Lecture 3 9/2/2020

DGP: $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ with mean θ, var σ^2

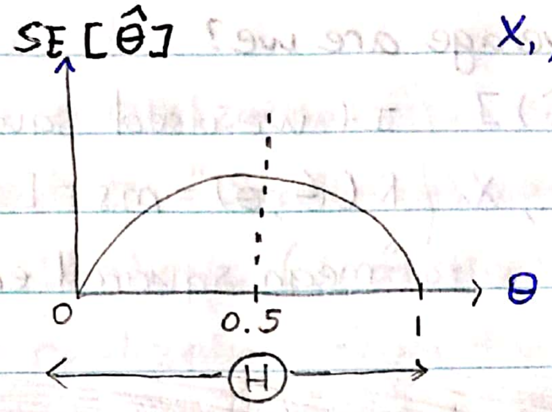
If $\hat{\theta} = \bar{x} \Rightarrow \hat{\theta}$ is unbiased $R(\hat{\theta}, \theta) = \frac{\theta(1-\theta)}{n}$ = MSE

$$SE[\hat{\theta}] = \sqrt{var[\tfrac{1}{n}(X_1 + \ldots + X_n)]} = \sqrt{\tfrac{1}{n^2}\sum var[X_i]}$$

$$= \sqrt{\tfrac{1}{n^{\cancel{2}}}\cancel{n}\sigma^2} = \frac{\sigma}{\sqrt{n}}$$

$$= \sqrt{\frac{\theta(1-\theta)}{n}}$$

$X_1, \ldots, X_n \overset{i.i.d.}{\sim} Bern(\theta)$



$$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \frac{1}{4n}$$

Supremum (maximum)

Goal #3 of inference: theory testing (hypothesis testing)

You have some well-specified mathematical theory about the DGP.

For example, in the iPhone survey, "I think the proportion of iPhone users in the population is NOT 52.4%." I want to prove my theory to the world (using my sample).

NOTE: It is absolutely impossible to prove / disprove my theory, because you can't see the whole population (or go inside of the DGP). We must use inference which is always a guess.

Two ways to go about "proving" my theory:

1) I assume I'm right and wait for other people to show me data that contradicts my theory.

2) I assume my theory is wrong. Then I adduce (bring) evidence (i.e. data) to the contrary until people are convinced my theory is right.

#2 is more intellectually honest and more likely to convince.

A "hypothesis" is a mathematical statement about the DGP e.g. $\theta = 0.9$, $\theta > 0.9$, θ is not equal to 0.9, or $\theta \leq 0.9$ or θ is in the set $[0.89, 0.91]$, etc.

The "alternative hypothesis" (H_a) is the theory you want to prove. The "null hypothesis" (H_0) is the opposite you assume in #2 for the purpose of contradicting it. Usual cases:

$H_0: \theta \leq \theta_0$, $H_a: \theta > \theta_0$ (right-tailed test)

$H_0: \theta \geq \theta_0$, $H_a: \theta < \theta_0$ (left-tailed test)

$H_0: \theta = \theta_0$, $H_a: \theta \neq \theta_0$ (two-tailed test)

How to perform this test? There are many, many options even for the same DGP. The protocol goes as follows:

1) You think of a "test statistic" that could measure the departure away from H_0

