DGP: $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

standardize the estimator

$$\hat{\theta} = \bar{X} \overset{\text{Math 241}}{\sim} N\left(\theta, \left(\underbrace{\frac{\sigma}{\sqrt{n}}}_{SE[\bar{X}]}\right)^2\right) \iff \frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

(exactly)

What if sigma is unknown?

S^2 estimates σ^2 $\Rightarrow$ $S = \sqrt{S^2}$ estimates σ.

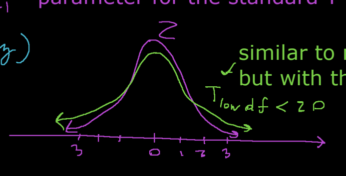
Does $\dfrac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \overset{?}{\sim} N(0,1)$ No! But close!

In 1907 Gosset proved:

$$\frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$$

(exactly)

Student's standard T distribution with n-1 "degrees of freedom" (the parameter for the standard T distr.



similar to normal but with thicker tails

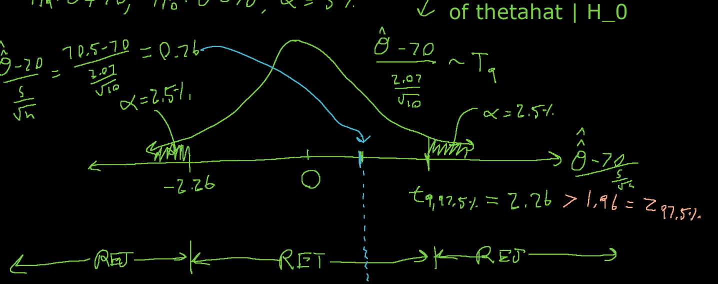
data from n=10 male student heights: xbar = 70.5, s = 2.07

$H_a: \theta \neq 70$, $H_0: \theta = 70$, $\alpha = 5\%$

the standardized distribution of thetahat | H_0

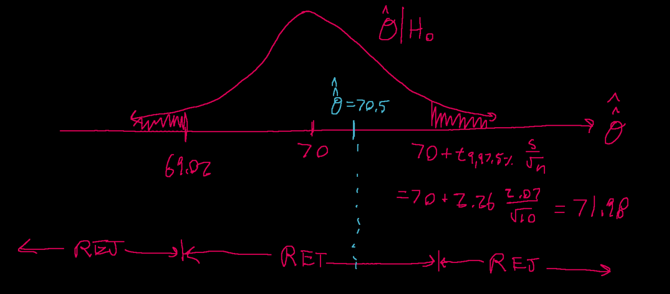
$$\frac{\hat{\theta} - 70}{\frac{s}{\sqrt{n}}} = \frac{70.5 - 70}{\frac{2.07}{\sqrt{10}}} = 0.76$$

$$\frac{\hat{\theta} - 70}{\frac{2.07}{\sqrt{10}}} \sim T_9$$



$t_{9, 97.5\%} = 2.26 > 1.96 = z_{97.5\%}$

$\Rightarrow$ Retain H_0

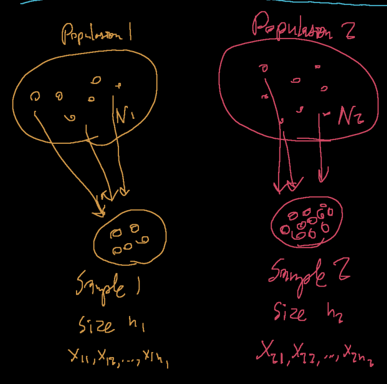


$70 + t_{9,97.5\%} \frac{s}{\sqrt{n}}$

$= 70 + 2.26 \frac{2.07}{\sqrt{10}} = 71.98$

$\Rightarrow$ Retain H_0

We just did our first "one-sample two-sided t test" (of a mean).



$N_1 \approx \infty$, $N_2 \approx \infty$ $\Rightarrow$ iid

Assume

$X_{11}, \dots, X_{1n_1} \overset{iid}{\sim} N(\theta_1, \sigma_1^2)$

independent of

$X_{21}, \dots, X_{2n_2} \overset{iid}{\sim} N(\theta_2, \sigma_2^2)$

σ_1^2, σ_2^2 are known but

θ_1, θ_2 are unknown

There are three types of tests that are usually done.

I. $H_a: \theta_1 \neq \theta_2 \Rightarrow H_0: \theta_1 = \theta_2$ equivalently..

$H_a: \theta_1 - \theta_2 \neq 0 \Rightarrow H_0: \theta_1 - \theta_2 = 0$

II. $H_a: \theta_1 < \theta_2 \Rightarrow H_0: \theta_1 \geq \theta_2$ equivalently...

$H_a: \theta_1 - \theta_2 < 0 \Rightarrow H_0: \theta_1 - \theta_2 \geq 0$

III. $H_a: \theta_1 > \theta_2 \Rightarrow H_0: \theta_1 \leq \theta_2$ equivalently...

$H_a: \theta_1 - \theta_2 > 0 \Rightarrow H_0: \theta_1 - \theta_2 \leq 0$

What is a test statistic? (For $\theta_1 - \theta_2$) $\hat{\theta}_1 - \hat{\theta}_2$ point estimate.

