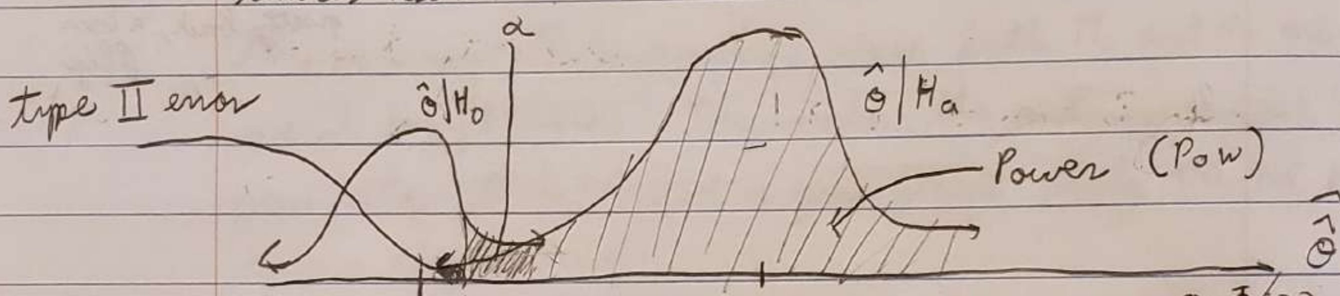
9/14/20

Let's look at power more generally (beyond two point hypotheses).

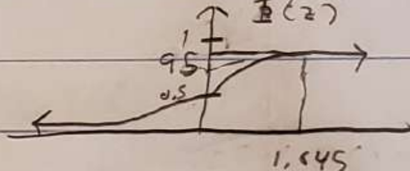
at size α

$H_0 : \theta \le \theta_0$, $H_a : \theta = \theta_1 > \theta_0$ right-tailed (test)

under iid Bern(theta) and the normal approximation



$\theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}$



CDF of N(0,1)

let $\Phi(z) : F_Z(z)$

$\Phi(z_{1-\alpha}) = 1-\alpha$

$\alpha = .5\% \Rightarrow z_{1-\alpha} = 1.645$

$$Pow = P\left(\hat\theta | H_a > \theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)$$

$$= P\left(\frac{\hat\theta|H_a - \theta_A}{\sqrt{\frac{\theta_A(1-\theta_A)}{n}}} > \frac{\theta_0 + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)} - \theta_a}{\sqrt{\frac{\theta_A(1-\theta_A)}{n}}}\right)$$

mult top/bot $\sqrt{n}$

z dist.

$$= P\left(Z > \frac{-\sqrt{n}(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}}\right)$$

realization

$= 1 - \Phi$

Power Function

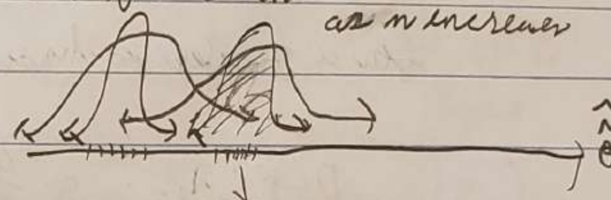
$$= 1 - \Phi\left(\frac{-\sqrt{n}\,(\theta_a - \theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_a(1-\theta_a)}} \right) = Pow(\theta_a, \theta_a, n, \alpha)$$

Observations about the power function:

trying to get lower standard error

If $n \to \infty \Rightarrow Pow \to 1$

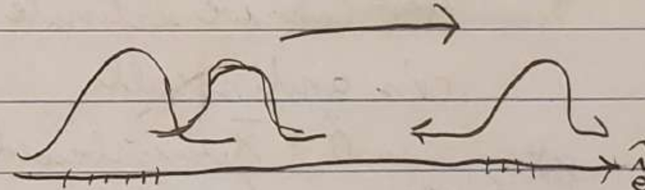
as n increases



If $\theta_a \to \infty \Rightarrow Pow \to 1$

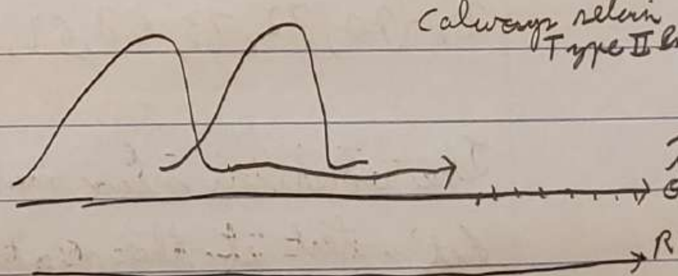
(If Ha is true)

