There are two options for computing the retrainment region (and ascertaining whether the estimate is inside of this region).

(1) on the scale (in the unit of the estimate)
(2) on a standardized scale

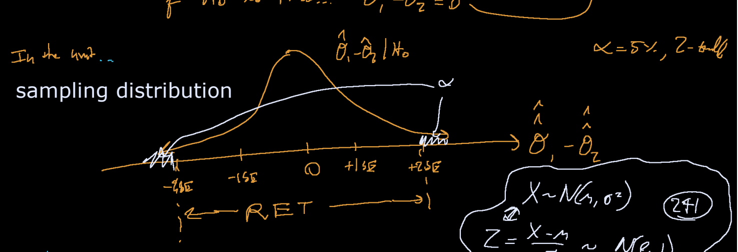
If X_1,1, ..., X_1,n_1 iid N(theta_1, sigsq_1) indep of
X_2,1, ..., X_2,n_2 iid N(theta_2, sigsq_2), σ_1^2, σ_2^2 are known

$$\Rightarrow \hat{\theta}_1 - \hat{\theta}_2 = \bar{X}_1 - \bar{X}_2 \sim N\left(\theta_1 - \theta_2, \left(\underbrace{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}_{SE}\right)^2\right) = N\left(0, \underbrace{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}_{SE}^{\,2}\right)$$

If H_0 is true... $\theta_1 - \theta_2 = 0$

In the unit...

sampling distribution

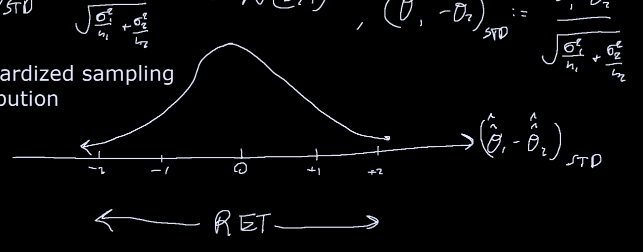


Standardized...

Under H_0, $\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \left(\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)^2\right)$

$$(\hat{\theta}_1 - \hat{\theta}_2)_{STD} := \frac{\hat{\theta}_1 - \hat{\theta}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1), \quad (\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2)_{STD} := \frac{\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

standardized sampling distribution

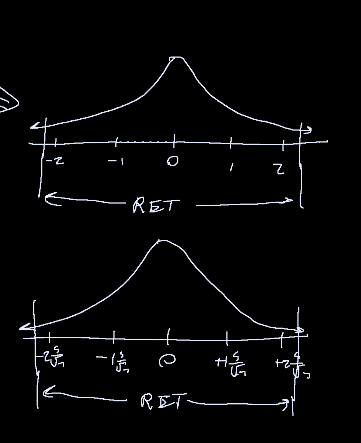


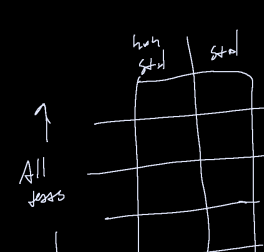
$P(\text{Type I err} \mid H_0)$

$$\hat{\theta}_{STD} = \frac{\hat{\theta} - 0}{SE} \sim T_{n-1}, \quad SE = \frac{s}{\sqrt{n}} \qquad s, n$$

NEVER...

$\hat{\theta} \sim T_{n-1}\left(0, \frac{s}{\sqrt{n}}\right)$





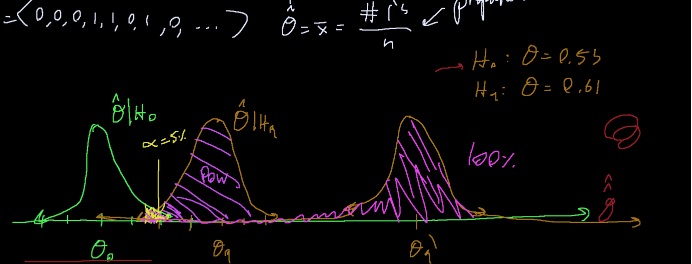
DGP: iid Bern(theta):
* One proportion z-test,
  (approx via CLT)

