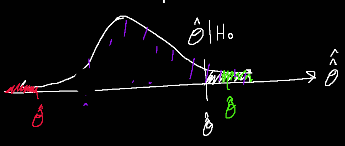
Here's a relevant fact about p-values that's important in our discussion about multiple comparisons. If H_0 is true, what is the distribution of the p-value?



consider is right-sided test

Proof for why p-vals under the null hypothesis are realizations from a U(0, 1) distribution. Assume left-sided test. The proof for right-sided and two-sided is similar.

$$P_{val} := F_{\hat{\theta}|H_0}(\hat{\theta})$$

rv model for p_{val}'s.

Let's examine the CDF of P_val to try and figure out its distribution. This is a proof from 368.

$$F_{P_{val}}(p_{val}) = P(P_{val} \le p_{val}) = P\left(F_{\hat{\theta}|H_0}(\hat{\theta}) \le p_{val}\right) = P\left(\hat{\theta} \le F^{-1}_{\hat{\theta}|H_0}(p_{val})\right)$$

Assume $\hat{\theta}|H_0$ is continuous

$$= F_{\hat{\theta}|H_0}\left(F^{-1}_{\hat{\theta}|H_0}(p_{val})\right) = p_{val} \Rightarrow P_{val} \sim U(0,1).$$

We will return to testing now. We previously proved...

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} \xrightarrow{d} N(0,1)$$

Wald test: for $H_a: \theta \neq \theta_0$, $RET \approx \left[\theta_0 \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\theta_0)^{-1}}{n}}\right]$

Wald CI via Richardization: $CI_{\theta, 1-\alpha} \approx \left[\hat{\theta}^{MLE} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$

We'll now derive a related means of testing H_a: theta is not theta_0
Recall for an iid DGP,

$$S(\theta; X_1, \ldots, X_n) = \sum_{i=1}^{n} \underbrace{\ell'(\theta; X_i)}_{W_i} \qquad \text{def 8, lec 9}$$

$$\Rightarrow \frac{1}{n} S(\theta; X_1, \ldots X_n) = \bar{W}$$

$E[W_i] = 0$ Fact 16, lec 9

$Var[W_i] = I(\theta)$ lec 9-10

$$\Rightarrow \frac{\frac{1}{n} S(\theta; X_1, \ldots, X_n)}{\sqrt{\frac{I(\theta)}{n}}} = \frac{\bar{W} - E[\bar{X}]}{SE[\bar{W}]} \xrightarrow{d} N(0,1)$$

$$\frac{S(\theta; X_1, \ldots X_n)}{\sqrt{n I(\theta)}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{S(\theta_0; X_1, \ldots X_n)}{\sqrt{n I(\theta_0)}} \overset{\cdot}{\sim} N(0,1)$$

Using this as a z test statistic was discovered by Rao in 1948 and is called the "score test" but others call is the "Lagrange multipler test"

At $\alpha = 5\%$

$$\Rightarrow \frac{S(\theta_0; X_1, \ldots X_n)}{\sqrt{n I(\theta_0)}} \in [-1.96, 1.96] \Rightarrow \text{Retain } H_0.$$

Note: this is "one-dimensional". There's only one theta being tested. You can derive the generalization with multiple thetas but we won't in this class.

This test statistic is really strange. Where is the estimator thetahat? You usually find an estimate that gauges the departure from H_0, and you find / approximate its distribution (the sampling distribution) and then check if thetahathat looks weird. If so, reject. But we don't do that here. The estimator is not in the epxression! And if you just want to test H_a: theta is not theta_0, you don't really need an estimator or an estimate.

Many times, it is the same as the Wald test when you actually algebraically solve for the test statistic (HW you'll do it for Bern).

Here's an example why you may care about this:

$\approx$ Normal

$$\text{DGP: iid Logistic}(\theta, 1) := \frac{e^{-(x-\theta)}}{\left(1 + e^{-(x-\theta)}\right)^2}$$

$$\mathcal{L} = \prod_{i=1}^{n} \frac{e^{-X_i} e^{+\theta}}{(1 + e^{-X_i} e^{+\theta})^2} = \frac{e^{-\sum X_i} e^{+n\theta}}{\prod_{i=1}^{n} (1 + e^{-X_i} e^{+\theta})^2}$$

$$\ell = -\sum X_i + n\theta - 2\sum \ln(1 + e^{-X_i} e^{+\theta})$$

$$S = \ell' = n - 2\sum \frac{e^{-X_i} e^{\theta}}{1 + e^{-X_i} e^{\theta}}$$

To get the MLE I set the above equal to zero and solve for theta. Good luck! It's not possible in closed form. You can use a computer to do a numerical solve if you wish.

