Lecture - 06 09/16/2020

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

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$\hat{\theta} = \bar{X} \sim N(\theta, (\sigma/\sqrt{n})^2) \iff \frac{\bar{X}-\theta}{\sigma/\sqrt{n}} \sim N(0,1)$ (exactly)

$SE[\bar{X}]$

standardize the estimator

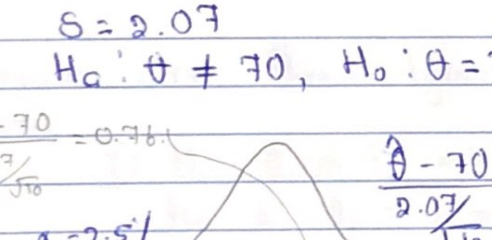
What if sigma is unknown?

S^2 estimates $\sigma^2 \Rightarrow S = \sqrt{S^2}$ estimates σ

Does $\frac{\bar{X}-\theta}{S/\sqrt{n}} \overset{?}{\sim} N(0,1)$ No! But close!

In 1907 Gosset proved:

$\frac{\bar{X}-\theta}{S/\sqrt{n}} \sim T_{n-1}$ (exactly) Student's standard T distribution with n-1 "degrees of freedom" (the parameter for the standard T distr.)



Similar to normal but with thicker tails $T_{low\ df < 2}$

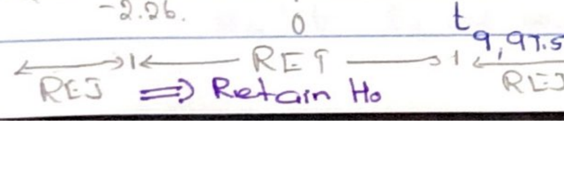
data from n = 10 male student heights: $\bar{x} = 70.5$, $s = 2.07$

$H_a: \theta \neq 70$, $H_0: \theta = 70$, $\alpha = 5\%$

$\frac{\hat{\theta}-70}{s/\sqrt{n}} = \frac{70.5-70}{2.07/\sqrt{10}} = 0.76$

the standardized distri. of $\hat{\theta} \mid H_0$

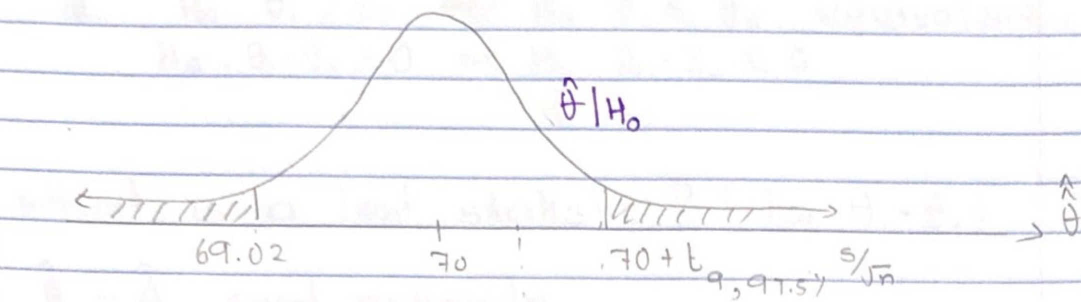
$\frac{\hat{\theta}-70}{2.07/\sqrt{10}} \sim T_9$



$\hat{\theta} - 70 / s/\sqrt{n}$

$= 2.26 > 1.96 = z_{97.5\%}$

$\Rightarrow$ Retain H_0

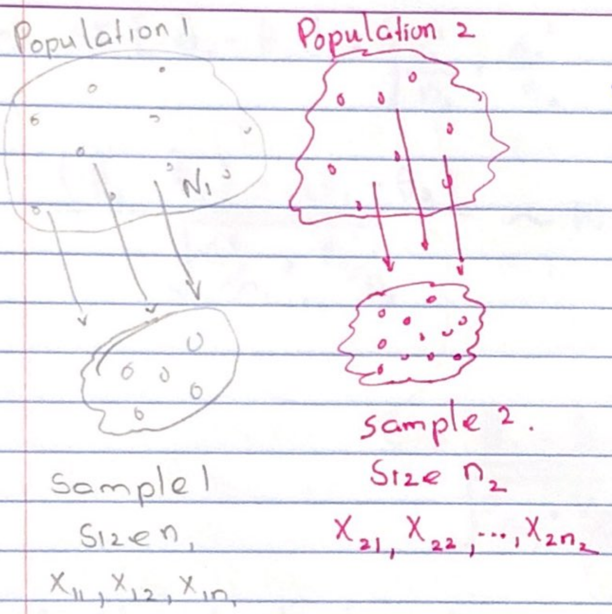


$= 70 + 2.26 \frac{2.07}{\sqrt{10}} = 71.98$

REJ — RET — REJ

$\Longrightarrow$ Retain H_0.

