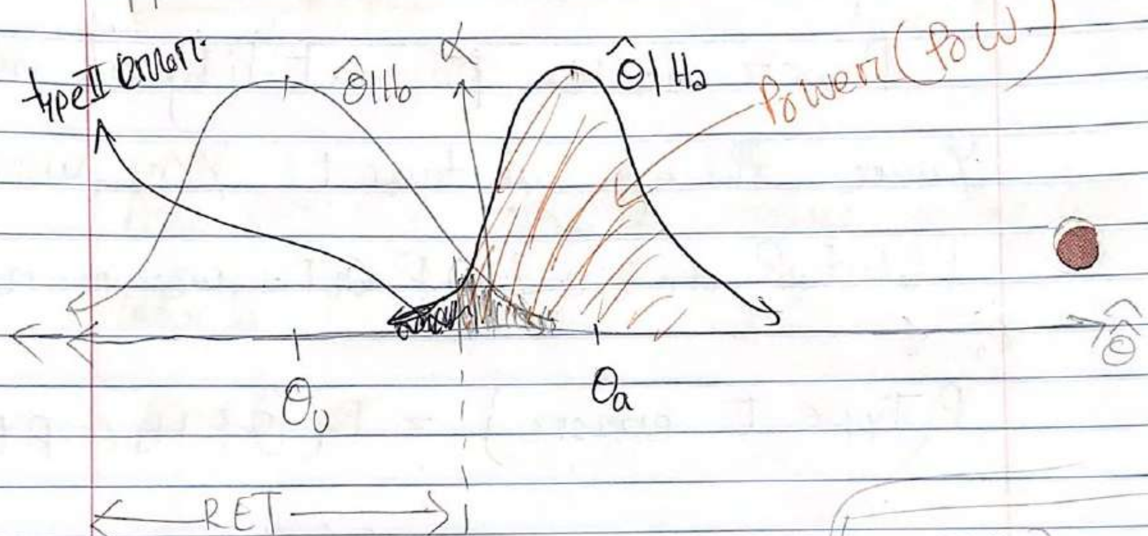
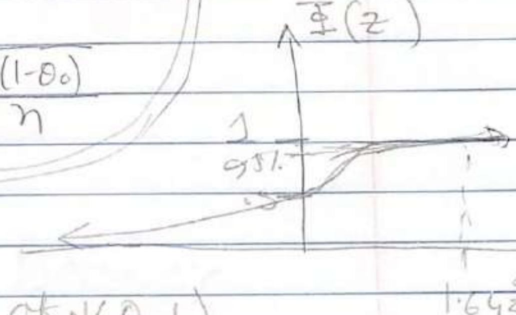
Lecture 5:

Let's Look at Power more generally (beyond two point hypotheses).

$H_0: \theta \le \theta_0$, $H_a: \theta > \theta_0$ right-tailed test

under iid Bern(theta) and the normal approximation:





CDF of N(0,1)

Let $\Phi(z) := F_Z(z)$

$\Phi(1-z) = 1-\alpha$

$\alpha = 5\% \Rightarrow z_{1-\alpha} = 1.645$

$$POW = P\left(\hat{\theta} \mid H_a > \theta_0 + Z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}} \right)$$

$$= P\left(\frac{\hat{\theta} \mid H_a - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}} > \frac{\theta_0 + Z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}} - \theta_a}{\sqrt{\frac{\theta_a(1-\theta_a)}{n}}} \right)$$

$$= P\left(Z > \frac{-\sqrt{n}\,(\theta_a - \theta_0) + Z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}} \right)$$

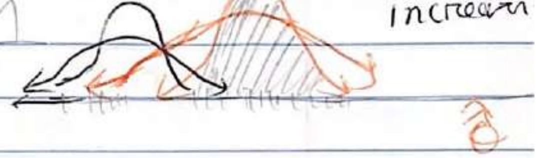
$$= 1 - \Phi\left(\frac{-\sqrt{n}\,(\theta_a - \theta_0) + Z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}} \right)$$

