MA369 09/02/2020.

Lecture 03

DGP : $X_1, \dots, X_n \overset{iid}{\sim}$ with mean θ, variance σ^2

If $\hat{\theta} = \bar{x} \Rightarrow \hat{\theta}$ is unbiased.

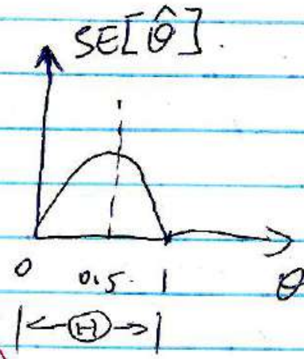
$$SE[\hat{\theta}] = \sqrt{Var[\tfrac{1}{n}(X_1 + \dots + X_n)]}$$

$$= \sqrt{\frac{1}{n^2} \sum Var[X_i]}$$

$$= \sqrt{\frac{1}{n^2} n \sigma^2} = \frac{\sigma}{\sqrt{n}}$$

$$= \sqrt{\frac{\theta(1-\theta)}{n}}$$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(\theta)$.



Supremum (maximum)

$$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \frac{1}{4n}$$

$$R(\hat{\theta}, \theta) = \frac{\theta(1-\theta)}{n} = MSE$$

Goal #3 of Inference: theory testing. (hypothesis testing)

You have some well-specified mathematical theory about the DGP. For example, in the iphone survey, "I think the proportion of iphone users in the population is NOT 52.4%. I want to prove my theory to the world (using my sample).

Note: It is absolutely impossible to prove or disprove my theory.

because you cannot see the whole population (or go inside of the DGP). We must use inference which is always a guess.

Two ways to go about "proving" my theory:
1) I assume I'm right and wait for other people to show me data that contradicts my theory.

2) I assume I'm wrong, then I adduce (bring) evidence (i.e. data) to the contrary until people are convinced my theory is right.

#2 is more intellectually honest and more likely to convince.

A "hypothesis" is a mathematical statement about the DGP e.g. $\theta = 0.9$, $\theta > 0.9$, θ is not equal to 0.9 or $\theta \leq 0.9$ or θ is in the set $[0.89, 0.91]$, etc.

The "alternative hypothesis" (H_a) is the theory you want to prove.
The "null hypothesis" (H_0) is the opposite you assume in #2 for the purpose of contradicting it.
Usual cases:

$H_0: \theta \leq \theta_0$, $H_a: \theta > \theta_0$ Right-tailed Test.

$H_0: \theta \geq \theta_0$, $H_a: \theta < \theta_0$. Left-tailed Test.

$H_0: \theta = \theta_0$, $H_a: \theta \neq \theta_0$ Two-tailed Test.

How to perform this test?

- There are many options even for the same DGP.
- The protocol goes as follows

1) You think of a "test statistic" that could measure the departure away from H_0.
2) Derive the statistical estimator's distribution under H_0.
3) Gauge the departure.

We begin with DGP: iid Bern(θ) and the "binomial exact test"

$H_0: \theta = \underbrace{0.524}_{\theta_0}$ $H_a: \theta \neq 0.524$.