We just did our first "one-sample two-sided t test" (of a mean)



$N_1 \approx \infty$, $N_2 \approx \infty \Rightarrow$ iid.

Assume
$X_{11}, \dots, X_{1n_1} \overset{iid}{\sim} N(\theta_1, \sigma_1^2)$
independent of.

$X_{21}, \dots, X_{2n_2} \overset{iid}{\sim} N(\theta_2, \sigma_2^2)$

σ_1^2, σ_2^2 are known but θ_1, θ_2 are unknown.

There are three types of tests that are usually done.

(I) $H_a: \theta_1 \neq \theta_2 \Rightarrow H_0: \theta_1 = \theta_2$ equivalently ...
$H_a: \theta_1 - \theta_2 \neq 0 \Rightarrow H_0: \theta_1 - \theta_2 = 0$

(II) $H_a: \theta_1 < \theta_2 \Rightarrow H_0: \theta_1 \geq \theta_2$ equivalently ...
$H_a: \theta_1 - \theta_2 < 0 \Rightarrow H_0: \theta_1 - \theta_2 \geq 0$

III $H_a: \theta_1 > \theta_2 \Rightarrow H_0: \theta_1 \leq \theta_2$ equivalently...

$H_a: \theta_1 - \theta_2 > 0 \Rightarrow H_0: \theta_1 - \theta_2 \leq 0$

What is a test statistic? (For $\theta_1 - \theta_2$)

$\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2$ point estimate

The estimator that produces these estimates is simply:

$\hat{\theta}_1 \sim N(\theta_1, \sigma_1^2/n_1)$ indep.
$\hat{\theta}_2 \sim N(\theta_2, \sigma_2^2/n_2)$

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$$\hat{\theta}_1 - \hat{\theta}_2 = \bar{X}_1 - \bar{X}_2 \underset{\text{exactly}}{\sim} N(\theta_1 - \theta_2, \sigma^2/n_1 + \sigma^2/n_2)$$

$$\Rightarrow SE[\hat{\theta}_1 - \hat{\theta}_2] = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Under H_0 (all three), $\theta_1 - \theta_2 = 0$

$$\frac{(\hat{\theta}_1 - \hat{\theta}_2) - (\theta_1 - \theta_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \sim N(0,1)$$

$$\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \sigma_1^2/n_1 + \sigma_2^2/n_2\right)$$

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \sim N(0,1)$$

Let's test if male mean height is different than female mean height.

$\vec{x} = \langle 60, 59, 64, 64, 64, 63 \rangle$ $n_2 = 6$ $\bar{X}_2 = 62.3$

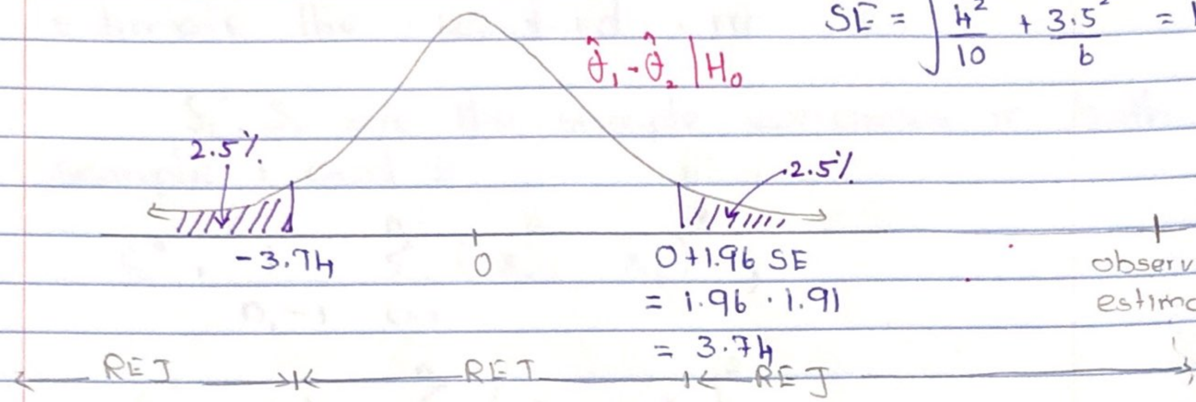
$n_1 = 10$, $\bar{X}_1 = 70.5$

$\hat{\theta}_1 - \hat{\theta}_2 = 70.5 - 62.3 = 8.2$

We assumed we knew the variances. So the variance for the men was assumed to be 4^2 and now the variance for the women is assumed to be 3.5^2. $\sigma_1^2 = 4^2$, $\sigma_2^2 = 3.5^2$; $\alpha = 5\%$

