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Math 369 9/9/20

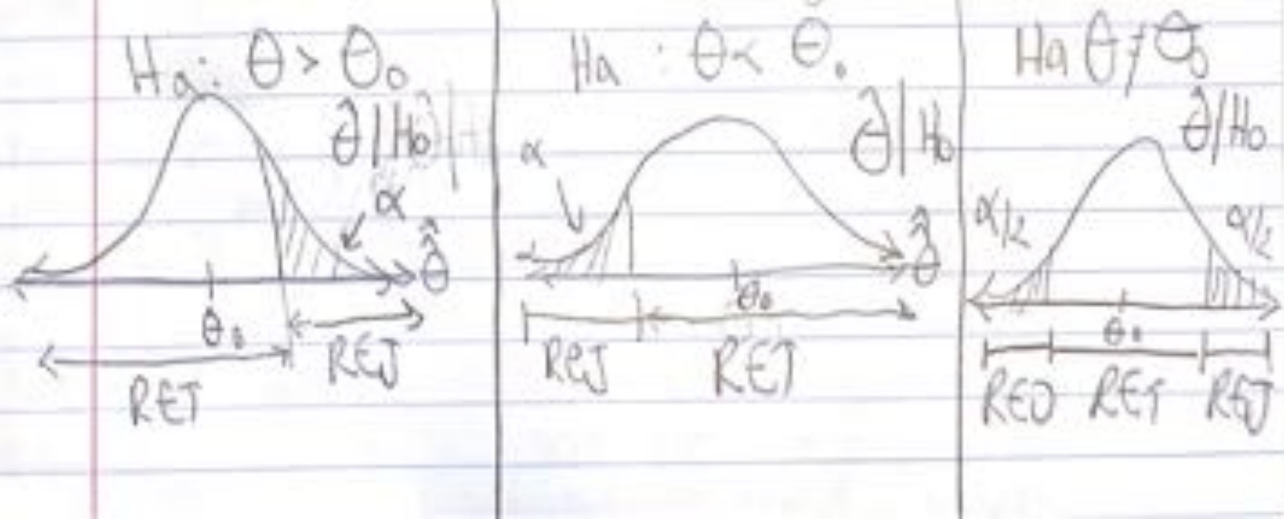
Lecture 4

I don't think I'll give exam q's on this:
"Level of a test" $\alpha$ is defined as P(Type 1 error) $\geq \alpha$
"Size of a test" is exactly P(Type 1 error).

In our ex., the level was 5% but the size was 7.06% since $\alpha = 5\%$ was unattainable

If $\hat{\theta} | H_0$ is continuous, then level = size = $\alpha$.
If it's discrete, some sizes won't be attainable.

If I want a level of $\alpha = 5\%$ and the size is lower, then I'm "cheating" (we'll see next class)



What we did in the previous lecture (2) was called a "binomial exact test of one proportion". Downsides: (1) you need a binomial pmf calculator and it's a lot of work to get the retainment region (2) not all sizes are attainable. This is the recommended test.

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Let $X_1, X_2, \dots X_n \overset{iid}{\sim}$ some distribution with mean mu and variance $\sigma^2$. The central limit theorem (CLT) shows that:

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

"convergence in distribution". It means as n gets large, the CDF of the LHS looks more and more like the CDF of the RHS

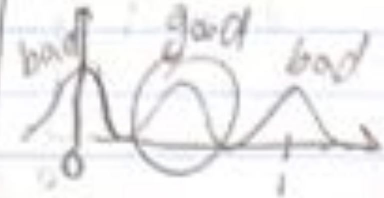
$* \Rightarrow \bar{X} \overset{\cdot}{\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$ ← approx distr.

and $T = X_1 + \dots + X_n \overset{\cdot}{\sim} N(n\mu, n\sigma^2)$

If $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$ and $n$ is "large" then:

$$\hat{\theta} = \bar{X} \overset{\cdot}{\sim} N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$

this is a pretty good approximation if $\theta$ is not too close to 0 or 1




How to perform an "approximate test"? There are many, many options even for the same DGP. The protocol goes as follows:

(1) you think of a "test statistic" that could measure the departure away from $H_0$

(2) Derive the statistical estimator's *approx* distribution under $H_0$, $\hat{\theta}|H_0$

(3) Gauge the departure of $\hat{\theta}$ from the bulk of the distribution $\hat{\theta}|H_0$ at level $\alpha$.

