09/09/2020 Lec 04 Math 621

Let $X, Y \overset{iid}{\sim} \text{Geom}(p)$

$P(X>Y) = ?$

$P(X>Y) = P(Y>X)$

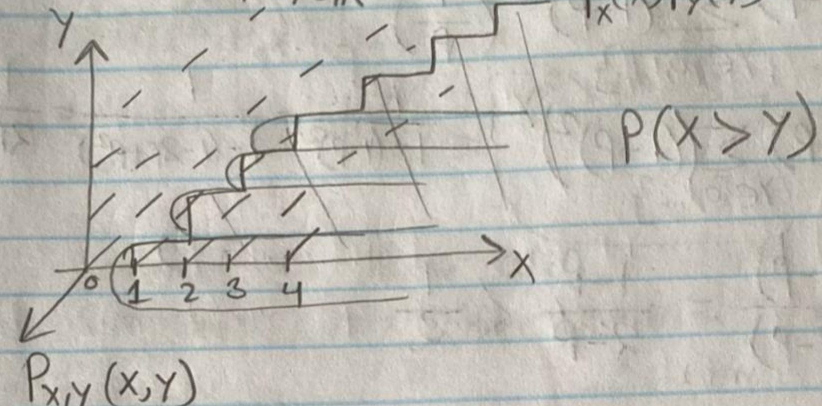
$P(X>Y) + P(Y>X) + P(X=Y) = 1$

$\Rightarrow 2P(X>Y) + P(X=Y) = 1$

$\Rightarrow P(X>Y) = \dfrac{1-P(X=Y)}{2} < \dfrac{1}{2}$ since $P(X=Y)$

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} \overbrace{P_{X,Y}(x,y)}^{\text{JMF}} \mathbb{1}_{x>y}$$

$P_{X,Y}(x,y) = P_X(x)P_Y(y)$



$$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} P_X(x) P_Y(y) \mathbb{1}_{x>y}$$

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$$= \sum_{y\in\mathbb{R}} P_Y(y) \sum_{x\in\mathbb{R}} P_X(x) \mathbb{1}_{x>y}$$

$$= \sum_{y\in\{0,1,\dots\}} p(1-p)^y \sum_{x\in\{0,1,\dots\}} p(1-p)^x \mathbb{1}_{x>y}$$

$x \geq y+1$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x\in\{y+1,y+2,\dots\}} (1-p)^x$$

let $x' = x-(y+1) \Rightarrow x' \in \{0,1,\dots\}$

$\Rightarrow x = x'+y+1$

$$= p^2 \sum_{y\in\{0,1,\dots\}} (1-p)^y \sum_{x'\in\{0,1,\dots\}} (1-p)^{x'}(1-p)^y(1-p)^1$$

$$= p^2(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y} \boxed{\sum_{x'\in\{0,1,\dots\}} (1-p)^{x'}}$$

$\leftarrow$ Geometric Series, $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$ $\quad a\in(-1,1)\setminus\{0\}$

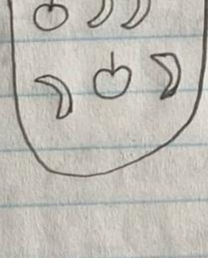
$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$= p^2(1-p) \sum_{y\in\{0,1,\dots\}} (1-p)^{2y}\left(\frac{1}{p}\right)$$

$$= p(1-p) \boxed{\sum_{y\in\{0,1,\dots\}} \left((1-p)^2\right)^y} = \frac{1}{1-(1-p)^2} = \frac{1}{1-(1-2p+p^2)} = \frac{1}{2p-p^2} = \frac{1}{p(2-p)}$$

$$= \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p} < \frac{1}{2}$$

Bag of Fruit of apples & bananas



Draw with replacement n times

Let $x = \#$ of apples, $p_1 = P(\text{apple})$

$\Rightarrow X_1 \sim \text{Bin}(n, p_1)$

Draw n with replacement

Let $X_1 = \#$ of apples, $X_2 = \#$ of banana

$X_1 \sim \text{Bin}(n, p_1)$, $X_2 \sim \text{Bin}(n, p_2)$

Are X_1 and X_2 independent?

Since $X_1 + X_2 = n \Rightarrow X_1, X_2$ dependent

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \vec{X} \sim p_{\vec{X}}(\vec{x}) = p_{\vec{X}}(x_1, x_2)$$

$$= \underbrace{\frac{n!}{x_1!\, x_2!}}\, p_1^{x_1} p_2^{x_2}\, \underbrace{\mathbb{1}_{x_1+x_2=n}}\, \underbrace{\mathbb{1}_{x_1 \in \{0,1,\ldots n\}}}\, \underbrace{\mathbb{1}_{x_2 \in \{0,1,\ldots n\}}}$$

$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ $\binom{n}{x_1, x_2}$ mutichoose notation

$$\Rightarrow \vec{X} \sim \text{Multi}(n, \vec{p}) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} \Leftarrow \text{Multinomial r.v. of dim} = 2$$

Since X_1, X_2 are dependent, we can't factor this JMF.

