9/16/20

DGP: $x_1, \dots, x_n \overset{iid}{\sim} N(\theta, \sigma^2)$ Standardize the estimator

math 241 SE[$\bar{x}$]

$\hat{\theta} = \bar{x} \overset{\cdot}{\sim} N\left(\theta, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \implies \dfrac{\bar{x} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ (exact)

What if sigma is unknown?

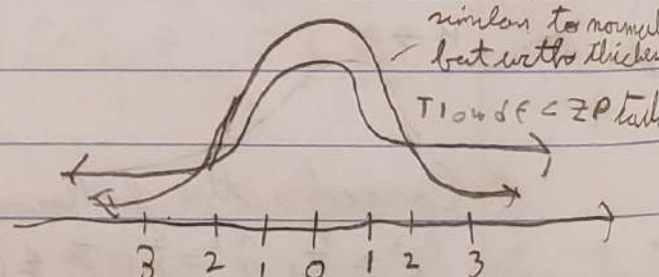
S^2 estimates $\sigma^2 \implies S = \sqrt{S^2}$ estimates σ

Does $\dfrac{\bar{x} - \theta}{\frac{s}{\sqrt{n}}} \overset{?}{\sim} N(0,1)$ No! but close!

In 1907 Gosset proved:

$\dfrac{\bar{x} - \theta}{\frac{s}{\sqrt{n}}} \sim T_{n-1}$ ← Student's T distribution with n-1 "degrees of freedom" (the parameter for the standard T distr

(exact)



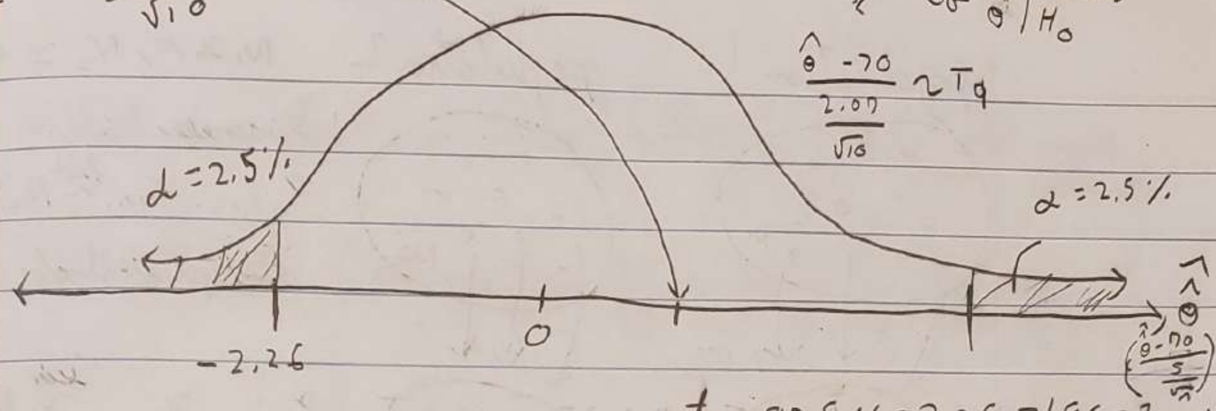
data from n=10 male student heights: $\bar{x} = 70.5$, $s = 2.07$

$H_a: \theta \neq 70$, $H_0: \theta = 70$, $\alpha = 5\%$.

