Lecture 4

I don't think I'll give you test on this

"Level of a test" alpha is defined as

P(Type I error) >= alpha

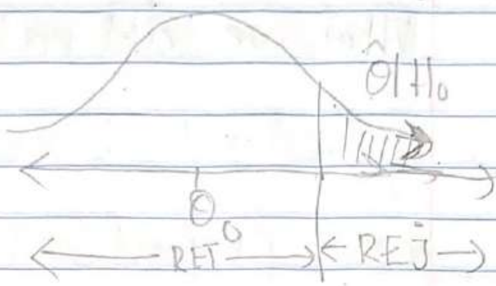
"Size of a test" is exactly P(Type I error).

In our example the level was 5% but
the size was 7.06%. Since
$\alpha = 5\%$ was "unattainable"

if $\hat{\theta} \mid H_0$ is continous, then level = Size = alpha. If It's discrete, some sizes won't be attainable.

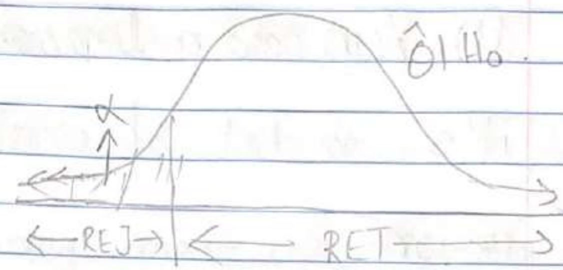
If I want a level of alph = 5% and the size is lower, then I am "Cheating" (we'll see why next class) it's a right tail so rejection on right

Ha: $\theta > \theta_0$

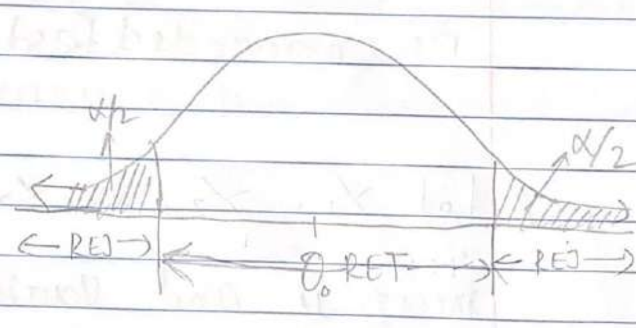



Left tail test:

Ha: $\theta < \theta_0$




Ha: $\theta \neq \theta_0$



What we did in the Previous lecture was called a "binomial exact test" of one proportion.

downsides:

(1) You need a binomial PMF calculator and it's a lot of work to get the rejection region.

(2) not all sizes are attainable. This is recommended test.

* Let $X_1, X_2, \ldots X_n$ iid some distribution mean $\mu$ and Variance $\sigma^2$. The central limit theorem (CLT) shows that:

convergence in distribution - means as n gets large, the CDF of the lhs looks more and more like the CDF of the rhs.

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

we will use this in... 

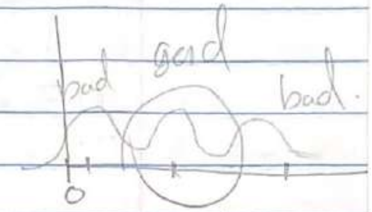
approx distr.

$$\Rightarrow \boxed{\bar{X} \overset{\cdot}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right)} \text{ and } T = X_1 + \ldots + X_n \overset{\cdot}{\sim} N(n\mu, n\sigma^2)$$

if $X_1, \dots, X_n \overset{iid}{\sim}$ Bern$(\theta)$ and $n$ is "large"

then:

$$\hat{\theta} = \bar{X} \overset{\cdot}{\sim} N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$




this is a pretty good approximation if theta is not too close to 0 or 1.

How to perform an "approximate test"?

There are many, many options even for the same DGP. The protocol goes as follows

(1) you think of a "test statistic" that could measure the departure away from $H_0$.

