Lecture 3

09/02/20

DGP: $X_1, \ldots, X_n \overset{iid}{\sim}$ with mean θ, Variance σ^2

If $\hat{\theta} = \bar{X} \Rightarrow \hat{\theta}$ is unbiased

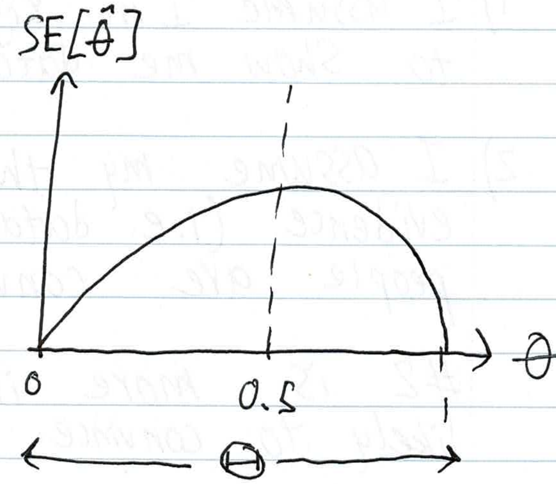
$$SE[\hat{\theta}] = \sqrt{Var \tfrac{1}{n}(X_1 + \ldots + X_n)} = \sqrt{\tfrac{1}{n^2} \sum Var[X_i]}$$

$$= \sqrt{\tfrac{1}{n^2}\, n\, \sigma^2} = \frac{\sigma}{\sqrt{n}} \qquad R(\hat{\theta}, \theta) = \frac{\theta(1-\theta)}{n} = MSE$$

$$= \sqrt{\frac{\theta(1-\theta)}{n}}$$

$\downarrow$

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(\theta)$



$$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \frac{1}{4n}$$

Supremum (maximum)

Goal # 3 of inference: Theory Testing (hypothesis testing)

You have some well specified mathematical theory about the DGP. For example, in the iPhone Survey, "I think the proportion of iPhone users in the population is NOT 52.4%"

I want to prove my theory to the world (using my sample)

Note: It is absolutely impossible to prove or disprove my theory, because you cannot see the whole population. We must use inference which is always a guess.

Two ways to go about "proving" my theory:

1) I assume I'm right and wait for other people to show me data that contradicts my theory

2) I assume my theory is wrong. Then I bring evidence (i.e. data) to the contrary until people are convinced my theory is right.

#2 is more intellectually honest and more likely to convince

A "hypothesis" is a mathematical statement about the DGP e.g. $\theta = 0.9$

$\theta > 0.9$, theta is not equal to 0.9, or $\theta \leq 0.9$ or θ is in the set $[0.89, 0.91]$ etc.

The "alternative hypothesis" (H_a) is the theory you want to prove.

The "null hypothesis" (H_0) is the opposite you assume in #2 for the purpose of contradicting it.

Usual cases:

$H_0: \theta \le \theta_0$, $H_a: \theta > \theta_0$ (right-tailed test)

$H_0: \theta \ge \theta_0$, $H_a: \theta < \theta_0$ (left-tailed test)

$H_0: \theta = \theta_0$, $H_a: \theta \ne \theta_0$ (two-tailed test)

How to perform this test? There are many, many options even for the same DGP. The protocol goes as follows:

1) You think of a "Test Statistic" that could measure the departure away from H_0

2) Derive the statistical estimators distribution under H_0

3) Gauge the departure

We begin with DGP: iid Bern (θ) and the "binomial exact test"

