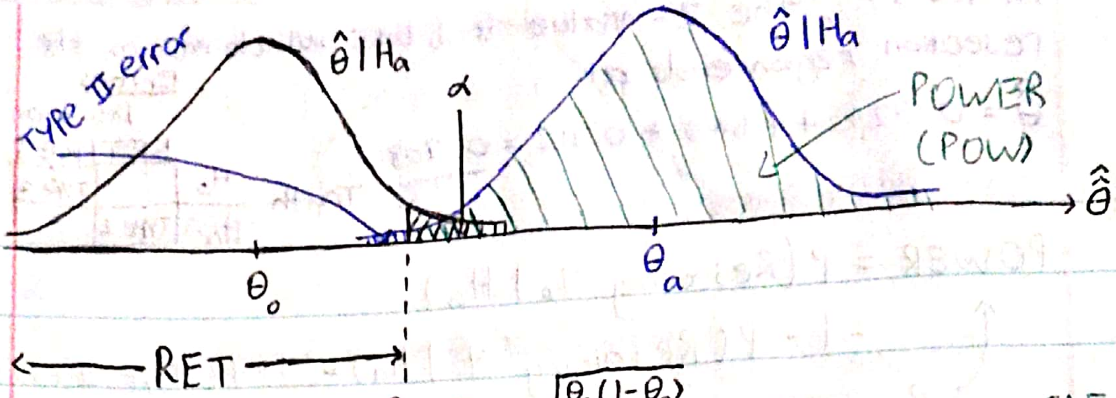
Lecture 5 9/14/2020

Let's look at power more generally (beyond two point hypotheses).

at size α

$H_0: \theta \leq \theta_0$, $H_a: \theta = \theta_a > \theta_0$ (Right-Tailed test)

Under i.i.d. Bern(θ) and the normal approximation,



$$\theta_0 + z_{1-\alpha}\underbrace{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}}_{\text{std error}}$$

CDF of N(0,1)

Let $\Phi(z) = F_z(z)$

$\Phi(z_{1-\alpha}) = 1-\alpha$

$\alpha = 5\% \Rightarrow z_{1-\alpha} = 1.645$

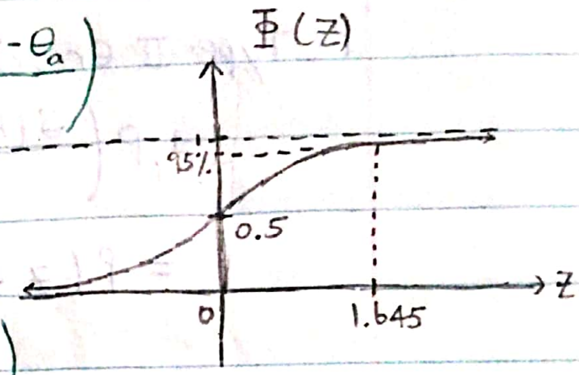
$$POW = P\left(\hat{\theta}|H_a > \theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)$$

$$= P\left(\frac{\hat{\theta}|H_a - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}} > \frac{\theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}} - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}}\right)$$

$$= P\left(Z > \frac{-\sqrt{n}(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}}\right)$$

$$= 1 - \Phi\left(\frac{-\sqrt{n}(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}}\right)$$

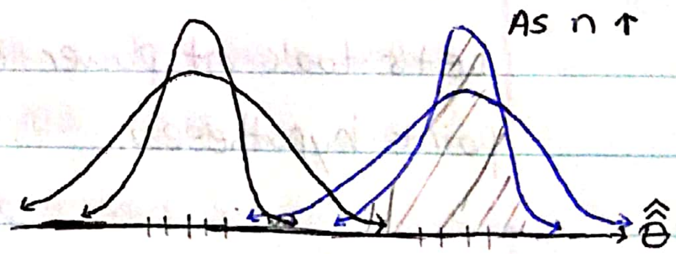
$$= POW(\theta_a, \theta_0, n, \alpha)$$



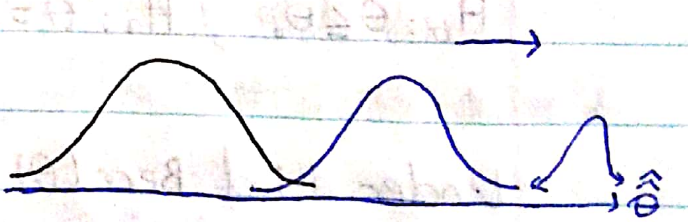
CDF of N(0,1)

Observations about the power function

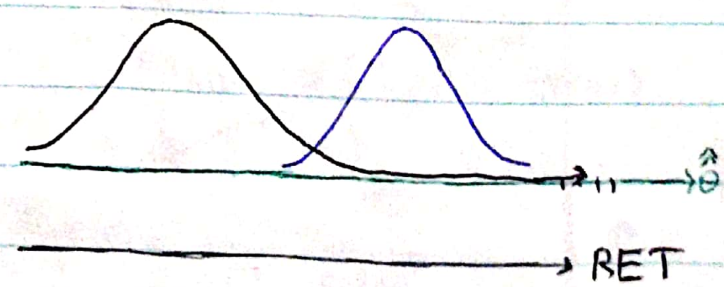
If $n \to \infty \Rightarrow POW \to 1$



If $\theta_a \to \infty \Rightarrow POW \to 1$



As $\alpha \to 0 \Rightarrow POW \to 0$



New type of survey. We ask "how tall are you (in inches)?" for men only. I'll ask 10 male students and get $X_1, \ldots, X_{10}$ (i.e. my data). The data is now continuous (no longer zeroes and ones). Height for a gender is known to be normally distributed.