(2) Derive the statistical estimator's "approx" distribution under $H_0$, $\hat{\theta} \mid H_0$

(3) Gauge the departure of $\hat{\theta}$ from the bulk of the distribution $\hat{\theta} \mid H_0$ at level $\alpha$.

$H_0 : \theta = .524, \quad H_a : \theta \neq .524,$

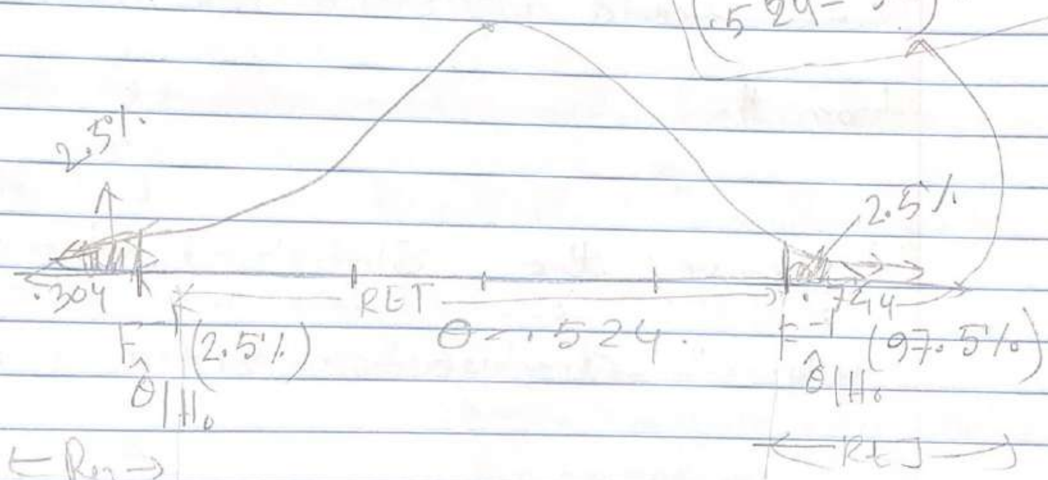
$n = 20, \quad \hat{\theta} = 0.6$ (same as last class)

$$\hat{\theta} \mid H_0 \sim N\left(.524, \frac{.524(1-.524)}{20}\right)$$

$$= N(.524, \underbrace{.112^2}_{\sigma}{}^{\sigma^2})$$

Set $\alpha = 5\%$ $\Rightarrow$ $\frac{\alpha}{2} = 2.5\%$

$\boxed{(.524 - .304) + .524}$




$P(\hat{\theta} \mid H_0 \leq \hat{\theta}) = 2.5\%$ Solve for $\hat{\theta}$.

$N(0,1) \rightarrow$ represnts an $Z$

$$\Rightarrow P\left(\frac{\hat{\theta} \mid H_0 - .524}{.112} \le \frac{\hat{\theta} - .524}{.112}\right) = 2.5\%$$

241 exercise

$$\Rightarrow P\left(Z \le \frac{\hat{\theta} - .524}{.112}\right) = 2.5\%$$

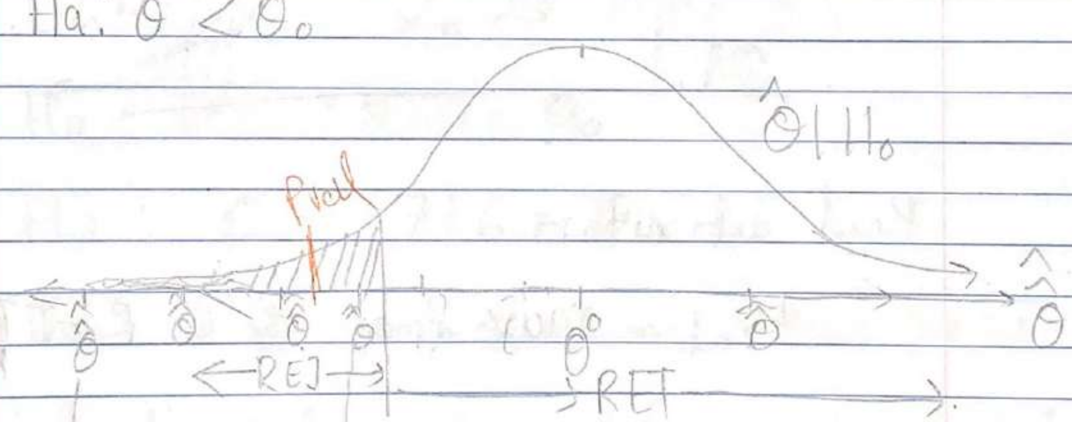
Look at table or use computer.