Power function

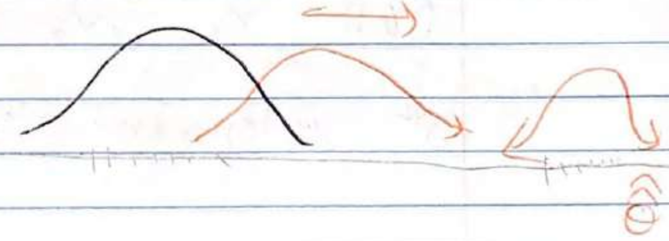
$$= Pow(\theta_a, \theta_0, n, \alpha)$$

Observations about the power function

if $n \to \infty \Rightarrow POW \to 1$

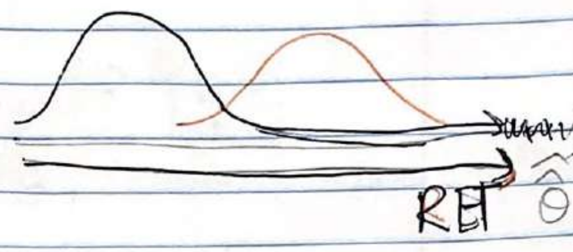


If $\theta_a \to \infty \Rightarrow Pow \to 1$



| Pow 100% win |

As $\alpha \to 0 \Rightarrow POW \to 0$



New Topic

New type of Survey. We ask "how tall are you (in inches)?" for men only. I'll ask 10 male students and get $x_1, \dots x_{10}$ (i.e my data). The data is now continous (no longer 0's and 1's). Height for a gender is known to be normally distributed

DGP: $x_1, \dots, x_n \overset{iid}{\sim} N(\theta, \sigma^2)$. Assume σ^2 is known $\sigma = 4^2$

How can we estimate θ? θ is the mean of the rv's. And recall

$\hat{\theta} = \bar{X}$ is unbiased. Let's use this estimator.

(X in inches) $\vec{x} = \langle 70, 72, 73, 68, 69, 70, 67, 72, 71, 73. \rangle$

$\hat{\theta} = \bar{X} = 70.5$

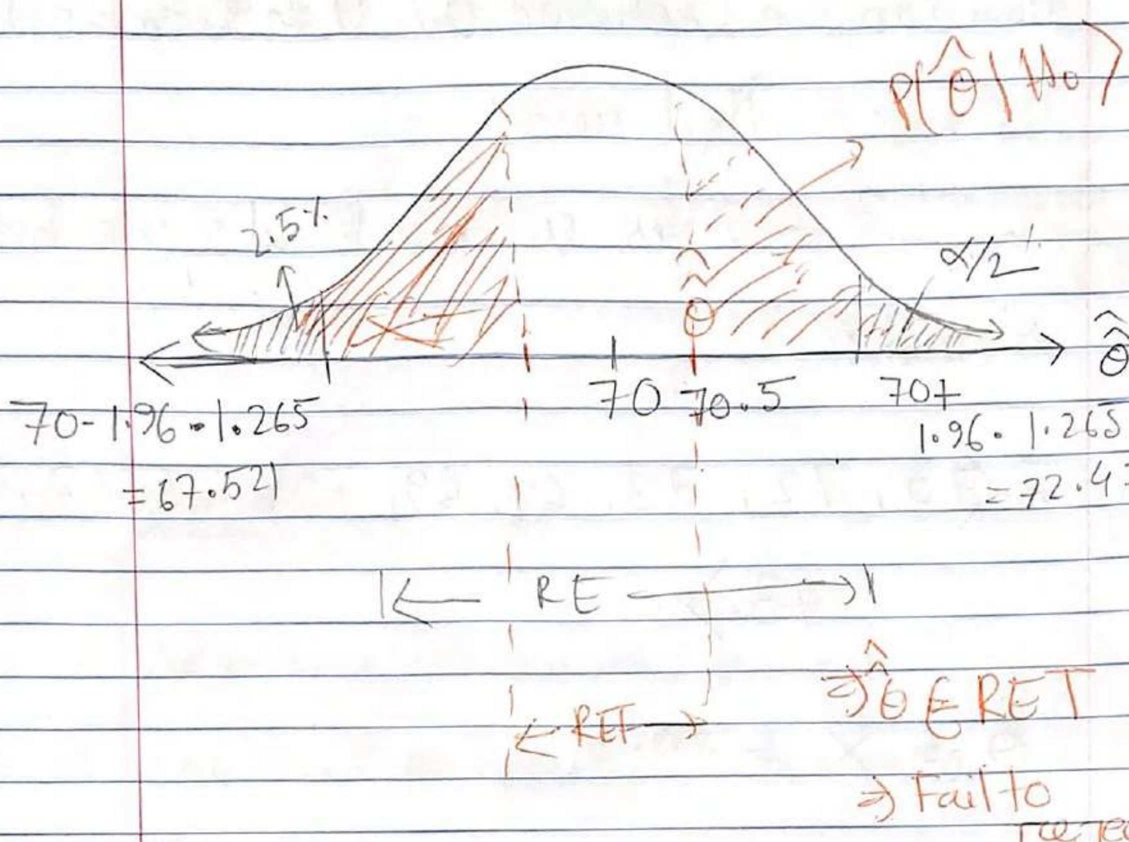
The american mean male adult height is $70''$

