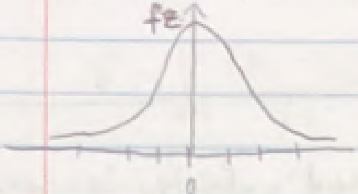
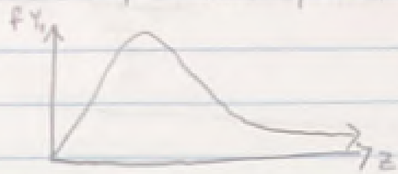
Lecture 15 10/28/20

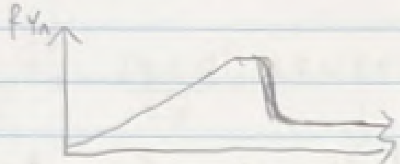
We need a couple of facts from Math 368 from distribution theory:

$Z \sim N(0,1) \Rightarrow Y_1 := Z \sim \chi_1^2$ chi-squared distribution with 1 degree of freedom (df)



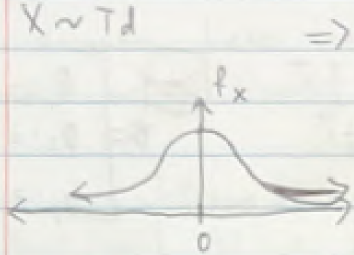


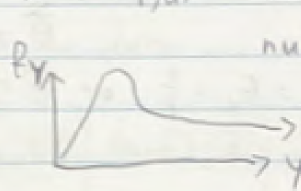
$Z_1, \dots, Z_n \overset{iid}{\sim} N(0,1) \Rightarrow Y_n := Z_1^2 + \dots + Z_n^2 \sim \chi_n^2$



* more n, more graph kicks out.

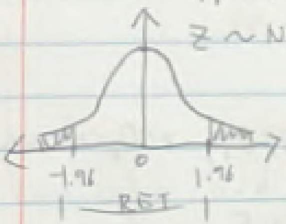
$X \sim T_d \Rightarrow Y = X^2 \sim F_{1,df}$ Fisher-Snecedor distribution with 1 numerator df and d denominator df.

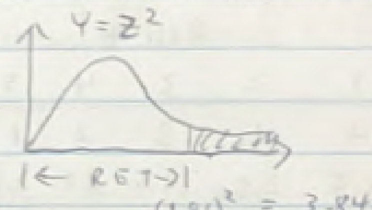




This means that every 2-sided z-test (exact or approx.) is equivalent to a chi-squared-1 test (exact or approx.) and every 2-sided t-test (exact or approx) is equivalent to an F test (exact or approx.).

$Z \sim N(0,1)$, $\alpha = 5\%$ $\Rightarrow$ $Y = Z^2$





$(1.96)^2 = 3.84$

DGP is iid Normal mean θ, variance σ^2, σ^2 known and the estimator is the sample average and you're testing $H_a: \theta \neq \theta_0$

$$\frac{\hat{\theta} - \theta_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \Rightarrow \frac{(\hat{\theta} - \theta_0)^2}{\frac{\sigma^2}{n}} \sim \chi_1^2$$

DGP is iid Bern (θ), and same as above,

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}} \overset{\cdot}{\sim} N(0,1) \Rightarrow \frac{(\hat{\theta} - \theta_0)^2}{\frac{\theta_0(1-\theta_0)}{n}} \overset{\cdot}{\sim} \chi_1^2$$

DGP is iid normal with both θ and σ^2 unknown, and same as above,

$$\frac{\hat{\theta}-\theta}{\frac{S}{\sqrt{n}}} \sim T_{n-1} \Rightarrow \frac{(\hat{\theta}-\theta_0)^2}{\frac{S^2}{n}} \dot{\sim} F_{1,n-1}$$

Let's say you want to prove a coin is weighted unfairly. So you assume it's flips have the DGP iid Bern(θ), and you test $H_a: \theta$ is not $\frac{1}{2}$.

$$\hat{\hat{\theta}}_{std} = \frac{\hat{\theta}-\frac{1}{2}}{\frac{\frac{1}{2}}{\sqrt{n}}} \overset{?}{\in} [-1.96, 1.96]$$

Let's say you want to prove a 6-sided die is unfair. At least one θ is not $1/6$.

H_a: die is unfair $\quad \exists j \;\; \theta_j \neq \frac{1}{6} \qquad \vec{\theta} \neq \vec{\theta}_0 = \frac{1}{6}\vec{1}$

H_0: die is fair $\quad \theta_1 = \theta_2 = \dots = \theta_6 = \frac{1}{6}$ or $\vec{\theta} = \vec{\theta}_0 = \frac{1}{6}\vec{1}$

$$\vec{\theta} = \begin{bmatrix} \theta_1 := P(\text{die}=1) \\ \theta_2 := P(\text{die}=2) \\ \vdots \\ \theta_6 := P(\text{die}=6) \end{bmatrix}$$

Given n rolls of the die $x_1, \dots, x_n$, how do we do our test? We need some way to measure/gauge departure from H_0 (a statistic or a set of statistics). Let's look at a frequency table e.g.

