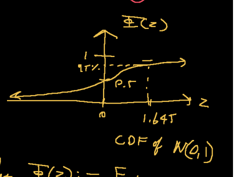
Let's look at power more generally (beyond two point hypotheses).

$H_0: \theta \le \theta_0 \;,\; H_1: \theta = \theta_1 > \theta_0$ at size α

right-tailed test

Under iid Bern(theta) and the normal approximation,



$$Pow = P\left(\hat\theta|H_1 > \theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)$$

let $\Phi(z) := F_Z(z)$

$\Phi(z_{1-\alpha}) = 1-\alpha$

$\alpha = 5\% \Rightarrow z_{1-\alpha} = 1.645$

$$= P\left(\frac{\hat\theta|H_1 - \theta_1}{\sqrt{\frac{\theta_1(1-\theta_1)}{n}}} > \frac{\theta_0 + z_{1-\alpha}\sqrt{\frac{\theta_0(1-\theta_0)}{n}} - \theta_1}{\sqrt{\frac{\theta_1(1-\theta_1)}{n}}}\right)$$

$$= P\left(Z > \frac{-\sqrt{n}(\theta_1-\theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_1(1-\theta_1)}}\right)$$

Power Function

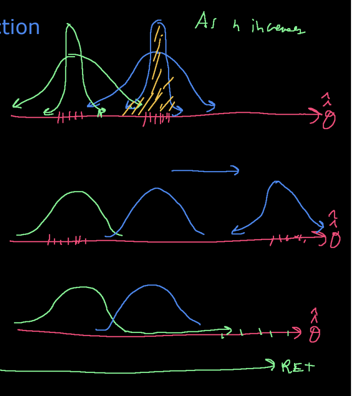
$$= 1 - \Phi\left(\frac{-\sqrt{n}(\theta_1-\theta_0) + z_{1-\alpha}\sqrt{\theta_0(1-\theta_0)}}{\sqrt{\theta_1(1-\theta_1)}}\right) = Pow(\theta_1, \theta_0, n, \alpha)$$

Observations about the power function

If $n \to \infty \Rightarrow Pow \to 1$

If $\theta_1 \to \infty \Rightarrow Pow \to 1$

As $\alpha \to 0 \Rightarrow Pow \to 0$



New type of survey. We ask "how tall are you (in inches)?" for men only. I'll ask 10 male students and get x_1, ..., x_{10} (i.e. my data). The data is now continuous (no longer zeroes and ones). Height for a gender is known to be normally distributed.

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$. Assume σ^2 is known and =4^2.

How can we estimate theta? Theta is the mean of the rv's. And recall $\hat\theta = \bar X$ is unbiased. Let's use this estimator.

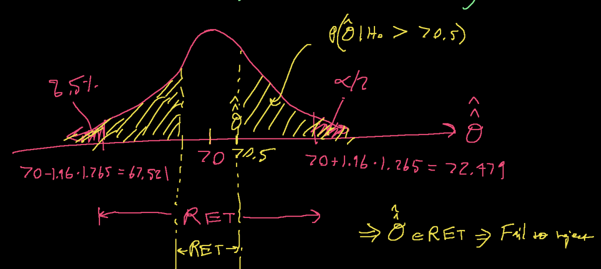
$\vec x = (70, 72, 73, 68, 69, 70, 67, 72, 71, 73)$ $\quad \hat\theta = \bar x = 70.5$

The american mean male adult height is 70".

Let's test if the mean of the population where this class is drawn from is different than 70".

$H_1: \theta \ne 70, \quad H_0: \theta = 70, \quad \alpha = 5\%$ one sample z-test

$$\hat\theta|H_0 \sim N\left(70, \frac{4^2}{10}\right) = N(70, 1.265^2)$$



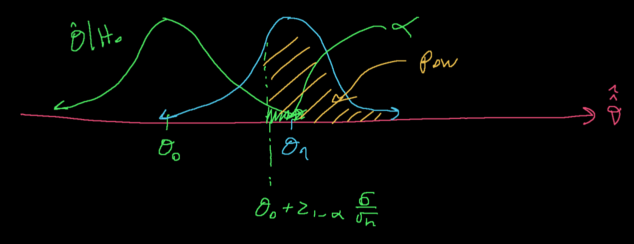
$\Rightarrow \hat\theta \in RET \Rightarrow$ Fail to reject