DGP: $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$. Assume σ^2 is known and $= 4^2$

How can we estimate θ? θ is the mean of the R.V.s. And recall

$\hat{\theta} = \bar{X}$ is unbiased. Let's use this estimator.

$\bar{x} = (70, 72, 73, 68, 69, 70, 67, 72, 71, 73)$

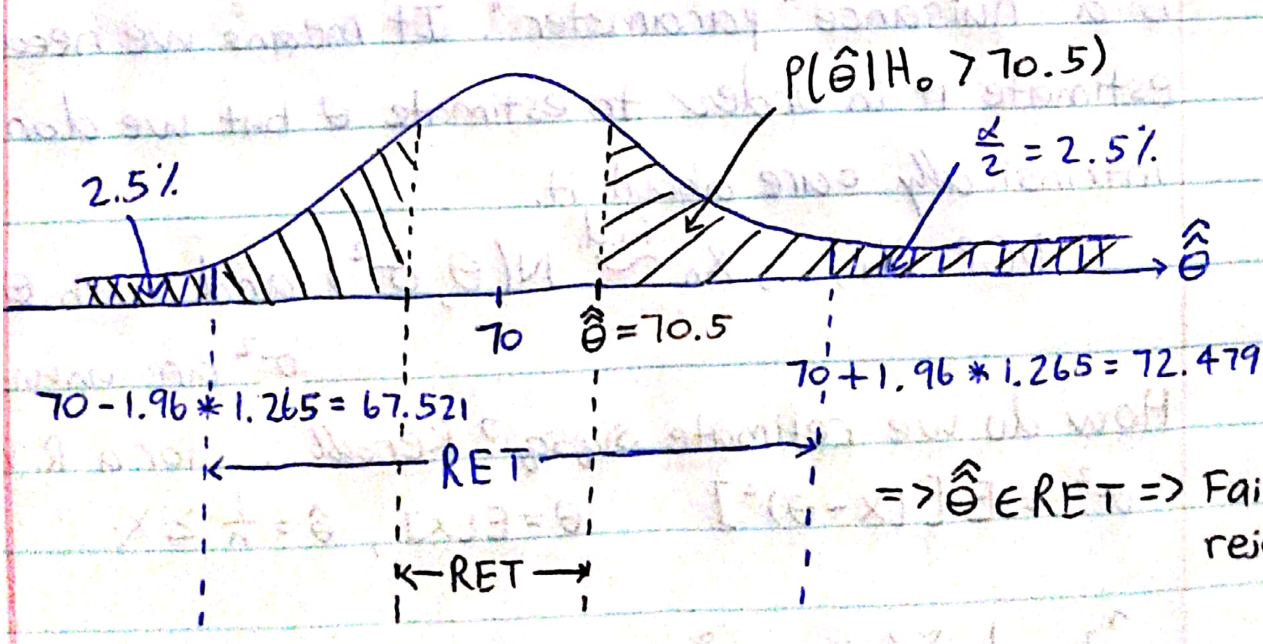
$\hat{\hat{\theta}} = \bar{x} = 70.5$ inches

The American mean male adult height is 70 inches.

Let's test if the mean of the population where this class is drawn from is different than 70 inches.

