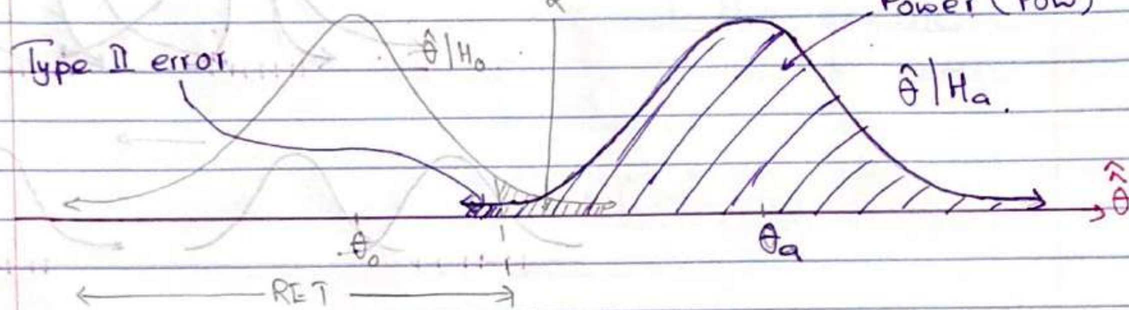
Lecture - 05 09/14/2020

Let's look at power more generally (beyond two point hypotheses).

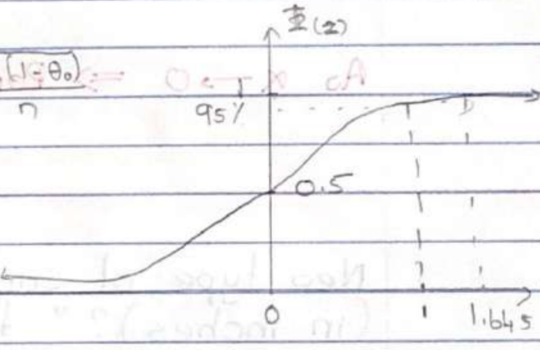
at size α

$H_0: \theta \leq \theta_0$, $H_a: \theta = \theta_a > \theta_0$, right-tailed test.

Under iid Bern(θ) and the normal approximation,



$\theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}$



CDF of $N(0,1)$

Let $\Phi(z) := F_z(z)$

$\Phi(z_{1-\alpha}) = 1-\alpha$

$\alpha = 5\% \Rightarrow z_{1-\alpha} = 1.645$

$$POW = P\left(\hat{\theta}|H_a > \theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)$$

$$= P\left(\frac{\hat{\theta}|H_a - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}} > \frac{\theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}} - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}}\right)$$

$$= P\left(z > \frac{-\sqrt{n}(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}}\right)$$

$$= 1 - \Phi\left(\frac{-\sqrt{n}\,(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}}\right)$$

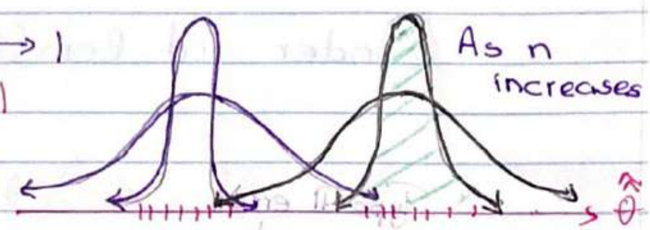
Power Function

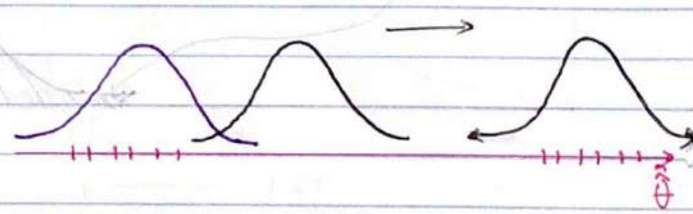
$$= Pow(\theta_a, \theta_0, n, \alpha)$$

Observations about the power function

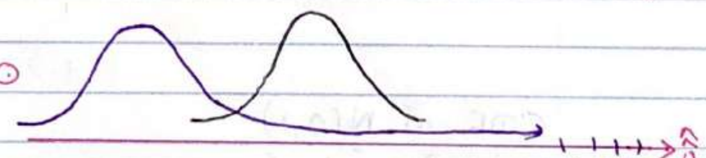
If $n \to \infty \implies Pow \to 1$

If $\theta_a \to \infty \implies Pow \to 1$





As $\alpha \to 0 \implies Pow \to 0$



New type of survey. We ask "how tall are you (in inches)?" for men only. I'll ask 10 male students and get $x_1, \ldots, x_{[10]}$. (i.e. my data). The data is now continuous (no longer zeroes and ones). Height for a gender is known to be normally distributed.

① DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$. Assume σ^2 is known and $= 4^2$

How can we estimate theta? θ is the mean of the r.v's. And recall

$\hat{\theta} = \bar{x}$ is unbiased. Let's use this estimator.

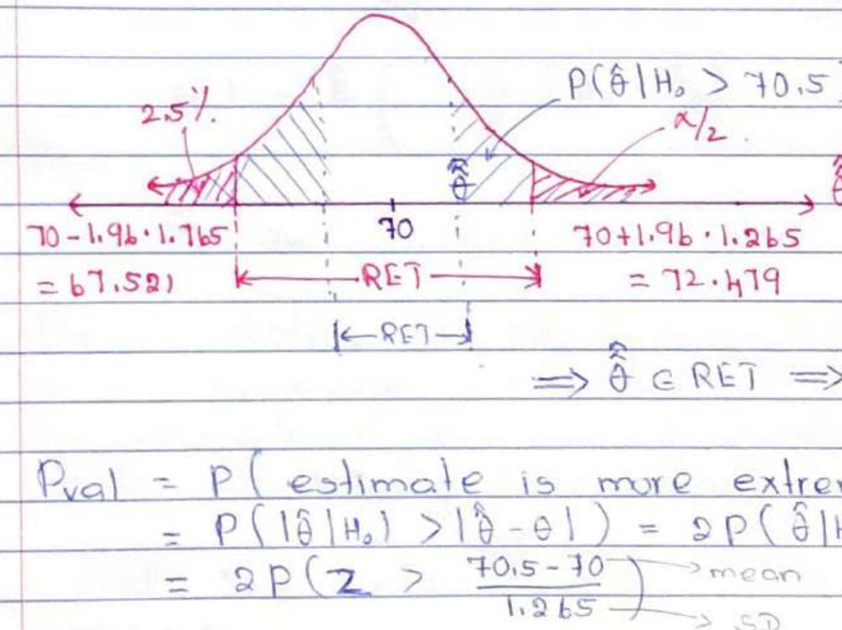
$\vec{x} = (70, 72, 73, 68, 69, 70, 67, 72, 71, 73)$ $\hat{\theta} = \bar{x} = 70.5$

The american mean male adult height is 70".

Let's test if the mean of the population where this class is drawn from is different than 70".

$H_a: \theta \neq 70, \quad H_0: \theta = 70$

$\hat{\theta} \mid H_0 \sim N(70, 4^2/10) = N(70, 1.265^2)$

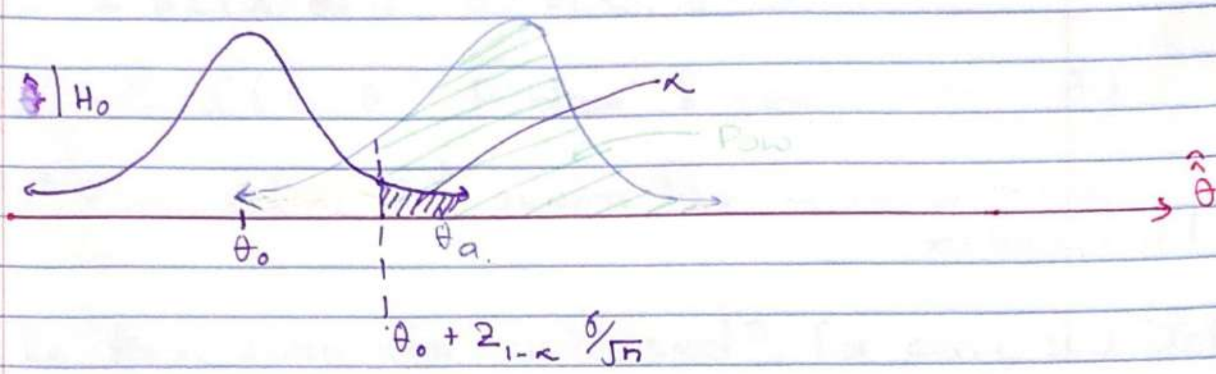


$\Rightarrow \hat{\hat{\theta}} \in RET \Rightarrow$ Fail to reject.

