide you with the following fact: $\Phi(-1.808) = 3.53\%$ where Φ is the CDF of the standard normal rv. (These are the kind of facts that will be provided to you on exams).

(f) [easy] How does the retainment region compare here to the retainment region in the binomial exact test from lecture 3? Is the normal approximation to the binomial a good approximation in this case?

(g) [difficult] Write about a scenario where this approximation to the exactness will provide the wrong outcome. Would practical advice does this scenario teach you?

(h) [easy] Why is this test named the *two-sided one-proportion z test*?

(i) [easy] Run the test and write your conclusion using and English sentence.

(j) [easy] What type of error could you have made?

(k) [easy] State the definition of Fisher's p-value.

(l) [harder] Find the p-value of our estimate as a function of Φ. Illustrate the p-value in the illustration in (b).

(m) [easy] Without computing the p-value explicitly, would it be above or below $\alpha = 7.06\%$? Is the estimate *statistically significant*?

Problem 2

In lecture 5, we did a *two-sided one sample z test*. Using the same data: $n = 10$ and $\bar{x} = 70.5$, we will now test if the population that this sample was drawn from has a *greater* mean than female height at $\alpha = 5\%$. According to this article, female height is $\sim \mathcal{N}(65, 3.5^2)$.

(a) [easy] Write the alternative and null hypotheses. Remember our theory we want to prove is: our population mean is greater than the mean female height.

(b) [easy] Write the distribution of our estimator under the null hypothesis which we denote $\hat{\theta} \mid H_0$. Assume the standard deviation is the same in our population as in the female height population. You will then calculate the standard error of $\bar{X}$. Provide your final answer in terms of standard error and round it to two decimal places. Then illustrate the sampling distribution. Label the x-axis and provide tick marks on the x-axis.

(c) [harder] Will this test be an *exact test* or instead an *approximate test*? Explain.

(d) [harder] Compute the retainment region and rejection region (remember $\Theta = \mathbb{R}$) and denote these two regions in your illustration in (b). To compute these regions, I'll provide you with the following fact: $\Phi(-1.645) = 5\%$.

(e) [easy] Why is this test called a *one-sided one-sample z test*?

(f) [easy] Run the test and write your conclusion using and English sentence.

(g) [easy] What type of error could you have made?

(h) [harder] Find the p-value of our estimate as a function of Φ. Illustrate the p-value in the illustration in (b).

(i) [easy] Without computing the p-value explicitly, would it be above or below $\alpha = 5\%$? Is the estimate *statistically significant*?

Problem 3

In the previous problem we did a *one-sided one-sample z test.* We will now do a *one-sided one-sample t test* Using the same data: $n = 10$ and $\bar{x} = 70.5$, we will again test if the population that this sample was drawn from has a *greater* mean than female height at $\alpha = 5\%$. According to this article, female height is $\sim \mathcal{N}(65, 3.5^2)$. In this test, you need to compute $s = 2.07$, the sample standard deviation.

(a) [easy] Write the alternative and null hypotheses. Same as 2(a).

(b) [easy] Write the distribution of the *standardized* estimator under the null hypothesis which we denote $\hat{\theta} \mid H_0$. Standardized means the mean is subtracted and you divide by the standard error. In contrast to 2(b), we do *not assume* the standard deviation is the same in our population as in the female height population. Illustrate the sampling distribution. Label the x-axis. You do *not* need to provide tick marks on the x-axis as this would require using a special calculator.

(c) [harder] Will this test be an *exact test* or instead an *approximate test*? Explain.

(d) [harder] Compute the retainment region and rejection region (remember $\Theta = \mathbb{R}$) and denote these two regions in your illustration in (b). To compute these regions, I'll provide you with the following fact: $F_{T_9}(-1.833) = 5\%$ where F_{T_9} is the CDF of Student's T distribution with 9 degree of freedom. (These are the kind of facts that will be provided to you on exams).

(e) [easy] Why is this test called a *one-sided one sample t test*?

(f) [easy] Why is σ^2 called a *nuisance parameter*? Explain.

(g) [easy] Run the test and write your conclusion using and English sentence.

(h) [easy] What type of error could you have made?

(i) [harder] Find the p-value of our estimate as a function of F_{t_9}. Illustrate the p-value in the illustration in (b).

(j) [easy] Without computing the p-value explicitly, would it be above or below $\alpha = 5\%$? Is the estimate *statistically significant*?

Problem 4

In the previous problem, we needed to estimate the nuisance paramter σ^2 in order to obtain Student's T distribution for the sampling distribution of the estimator, the sample average. If we naively estimated the population $\mathbb{E}[X]$ with the sample average, the corresponding estimator for the population variance $\mathbb{V}\text{ar}[X]$ would be

$$\hat{\sigma}^2 := \frac{1}{n}\sum_{i=1}^{n}(x_i - \theta)^2.$$

(a) [easy] Given only the data $x_1, \ldots, x_n$, what is the practical problem with $\hat{\sigma}^2$ above?