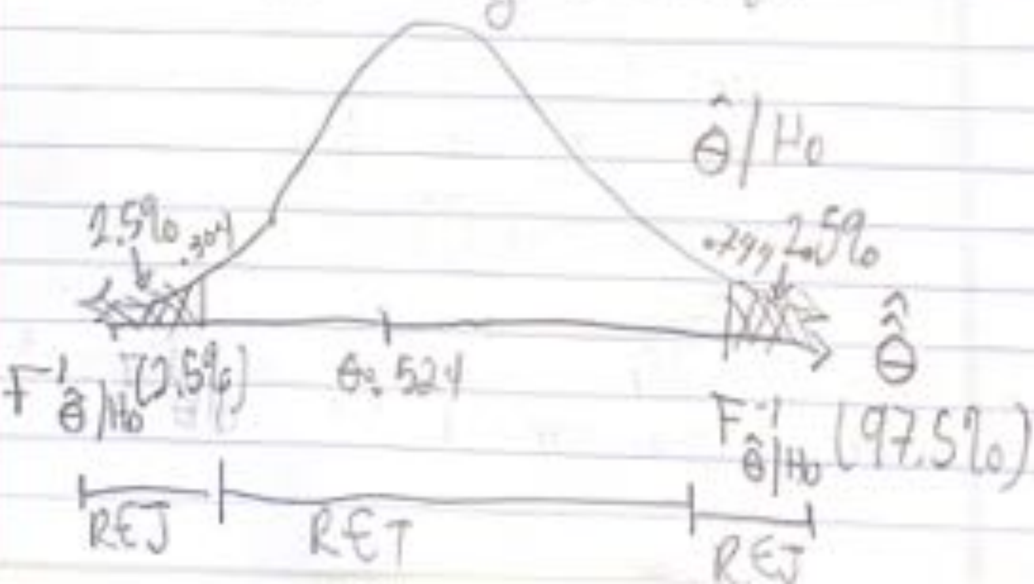
$H_0: \theta = .524$, $H_a: \theta \neq .524$ $n=20$ $\hat{\theta} = 0.6$

(same as last class)

$$\hat{\theta}|H_0 \sim N\Big(.524, \underbrace{\frac{.524(1-.524)}{20}}_{\sigma^2}\Big)$$

$$= N(.524, \underbrace{.112}_{\sigma}{}^2)$$

set $\alpha = 5\% \Rightarrow \frac{\alpha}{2} = 2.5\%$



$F^{-1}_{\hat\theta|H_0}(2.5\%)$ 4

$P(\hat\theta|H_0 \le \hat{\hat\theta}) = 2.5\%$ solve for $\hat{\hat\theta}$

$\Rightarrow P\left(\underbrace{\frac{\hat\theta|H_0 - .524}{.112}}_{N(0,1)} \le \frac{\hat{\hat\theta} - .524}{.112}\right) = 2.5\%$

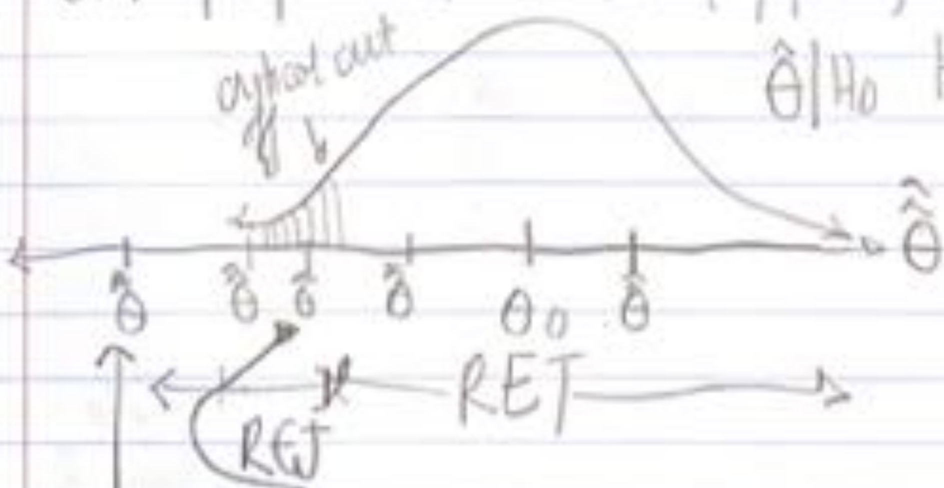
$\Rightarrow P\left(Z \le \frac{\hat{\hat\theta} - .524}{.112}\right) = 2.5\%$

Look at table or use computer

$\Rightarrow \frac{\hat{\hat\theta} - .524}{.112} \approx 1.96 \approx 2$

$\Rightarrow \hat{\hat\theta} = .304$

One proportion z-test (approximate test).

