lecture 12:

In lec 10;

$$\frac{\hat{\theta}-\theta}{SE[\hat{\theta}]} \xrightarrow{d} N(0,1) \Rightarrow \frac{\hat{\theta}-\theta}{\hat{SE}[\hat{\theta}]} \xrightarrow{d} N(0,1)$$

We can use this now in our situation:

$$\frac{\hat{\theta}_1-\hat{\theta}_2}{SE[\hat{\theta}_1-\hat{\theta}_2]} \xrightarrow{d} N(0,1) \Rightarrow \frac{\hat{\theta}_1-\hat{\theta}_2}{\hat{SE}[\hat{\theta}_1-\hat{\theta}_2]} \xrightarrow{d} N(0,1)$$

$$SE[\hat{\theta}_1-\hat{\theta}_2] = \sqrt{\theta_{shared}(1-\theta_{shared})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

$$\hat{SE}[\hat{\theta}_1-\hat{\theta}_2] = \sqrt{\hat{\theta}_{shared}(1-\hat{\theta}_{shared})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$ if $\hat{\theta}_{shared}$ is consistant

$\hat{\theta}_{shared}$ = avg. over both samples

$$= \frac{\Sigma X_{1i} + \Sigma X_{2i}}{n_1+n_2}$$

$$\Rightarrow \frac{\hat{\theta}_1-\hat{\theta}_2}{\sqrt{\frac{\Sigma x_{1i}+\Sigma x_{2i}}{n_1+n_2}\left(1-\frac{\Sigma x_{1i}+2_{1i}}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \dot{\sim} N(0,1)$$

eg. $H_a: \theta_1 - \theta_2 \neq 0$, $H_0: \theta_1 - \theta_2 = 0$, $\alpha = 5\%$

Control | $n_1 = 81$, $\Sigma X_{1i} = 27 \Rightarrow \hat{\theta}_1 = \frac{27}{81} = 0.333$

Experimental | $n_2 = 79$, $\Sigma X_{2i} = 12 \Rightarrow \hat{\theta}_2 = \frac{12}{79} = .152$

$$\hat{\theta}_{shared} = \frac{27+12}{81+79} = .244$$

$$(\hat{\theta}_1 - \hat{\theta}_2)_{std} = \frac{.333 - 0.152}{\sqrt{0.244(1-0.244)\left(\frac{1}{81}+\frac{1}{79}\right)}}$$

$$= 2.66 \notin [-1.96, +1.96]$$

$\Rightarrow$ Reject H_0.

Another (obvious) Wald Test: if X_1, X_n

iid DAP with mean θ and variance sigma-squared (σ^2) and the estimator $\hat{\theta}$ is $\bar{X}$, then the

CLT implies that

$$\frac{\hat{\theta} - \theta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1) \quad \text{if } \sigma \text{ is know}$$

if σ is unknown... I can replace Sigma with any consistent estimator eg. S, Sigma-hat and

$$\frac{1}{n} \Sigma (x_i - \theta)^2$$

$$\Rightarrow \frac{\hat{\theta} - \theta}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

Are you allowed to just use the T-test ~~test~~ here? Many people just use the T-test here. Technically it's wrong because you need the DGP to be normal iid. But if you use the T-test... it's "not so bad" I did this on problem 11 of the midterm:

Ha: $\theta > 2$, $n = 30$, $\bar{X} = 2.57$, $S = 1.00$

$$\hat{\theta}_{std} = \frac{2.57 - 2}{\frac{1.00}{\sqrt{30}}} = 3.12 \notin [-\infty, 1.645]$$

$\Rightarrow$ Reject H_0

* Another Wald test for two independent Samples with unknown variances and You wish to test a diffent in means.

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{\theta_1^2}{n_1} + \frac{\theta_2^2}{n_2}}} \xrightarrow[\text{from last class}]{d} N(0,1) \Rightarrow \frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \xrightarrow{d} N(0,1)$$

if You use the Satterthwaite t-test, it "wouldn't be so bad" because unless Your population distributions were so very skewed it should be fine.

