MATH 369/650 Fall 2020 Homework #4

Professor Adam Kapelner

Due by email 11:59PM Saturday, October 24, 2020

(this document last updated Tuesday 27^{th} October, 2020 at 4:20pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning the normal distribution. Then read about the class topics (e.g. CRLB, Fisher Information, asymptotically normal estimators, etc) in the two recommended textbooks and online.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for the 600-level class and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 7 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____________________

Problem 1

We will prove the CRLB and some other facts here.

(a) [easy] Prove the CRLB from scratch. Justify each step. List assumptions.

(b) [easy] State Thm 5.5.4 p233 C&B.

(c) [easy] Use Thm 5.5.4 to prove that if $X \xrightarrow{p} c$ then $\frac{X}{c} \xrightarrow{p} 1$ and $\frac{c}{X} \xrightarrow{p} 1$.

(d) [easy] State Slutsky's Theorem.

(e) [difficult] Assume an iid DGP with mean θ and variance σ^2. Recall the CLT,

$$\frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} \mathcal{N}\left(0,\, 1\right).$$

Let S^2 be the estimator for σ^2 we discussed in class. In a more advanced probability class, you can show that $S \xrightarrow{p} \sigma$. Use Thm 5.5.4 and Slutsky's Theorem to show that the usual t statistic is asymptotically normal i.e. prove that

$$\frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \xrightarrow{d} \mathcal{N}\left(0,\, 1\right)$$

Hint: it is two steps. Don't overthink it!

(f) [easy] Prove that the MLE is asymptotically normal and asymptotically efficient from scratch. Justify each step.

(g) [difficult] Prove that the score function over n is asymptotically normal using the CLT. The facts should be found in the previous problem which you got from the notes.

(h) [difficult] Prove that Fisher Information which is defined as $I(\theta) := \mathbb{E}\left[\ell'(\theta; X)^2\right]$, the expected score squared, is equal to $\mathbb{E}\left[-\ell''(\theta; X)\right]$. If you make any assumptions proving this, indicate it so. This is not easy. There will be lots of hints on slack given.

Problem 2

We will get practice finding MLE's, their asymptotic distributions and proving estimators are UMVUE's.

(a) [easy] Consider the DGP $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Bern}\,(\theta)$. Find $\hat{\theta}^{\text{MLE}}$.

(b) [easy] Prove that the MLE is the UMVUE for θ.

(c) [easy] Write the approximate distribution of the MLE.

(d) [harder] Consider the DGP $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(0,\, \theta)$. Find $\hat{\theta}^{\text{MLE}}$.

(e) [harder] Prove that the MLE is the UMVUE for θ. You need to use the fact that $\mathbb{E}\left[X^4\right] = 3\theta^2$ for this DGP, i.e. when $X \sim \mathcal{N}(0,\, \theta)$.

(f) [easy] Write the approximate distribution of the MLE.

(g) [harder] Consider the DGP $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta) := \theta^x e^{-\theta}/x!$ where $\mathbb{E}[X] = \theta$ and $\mathbb{V}\text{ar}[X] = \theta$. Find $\hat{\theta}^{\text{MLE}}$.

(h) [easy] Prove that the MLE is the UMVUE for θ.

(i) [easy] Write the approximate distribution of the MLE.

(j) [difficult] Consider the DGP $X_1, \ldots, X_n \overset{iid}{\sim} \text{Log}\mathcal{N}(\theta,\, 1) := \frac{1}{x\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(x)-\theta)^2}$ and find $\hat{\theta}^{\text{MLE}}$.

(k) [harder] Write the approximate distribution of the MLE.

(l) [difficult] Consider the DGP $X_1, \ldots, X_n \overset{iid}{\sim} \text{Gumbel}(\theta, 1) := e^{-(x-\theta)-e^{-(x-\theta)}}$ and find $\hat{\theta}^{\text{MLE}}$.

(m) [difficult] Write the approximate distribution of the MLE. You will need the following fact from Math 368 to find the answer: $\mathbb{E}\left[e^{-X}\right] = e^{-\theta}$.

(n) [difficult] Consider the DGP $X_1, \ldots, X_n \overset{iid}{\sim} \text{Logistic}(\theta, 1) := e^{-(x-\theta)}/(1+e^{-(x-\theta)})$. Find an expression for when solved will find $\hat{\theta}^{\text{MLE}}$. There is no closed form expression that exists.

(o) [harder] In a few sentences, explain why it would be difficult to provide the approximate distribution of the MLE.