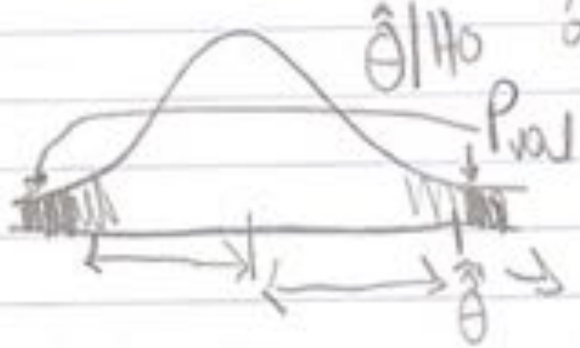



the estimate should imply a "stronger" rejection than this estimate

To measure the "strength" of a rejection (or "weakness" of a retainment), Fisher introduced the "p-value" also called the level of statistical significance as: 5

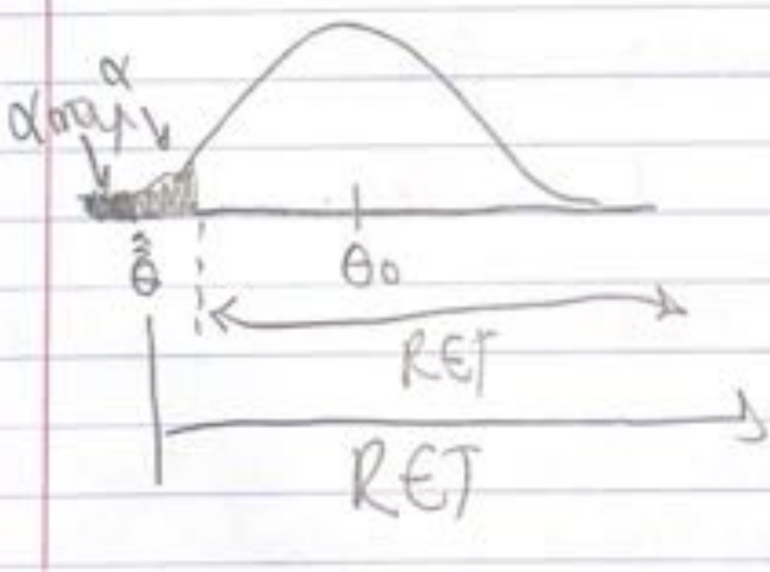
$P_{val} := P(\text{estimate is more extreme than one observed} \mid H_0)$