H- 41mm

Let's use the asymptotic normality of the MLE thm (last class) to do a wald Test. Hw 4l,m has DGP: $X_1, \dots X_n$ iid Gumbel(0,1) The Gumbel is a rv model for "extreme events" think maximum rainfall per month.

$$l'(\theta; x_1, \dots, x_k) = n - e^{\theta} \sum e^{-x_i} \overset{set}{=} 0 \Rightarrow \hat{\theta}^{MLE} = \ln\left(\frac{n}{\sum e^{-x_i}}\right)$$

$$\Updownarrow$$

$$l'(\theta; x) = 1 - e^{-\theta} e^{-x} \Rightarrow l''(\theta; x) = -e^{-\theta} e^{-x} \, ?$$

$$I(\theta) = E\left[-l''(\theta, x)\right]$$

$$= E\left[e^{-\theta} e^{-x}\right] = e^{\theta} E\left[e^{-x}\right] = e^{-2\theta}$$

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} = \frac{\hat{\theta}^{MLE} - \theta}{\frac{e^{\theta}}{\sqrt{n}}}$$

$$= \frac{\ln\left(\frac{n}{\sum e^{-x}}\right) - \theta}{\frac{e^{\theta}}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

$x_1 = 2.15$, $x_2 = 1.91$, 3.66, 4.85, 3.03, 1.03, 3.58

$n = 7$.

$\hat{\theta}^{MLE} = 2.26$

Test $H_a: \theta > 2$ $\quad \alpha = 5\%$.

$$\hat{\theta}^{MLE}_{std} = \frac{2.26 - 2}{\frac{e^2}{\sqrt{7}}} = \frac{.26}{2.79} = 0.09 \in [\infty, 1.645]$$

$\Rightarrow$ Retain H_0

Next unit

There are three goals of statistical inference

(1) Point Estimation

Goal here is to Provide a best guess, $\hat{\theta}$ of the value of θ. You don't know if your specific guess is good, is close, is bad, is far ... How do we ask the question "is it good / bad"? We imagined $\hat{\theta}$ Coming from from a distribution $\hat{\theta}$; the "sampling distribution". There are properties about the sampling distribution e.g. Some good properties are unbiasedness, consistency, low MSE, low risk (for general loss functions)

(2) Testing.

Goal Here is to test a theory about a specific θ. We used hypothesis testing. What makes a "good test"? One property is "power". There are other propertijes we did not discuss.

(3) Confidence Sets

The "Goal here is to create a set of values for θ that you are "Confident in". The approch we use here is the "Confidence interval".

Definition: an "interval estimate" are two statistics:

$W_L(x_1 .. x_n)$ & $W_u(x_1 .. x_n)$ s.t. $W_L < W_u$ for all data sets.

estimate is a number

Combined in an interval: $[W_L(x_1 .. x_n), W_u(x_1 .. x_n)]$

e.g $[1.789, 2.463]$

estimator is a random variable.

and of course, the "interval estimator" is.

