In Lec10 $\quad \dfrac{\hat\theta-\theta}{SE[\hat\theta]} \xrightarrow{d} N(0,1) \Rightarrow \dfrac{\hat\theta-\theta}{\hat{SE}[\hat\theta]} \xrightarrow{d} N(0,1)$

We can use this now in our situation:

$$\frac{\hat\theta_1-\hat\theta_2}{SE[\hat\theta_1-\hat\theta_2]} \xrightarrow{d} N(0,1) \implies \frac{\hat\theta_1-\hat\theta_2}{\hat{SE}[\hat\theta_1-\hat\theta_2]} \xrightarrow{d} N(0,1)$$

$$SE[\hat\theta_1-\hat\theta_2] = \sqrt{\theta_{shared}(1-\theta_{shared})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

$$\hat{SE}[\hat\theta_1-\hat\theta_2] = \sqrt{\hat\theta_{shared}(1-\hat\theta_{shared})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} \quad \text{if } \hat\theta_{shared} \text{ is consistent}$$

$$\hat\theta_{shared} = \text{avg. over both samples} = \frac{\sum X_{1i} + \sum X_{2i}}{n_1+n_2}$$

$$\Rightarrow \frac{\hat\theta_1-\hat\theta_2}{\sqrt{\frac{\sum X_{1i}+\sum X_{2i}}{n_1+n_2}\left(1-\frac{\sum X_{1i}+\sum X_{2i}}{n_1+n_2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \stackrel{\cdot}{\sim} N(0,1)$$

e.g. $H_a: \theta_1-\theta_2 \neq 0$, $H_0: \theta_1-\theta_2=0$, $\alpha = 5\%$

Control $n_1 = 81$, $\sum X_{1i} = 27 \Rightarrow \hat\theta_1 = \frac{27}{81} = 0.333$

experimental $n_2 = 79$, $\sum X_{2i} = 12 \Rightarrow \hat\theta_2 = \frac{12}{79} = 0.152 \Rightarrow \hat\theta_{shared} = \frac{27+12}{81+79} = 0.244$

$$(\hat\theta_1-\hat\theta_2)_{std} = \frac{0.333-0.152}{\sqrt{0.244(1-0.244)\left(\frac{1}{81}+\frac{1}{79}\right)}} = 2.66 \notin [-1.96, +1.96]$$

$\Rightarrow$ Reject H_0.

Another (obvious) Wald Test: If X_1, ..., X_n iid DGP with mean theta and variance sigma-squared and the estimator thetahat is Xbar, then the CLT implies that:

$$\frac{\hat\theta-\theta}{\frac{\sigma}{\sqrt n}} \xrightarrow{d} N(0,1) \quad \text{if } \sigma \text{ is known}$$

If σ is unknown.... I can replace sigma with any consistent estimator e.g. S, sigma-hat and

$$\frac{1}{n}\sum(X_i-\theta)^2$$

$$\Rightarrow \frac{\hat\theta-\theta}{\frac{S}{\sqrt n}} \xrightarrow{d} N(0,1)$$

Are you allowed to just use the T-test here? Many people just use the T-test here. Technically it's wrong because you need the DGP to be normal iid. But if you use the T-test... it's "not so bad". I did this on problem 11 of the midterm:

$H_a: \theta > 2$, $n = 30$, $\bar x = 2.57$, $s = 1.00$

$$\hat\theta_{std} = \frac{2.57-2}{\frac{1.00}{\sqrt{30}}} = 3.12 \notin (-\infty, 1.645] \Rightarrow \text{Reject } H_0.$$

Another Wald test for two independent samples with unknown variances and you wish to test a different in means.

$$\frac{\hat\theta_1-\hat\theta_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} \xrightarrow{d} N(0,1) \implies \frac{\hat\theta_1-\hat\theta_2}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} \xrightarrow{d} N(0,1)$$

from last class

If you use the Satterthwaite t-test, it "wouldn't be so bad" because unless your population distributions were so very skewed, it should be fine.

Let's use the asymptotic normality of the MLE thm (last class) to do a Wald Test. HW4l,m has DGP: X_1, ..., X_n iid Gumbel(theta, 1). The Gumbel is a rv model for "extreme events" think maximum rainfall per month.

$$\ell'(\theta; x_1,\dots,x_n) = n - e^{-\theta}\sum e^{-x_i} \stackrel{set}{=} 0 \Rightarrow \hat\theta^{MLE} = \ln\left(\frac{n}{\sum e^{-x_i}}\right)$$

$\Downarrow$

$$\ell'(\theta; x) = 1 - e^{-\theta}e^{-x} \Rightarrow \ell''(\theta; x) = -e^{-\theta}e^{-x} \;?$$

$$I(\theta) = E[-\ell''(\theta; X)] = E[e^{-\theta}e^{-X}] = e^{-\theta}E[e^{-X}] = e^{-2\theta}$$

$$\frac{\hat\theta^{MLE}-\theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} = \frac{\hat\theta^{MLE}-\theta}{\frac{e^{\theta}}{\sqrt n}} = \frac{\ln\left(\frac{n}{\sum e^{-X}}\right)-\theta}{\frac{e^{\theta}}{\sqrt n}} \xrightarrow{d} N(0,1).$$