$$\Rightarrow \frac{\hat{\theta} - .524}{.112} = 1.96 = 2$$

$$\Rightarrow \hat{\theta} = .304$$

One Proportion Z-test (approximate test):

Ha: $\theta < \theta_0$

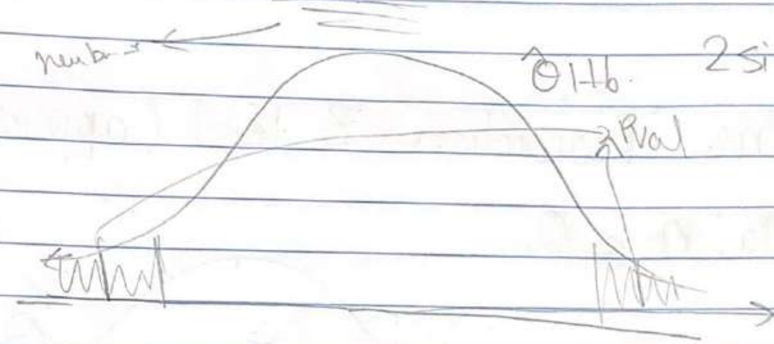



This estimate should imply a "stronger" rejection than this estimate.

To measure the "strength" of a rejection (or "weakness" of a retainment), Fisher introduced the "P-Value" also called the level of statistical significance as:

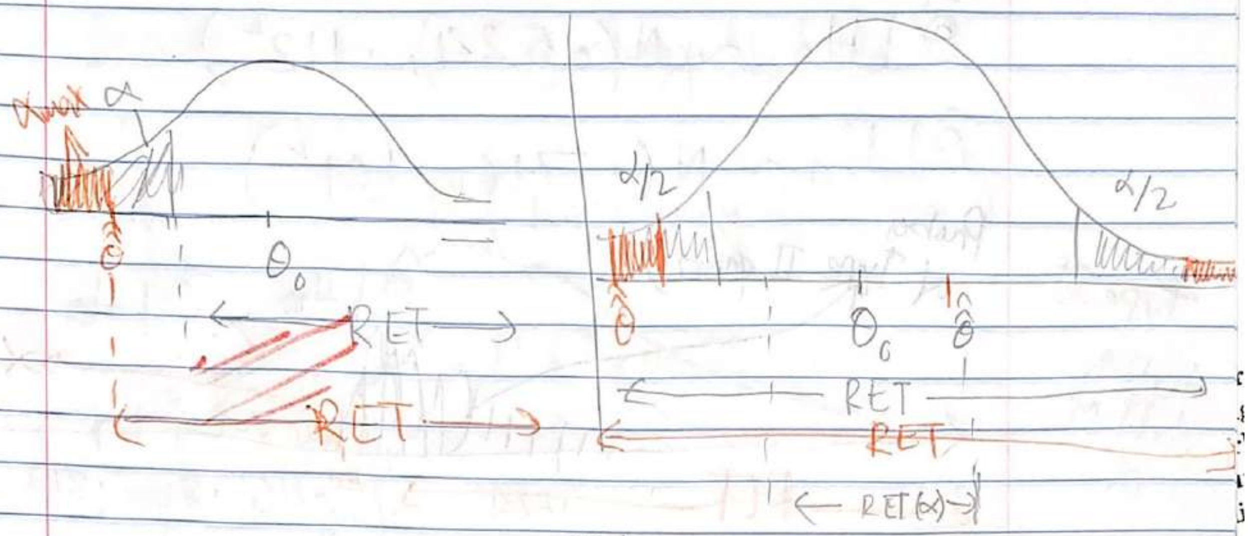
$P_{val} = P($estimate is more extreme than the one observed $| H_0)$

