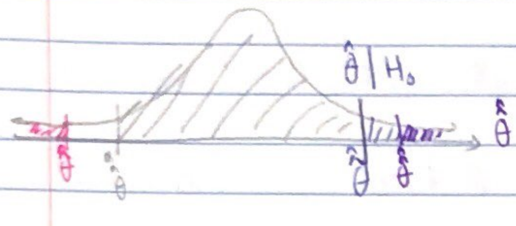
Lecture 19 11/18/2020

Here's a relevant fact about p-values that's important in our discussion about multiple comparisons. If H_0 is true, what is the distribution of the p-value?



Consider is right-sided test.

Proof for why p-vals under the null hypothesis are realizations from a $U(0,1)$ distribution. Assume left-sided test. The proof for right-sided and two-sided is similar.

$P_{val} := F_{\hat{\theta}|H_0}(\hat{\theta})$

rv model for Pval's

Let's examine the CDF of Pval to try and figure out it's distribution. This is a proof from 368.

$F_{P_{val}}(pval) = P(P_{val} \le pval) = P(F_{\hat{\theta}|H_0}(\hat{\theta}) \le pval)$

$= P(\hat{\theta} \le F^{-1}_{\hat{\theta}|H_0}(pval)) = F_{\hat{\theta}|H_0}(F^{-1}_{\hat{\theta}|H_0}(pval)) = pval \Longrightarrow$

Assume $\hat{\theta}|H_0$ is constant

$P_{val} \sim U(0,1)$.

We will return to testing now. We previously proved

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} \xrightarrow{d} N(0,1)$$

wald test → for $H_a: \theta \ne \theta_0$, $RET \approx \left[\theta_0 \pm z_{1-\alpha/2}\sqrt{\frac{I(\theta)^{-1}}{n}}\right]$

wald CI via Richardization → $CI_{\theta,1-\alpha} \approx \left[\hat{\theta}^{MLE} \pm z_{1-\alpha/2}\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$

We'll now derive a related means of testing $H_a: \theta$ is not θ_0. Recall for an iid DGP,

$$S(\theta; X_1, \dots X_n) = \sum_{i=1}^{n} \underbrace{l'(\theta; X_i)}_{W_i} \dots \quad \text{def 8, lec 9}$$

$$\Rightarrow \frac{1}{n} S(\theta; X_1, \dots, X_n) = \bar{W}$$

$E[W_i] = 0$ fact 16, lec 9

$Var[W_i] = I(\theta)$ lec 9-10

$$\Rightarrow \frac{\frac{1}{n} S(\theta; X_1, \dots X_n)}{\sqrt{\frac{I(\theta)}{n}}} = \frac{\bar{W} - E[\bar{X}]}{SE[\bar{W}]} \xrightarrow{d} N(0,1)$$

$$\frac{S(\theta; X_1, \dots X_n)}{\sqrt{n I(\theta)}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{S(\theta_0; X_1, \dots, X_n)}{\sqrt{n I(\theta_0)}} \dot{\sim} N(0,1)$$

using this as a z test statistic was discovered by Rao in 1948 and is called the "score test" but others call is the "Lagrange multiplier test"

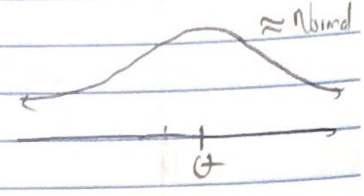
$\Rightarrow$ $\frac{S(\theta_0; X_1, \dots X_n)}{\sqrt{n I(\theta)}} \in [-1.96, 1.96] \rightarrow$ Retain H_0.

Note: this is "one-dimensional". There's only one θ being tested. You can derive the generalization with multiple θ's but we won't in this class.

This test statistic is really strange. Where is the estimator $\hat{\theta}$? You usually find an estimate that gauges the departure from H_0, & you find/approximate its distribution (the sampling distribution) and then check if $\hat{\theta}$ looks weird. If so, reject. But we don't do that here. The estimator is not in the expression! And if you just want to test $H_a: \theta$ is not θ_0, you don't really need an estimator or an estimate.

Many times, it is the same as the Wald test when you actually algebraically solve for the test statistic (HW you'll do it for Bern).

Here's an example why you may care about this:



DGP: iid Logistic$(\theta,1) := \dfrac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}$

$$\mathcal{L} = \prod_{i=1}^{n} \frac{e^{-x_i} e^{+\theta}}{(1+e^{-x_i}e^{-\theta})^2} = \frac{e^{-\Sigma x_i} e^{+n\theta}}{\prod_{i=1}^{n}(1+e^{-x_i}e^{-\theta})^2}$$

$$\ell = -\Sigma x_i + n\theta - 2\Sigma \ln(1+e^{-x_i}e^{+\theta})$$

$$S = \ell' = +n - 2\Sigma \frac{e^{-x_i}e^{\theta}}{1+e^{-x_i}e^{\theta}}$$

To get the MLE $\hat{\theta}$ set the above equal to zero and solve for θ. Good Luck! It's not possible ~~to~~ in closed form. You can use a computer to do a numerical solve if you wish.