$$\frac{\hat{\theta}-70}{\frac{s}{\sqrt{n}}} = \frac{70.5-70}{\frac{2.07}{\sqrt{10}}} = 0.76$$

the standardized distribution of $\hat{\theta} \mid H_0$

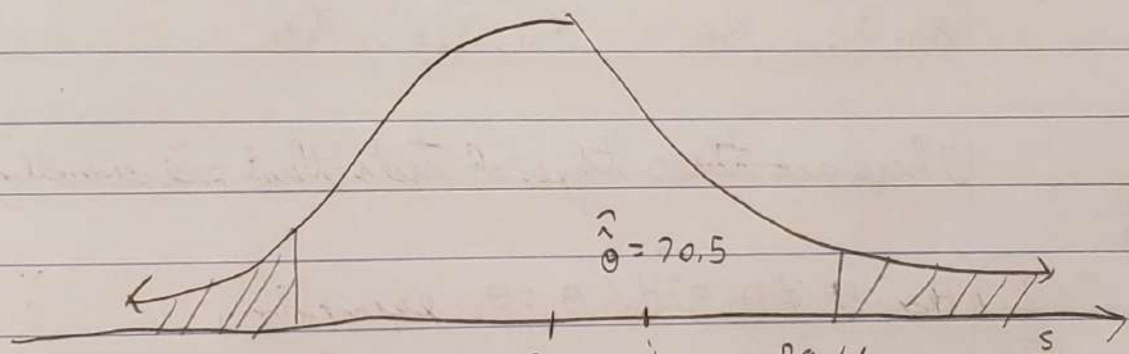
$$\frac{\hat{\theta}-70}{\frac{2.07}{\sqrt{10}}} \sim T_9$$



$t_{9,\,97.5\%} = 2.26 > 1.96 = z_{97.5\%}$

|— REJ —|———— RET ————|— REJ —|

$\Rightarrow$ Retain H_0



$70 + t_{9,\,97.5\%}\frac{s}{\sqrt{n}}$

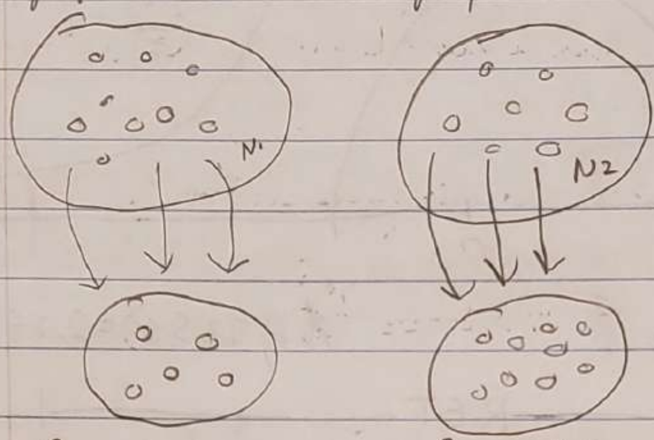
$= 70 + 2.26\frac{2.07}{\sqrt{10}} = 71.98$

← REJ ——✱———— RET ————✱—— REJ ——→

$\Rightarrow$ Retain H_0

We just did our first "one-sample two-sided t test" (of a mean).

population 1 population 2 $N_1 \simeq \infty$, $N_2 \simeq \infty \Rightarrow$ iid



assume
$x_{11}, \dots x_{1n_1} \overset{iid}{\sim} N(\theta_1, \sigma_1^2)$
independent of

$x_{21}, \dots x_{2n_2} \overset{iid}{\sim} N(\theta_2\ \sigma_2^2)$
σ_1^2, σ_2^2 are known but
θ_1, θ_2 are unknown

Sample 1 Sample 2
size n_1 size n_2
$x_{11}, x_{12}, \dots x_{1n_1}$ $x_{21}, x_{22} \dots x_{2n_2}$

There are three types of tests that are usually done.