$$\ell'(\theta; x) = 1 - 2\frac{e^{-x} e^{\theta}}{1 + e^{-x} e^{\theta}}, \quad -\ell''(\theta; x) = 2\frac{(1 + e^{-x} e^{\theta}) e^{-x} e^{\theta} - (e^{-x} e^{\theta})^2}{(1 + e^{-x} e^{\theta})^2}$$

$$= 2\frac{e^{-x} e^{\theta}}{(1 + e^{-x} e^{\theta})^2}$$

$$I(\theta) = E\left[2\frac{e^{-x} e^{\theta}}{(1 + e^{-x} e^{\theta})^2}\right] = 2\int_{\mathbb{R}} \frac{e^{-x} e^{\theta}}{(1 + e^{-x} e^{\theta})^2} f_X(x)\, dx = 2\int_{\mathbb{R}} \frac{(e^{-x} e^{\theta})^2}{(1 + e^{-x} e^{\theta})^4} dx = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

$$\int_{\mathbb{R}} \left(\frac{1}{1 + e^{-x} e^{\theta}}\right)^2 \left(\frac{e^{-x} e^{\theta}}{1 + e^{-x} e^{\theta}}\right)^2 dx = \int_0^1 u^{\cancel{2}} (1-u)^{\cancel{2}} \frac{1}{\cancel{(1-u) u}} du = \int_0^1 (u - u^2)\, du = \left[\frac{u^2}{2} - \frac{u^3}{3}\right]_0^1 = \frac{1}{6}$$

$$\text{let } u = \frac{1}{1 + e^{-x} e^{\theta}} \Rightarrow 1 - u = \frac{e^{-x} e^{\theta}}{1 + e^{-x} e^{\theta}} \Rightarrow \frac{du}{dx} = -(1 + e^{-x} e^{\theta})^{-2} (-e^{-x} e^{\theta}) = \frac{e^{-x} e^{\theta}}{1 + e^{-x} e^{\theta}} \frac{1}{1 + e^{-x} e^{\theta}}$$

$$= (1-u) u$$

$$\Rightarrow dx = \frac{1}{(1-u) u} du, \quad x \to -\infty \Rightarrow u = 0, \quad x \to \infty \Rightarrow u = 1$$

Under $H_0: \theta = \theta_0$,

$$\Rightarrow \text{Score statistic is } \frac{n - 2\sum \frac{e^{-X_i} e^{\theta_0}}{1 + e^{-X_i} e^{\theta_0}}}{\sqrt{n/3}} \overset{\cdot}{\sim} N(0,1)$$

In our data example, we get $\frac{10 - 2 \cdot 0.646}{\sqrt{10/3}} = 4.77 \notin [-1.96, 1.96]$

$\Rightarrow$ Reject H_0

Here's another also related testing procedure to the Wald and Score. Here too we wish to test against H_0: theta = theta_0. Remember, we want an estimate that gauges deaprture from this. How about...

iid DGP

$$\hat{LR} := \frac{\mathcal{L}(\hat{\theta}^{MLE}; X_1, \ldots X_n)}{\mathcal{L}(\theta_0; X_1, \ldots X_n)} = \frac{\prod_{i=1}^{n} \mathcal{L}(\hat{\theta}^{MLE}; X_i)}{\prod_{i=1}^{n} \mathcal{L}(\theta_0; X_i)} = \prod_{i=1}^{n} \frac{\mathcal{L}(\hat{\theta}^{MLE}; X_i)}{\mathcal{L}(\theta_0; X_i)}$$

Likelihood Ratio. If it's significantly greater than one, then we reject H_0. Now we just need LR-hat, the sampling distribution. You can prove that:

$$\hat{\Lambda} := 2\ln(\hat{LR}) \xrightarrow{d} \chi^2_1$$

Recall $F_{\chi^2_1}(3.84) = 95\%$

E.g. iid Bern(θ). $H_a: \theta \neq \theta_0$.

$$\hat{LR} = \prod_{i=1}^{n} \frac{\mathcal{L}(\bar{x}; x_i)}{\mathcal{L}(\theta_0; x_i)} = \prod_{i=1}^{n} \frac{\bar{x}^{x_i} (1 - \bar{x})^{1 - x_i}}{\theta_0^{x_i} (1 - \theta_0)^{1 - x_i}} = \left(\frac{\bar{x}}{\theta_0}\right)^{\sum x_i} \left(\frac{1 - \bar{x}}{1 - \theta_0}\right)^{n - \sum x_i}$$

Discrete KL-divergence

$$\hat{\Lambda} = 2\left(\sum x_i \ln\left(\frac{\bar{x}}{\theta_0}\right) + (n - \sum x_i) \ln\left(\frac{1 - \bar{x}}{1 - \theta_0}\right)\right) = 2\left(O_1 \ln\left(\frac{O_1}{E_1}\right) + O_2 \ln\left(\frac{O_2}{E_2}\right)\right)$$

let $O_1 :=$ # ones, $O_2 :=$ # zeroes, $E_1 :=$ # exp ones $= n\theta_0$, $E_2 :=$ # exp zeroes $= n(1 - \theta_0)$