Lecture - 03 09/02/2020

DGP: $X_1, \dots, X_n \overset{iid}{\sim}$ with mean $\theta$, variance $\sigma^2$

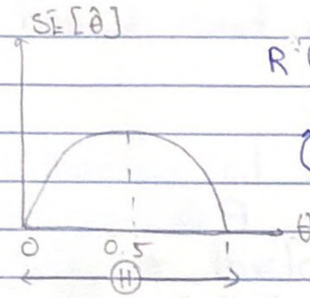
If $\hat{\theta} = \bar{x} \implies \hat{\theta}$ is unbiased

$$SE[\hat{\theta}] = \sqrt{Var[1/n(X_1 + \dots + X_n)]}$$

$$= \sqrt{\frac{1}{n^2} \sum Var[X_i]} = \sqrt{\frac{1}{n^2} n \sigma^2} = \frac{\sigma}{\sqrt{n}}$$

$$= \sqrt{\frac{\theta(1-\theta)}{n}}$$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(\theta)$




$R(\hat{\theta}, \theta) = \frac{\theta(1-\theta)}{n} = MSE$

$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \frac{1}{4n}$

supremum (maximum)

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Goal # 3 of inference: theory testing (hypothesis testing)

You have some well-specified mathematical theory about the DGP. for example, in the iphone survey, 'I think the proportion of iphone users in the population is NOT 52.4%. I want to prove my theory to the world (using my sample).

Note:- It is absolutely impossible to prove or disprove my theory because you cannot see the whole population (or go inside of the DGP). We must use inference which always a guess.

Two ways to go about "proving" my theory:
(1) I assume I'm right and wait for other people to show me data that contradicts my theory.

(2) I assume my theory is wrong. Then I adduce (bring) evidence (i.e. data) to the contrary until people are convinced my theory is right.

#2 is more intellectually honest and more likely to convince

A "hypothesis" is a mathematical statement about the DGP e.g. theta = 0.9, theta > 0.9, theta is not equal to 0.9, or theta ≤ 0.9 or theta is in the set [0.89, 0.91], etc.

The "alternative hypothesis" (H_a) is the theory you want to prove. The "null hypothesis" (H_0) is the opposite you assume in #2 for the purpose of contradicting H. Usual cases:

$H_0: \theta \leq \theta_0$, $H_a: \theta > \theta_0$ (right-tailed test)

$H_0: \theta \geq \theta_0$, $H_a: \theta < \theta_0$ (left-tailed test)

$H_0: \theta = \theta_0$, $H_a: \theta \neq \theta_0$ (two-tailed test)

How to perform this test? There are many, many options even for the same DGP. The protocol goes as follows.

(1) You think of a "test statistic" that could measure the departure away from $H_0$.

(2) Derive the statistical estimator's distribution

(3) Gauge the departure.

We begin with DGP: iid Bern(theta) and the "binomial exact test"

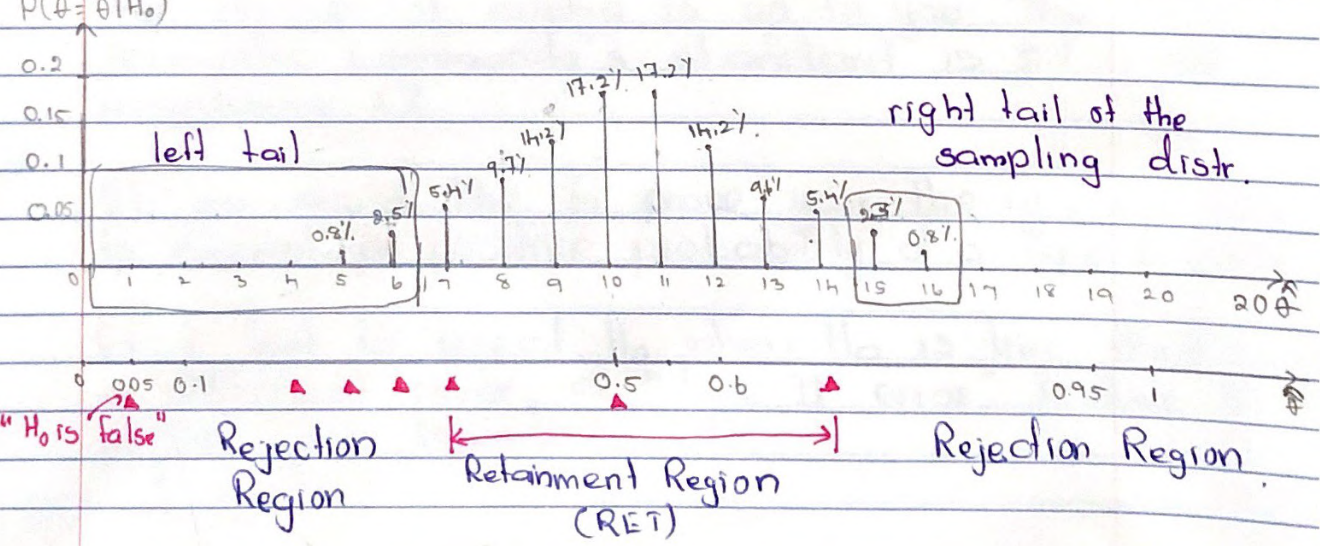
$H_a: \theta \neq 0.524$ , $H_0: \theta = 0.524$

① My test statistic is .. $\hat{\hat{\theta}} = \bar{x}$ . $\hat{\hat{\theta}}$ is a realization from $\hat{\theta}$.

② $\hat{\theta} \mid H_0 \sim ?$ $n = 20$

$$\hat{\theta} = \frac{X_1 + \ldots + X_{20}}{20} \Rightarrow 20\,\hat{\theta} \mid H_0$$

$$= X_1 + \ldots + X_{20} \sim \text{Binom}(20, \theta = 0.524)$$



$\hat{\theta} \in RET \Rightarrow$ Retain $H_0$ (fail to reject $H_0$). Not enough evidence to reject $H_0$. Some authors say "accept $H_0$".

$\hat{\theta} \notin RET \Rightarrow$ Reject $H_0$ / Accept $H_a$. My estimate is "~~sta~~ statistically significant".

Let's say we rejected $H_0$ but it was true. This is called a Type I error. Where is the P(Type I error) on our plot?

$$\alpha := P(\text{Type I error}) = P(\hat{\theta} \notin RET \mid H_0)$$

Then in a 2-tailed test, I apportion about alpha/2 to the left tail and about alpha/2 to the right tail.

In my RET,

$$\alpha = P(\hat{\theta} = 0 \mid H_0) + \ldots + P(\hat{\theta} = 0.3 \mid H_0) + P(\hat{\theta} = 0.75 \mid H_0) + \ldots + P(\hat{\theta} = 1 \mid H_0) = 7.06\%$$

The choice of alpha is up to you. The scientific community's standard is 5% and sometimes 1%.

If you would like to prove your theory, you have to accept a positive probability of a Type I error.

If I fail to reject $H_0$ when $H_a$ is true that's a different error, a "type II error". Failure to prove your theory.

The smaller the alpha, the larger the P( Type II error).

| | | Decision | |
|---|---|---|---|
| | | Retain $H_0$ | Reject $H_0$ |
| Truth | $H_0$ | ✓ | Type I error |
| | $H_a$ | Type II error | ✓ |

As of now, we cannot calculate the P( Type II error)