$p_{val} = P($estimate is more extreme than observed | H_0$)$

$$= P(|\hat\theta|H_0| > |\hat\theta - \theta|) = 2P(\hat\theta|H_0 > 70.5)$$

$$= 2P\left(Z > \frac{70.5-70}{1.265}\right) = 69.3\% \not< \alpha$$ statistically insignificant

$H_0: \theta \le \theta_0, \; H_1: \theta = \theta_1 > \theta_0,$ size α



$$Pow = P\left(\hat\theta|H_1 > \theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt n}\right)$$

$$= P\left(\frac{\hat\theta|H_1 - \theta_1}{\frac{\sigma}{\sqrt n}} > \frac{\theta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt n} - \theta_1}{\frac{\sigma}{\sqrt n}}\right)$$

$$= 1 - \Phi\left(-\frac{\sqrt n}{\sigma}(\theta_1 - \theta_0) + z_{1-\alpha}\right) = Pow(\theta_1, \theta_0, n, \alpha, \sigma)$$

More realistic: we don't know sigsq. But... sigsq is a "nuisance parameter". It means we need to estimate it in order to estimate theta but we don't intrinsically care about it.

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ and both θ, σ^2 are unknown.

How do we estimate sigsq? Recall... for a rv X,

$\sigma^2 := E[(X-\theta)^2]$ $\qquad \theta = E[X], \; \hat\theta = \frac{1}{n}\sum X_i$

$\hat\sigma^2 = \frac{1}{n}\sum (X_i - \theta)^2$ Problem: I need to know theta!

$\hat\sigma^2 = \frac{1}{n}\sum (X_i - \bar X)^2$ Seems like a reasonable estimator!

Is this estimator unbiased? For any iid DGP...

$$E[\hat\sigma^2] = E\left[\frac{1}{n}\sum (X_i - \bar X)^2\right] = \frac{1}{n}\sum E(X_i - \bar X)^2 \overset{iid}{=} \cancel{\frac{1}{n}}\,\cancel{n}\,E[(X_1 - \bar X)^2]$$

$$= E[X_1^2 - 2X_1\bar X + \bar X^2] = E[X_1^2] - 2E\left[X_1 \cdot \frac{X_1 + \ldots + X_n}{n}\right] + E[\bar X^2]$$

Recall Var[X] = E[X^2] - E[X]^2

$$= \sigma^2 + \theta^2 - \frac{2}{n}E[\underbrace{X_1^2 + X_1X_2 + \ldots + X_1X_n}_{n}] + \frac{\sigma^2}{n} + \theta^2$$

$$= \frac{n+1}{n}\sigma^2 + 2\theta^2 - \frac{2}{n}(\sigma^2 + \theta^2 + \theta^2 + \ldots + \theta^2)$$

$$= \frac{n-1}{n}\sigma^2 \ne \sigma^2$$ $\Rightarrow$ It's a little bit biased...

However, it is "asymptotically unbiased" meaning...

$\lim_{n\to\infty} E[\hat\theta] = \theta$ $\qquad$ e.g. $\lim_{n\to\infty} E[\hat\sigma^2] = \lim \frac{n-1}{n}\sigma^2 = \sigma^2$ ✓

Consider the following estimator:

$$S^2 := \frac{n}{n-1}\hat\sigma^2 = \frac{n}{n-1}\frac{1}{n}\sum (X_i - \bar X)^2 = \frac{1}{n-1}\sum (X_i - \bar X)^2$$

The beauty of this estimator is that

$$E[S^2] = E\left[\frac{n}{n-1}\hat\sigma^2\right] = \frac{n}{n-1}E[\hat\sigma^2] = \frac{\cancel{n}}{\cancel{n-1}}\frac{\cancel{n-1}}{\cancel{n}}\sigma^2$$ i.e. unbiased

And it's the default estimator for sigsq (variances in DGP's) and it's really important in normal theory...