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<u>Math 369</u> 9/2/20

<u>Lecture 3</u>

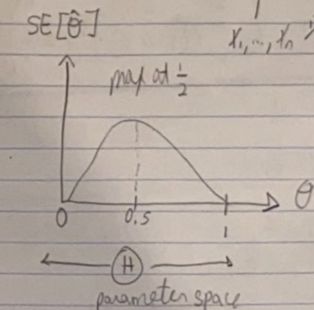
DGP $X_1, \dots, X_n \overset{iid}{\sim}$ with mean θ, variance σ^2

If $\hat{\theta} = \bar{X} \Rightarrow \hat{\theta}$ is unbiased

$$SE[\hat{\theta}] = \sqrt{Var\left[\tfrac{1}{n}(X_1 + \dots + X_n)\right]} = \sqrt{\tfrac{1}{n}\sum var[X_i]}$$

$$= \sqrt{\tfrac{1}{n^2} n \sigma^2} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{\theta(1-\theta)}{n}}$$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(\theta)$



$$\sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \frac{1}{4n}$$

<u>Recall</u>: $R(\hat{\theta}, \theta) = \frac{\theta(1-\theta)}{n} = MSE$

supremum (maximum)

Goal #3 of inference: theory testing (hypothesis testing).

You have some well-specified mathematical theory about the DGP.

For example, in the iphone survey, "I think the proportion of iPhone users in the population is <u>NOT</u> 52.4%. I want to prove my theory to the world (using my sample).

NOTE: it is absolutely impossible to prove or disprove my theory because you cannot see the whole population (or go inside of the DGP). We must use inference which is always a guess.

Two ways to go about "proving" my theory:

(1) I assume I'm right and wait for other people to show me data that contradicts my theory

(2) I assume my theory is wrong. Then I adduce (bring) evidence (i.e. data) to the contrary until people are convinced my theory is right.

#2 is more intellectually honest and more likely to convince

A "hypothesis" is a mathematical statement about DGP e.g. $\theta = 0.9$, $\theta > 0.9$, $\theta \neq 0.9$, $\theta \leq 0.9$, θ is in the set $[0.89, 0.91]$, etc

The "alternative hypothesis (H_a) is the theory you want to prove. The "null hypothesis" (H_0) is the opposite you assume in HT for the purpose of contradicting it. Usual cases:

$H_0: \theta \le \theta_0, \ H_a: \theta > \theta_0$ (right-tailed test)
I start here

$H_0: \theta \ge \theta_0, \ H_a: \theta < \theta_0$ (left-tailed test)

$H_0: \theta = \theta_0 \quad H_a: \theta \ne \theta_0$ (two tailed test)

How to perform this test? There are many, many options even for the same DGP. The protocol goes as follows

(1) you think of a "test statistic" that could measure the departure away from H_0

(2) derive the statistical estimator's distribution under H_0

(3) gauge the departure

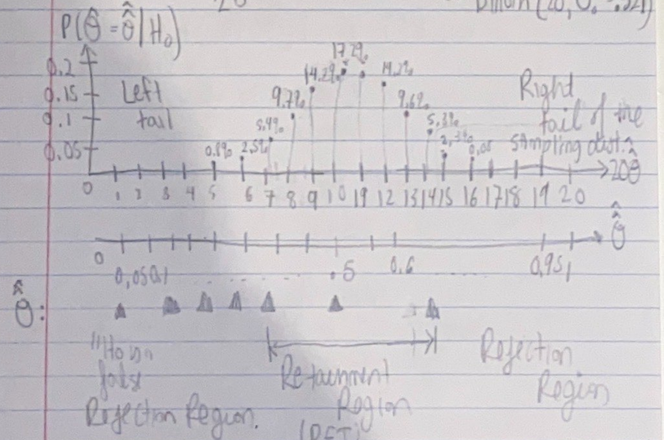
We begin with DGP: iid Bern(θ) and the "binomial exact test"

$H_a: \theta \ne .524, \ H_0: \theta = .524 = \theta_0$

(1) my test statistic is ... $\hat{\hat{\theta}} = \bar{x}$. $\hat{\hat{\theta}}$ is a realization from $\hat{\theta}$

(2) $\hat{\theta} \mid H_0 \sim ?$ $\quad n = 20$

$$\hat{\theta} = \frac{X_1 + \dots + X_{20}}{20} \Rightarrow 20\hat{\theta} \mid H_0 = X_1 + \dots + X_{20} \sim \text{Binom}(20, \theta_0 = .524)$$



$\hat{\hat{\theta}} \in RET \Rightarrow$ Retain H_0 (fail to reject H_0). Not enough evidence to reject H_0. Some authors say "accept H_0".

$\hat{\hat{\theta}} \notin RET \Rightarrow$ Reject H_0 / Accept H_a. My estimate is "statistically significant".

Lets say we rejected H_0 but it really was true. This is called a Type 1 error. Where is the P(Type 1 error) on our plot?

$\alpha := P(\text{Type 1 error}) = P(\hat{\theta} \notin RET \mid H_0)$

Then in a 2-tailed test, I apportion about $\alpha/2$ to the left tail and about $\alpha/2$ to the right tail.

In my RET, $\alpha = P(\hat{\theta}=0 \mid H_0) + \ldots + P(\hat{\theta}=0.3 \mid H_0) + P(\hat{\theta}=0.75 \mid H_0)$
$+ \ldots + P(\hat{\theta}=1 \mid H_0) = 7.06\%$

The choice of α is up to you. The scientific community's standard is 5% (sometimes 1%)

If you would like to prove your theory, you have to accept a positive probability of a Type I error

If I fail to reject H_0 when H_a is true, that's a different error, a "type II error". Failure to prove your theory

The smaller the α, the larger the P(Type II error).

Decision

Truth		Retain H_0	Reject H_0
	H_0	✓	Type I error
	H_a	Type II error	✓

As of now, we cannot calculate the P(Type II error)