DGP: iid Normal:
(all except last are exact)tn
* One sample z-test,
* One sample t-test,
* Two sample z-test unequal variances
* Two sample z-test equal variances
* Two sample t-test equal variances
  (you use the s_pooled)
* Two sample t-test unequal variances
  (approximate via Welch-Satterthwaite's T approx with weird-looking df formula)

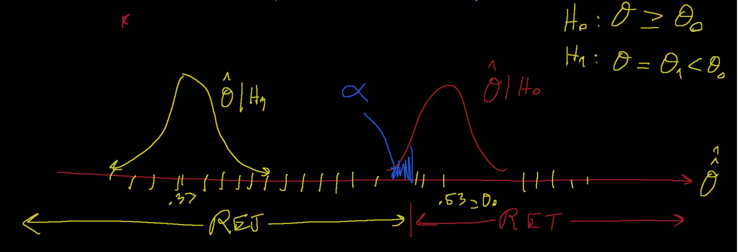
iid Bern(θ)

$\vec{x} = \langle 0, 0, 0, 1, 1, 0, 1, 0, \ldots \rangle$ $\quad \hat{\theta} = \bar{x} = \frac{\#1's}{n}$ ← "proportion"

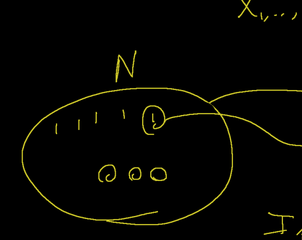
$H_0: \theta = 0.53$
$H_1: \theta = 0.61$



$H_0: \theta \geq \theta_0$
$H_1: \theta = \theta_1 < \theta_0$



$X_1, \ldots, X_n$



$P(X_1 = 1) = \frac{6}{6+3}$ $\quad P(X_2 = 1 \mid X_1 = 1) = \frac{5}{5+3}$

If $N \to \infty$ $\quad P(X_1 = 1) = \theta$ $\quad P(X_2 = 1 \mid X_1 = 1) = \theta$

$\theta = 0.4$

$$\frac{400{,}000}{1{,}000{,}000} \neq \frac{399{,}999}{999{,}999}$$

$\binom{N}{n}$ $\quad n < N$ $\quad n \lll N$

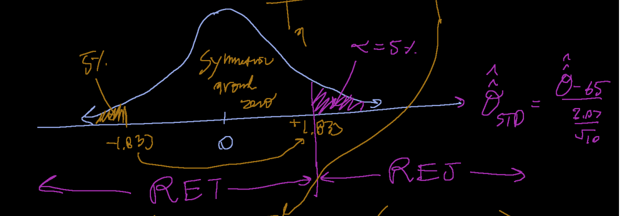
HW2, 3(d) $\quad X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$. σ^2 unknown

$n = 10$, $\bar{x} = 70.5$, $s = 2.07$, $\alpha = 5\%$

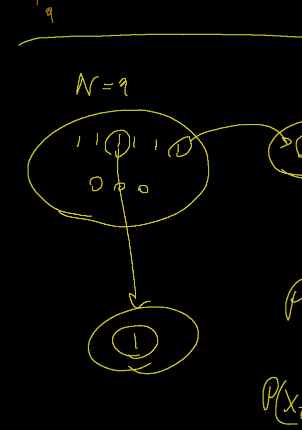
$H_a: \theta > 65$, $\quad H_0: \theta \leq 65$

Under H_0

$$\hat{\theta}_{STD} := \frac{\hat{\theta} - \theta_0}{SE} = \frac{\hat{\theta} - \theta_0}{\frac{s}{\sqrt{n}}} = \frac{\hat{\theta} - 65}{\frac{2.07}{\sqrt{10}}} \sim T_9$$



$F_{T_9}(-1.833) = 5\%$



$X_1 \sim \text{Bern}\left(\frac{6}{9}\right)$

$P(X_1 = 1) = \frac{6}{9}$ $\quad \Rightarrow$ DEPENDENT

$P(X_2 = 1 \mid X_1 = 1) = \frac{5}{8}$

$P(X_2 = 1) = \frac{6}{9}$ $\Rightarrow$ ident. distr.