$P_{val} = P(\text{estimate is more extreme than observed} \mid H_0)$
$= P(|\hat{\theta}|H_0| > |\hat{\hat{\theta}} - \theta|) = 2P(\hat{\theta}|H_0 > 70.5)$
$= 2P\left(Z > \frac{70.5 - 70}{1.265}\right)$ → mean, → SD

$= 69.3\% \not< \alpha$ statistically insignificant.

$H_0: \theta \leq \theta_0$, $H_a: \theta = \theta_a > \theta_0$, size α



$$\text{Pow} = P(\hat{\theta}|H_a > \theta_0 + z_{1-\alpha} \sigma/\sqrt{n})$$

$$= P\left(\frac{\hat{\theta}|H_a - \theta_a}{\sigma/\sqrt{n}} > \frac{\theta_0 + z_{1-\alpha} \sigma/\sqrt{n} - \theta_a}{\sigma/\sqrt{n}} \right)$$

$$= 1 - \Phi\left(\frac{-\sqrt{n}}{\sigma} (\theta_a - \theta_0) + z_{1-\alpha} \right)$$

$$= \text{Pow}(\theta_a, \theta_0, n, \alpha, \sigma)$$

More realistic: We don't know sigsq. But .. sigsq is a "nuisance parameter". It means we need to estimate it in order to estimate θ but we don't intrinsically care about it.

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ and both θ, σ^2 are unknown.

How do we estimate sigsq? Recall... for a rv X,

$\sigma^2 := E[(X-\theta)^2] \quad \theta = E[X], \ \hat{\theta} = 1/n \sum X_i.$

$\hat{\sigma}^2 = 1/n \sum (X_i - \theta)^2$ Problem: I need to know θ!

$\hat{\sigma}^2 = 1/n \sum (X_i - \bar{X})^2$ seems like a reasonable estimator!

Is this estimator unbiased? For any iid DGP...

$$E[\hat{\sigma}^2] = E[1/n \sum (X_i - \bar{X})^2]$$

$$= 1/n \sum E(X_i - \bar{X})^2 \overset{iid}{=} 1/\cancel{n}\, \cancel{n}\, E[(X_i - \bar{X})^2]$$

$$= E[X_i^2 - 2X_i\bar{X} + \bar{X}^2]$$

$$= E[X_i^2] - 2E\left[X_i \cdot \frac{X_1 + \dots + X_n}{n}\right] + \underbrace{E[\bar{X}^2]}$$

Recall $Var[X] = E[X^2] - E[X]^2$

$$= \sigma^2 + \theta^2 - 2/n\, E[X_1^2 + X_1X_2 + \dots + X_1X_n] + \overbrace{\sigma^2/n + \theta^2}.$$

$$= \frac{n+1}{n}\sigma^2 + 2\theta^2 - \frac{2}{n}\underbrace{(\sigma^2 + \theta^2 + \theta^2 + \dots + \theta^2)}_{n}$$

$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2 \implies$$ It's a little bit biased...

However, it is "asymptotically unbiased" meaning...

$\lim_{n \to \infty} E[\hat{\theta}] = \theta \quad$ e.g. $\lim_{n\to\infty} E[\hat{\sigma}^2] = \lim \frac{n-1}{n}\sigma^2$

$= \sigma^2$ ✓

Consider the following estimator:

$$S^2 := \frac{n}{n-1}\hat{\sigma}^2 = \frac{n}{n-1}\frac{1}{n}\sum(x_i - \bar{x})^2$$

$$= \frac{1}{n-1}\sum(x_i - \bar{x})^2.$$

The beauty of this estimator is that

$$E[S^2] = E\left[\frac{n}{n-1}\hat{\sigma}^2\right] = \frac{n}{n-1}E[\hat{\sigma}^2]$$

$$= \frac{n}{n-1}\frac{n-1}{n}\sigma^2 \quad \text{i.e. unbiased.}$$

And it's the default estimator for sigsq (variances in DGP's) and it's really important in normal theory ...