The smaller the P value the stronger rejection




Real definition.

$$P_{val} = \max\{\alpha : \hat{\theta} \in RET(\alpha)\}$$




if $H_0$ is retained $\overset{\text{that means}}{\Longrightarrow}$ $P_{val} \geq \alpha$

if $H_0$ is rejected $\Rightarrow$ $P_{val} < \alpha$.

Type II errors and POWER

DGP: $X_1, \ldots X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$H_0: \theta = .524 = \theta_0$

$H_a: \theta = .716 = \theta_a$

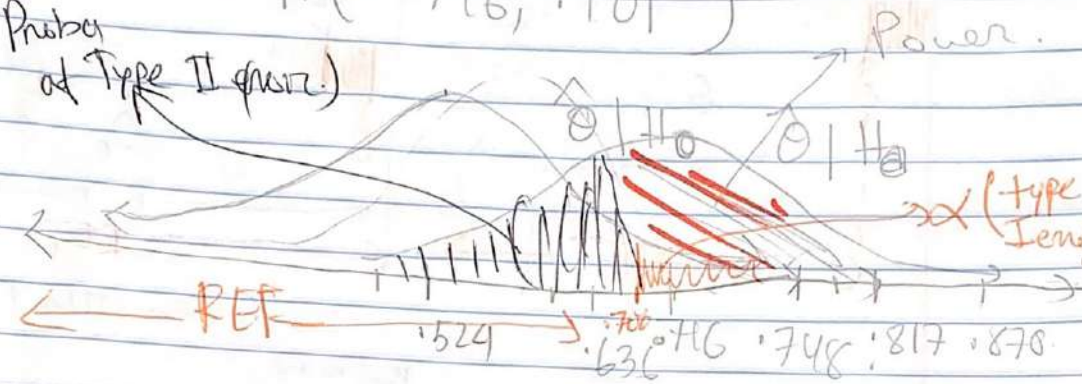
This is a non-standard setup since both $H_0$ and $H_a$ are "Point Hypothesis"

This makes the outcomes weird. either You retain theta = 0.524 or You accept $\theta = .716$. But ignore this for now

$\hat{\theta} \mid H_0 \sim N(.524, .112^2)$

$\hat{\theta} \mid H_a \sim N(.716, .101^2)$

Type II to the left of $\alpha$.




At $\alpha = 5\%$ the Z value is 1.645 which means the rejection region ends at

$$\hat{\theta} = .524 + 1.645 * .122$$
$$= .708$$

| | | Decision | |
|---|---|---|---|
| | | RET | REJ. |
| Truth | $H_0$ | | Type I |
| | $H_a$ | Type II | |

$POWER = P(\text{Rejecting } H_0 \mid H_a)$

$= 1 - P(\text{Retaining } H_0 \mid H_a)$

$= 1 - P(\text{Type II error})$

Power is the probability of proving your theory is true!! You want POWER to be LARGE i.e. near 100%.

$P(\text{Type II error}) = P(\hat{\theta} \mid H_a \in RET)$

$= P(\hat{\theta} \mid H_a \leq .708)$

$= P\left(\frac{\hat{\theta} \mid H_a - .716}{.101} \leq \frac{.708 - .716}{.101}\right)$

$= P(z \leq .079)$

$\approx 47\% \rightarrow$ POWER $\approx 53\%$