Roll #

	1	2	3	4	5	6	n
observed Quantity	4	1	3	2	1	4	n = 15
expected Quantity	2.5	2.5	2.5	2.5	2.5	2.5	n = 15

$O_1, O_2, \dots O_6$ r.v.'s

$E_1, E_2, \dots E_6$ constants

$\hat{\hat{\Phi}} = (O_1 - E_1) + (O_2 - E_2) + \dots + (O_6 - E_6)$ if $\hat{\hat{\Phi}}$ large $\Rightarrow$ Reject H_0

maybe...

$\hat{\hat{\Phi}} = |O_1 - E_1| + \dots + |O_6 - E_6|$ this is a good estimator for departure from the null hypothesis...

but we don't know its sampling distibution making it unusable in practice

$$\hat{\Phi} = \frac{(O_1-E_1)^2}{E_1} + \dots \frac{(O_6-E_6)^2}{E_6} \xrightarrow{d} \chi^2_5$$

this fact is proved in Math 368 if we had more time

Karl Pearsen (1900) and it's named the "χ^2 goodness of fit test" In general, if there are K categories (e.g. here $K=6$), then the following:

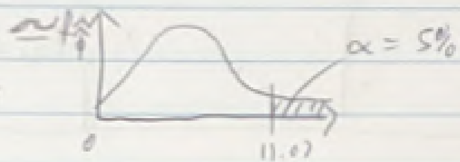
$$\hat{\phi} = \sum_{k=1}^{K} \frac{(O_K - E_K)^2}{E_K} \xrightarrow{d} \chi^2_{K-1}$$

Let's run over "die unfair test" for the data above at alpha = 5%.

$F_{\chi^2_5}(11.07) = 95\%$

$$\hat{\hat{\phi}} = \frac{(4-2.5)^2}{2.5} + \dots \frac{(4-2.5)^2}{2.5} = 3.8 \in RET$$

Retain H_0



New situation. Let's look at data for $n=279$ men and record their hair color and eye color. Here's the raw data as a "contingency table" or "cross tabulation."

		Eye Color				
		Brown (EB)	Blue (EL)	Hazel (EZ)	Green (EG)	Total
Hair Color	Black (HB)	$32 = O_{11}$	11	10	3	$56 = n_{HB} = n_{1\cdot}$
	Brown (HO)	$53 = O_{21}$	50	27	$15 = O_{24}$	$143 = n_{HO} = n_{2\cdot}$
	Red (HR)	10	10	$7 = O_{33}$	7	$30 = n_{HR} = n_{3\cdot}$
	Blonde (HL)	3	30	5	8	$46 = n_{HL} = n_{4\cdot}$
		$98 = n_{EB} = n_{\cdot 1}$	$101 = n_{EL} = n_{\cdot 2}$	$47 = n_{EZ} = n_{\cdot 3}$	$33 = n_{EG} = n_{\cdot 4}$	$n = 279$

I want to test

H_a: hair color and eye color are dependent events

H_0: hair color and eye color are independent events

Let θ denote a true population probability e.g.

$\theta_{HB,EB} = \theta_{1,1} = P$ (Black hair and brown eyes),

$\theta_{HB} = \theta_{1,\cdot} = P$ (Black hair)

H_a: $\exists j,k$ such that $\theta_{jk} \neq \theta_{j\cdot}\theta_{\cdot k}$ i.e. at least one is unequal

H_0: $\theta_{1,1} = \theta_{1\cdot}\theta_{\cdot 1}$, $\theta_{1,2} = \theta_{1\cdot}\theta_{\cdot 2}$, $\dots$ $\theta_{4,4} = \theta_{4\cdot}\theta_{\cdot 4}$

H_0 is $r \times c = 4 \times 4 = 16$ equalities.

We need a statistic to gauge the departure from H_0. Let's follow the reasoning from the previous example. We first looked at the data we expect if H_0 was true.

Hair Color \ Eye Color	1	2	3	4	Total
1	$E_{11} = n\theta_{1\cdot}\theta_{\cdot 1}$	$E_{12} = n\theta_{1\cdot}\theta_{\cdot 2}$	...	...	
2	...	...	...	...	
3	...	...	$E_{33} = n\theta_{3\cdot}\theta_{\cdot 3}$	...	
4	...	...	...	...	
Tot					

$$\hat{\Phi} = \frac{(O_{11} - E_{11})^2}{E_{11}} + \dots \frac{(O_{44} - E_{44})^2}{E_{44}}$$

$$= \frac{(O_{11} - n\theta_{1\cdot}\theta_{\cdot 1})^2}{n\theta_{1\cdot}\theta_{\cdot 1}} + \dots + \frac{(O_{44} - n\theta_{4\cdot}\theta_{\cdot 4})^2}{n\theta_{4\cdot}\theta_{\cdot 4}}$$

Can we compute $\hat{\hat{\Phi}}$? No. You do not know any of the $\theta_{i\cdot}$ or any of the $\theta_{\cdot j}$

How about we [Rschondify and] replace the $\theta_{i\cdot}$'s and $\theta_{\cdot j}$'s with $\hat{\theta}_{i\cdot}$ and $\hat{\theta}_{\cdot j}$. Yes...