(no overlap)



As $\alpha \to 0 \Rightarrow Pow \to 1$

(always retain Type II Error)



New type of survey. We ask "How tall are you (in inches)?" for men only. I'll ask 10 male students and get $x_1, \dots, x_{\{10\}}$ (i.e. my data) The data is now continuous (no longer zeroes and ones). Heights for a gender is known to be normally distributed

DGP: $x_1, \dots, x_n \overset{iid}{\sim} N(\theta, \sigma^2)$. Assume σ^2 is known and 4^2

How can we estimate theta? Theta is the mean of the rv's and recall

$\hat{\theta} = \bar{X}$ is unbiased, Let's use this estimator

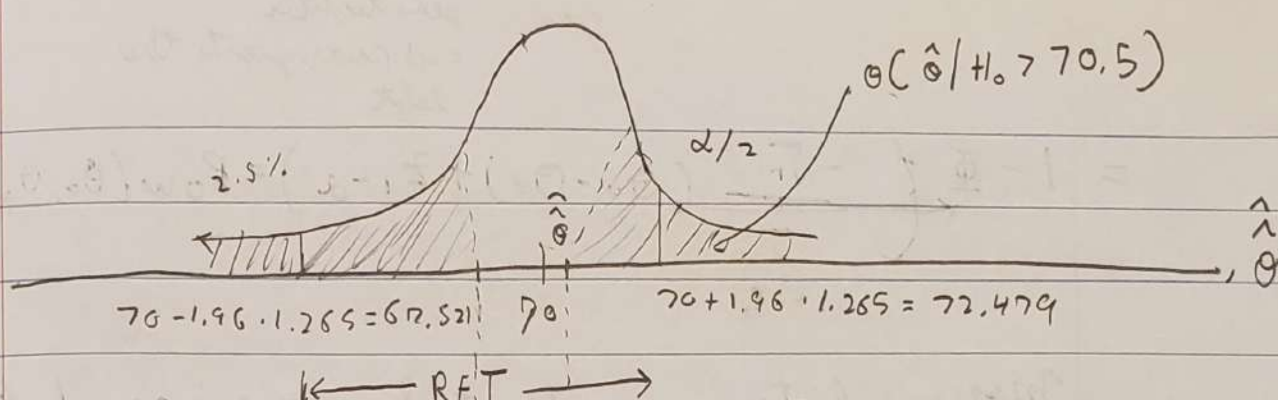
test ← <u>one sample z-test</u>

$\bar{x}_1 = (70, 72, 73, 68, 69, 70, 67, 72, 71, 73)$ $\hat{\theta} = \bar{x} = 70.5$

The american mean male adult height is "70".
Let's test if the mean of the population where this class is drawn from is different than "70"

$H_a: \theta \neq 70$, $H_0: \theta = 70$ $\alpha = 5\%$

$\hat{\theta} \mid H_0 \sim N(70, \frac{4^2}{10}) = N(70, 1.265^2)$



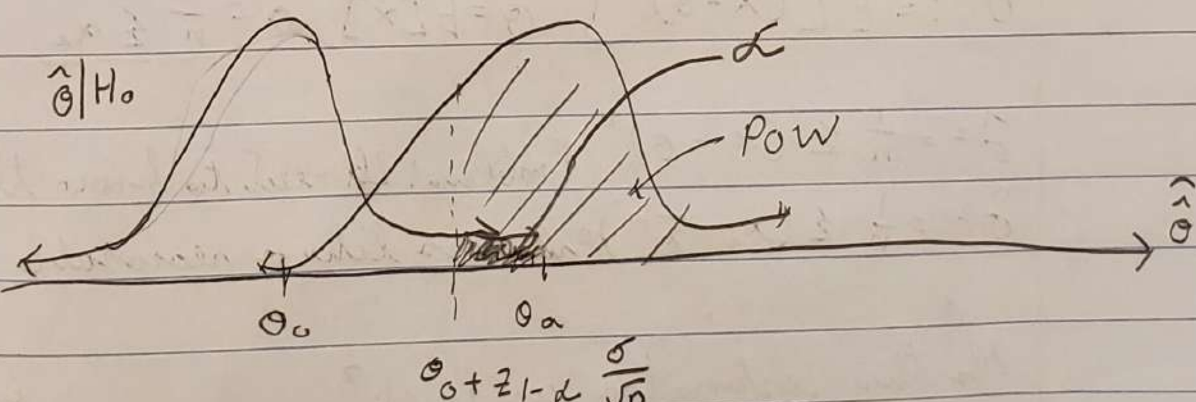
$\Rightarrow \hat\theta \in RET \Rightarrow$ Fail to Reject