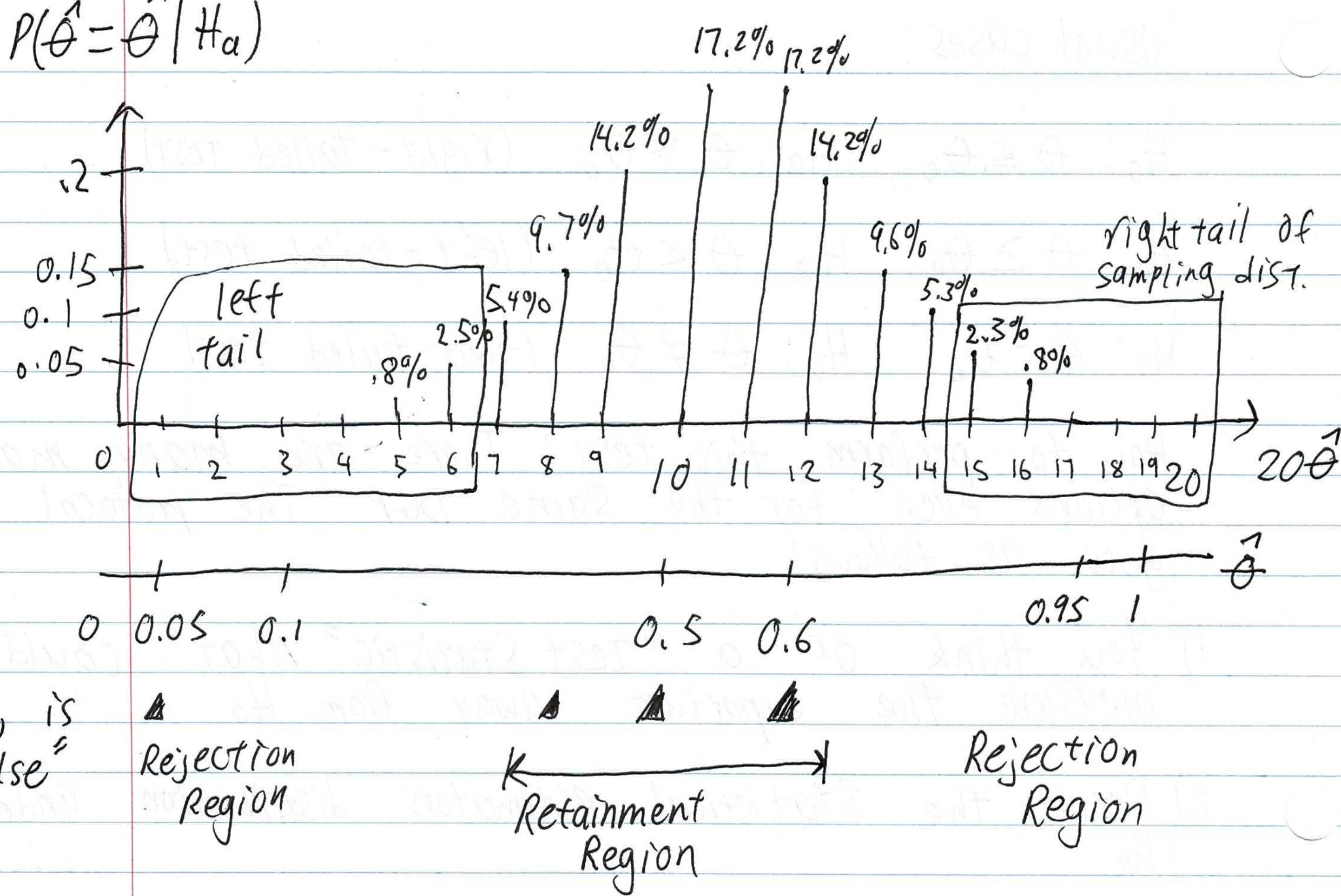
$H_a: \theta \ne .524$, $H_0: \theta = .524$

1) My test statistic is... $\hat{\theta} = \bar{X}$
$\hat{\theta}$ is a realization from $\hat{\theta}$

2) $\hat{\theta} \mid H_0 \sim ?$ $n = 20$

$$\hat{\theta} = \frac{X_1 + \ldots + X_{20}}{20}$$

$$20\hat{\theta} \mid H_0 = X_1 + \ldots + X_{20} \sim \text{Binom}(20, \theta_0 = .524)$$



(RET)

$\hat{\theta} \in RET \Rightarrow$ Retain H_0 (fail to reject H_0)
Not enough evidence to reject H_0. Some authors say "accept H_0"

$\hat{\theta} \notin RET \Rightarrow$ reject H_0 / Accept H_a

My estimate is "Statistically significant"

Let's say we rejected H_0 but it really was true. This is called a Type I error.
Where is the P(Type I error)

$$\alpha = P(\text{Type I error}) = P(\hat{\theta} \notin RET \mid H_0)$$

Then in a 2-tailed test, I apportion about $\frac{\alpha}{2}$ to the left tail and about $\frac{\alpha}{2}$ to the right tail

In my RET,

$$\alpha = P(\hat{\theta} = 0 | H_0) + \ldots + P(\hat{\theta} = 0.3 | H_0)$$

$$+ P(\hat{\theta} = 0.75 | H_0) + \ldots + P(\hat{\theta} = 1 | H_0) = 7.06\%$$

The choice of alpha is up to you. The scientific community's standard is 5% and sometimes 1%.
If you would like to prove your theory, you have to accept a positive probability of a Type I error.
If I fail to reject H_0 when H_a is true, that's a different error, a "type II error" Failure to prove your theory. The smaller the alpha, the larger the P(Type II error)

		Decision	
		Retain H_0	Reject H_0
Truth	H_0	✓	Type I error
	H_a	Type II error	✓

As of now, we cannot calculate the P(Type II error)