I) $H_a: \theta_1 \neq \theta_2 \Rightarrow H_0: \theta_1 = \theta_2$ equivlaly

$H_a: \underbrace{\theta_1 - \theta_2} \neq 0 \Rightarrow H_0: \theta_1 - \theta_2 = 0$

II) $H_a: \theta_1 < \theta_2 \Rightarrow H_0: \theta_1 \geq \theta_2$ equalivly
$H_a: \underbrace{\theta_1 - \theta_2} < 0 \Rightarrow H_0: \theta_1 - \theta_2 \geq 0$

III) $H_a: \theta_1 > \theta_2 \Rightarrow H_0: \theta_1 \leq \theta_2$ equality
$H_a: \underbrace{\theta_1 - \theta_2} > 0 \Rightarrow H_0: \theta_1 - \theta_2 \leq 0$

What is a test statistic? (For $\theta_1 - \theta_2$) $\hat{\theta}_1 - \hat{\theta}_2$ point estimate

The estimator that produces these estimates is simply:

math241 $\hat{\theta}_1 \sim N(\theta_1, \frac{\sigma_1^2}{n_1})$ indep. $\hat{\theta}_2 \sim N(\theta_2, \frac{\sigma_2^2}{n_2})$

$$\hat{\theta}_1 - \hat{\theta}_2 = \bar{X}_1 - \bar{X}_2 \underset{exact}{\sim} N\left(\theta_1 - \theta_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)$$

$$\Rightarrow SE[\hat{\theta}_1 - \hat{\theta}_2] = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(null)
under Ho (all three), $\theta_1 - \theta_2 = 0$

$$\frac{(\hat{\theta}_1 - \hat{\theta}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

$$\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Let's test if male mean height is different than female mean height.

$X_2 = \langle 60, 59, 64, 64, 64, 63 \rangle$ $n_2 = 6$

$\bar{x}_2 = 62.3$

$n_2 = 6$, $\bar{x}_2 = 62.3$

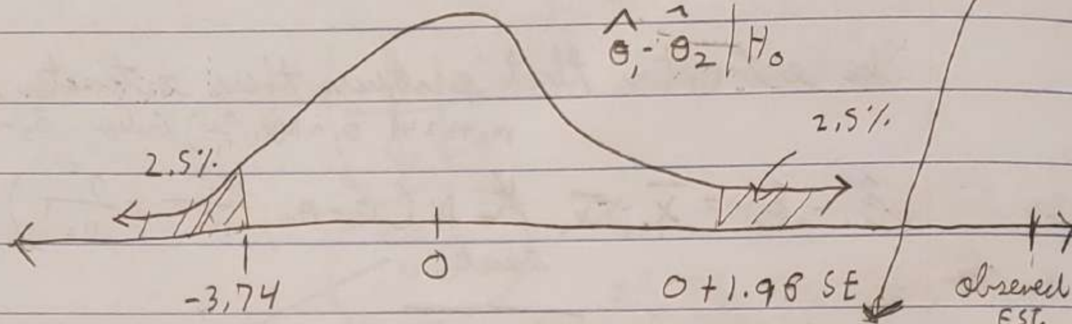
$n_1 = 10$, $\bar{x}_1 = 70.5$

$\hat{\theta}_1 - \hat{\theta}_2 = 70.5 - 62.3 = 8.2$

We assumed we knew the variances. So the variance for the men was assumed to be 4^2 and now the variance for the women is assumed to be 3.5^2 $\sigma_1^2 = 4^2$, $\sigma_2^2 = 3.5^2$

$$SE = \sqrt{\frac{4^2}{10} + \frac{3.5^2}{6}} = 1.91$$

We can now do our 2-sample 2-sided z test



$0 + 1.96\,SE$
$= 1.96 \cdot 1.91$
$= 3.74$

← REJ → | ← RET → | ← REJ →

Observed estimate $\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2 \notin RET \Rightarrow$ Reject H_0

$$P_{val} = 2P\left(\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2 > 8.2\right) = 2P\left(\frac{\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2}{SE} > \frac{8.2}{1.91}\right) = 2P(z > 4.29) = 1.8 \times 10^{-5} < \alpha$$

$\Rightarrow$ Reject

Let's sample from two populations again however, this time we have the same variance, SigSq which we still assume known.
$= SigSq_1 = SigSq2$

$$x_1, \ldots, x_{n_1} \overset{iid}{\sim} N(\theta_1, \sigma^2) \text{ indep. of } x_{21} \ldots x_{2n_2} \overset{iid}{\sim} N(\theta_2, \sigma^2)$$

under H_0, $\hat{\theta}_1 - \hat{\theta}_2 \sim N(0, \sqrt{\sigma(\frac{1}{n_1}+\frac{1}{n_2})}^{\,2})$

also $\dfrac{\hat{\theta}_1 - \hat{\theta}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim N(0,1)$ 2-Sample, 2-sided z test of equal variance

The test can be run again, you can probably assume sigma = 3.75.