(b) [easy] Solving this practical problem using an additional estimate, how would you define $\hat{\sigma}^2$?

(c) [easy] Let $S^2 = \frac{n}{n-1}\hat{\sigma}^2$. The constant $\frac{n}{n-1}$ is known as Bessel's correction. Write the formula for S^2 only as a function of the data (the quantity n is considered a function of the data, because it is the sample size which is the data's dimensionality).

(d) [harder] Prove S^2 is unbiased to directly without computing $\mathbb{E}\left[\hat{\sigma}^2\right]$ first (which is in the lecture). This is remarkably similar to the computation of $\mathbb{E}\left[\hat{\sigma}^2\right]$ but different enough to make you think. Somewhere in this proof, you need to use the assumption that the DGP is $\overset{iid}{\sim}$. Mark that location clearly.

(e) [easy] S^2 is an estimator for σ^2. A natural estimator for σ then is $S := \sqrt{S^2}$. Write the formula for S below.

(f) [harder] Even though S^2 is unbiased, it turns out that S is biased due to Jensen's Inequality (which is taught in 368). Bessel's correction was a "one-size fits all" correction to the bias for sample variance. Unfortunately, there's no analogous "one-size fits all" correction to the bias for sample standard deviation; for every DGP it's different. Let the DGP be $\overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$ where neither parameter are known. In 368, an extra credit problem shows that

$$\mathbb{E}[S] = \sqrt{\frac{2}{n-1}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \sigma$$

where $\Gamma()$ is called the *gamma function*. Create a new estimator S_* that's unbiased for the DGP $\overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$ and write its formula as a function of the data only (the quantity n is considered a function of the data, because it is the sample size which is the data's dimensionality).

This looks harder than it is! You just need to mirror the trick used in class when we derived Bessel's Correction. You don't even need to know anything about the Gamma function.

(g) [difficult] [MA] Prove that S is asymptotically unbiased (a calculus exercise).

(h) [E.C.] Assume squared error loss and compute $R(S^2, \sigma^2)$. Do this on a separate page.

Problem 5

We will now explore the concept of power and Type II errors. Let the DGP be $\overset{iid}{\sim} \text{Bern}(\theta)$.

(a) [easy] Let $H_0 : \theta = .524$ and $H_a : \theta = .284$ and let $n = 20$. Find the approximate sampling distribution of $\hat{\theta} = \bar{X}$ using the CLT for both the null and alternative hypotheses. Leave in terms of standard error and compute to 3 decimal places.

(b) [easy] Would this be a left-sided test, a right-sided test or a two sided test? Explain.

(c) [easy] Illustrate the sampling distributions from the previous problem on one plot. Label the x-axis and provide tick marks.

(d) [easy] For $\alpha = 5\%$, compute the retainment region and rejection region (remember $\Theta = [0, 1]$) and denote these two regions in your illustration in (c). To compute these regions, I'll provide you with the following fact: $\Phi(-1.645) = 5\%$.

(e) [easy] Write the definition of power (POW).

(f) [easy] Explain why, in general, you want POW to be large and close to 100%.

(g) [harder] Calculate POW as a function of Φ.

(h) [easy] Assuming H_a, in the illustration in (c), shade in the area corresponding to the integral that computes POW using vertical stripes and shade in the area corresponding to the integral that computes Type II error using horizontal stripes. Don't write anything to answer this question.

(i) [harder] Let's do this generally. Let $H_0 : \theta \geq \theta_0$ and $H_a : \theta = \theta_a < \theta_0$. Derive the power function $POW(\theta_0, \theta_a, n, \alpha)$ which uses the Φ function.

(j) [harder] Prove that $\lim_{n \to \infty} POW(\theta_0, \theta_a, n, \alpha) = 1$.

(k) [harder] Assume the DGP $\overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. In a right-tailed test, we derived the power function in lecture 5 as

$$POW(\theta_0, \theta_a, n, \sigma, \alpha) = 1 - \Phi\left(-\frac{\sqrt{n}}{\sigma}(\theta_a - \theta_0) + z_{1-\alpha}\right).$$

Prove that $\lim_{\sigma \to 0} POW(\theta_0, \theta_a, n, \sigma, \alpha) = 1$.

(l) [E.C.] Let $H_0 : \theta \geq \theta_0$ and $H_a : \theta = \theta_a < \theta_0$. Calculate the power function for the binomial exact test at size α. Do this on a separate page.