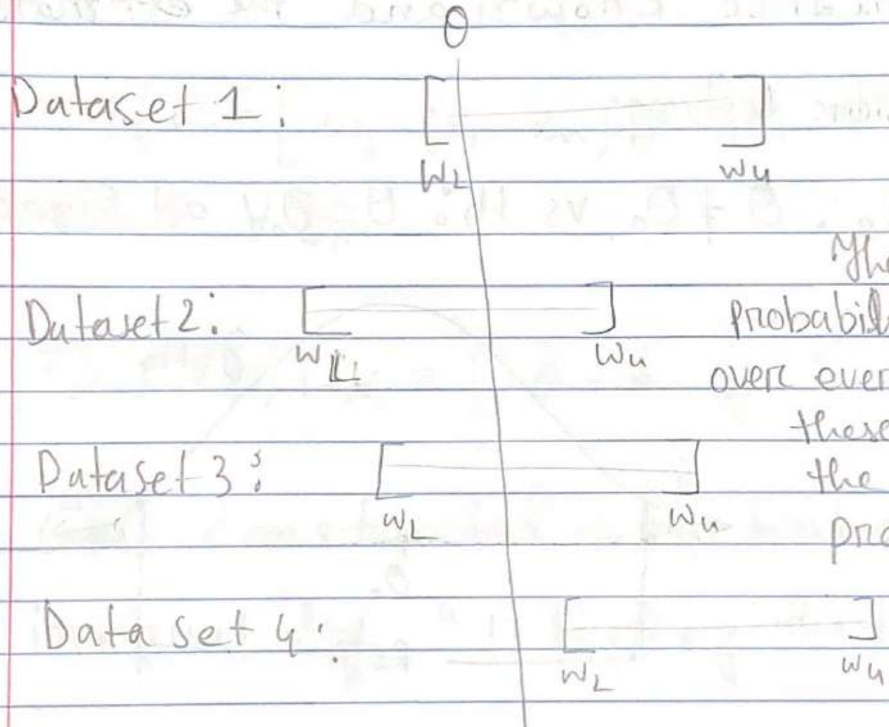
$$[W_L(X_1 .. X_n), W_u(X_1, .. X_n)]$$

Which is a "random interval".

Definition: An interval estimator has "Coverage probability"

$$P\left(\theta \in [W_L(X_1, \dots, X_n), W_u(X_1, \dots, X_n)] \mid \theta\right).$$

An illustration:



The coverage probability is computed over every dataset. For these four datasets, the coverage probability would be $3/4 = 75\%$

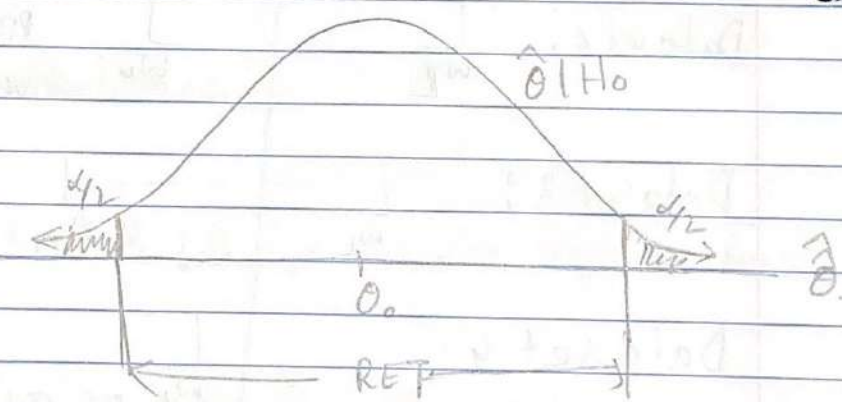
We define the "Confidence interval" with the coverage probability $1-\alpha$ for parameter θ as this interval estimate and interval estimator (depending on context)

Demoset $CI_{\theta, 1-\alpha}$.

Given α, how do we find the confidence interval? Let's begin with the DGP iid normal mean θ, variance σ^2 and Variance known and the estimator $\bar{X}$

Considere testing:

$H_a: \theta \neq \theta_0$ vs $H_0: \theta = \theta_0$ at size α



$P(\hat{\theta} \in RET | H_0) = 1 - \alpha$ $\quad Z_{1-\frac{\alpha}{2}} := F_Z^{-1}(1-\frac{\alpha}{2})$

$\|$

$P(\hat{\theta} \in [\theta_0 - Z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \theta_0 + Z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}] | \theta = \theta_0)$

$$= P\left(\hat{\theta} - \theta_0 \in \left[-z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, +z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right] \middle| \theta = \theta_0\right)$$

$$= P\left(\theta_0 - \hat{\theta} \in \left[-z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, +z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right] \middle| \theta = \theta_0\right)$$

$$= P\left(\theta_0 \in \left[\underbrace{\hat{\theta} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{n}}_{w_L}, \underbrace{\hat{\theta} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}_{w_u}\right] \middle| \theta = \theta_0\right)$$

$= P\left(\theta_0 \in \left[w_L(x_1 \dots x_n), w_u(x_1 \dots x_n)\right] \middle| \theta = \theta_0\right)$ since valid $\forall \theta_0$ ...

$$\Rightarrow CI_{\theta, 1-\alpha} = \left[\hat{\theta} - z_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \hat{\theta} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right]$$

We constructed our first confidence interval by "inverting the test".