Same as above but sigsq unknown. How can we estimate the standard error?

S_1^2, S_2^2 are the sample variance in both samples 1 and 2.

$$S_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_{1,i}-\overline{X}_1)^2, \quad S_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(X_{2,i}-\overline{X}_2)^2$$

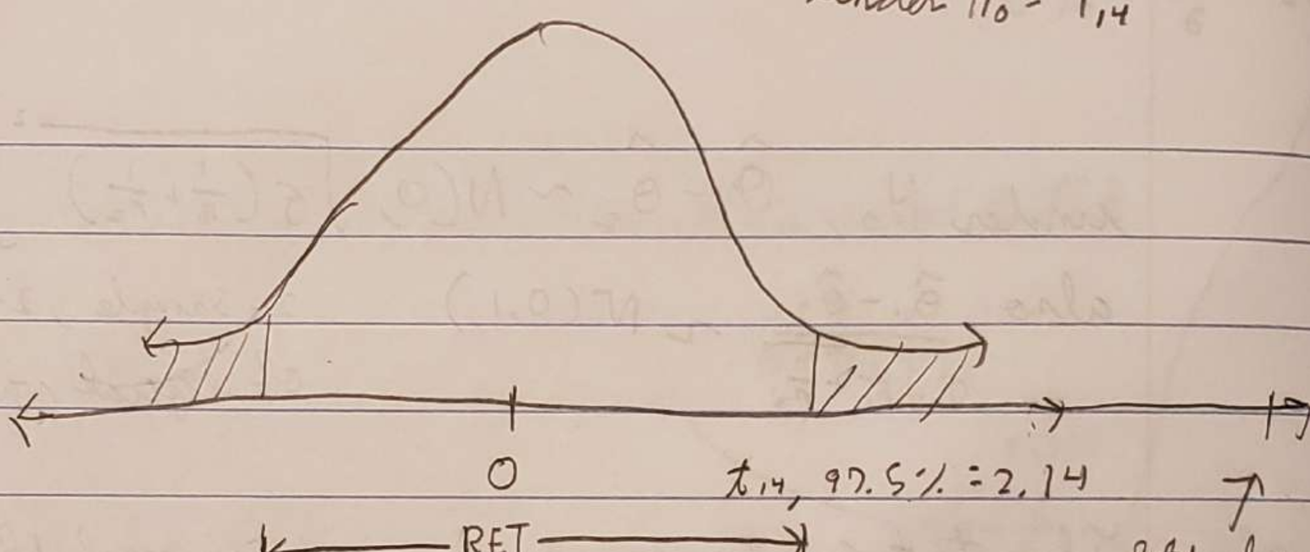
$$S^2_{pooled} := \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$ weight average

you can prove that

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{S_{pooled}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim T_{n_1+n_2-2}$$

this allows you to do the "2-sample t test of equal variance"

Std estimator under $H_0 = T_{14}$



Std observed estimate

$$\hat{\hat{\theta}} = \frac{70.5 - 62.3}{S_{pooled}\sqrt{\frac{1}{10}+\frac{1}{8}}} = \frac{8.2}{2.14 \cdot 0.51} = 7.42$$

$$S_{pooled}^2 = \frac{9 \cdot 2.07^2 - 5 \cdot 2.25^2}{14} = \sqrt{4.58}$$

$\Rightarrow$ Reject H_0