Let's test if the mean of the population where the class is drawn from is different than $70''$ (one sample z-test)

$N(\theta, \sigma^2)$

Ha: $\theta \neq 70$, Ho: $\theta = 70.$, $\alpha = 5\%$

$\hat{\theta} \mid H_0 \sim N\left(70, \frac{4^2}{10}\right) = N(70, 1.265^2)$



$P_{val} \approx P$(estimate is more extreme than observe $| H_0$)

$$= P\left(|\hat{\theta} | H_0| > |\hat{\theta} - \theta| \right)$$

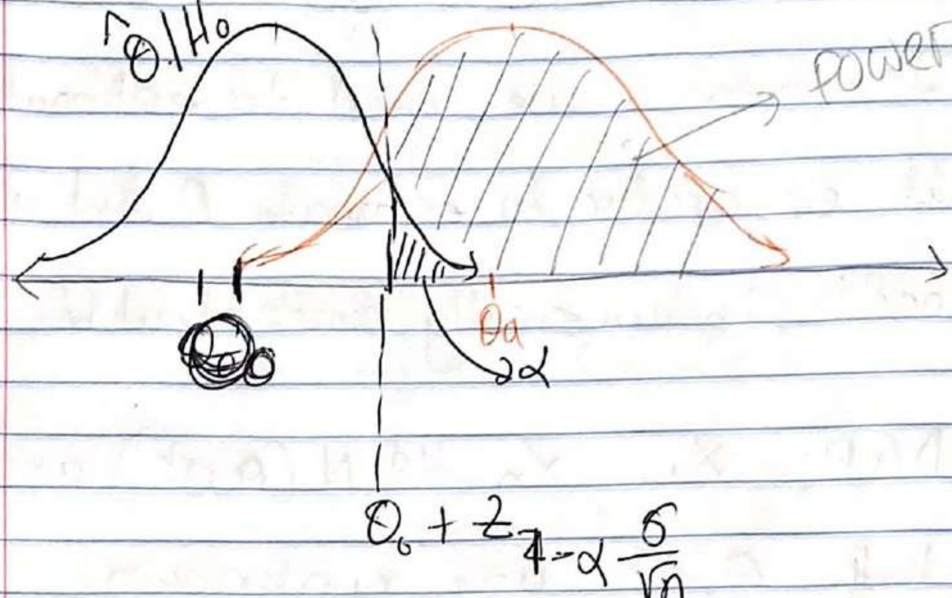
$$= 2P\left(\hat{\theta} \mid H_0 > 70.5 \right)$$

$$= 2P\left(Z > \frac{\overbrace{70.5 - 70}^{\text{mean}}}{\underbrace{1.265}_{\text{SD}}} \right)$$