$P_{val} = P($estimate is more extreme than observed$| H_0)$

$$= P\left(\hat\theta | H_0 > |\hat{\hat\theta} - \theta|\right) = 2p(\hat\theta | H_0 > 70.5)$$

$$= 2p\left(Z > \underbrace{\frac{70.5 - 70}{1.265}}_{0.39525}\right) = 69.3\% \nless \alpha \text{ statistically insignificant}$$

$H_0: \theta \le \theta_0$, $H_a: \theta = \theta_a > \theta_0$ size α



$$Pow = p\left(\hat\theta | H_a > \theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\right)$$

$$= P\left(\frac{\hat\theta | H_a - \theta_a}{\frac{\sigma}{\sqrt{n}}} > \frac{\theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} - \theta_a}{\frac{\sigma}{\sqrt{n}}}\right)$$

$\sigma \to$ gets small,
gets tighter
and power goes to the
left

$$= 1 - \Phi\left(\frac{-\sqrt{n}}{\sigma}(\theta_a - \theta_0) + z_{1-\alpha}\right) = Pow(\theta_a, \theta_0, n, \alpha, \sigma).$$

More realistic: we don't know sigsq. but... sigsq is a "nuisance parameter", It means we need to estimate it in order to estimate theta but we don't intrinsically care about it

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ and both θ, σ^2 are unknown

How do we estimate sigsq? Recall... for a rv X,

$$\sigma^2 := E[(X-\theta)^2] \quad \theta = E[X], \ \hat{\theta} = \frac{1}{n}\sum x_i$$

$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \theta)^2$ Problem: I need to know theta!
$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{x}^2)$ Seems like a reasonable estimator!

Is this estimator unbiased?

$$E[\hat{\sigma}^2] = E\left[\frac{1}{n}\sum(x_i - \bar{x}^2\right] = \frac{1}{n}\sum E(x_i - \bar{x})^2 \overset{iid}{=} \frac{1}{n} \times E[(x_1 - \bar{x})^2]$$

$= E[x_1^2 - 2x_1\bar{x} + \bar{x}^2] = E[x_1^2] - 2E[x_1 \cdot \frac{x_1 + \dots + x_n}{n}] + E[\bar{x}^2]$

Recall $Var[x] = E[x^2] - E[x]^2$

$= \sigma^2 + \theta^2 - \frac{2}{n} E[x_1^2 + x_1 x_2 + \dots + x_1 x_n] + \frac{\sigma^2}{n} + \theta^2$

$= \frac{n+1}{n}\sigma^2 + 2\theta^2 - \frac{2}{n}(\sigma^2 + \overbrace{\theta^2 + \theta^2 + \dots + \theta^2}^{n})$

cancels out

$= \frac{n-1}{n}\sigma^2 \neq \sigma^2 \Rightarrow$ its a little bit biased . . .
(towards 0)

However, it is "asymptotically unbiased" meaning . . .

$\lim_{n\to\infty} E[\hat{\theta}] = \theta$ eg $\lim_{n\to\infty} E[\hat{\sigma}^2] = \lim \frac{n-1}{n}\sigma^2 = \sigma^2$

Consider the following estimator:

$S^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{n}{n-1}\frac{1}{n}\sum(x_i - \bar{x})^2 = \frac{1}{n-1}\sum(X_i - \bar{x})^2$

The beauty of this estimator is that

$E[S^2] = E[\frac{n}{n-1}\hat{\sigma}^2] = \frac{n}{n-1}E[\hat{\sigma}^2] = \frac{\cancel{n}}{\cancel{n-1}}\frac{\cancel{n-1}}{\cancel{n}}\sigma^2$ ie unbiased

and its the default estimator for sig sq (variances in DGP

and it's really important in normal theory. . .