$$\ell'(\theta;x) = 1 - 2\frac{e^{-x}e^{\theta}}{1+e^{-x}e^{\theta}}, \quad -\ell''(\theta;x) = 2\frac{(1+e^{-x}e^{\theta})e^{-x}e^{\theta} - (e^{-x}e^{\theta})^2}{(1+e^{-x}e^{\theta})^2}$$

$$= 2\cdot\frac{e^{-x}e^{\theta}}{(1+e^{-x}e^{\theta})^2}$$

$$\mathcal{I}(\theta) = E\left[2\frac{e^{-x}e^{\theta}}{(1+e^{-x}e^{\theta})^2}\right] = 2\int_{\mathbb{R}} \frac{e^{-x}e^{\theta}}{(1+e^{-x}e^{\theta})^2} f_X(x)\,dx = 2\int_{\mathbb{R}} \frac{(e^{-x}e^{\theta})^2}{(1+e^{-x}e^{\theta})^4}dx$$

$$= 2\cdot 1/6 = 1/3$$

$$\int_{\mathbb{R}} \left(\frac{1}{1+e^{-x}e^{\theta}}\right)^2 \left(\frac{e^{-x}e^{\theta}}{1+e^{-x}e^{\theta}}\right)^2 dx = \int_0^1 u^2(1-u)^2 \frac{1}{(1-u)u}\,du = \int_0^1 (u-u^2)\,du = \left[\frac{u^2}{2} - \frac{u^3}{3}\right]_0^1 = 1/6$$

let $u = \dfrac{1}{1+e^{-x}e^{\theta}} \Rightarrow 1-u = \dfrac{e^{-x}e^{\theta}}{1+e^{-x}e^{\theta}} \Rightarrow \dfrac{du}{dx} = \cancel{-}(1+e^{-x}e^{\theta})^{-2}(\cancel{-}e^{-x}e^{\theta})$

$$= \frac{e^{-x} e^{\theta}}{1+e^{-x}e^{\theta}} \frac{1}{1+e^{-x}e^{\theta}} = (1-u)u.$$

$$\Rightarrow dx = \frac{1}{(1-u)u} du, \quad x \to -\infty \Rightarrow u = 0, \quad x \to \infty \Rightarrow u = 1$$

under $H_0: \theta = \theta_0$,

$\Rightarrow$ score statistic is $\dfrac{n - 2\sum \frac{e^{-x_i}e^{\theta_0}}{1+e^{-x_i}e^{\theta_0}}}{\sqrt{n/3}} \stackrel{\cdot}{\sim} N(0,1)$

In our data example, we get $\dfrac{10 - 2 \times 0.646}{\sqrt{10/3}}$

$$= 4.77 \notin [-1.96, 1.96] \Rightarrow \text{Reject } H_0$$

Here's another also related testing procedure to the Wald and Score. Here too we wish to test against $H_0: \theta = \theta_0$. Remember, we want an estimate that gauges departure from this. How about ...

iid DGP

$$\hat{LR} := \frac{\mathcal{L}(\hat{\theta}^{MLE}; x_1, \ldots, x_n)}{\mathcal{L}(\theta_0; x_1, \ldots, x_n)} = \frac{\prod_{i=1}^{n} \mathcal{L}(\hat{\theta}^{MLE}; x_i)}{\prod_{i=1}^{n} \mathcal{L}(\theta_0; x_i)} = \prod_{i=1}^{n} \frac{\mathcal{L}(\hat{\theta}^{MLE}; x_i)}{\mathcal{L}(\theta_0; x_i)}$$

Likelihood Ratio. If it's significantly greater than one, then we reject H_0. Now we just need $\hat{LR}$, the sampling distribution. You can prove that:

$$\hat{\Lambda} := 2 \ln(\hat{LR}) \xrightarrow{d} \chi^2_1 \quad \text{Recall } F_{\chi^2_1}(3.84) = 95\%.$$

E.g. iid $\text{Bern}(\theta)$. $H_a: \theta \neq \theta_0$

$$\hat{LR} = \prod_{i=1}^{n} \frac{\mathcal{L}(\bar{x}; x_i)}{\mathcal{L}(\theta_0; x_i)} = \prod_{i=1}^{n} \frac{\bar{x}^{x_i}(1-\bar{x})^{1-x_i}}{\theta_0^{x_i}(1-\theta_0)^{1-x_i}} = \left(\frac{\bar{x}}{\theta_0}\right)^{\sum x_i} \left(\frac{1-\bar{x}}{1-\theta_0}\right)^{n-\sum x_i}$$

$$\hat{\Lambda} = 2\left(\sum x_i \ln\left(\bar{x}/\theta_0\right) + \left(n - \sum x_i\right)\ln\left(\frac{1-\bar{x}}{1-\theta_0}\right)\right)$$

$$= 2\underbrace{\left(O_1 \ln\left(O_1/E_1\right) + O_2 \ln\left(O_2/E_2\right)\right)}^{\text{Discrete KL-divergence}}$$

let $O_1 :=$ # ones, $O_2 :=$ # zeroes, $E_1 :=$ # exp ones $= n\theta_0$, $E_2 :=$ # exp zeroes $= n(1-\theta_0)$