$= 69.3\% \not< \alpha$ Stastically insifigicant

Power:

$H_0: \theta \le \theta_0, \quad H_a: \theta_a > \theta_0, \quad \text{Size } \alpha$



$$Pow = P\left(\hat{\theta}|H_a > \theta_0 + Z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$= P\left(\frac{\hat{\theta}|H_a - \theta_a}{\sigma/\sqrt{n}} > \frac{\theta_0 + Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \theta_0}{\sigma/\sqrt{n}}\right)$$

$$= 1 - \Phi\left(-\frac{\sqrt{n}}{\sigma}(\theta_a - \theta_0) + Z_{1-\alpha}\right)$$

$$= Pow(\theta_a, \theta_0, n, \alpha, \sigma).$$

σ gets smaller
Power gets higher

More realistic: We don't know.

But σ^2 is a "nuisance parameter". It means we need to estimate it in order to estimate θ but we don't intrinsically care about it.

DGP: $X_1, \dots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ and both θ, σ^2 are unknown

How do we estimate σ^2?

Recall for a rv X

$\sigma^2 := E\left[(X-\theta)^2\right]$ $\quad \theta = E[X]$,

$\hat{\theta} = \frac{1}{n} \Sigma X_i$

$\hat{\sigma}^2 = \frac{1}{n} \Sigma (X_i - \theta)^2$ Problem: I need to know θ!

$\hat{\sigma}^2 = \frac{1}{n} \Sigma (X_i - \bar{X})$ Seems like a reasonable estimator!

is this estimator unbiased? For any iid DGP

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum(X_i - \bar{X})^2\right]$$

$$= \frac{1}{n}\sum E(X_i - \bar{X})^2$$

$$\overset{iid}{\uparrow}{=} \frac{1}{\cancel{n}}\cancel{n}E\left[(X_1 - \bar{X})^2\right]$$

$$= E\left[X_1^2 - 2X_1\bar{X} + \bar{X}^2\right]$$

$$= E[X_1]^2 - 2E\left[X_1 \cdot \frac{X_1 + \dots + X_n}{n}\right]$$

$$+ E\left[\bar{X}^2\right]$$

$$\boxed{\begin{array}{l}\text{RECall} \\ Var[X] = E[X^2] - E[X]^2\end{array}}$$

$$= \overbrace{\sigma^2 + \theta^2} - \frac{2}{n}\left(E\left[X_1^2 + X_1X_2 + \dots + X_1X_n\right]\right.$$

$$+ \overbrace{\frac{\sigma^2}{n} + \theta^2}$$

$$= \frac{n+1}{n}\sigma^2 + 2\theta^2 - \frac{2}{n}(\sigma^2 + \overbrace{\theta^2 + \theta^2 + \dots \theta^2}^{n})$$

$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2 \Rightarrow \text{it's a little bit biased...}$$

(towards 0)

However, it is "asymptotically unbiased" meaning...

$$\lim_{n \to \infty} E[\hat{\theta}] = \theta$$

e.g. $\lim E[\hat{\sigma}^2] = \lim \frac{n-1}{n}\sigma^2 = \sigma^2$

Consider the following estimator:

$$s^2 = \frac{n}{n-1}\hat{\sigma}^2$$

$$= \frac{n}{n-1} \cdot \frac{1}{n} \Sigma (x_i - \bar{x})^2$$

$$= \frac{1}{n-1} (\Sigma x_i - \bar{x})^2$$

The beauty of this estimator is that

$$E[s^2] = E\left[\frac{n}{n-1}\hat{\sigma}^2\right]$$

$$= \frac{n}{n-1} E[\hat{\sigma}^2]$$

$$= \frac{\cancel{n}}{\cancel{n-1}} \cdot \frac{\cancel{n-1}}{\cancel{n}} \sigma^2$$

i.e unbiased.

And it's the default estimator for σ^2 (variances in DGP's) and it's really important in more