The estimator that produces these estimates is simply:

Math 241: $\hat{\theta}_1 \sim N(\theta_1, \frac{\sigma_1^2}{n_1})$ indep. $\hat{\theta}_2 \sim N(\theta_2, \frac{\sigma_2^2}{n_2})$

$$\hat{\theta}_1 - \hat{\theta}_2 = \bar{X}_1 - \bar{X}_2 \underset{\text{exactly}}{\sim} N\left(\theta_1 - \theta_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$\Rightarrow SE[\hat{\theta}_1 - \hat{\theta}_2] = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\frac{(\hat{\theta}_1 - \hat{\theta}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Under H_0 (all three), $\theta_1 - \theta_2 = 0$

$$\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

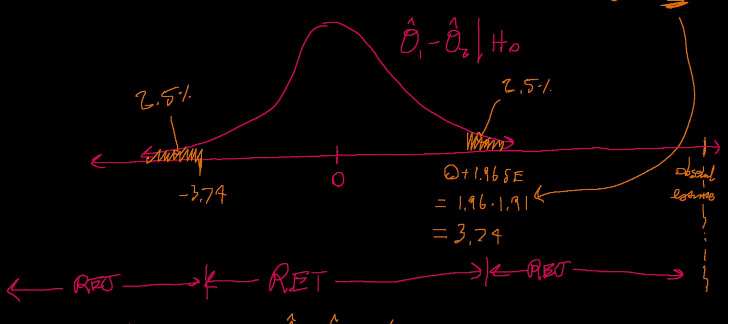
Let's test if male mean height is different than female mean height. $\vec{x}_2 = \langle 60, 59, 64, 64, 64, 63 \rangle$ $n_2 = 6$ $\bar{x}_2 = 62.3$

$n_1 = 10$, $\bar{x}_1 = 70.5$

$\hat{\theta}_1 - \hat{\theta}_2 = 70.5 - 62.3 = 8.2$

We assumed we knew the variances. So the variance for the men was assumed to be 4^2 and now the variance for the women is assumed to be 3.5^2. $\sigma_1^2 = 4^2$, $\sigma_2^2 = 3.5^2$, $\alpha = 5\%$

We can now do our 2-sample 2-sided z test. $SE = \sqrt{\frac{4^2}{10} + \frac{3.5^2}{6}} = 1.91$



observed estimate $\hat{\theta}_1 - \hat{\theta}_2 \notin RET \Rightarrow$ Reject H_0.

$$p\text{-value} = 2P(\hat{\theta}_1 - \hat{\theta}_2 > 8.2) = 2P\left(\frac{\hat{\theta}_1 - \hat{\theta}_2}{SE} > \frac{8.2}{1.91}\right) = 2P(Z > 4.29)$$

$= 1.8 \times 10^{-5} < \alpha$

$\Rightarrow$ Reject

Let's sample from two populations again however, this time we have the same variance sigsq = sigsq_1 = sigsq_2 which we still assume known.

$X_{11}, \dots, X_{1n_1} \overset{iid}{\sim} N(\theta_1, \sigma^2)$ indep. of $X_{21}, \dots, X_{2n_2} \overset{iid}{\sim} N(\theta_2, \sigma^2)$

Under H_0, $\hat{\theta}_1 - \hat{\theta}_2 \sim N\left(0, \sqrt{\sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}^2\right)$

also $$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

2-sample 2-sided z test of equal variance

The test can be run again, you can probably assume sigma = 3.75.

Same as above but sigsq unknown. How can we estimate the standard error?

S_1^2, S_2^2 are the sample variances in both samples 1 and 2.

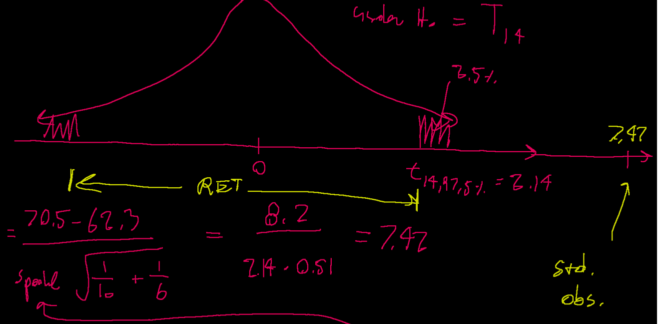
$$S_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_{1,i} - \bar{X}_1)^2, \quad S_2^2 = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (X_{2,i} - \bar{X}_2)^2$$

$$S_{pooled}^2 := \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}$$ weighted average

You can prove that

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{S_{pooled}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T_{n_1 + n_2 - 2}$$

this allows you to do the "2-sample t test of equal variance"



$$\hat{\theta} = \frac{70.5 - 62.3}{s_{pooled}\sqrt{\frac{1}{10} + \frac{1}{6}}} = \frac{8.2}{2.14 \cdot 0.51} = 7.42$$

$$s_{pooled}^2 = \frac{9 \cdot 2.07^2 + 5 \cdot 2.25^2}{14} = 4.56$$

$\Rightarrow$ Reject H_0