$x_1 = 2.15$, $x_2 = 1.91$, x_3 "3.66, x_4 4.85, 3.03, 1.03, 3.58, $n = 7$

$\hat\theta^{MLE} = 2.26$. Test $H_a: \theta > 2$. $\alpha = 5\%$

$$\hat\theta^{MLE}_{std} = \frac{2.26-2}{\frac{e^2}{\sqrt 7}} = \frac{0.26}{2.79} = 0.09 \in (-\infty, 1.645]$$

$\Rightarrow$ Retain H_0

There are three goals of statistical inference
(1) Point Estimation
Goal here is to provide a best guess, thetahathat of the value of theta. You don't know if your specific guess is good, is close, is bad, is far... How do we ask the question "is it good / bad"? We imagined thetahahat coming from a distribution thetahat, the "sampling distribution". There are properties about the sampling distribution e.g. some good properties are unbiasedness, consistency, low MSE, low risk (for general loss functions).
(2) Testing
Goal here is to test a theory about a specific theta. We used hypothesis testing. What makes a "good test"? One property is "power". There are other properties we didn't discuss.
(3) Confidence Sets
The goal here is to create a set of values for theta that you're "confident in". The approach we use here is the "confidence interval".

Definition: an "interval estimate" are two statistics:

$w_L(x_1,\dots,x_n)$ & $w_U(x_1,\dots,x_n)$ s.t. $w_L < w_U$ for all data sets

combined in an interval: $[w_L(x_1,\dots,x_n), w_U(x_1,\dots,x_n)]$

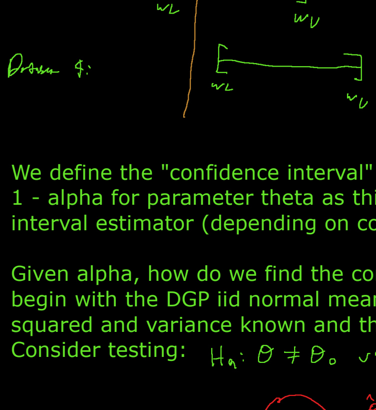
e.g. $[1.789, 2.463]$

and of course, the "interval estimator" is: $[w_L(X_1,\dots,X_n), w_U(X_1,\dots,X_n)]$

which is a "random interval".

Definition: An interval estimator has "coverage probability"

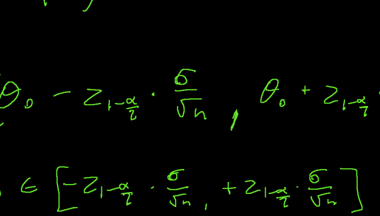
$P(\theta \in [w_L(X_1,\dots,X_n), w_U(X_1,\dots,X_n)] \mid \theta)$. An illustration:



The coverage probability is computed over every dataset. For these four datasets, the coverage probability would be 3/4 = 75%.

We define the "confidence interval" with coverage probability 1 - alpha for parameter theta as this interval estimate and interval estimator (depending on context). Denoted $CI_{\theta, 1-\alpha}$

Given alpha, how do we find the confidence interval? Let's begin with the DGP iid normal mean theta, variance sigma-squared and variance known and the estimator = Xbar. Consider testing: $H_a: \theta \neq \theta_0$ vs. $H_0: \theta = \theta_0$ at size α.



$$z_{1-\frac{\alpha}{2}} := F_Z^{-1}\left(1-\frac{\alpha}{2}\right)$$

$$P(\hat\theta \in RET \mid H_0) = 1-\alpha$$

$\parallel$

$$P\left(\hat\theta \in \left[\theta_0 - z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}, \theta_0 + z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}\right] \,\middle|\, \theta=\theta_0\right)$$

$$= P\left(\hat\theta - \theta_0 \in \left[-z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}, +z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}\right] \,\middle|\, \theta=\theta_0\right)$$

$$= P\left(\theta_0 - \hat\theta \in \left[-z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}, +z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt n}\right] \,\middle|\, \theta=\theta_0\right)$$

$$= P\left(\theta_0 \in \left[\underbrace{\hat\theta - z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}}_{w_L}, \underbrace{\hat\theta + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt n}}_{w_U}\right] \,\middle|\, \theta=\theta_0\right)$$

$$= P(\theta_0 \in [w_L(X_1,\dots,X_n), w_U(X_1,\dots,X_n)] \mid \theta=\theta_0) \quad \text{Since valid } \forall\, \theta_0 \dots$$

$$\Rightarrow CI_{\theta, 1-\alpha} = \left[\hat\theta - z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}, \hat\theta + z_{1-\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt n}\right]$$

we constructed our first confidence interval by "inverting the test"