$H_a: \theta \neq 70$, $H_0: \theta = 70$, $\alpha = 5\%$ One-Sample Z-test

$\hat{\theta} | H_0 \sim N(70, \frac{4^2}{10}) = N(70, 1.265^2)$



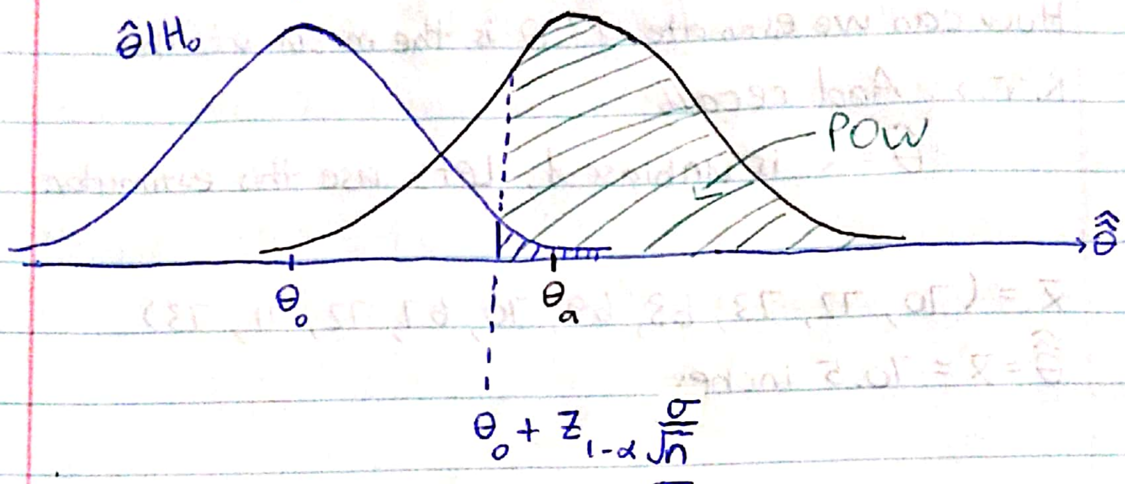
$\Rightarrow \hat{\hat{\theta}} \in RET \Rightarrow$ Fail to reject

P-value = P(estimate is more extreme than observed | H_0)

$= P(|\hat{\theta}|H_0| > |\hat{\theta}-\theta|) = 2P(\hat{\theta}|H_0 > 70.5)$

$= 2P\left(Z > \frac{\overset{\bar{x}}{70.5} - \overset{\mu}{70}}{\underset{\sigma}{1.265}}\right) = 69.3\% \not< \alpha$ statistically insignificant

$H_0 : \theta \le \theta_0$, $H_a : \theta = \theta_a > \theta_0$, size α



$POW = P(\hat{\theta}|H_a > \theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}})$

$= P\left(\frac{\hat{\theta}|H_a - \theta_a}{\frac{\sigma}{\sqrt{n}}} > \frac{\theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \theta_a}{\frac{\sigma}{\sqrt{n}}}\right)$

$= 1 - \Phi\left(-\frac{\sqrt{n}}{\sigma}(\theta_a - \theta_0) + z_{1-\alpha}\right) = POW(\theta_a, \theta_0, n, \alpha, \sigma)$

More realistic: we don't know sigsq. But... sigsq is a "nuisance" parameter". It means we need to estimate it in order to estimate θ but we don't intrinsically care about it.

DGP: $X_1, \dots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$ and both θ and σ^2 are unknown.

How do we estimate sigsq? Recall... for a R.V. X,

$\sigma^2 := E[(X-\theta)^2]$ $\theta = E[X]$, $\hat{\theta} = \frac{1}{n}\sum X_i$

$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \theta)^2$ Problem: I need to know θ!

$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{x})^2$ Seems like a reasonable estimator!

Is this estimator unbiased?

$$E[\hat{\sigma}^2] = E[\tfrac{1}{n}\Sigma(X_i-\bar{x})^2] = \tfrac{1}{n}\Sigma E(X_i-\bar{x})^2 \overset{\text{i.i.d.}}{=} \tfrac{1}{\cancel{n}}\cancel{n}\, E[(X_1-\bar{x})^2]$$

$$= E[X_1^{\,2} - 2X_1\bar{x} + \bar{x}^{\,2}] = \underline{E[X_1^{\,2}]} - 2E[X_1 \tfrac{X_1+\dots+X_n}{n}] + \underline{E[\bar{x}^2]}$$

Recall $Var(X) = E(X^2) - E(X)^2$

$$= \sigma^2+\theta^2 - \tfrac{2}{n} E[X_1^{\,2} + X_1X_2 + \dots + X_1X_n] + \tfrac{\sigma^2}{n} + \theta^2$$

$$= \frac{n+1}{n}\sigma^2 + 2\theta^2 - \frac{2}{n}(\sigma^2 + \underbrace{\theta^2+\theta^2+\dots+\theta^2}_{n})$$

$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2$$ => It's a little bit biased...

However, it is "asymptotically unbiased" meaning...

$$\lim_{n\to\infty} E[\hat{\theta}] = \theta \qquad \text{e.g. } \lim_{n\to\infty} E[\hat{\sigma}^2] = \lim \frac{n-1}{n}\sigma^2 = \sigma^2 \checkmark$$

Consider the following estimator:

$$S^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{n}{n-1}\frac{1}{n}\Sigma(X_i-\bar{x})^2 = \frac{1}{n-1}\Sigma(X_i-\bar{x})^2$$

The beauty of this estimator is that

$$E[S^2] = E\left[\frac{n}{n-1}\hat{\sigma}^2\right] = \frac{n}{n-1}E[\hat{\sigma}^2] = \frac{\cancel{n}}{\cancel{n-1}}\frac{\cancel{n-1}}{\cancel{n}}\sigma^2$$ i.e. unbiased

And it's the default estimator for sigsq (variances in DGP's) and it's really important in normal theory...