2) Derive the statistical estimator's distribution under H_0

3) Gauge the departure

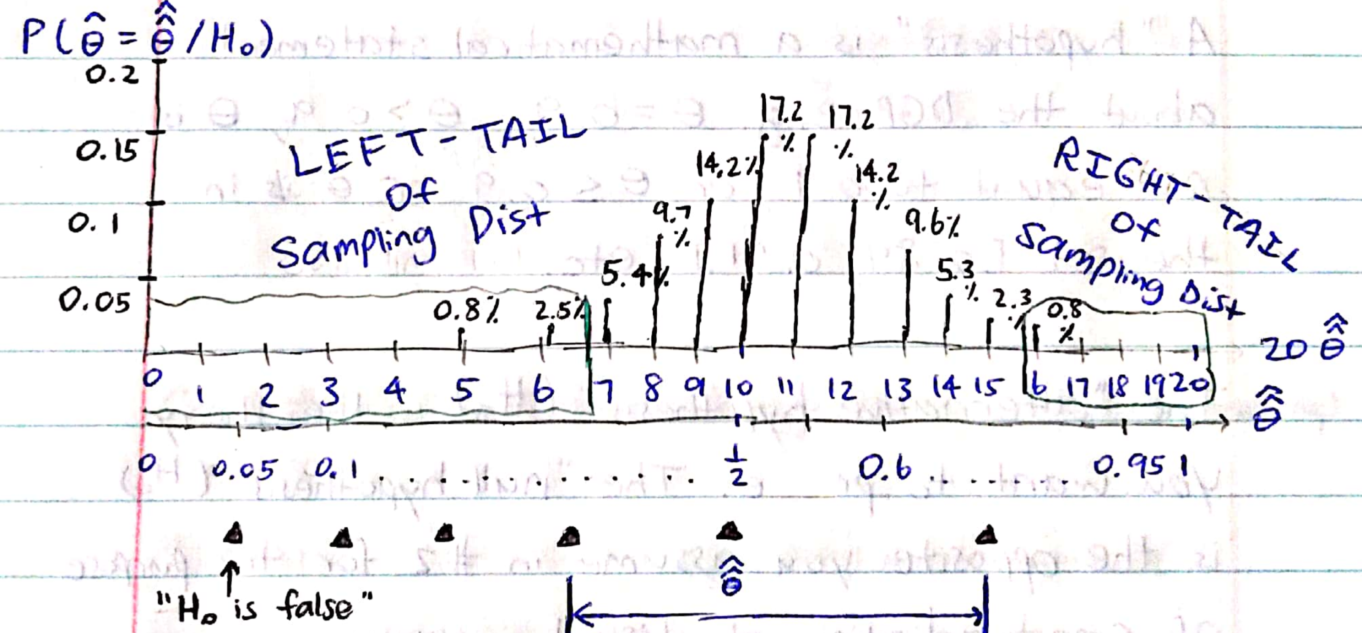
We begin w/ DGP: i.i.d. Bern(θ) and the "binomial exact test"

$H_a: \theta \neq 0.524 = \theta_0$, $H_0: \theta = 0.524 = \theta_0$

1) My test statistic is $\hat{\theta} = \bar{X}$, $\hat{\hat{\theta}}$ is a realization from $\hat{\theta}$

2) $\hat{\theta} / H_0 \sim ?$ $n = 20$

$$\hat{\theta} = \frac{X_1 + \ldots + X_{20}}{20} \Rightarrow 20\hat{\theta}/H_a = X_1 + \ldots + X_{20} \sim \text{Binom}(20, \theta_0 = 0.524)$$



Rejection Region | Retainment Region (RET) | Rejection Region

$\hat{\hat{\theta}} \in$ RET $\Rightarrow$ Retain H_0 (fail to reject H_0). Not enough evidence to reject H_0. Some authors say "accept H_0".

$\hat{\hat{\theta}} \notin$ RET $\Rightarrow$ Reject H_0 / accept H_a. My estimate is "statistically significant".

Let's say we rejected H_0 but it was really true. This is called a Type 1 error. Where is the P(Type 1 error) on our plot?

$\alpha := P(\text{Type 1 error}) = P(\hat{\theta} \notin RET / H_0)$

Then in a 2-tailed test, I apportion about $\frac{\alpha}{2}$ to the left-tail and about $\frac{\alpha}{2}$ to the right-tail.

In my RET,

$\alpha = P(\hat{\theta} = 0 / H_0) + \ldots + P(\hat{\theta} = 0.3 / H_0) + P(\hat{\theta} = 0.75 / H_0) + \ldots$

$P(\hat{\theta} = 1 / H_0) = 7.06\%$

The choice of α is up to you. The scientific community's std is 5% and sometimes 1%.

If you would like to prove your theory, you have to accept a positive prob of a Type I error.

If I fail to reject H_0 when H_a is true, that's a different error, a "Type II error". Failure to prove your theory.

The smaller the α, the larger the P(Type II error).

Decision

Trust	Retain H_0	Reject H_0
H_0	✓	Type I error
H_a	Type II error	✓

As of now, we can't calculate P(Type II error).