We can now do our 2-sample 2-sided z-test.

$$SE = \sqrt{\frac{4^2}{10} + \frac{3.5^2}{6}} = 1.91$$



observed estimate $\hat{\theta}_1 - \hat{\theta}_2 \notin$ RET $\Longrightarrow$ Reject H_0

$$P_{val} = 2P(\hat{\theta}_1 - \hat{\theta}_2 > 8.2) = 2P\left(\frac{\hat{\theta}_1 - \hat{\theta}_2}{SE} > \frac{8.2}{1.91}\right) = 2P(\underset{\text{standard normal}}{Z} > 4.29)$$

$$= 1.8 \times 10^{-5} < \alpha$$

= Reject.

Let's sample from two populations again however, this time we have the same variance, σ^2 which we still assume known. $(\sigma^2 = \sigma_1^2 = \sigma_2^2)$

$$X_{11}, \ldots, X_{1n_1} \overset{iid}{\sim} N(\theta_1, \sigma^2) \text{ indep. of } X_{21}, \ldots, X_{2n_2} \overset{iid}{\sim} N(\theta_2, \sigma^2)$$

Under H_0,
$$\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \sqrt{\sigma^2(1/n_1 + 1/n_2)}^{\,2}\right)$$

also,
$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sigma\sqrt{1/n_1 + 1/n_2}} \sim N(0,1)$$

The test can be run again, you can probably assume $\sigma = 3.75$.

Same as above but σ unknown. How can we estimate the standard error?

S_1^2, S_2^2 are the sample variances in both sample 1 and 2.

$$S_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_{1,i} - \bar{X}_1)^2,$$

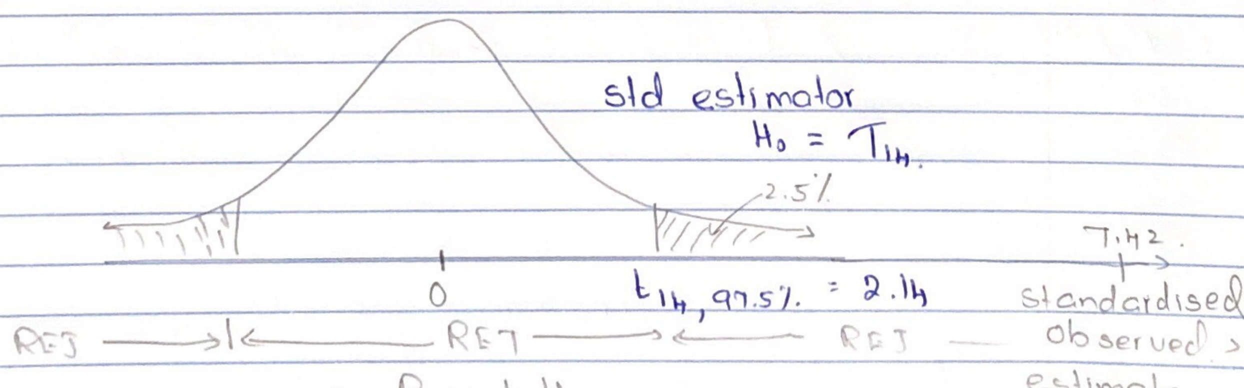
$$S_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (X_{2,i} - \bar{X}_2)^2.$$

$$S_{pooled}^2 = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}.$$ weighted average.

You can prove that

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{S_{pooled}\sqrt{1/n_1 + 1/n_2}} \sim T_{n_1 + n_2 - 2}$$

this allows you to do the "2-sample t test of equal variance"



$\Longrightarrow$ Reject H_0.

$$\hat{\theta} = \frac{70.5 - 62.3}{S_{pooled}\sqrt{1/10 + 1/6}} = \frac{8.2}{2.14 \cdot 0.51} = 7.42$$

$$S_{pooled} = \frac{9 \cdot 2.07^2 + 5 \cdot 2.25^2}{14}$$

$$= 4.56$$