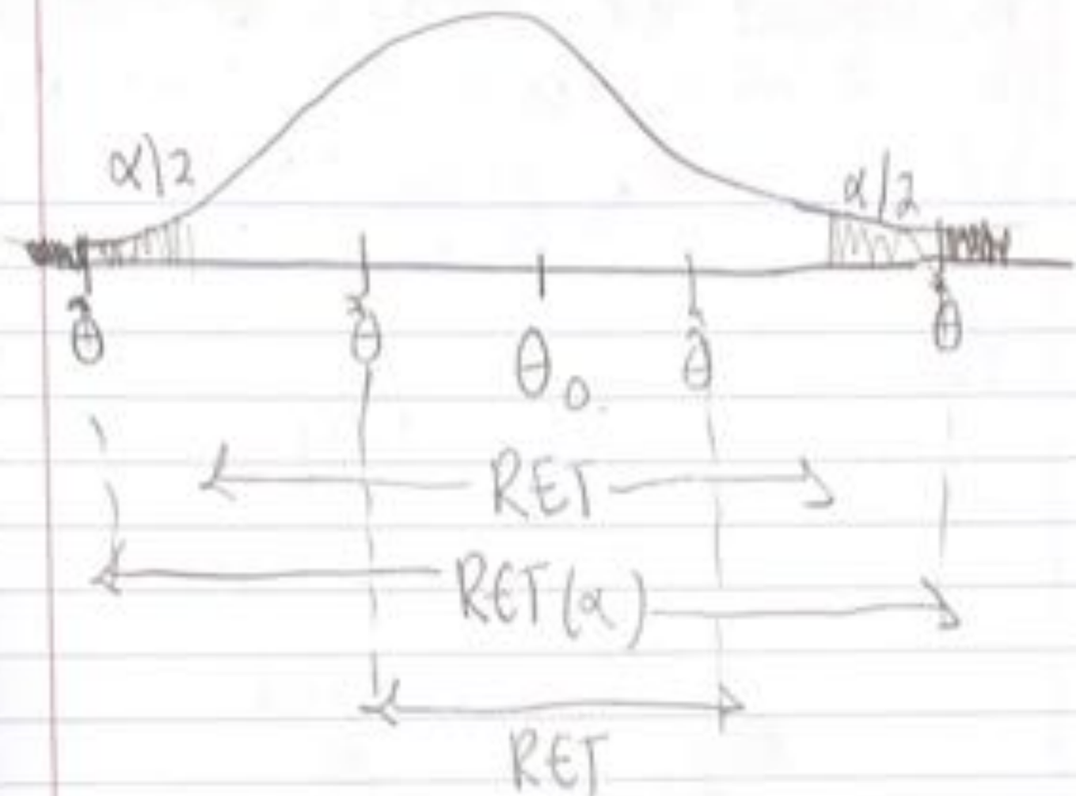
$\hat{\theta}$




Real definition

$P_{val} = \max \{ \alpha : \hat{\theta} \in RET(\alpha) \}$






If $H_0$ is retained $\Rightarrow$ pval $\geq \alpha$

If $H_0$ is rejected $\Rightarrow$ pval $< \alpha$

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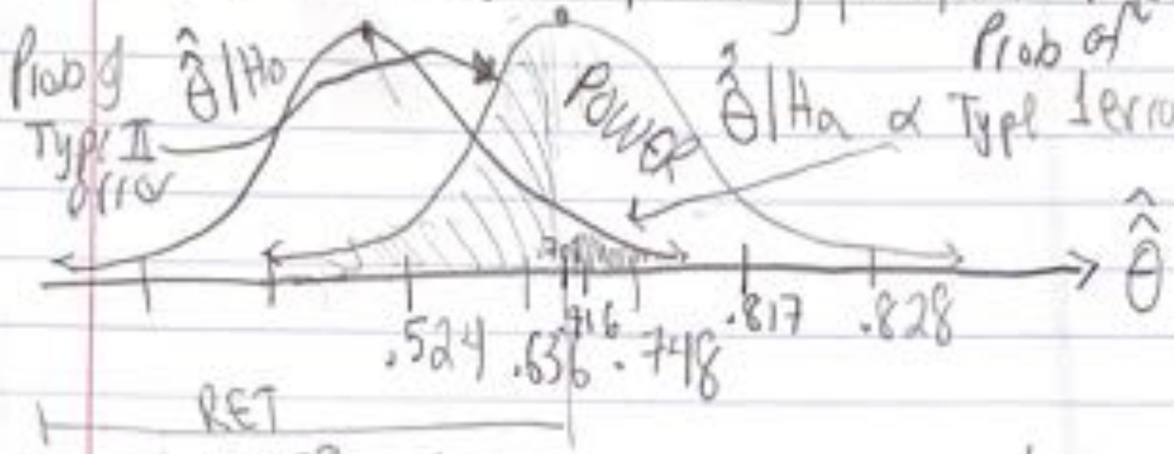
Type II errors to POWER

DGP $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$H_0$: $\theta = .524 = \theta_0$ but $H_a$: $\theta = .716 = \theta_a$

This is a non-standard setup since both $H_0$ and $H_a$ are "point hypotheses". This makes the outcome weird: either you retain $\theta = .524$ or you accept $\theta = .716$. But ignore this for now

$\hat{\theta}|H_0 \sim N(.524, .112^2)$, $\hat{\theta}|H_a \sim N(.716, .101^2)$




At $\alpha = 5\%$, the z value is 1.645 which means the rejection region ends at $\hat{\theta} = .524 + 1.645(.112) = .708$

$P(\text{Type II error}) = P(\hat{\theta}|H_a \in RET)$

$= P\left(\frac{\hat{\theta}|H_a - .716}{.101} \leq \frac{.708 - .716}{.101}\right)$

$= P(Z \leq -.079) \approx 47\% \Rightarrow$ Power $\approx 53\%$ pretty bad

Errors

| Truth \ Decision | RET | REJ |
|---|---|---|
| $H_0$ | | Type I |
| $H_a$ | Type II | |

POWER = P(Rejecting $H_0|H_a$)

= 1 − P(Retaining $H_0|H_a$)

= 1 − P(Type II error)

Power is the probability of proving your theory is true!! You want Power to be Large i.e near 100%