$= P(X_2 = 1 \mid X_1 = 1) P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 0) P(X_1 = 0)$

$= \frac{6}{9}$

$\theta, \hat{\hat{\theta}}, \hat{\theta}, E[\hat{\theta}], \ell(\hat{\theta}, \theta), \ell(\hat{\theta}, \theta)$

$R(\hat{\theta}, \theta) := E[\ell(\hat{\theta}, \theta)]$

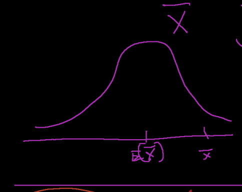
DGP: $\overset{iid}{\sim}$ Bern(θ)

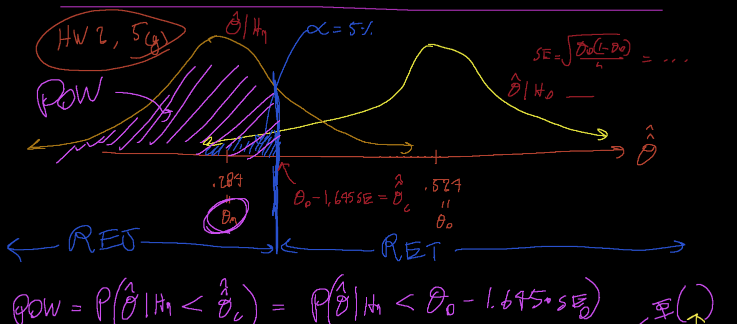
POW($\theta_0 = 0.5$, $\theta_a = 0.7$, $n = 100$, $\alpha = 1\%$)

$\bar{x}$, $\bar{X}$ ✓
s, S^2

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$H_0: \theta_1 - \theta_2 \neq 0$
$H_0: \theta_1 - \theta_2 \geq 0$
$H_0: \theta_1 - \theta_2 \leq 0$





$$POW = P(\hat{\theta} \mid H_a < \hat{\hat{\theta}}_c) = P(\hat{\theta} \mid H_a < \theta_0 - 1.645 \cdot SE_0)$$

$$= P\left(\frac{\hat{\theta} \mid H_a - \theta_a}{SE_a} < \frac{\theta_0 - 1.645 \cdot SE_0 - \theta_a}{SE_a}\right) = \Phi(\;)$$

$SE_a = \sqrt{\frac{\theta_a(1-\theta_a)}{n}}$

$$\Rightarrow P\left(\frac{\hat{\theta} \mid H_0 - \theta_0}{SE}\right)$$

$$\Rightarrow P\left(Z < \frac{\hat{\hat{\theta}}_c - \theta_0}{SE}\right) = 5\% \Rightarrow \Phi\left(\frac{\hat{\hat{\theta}}_c - \theta_0}{SE}\right) = 5\%$$

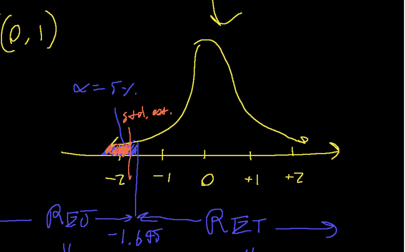
$$-1.645 = \frac{\hat{\hat{\theta}}_c - \theta_0}{SE}$$

$$\Rightarrow \hat{\hat{\theta}}_c = \theta_0 - 1.645\,SE$$

$\sigma_1^2 = 3.5^2$, $\sigma_2^2 = 4^2$

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{SE} \Big|_{H_0} \sim N(0,1)$$

$$SE = \sqrt{\frac{3.5^2}{6} + \frac{4^2}{10}} = 1.908$$



For practice... in alternate universe...

$\bar{x}_1 = 66.1$
$\bar{x}_2 = 69.7$

$$\Rightarrow \hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2 = -3.6 \qquad (\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2)_{STD} = \frac{-3.6}{1.908} = -1.886$$

$$p_{val} = P(\hat{\theta} \mid H_0 < \hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2)$$

$$= P\left(Z < \frac{\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2}{SE}\right) = P(Z < -1.886) = \Phi(-1.886) < 5\%$$

$$P(Z > 1.886) = P(Z < -1.886) = \Phi(-1.886) = 1 - \Phi(1.886)$$