Bag of fruit now has cantaloupes. You draw cantaloupes with probability p_3 and X_3 is the count of canptaloupes.

$$\vec{X} \sim \text{Multi}(n, \vec{p}) = \binom{n}{x_1, x_2, x_3} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

$$= \frac{n!}{x_1!\, x_2!\, x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}\, \mathbb{1}_{x_1+x_2+x_3=n}\, \mathbb{1}_{x_1 \in \{0,\ldots n\}}\, \mathbb{1}_{x_2 \in \{0,\ldots n\}}\, \mathbb{1}_{x_3 \in \{0,\ldots n\}}$$

In general, if there are K types of fruits (# categories) then the general r.v. of dim K is:

$$\begin{bmatrix} p_1 \\ \vdots \\ k \end{bmatrix} \quad \vec{X} \sim \text{Multi}(n, \vec{p}) = \binom{n}{x_1, x_2, \ldots x_K} \prod_{k=1}^{K} p_k^{x_k}$$

Parameter space: $n \in \mathbb{N}$, $\vec{p} \in \{\vec{v} : \vec{v} \cdot \vec{1} = 1, v_1 \in (0,1) \ldots v_k \in (0,1)\}$

Support: $\text{Supp}[\vec{X}] = \{\vec{x} : \vec{x} \cdot \vec{1} = n,\ x_1 \in \{0,1,\ldots n\}, \ldots, x_K \in \{0,1,\ldots n\}\}$

$$\vec{X} \sim \text{Multi}\left(n, \overset{K=2}{\begin{bmatrix} p \\ 1-p \end{bmatrix}}\right) = \binom{n}{x_1, x_2} p^{x_1} (1-p)^{x_2}$$

$$\underset{\substack{\parallel \\ \text{Deg}(n-x_2)}}{P(X_1 = x_1 \mid X_2 = x_2)} \overset{?}{=} P(X_1 = x_1) = \overset{\text{Dependent?}}{\text{Bin}(n, p_1)} \Rightarrow \text{Conditional PMF}$$

$\text{Deg}(n - x_2) \Rightarrow$ Dependent!

$$P_{X_1|X_2}(x_1,x_2) := \frac{P_{X_1,X_2}(x_1,x_2)}{P_{X_2}(x_2)}$$

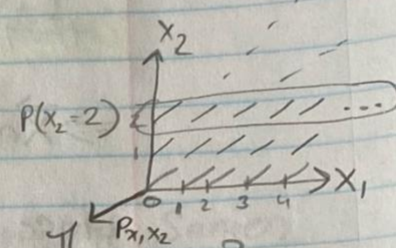
← JMF

← Marginal PMF of X_2

want to show $X_2 \sim \text{Bin}(n, P_2)$

$$P_{X_2}(x_2) = P(X_2 = x_2) = \sum_{x_1 \in \mathbb{R}} P_{X_1,X_2}(x_1,x_2)$$

"Margining out x_1"



$$= \sum_{x_1 \in \mathbb{R}} \binom{n}{x_1, x_2} p^{x_1}(1-p)^{x_2}$$

$$= \sum_{x_1 \in \mathbb{R}} \frac{n!}{x_1!\,x_2!} p^{x_1}(1-p)^{x_2}\, \mathbb{1}_{x_1+x_2=n}\, \mathbb{1}_{x_1 \in \{0,\dots n\}}\, \mathbb{1}_{x_2 \in \{0,\dots n\}}$$

$$= \frac{n!}{x_2!}(1-p)^{x_2}\, \mathbb{1}_{x_2 \in \{0,1,\dots n\}} \sum_{x_1 \in \{0,1,\dots n\}} \frac{p^{x_1}}{x_1!}\, \mathbb{1}_{x_1 = n - x_2}$$

$$= \boxed{\frac{n!}{x_2!}} (1-p)^{x_2}\, \mathbb{1}_{x_2 \in \{0,1,\dots n\}} \frac{p^{n-x_2}}{(n-x_2)!} = \binom{n}{x_2}$$

$$= \binom{n}{x_2} p^{n-x_2}(1-p)^{x_2} = \text{Bin}(n, 1-p)$$

Margining a multinomial to yield one dimension is binomial.