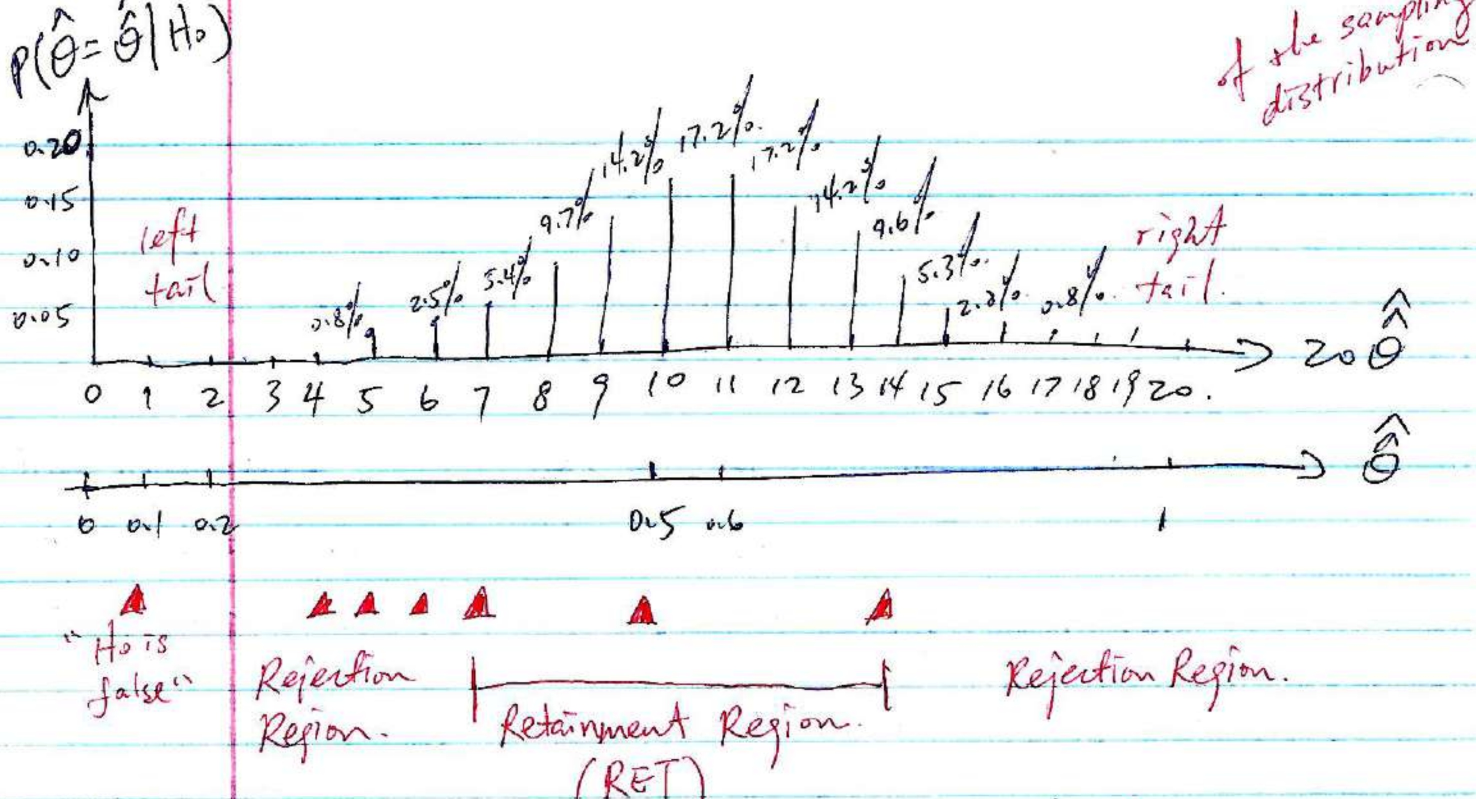
① My test statistic is ... $\hat{\theta} = \bar{x}$, $\hat{\theta}$ is a realization from $\hat{\Theta}$.

② $\hat{\Theta} | H_0 \sim ?$ $n = 20$.

$$\hat{\Theta} = \frac{X_1 + \dots + X_{20}}{20} \Rightarrow 20\hat{\Theta} | H_0 = X_1 + \dots + X_{20}.$$

$$\sim \text{Binom}(20, \theta_0 = 0.524)$$

of the sampling distribution.



$\hat{\theta} \in RET \Rightarrow$ Retain Ho or Fail to reject Ho.
or Not enough evidence to reject Ho.
Some authors say "accept Ho"

$\hat{\theta} \notin RET \Rightarrow$ Reject Ho / Accept Ha.
My estimate is "statistically significant".

Let's say we rejected Ho but it was really true.
This is called a Type I error. Where is the P(Type I error) on our plot?

$\alpha = P(\text{Type I error}) = P(\hat{\theta} \notin RET \mid H_0)$.

Then in a 2 tailed test, I apportion about $\frac{\alpha}{2}$ to the left tail and about $\frac{\alpha}{2}$ to the right tail.

In my RET,

$$\alpha = P(\hat{\theta} = 0 \mid H_0) + \cdots + P(\hat{\theta} = 0.3 \mid H_0) + P(\hat{\theta} = 0.75 \mid H_0) + \cdots + P(\hat{\theta} = 1 \mid H_0) = 7.06\%.$$

The choice of α is up to you. The scientific community's standard is 5% and sometimes 1%.

If you would like to prove your theory, you have to accept a positive probability of a Type I error.

If I fail to reject Ho when Ha is true that's a different error, a "Type II error". Failure to prove your theory.

The smaller the alpha, the larger the P(Type II error).

Truth.	Decision: Retain Ho	Decision: Reject Ho
Ho	✓	Type I error
Ha	Type II error	✓

As of now, we cannot calculate the P(Type II error).