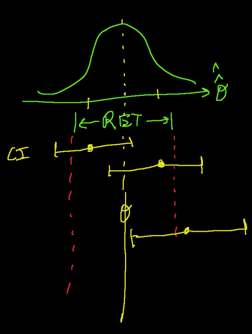


theta_0 being in the confidence interval with coverage probability 1 - alpha, is equivalent to the test at size alpha retaining.

$$\hat{\hat{\theta}} \in RET_{\theta_0,\alpha} \iff \theta_0 \in CI_{\theta,1-\alpha}$$

p421 C&B: both hypothesis testing and interval construction look for consonance between the sample statistic (thetahathat) and the population parameter (theta).

Hypothesis tests fix the value of the parameter theta (H_0) and ask "is the estimate thetahathat in agreement?" If no => Reject.

Confidence sets fixes the estimate (thetahathat) and asks "which values of the parameter (theta) are in agreement"?

We inverted a 2-sided hypothesis test to get a 2-sided CI. You can also have a 1-sided CI e.g.

$$CI_{L,\theta,1-\alpha} := [w_L(X_1,\ldots,X_n), \infty) \text{ or } CI_{R,\theta,1-\alpha} := (-\infty, w_U(X_1,\ldots,X_n)]$$

but we won't do this in class only for the interest of saving time and moving on to other topics.

Sometimes the sampling distribution was approximate. Inverting that test will yield CI's with approximate coverage i.e. "approximate CI's". Let's build some popular CI's!

DGP: $\stackrel{iid}{\sim} N(\theta, \sigma^2)$ with σ^2 unknown, $\hat{\hat{\theta}} = \bar{X}$

$$CI_{\theta,1-\alpha} = \left[\hat{\hat{\theta}} \pm t_{1-\frac{\alpha}{2},n-1} \frac{S}{\sqrt{n}}\right] = [\ \hat{\hat{\theta}}\ ]$$

margin of error

DGP: $X_{11},\ldots,X_{1n_1} \stackrel{iid}{\sim} N(\theta_1, \sigma_1^2)$ indep of $X_{21},\ldots,X_{2n_2} \stackrel{iid}{\sim} N(\theta_2,\sigma_2^2)$, $\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2 = \bar{X}_1 - \bar{X}_2$

if σ_1^2, σ_2^2 known

$$CI_{\theta_1-\theta_2,1-\alpha} = \left[(\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right]$$

if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ known

$$= \left[(\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2) \pm z_{1-\frac{\alpha}{2}}\, \sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$$

if $\sigma_1^2 = \sigma_2^2$ but unknown

$$= \left[(\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2) \pm t_{1-\frac{\alpha}{2},n_1+n_2-2}\, s_{pooled} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$$

see lec. 6

if $\sigma_1^2 \neq \sigma_2^2$ and unknown

$$\approx \left[(\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2) \pm t_{1-\frac{\alpha}{2},df} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right]$$

see lec 7 for the Satterthwaite approximation

DGP: $\stackrel{iid}{\sim} Bern(\theta)$, $\hat{\theta} = \bar{X}$. via the CLT, $\frac{\hat{\theta}-\theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \xrightarrow{d} N(0,1)$.

$$\Rightarrow \frac{\hat{\theta}-\theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \mathrel{\dot\sim} N(0,1) \xRightarrow[\text{Slutsky's}]{\text{via Thm 5.5.4 \&}} \frac{\hat{\theta}-\theta}{\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}} \xrightarrow{d} N(0,1).$$

using this fact and following through...

$$\Rightarrow P\left(\frac{\hat{\theta}-\theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \in \left[-z_{1-\frac{\alpha}{2}}, +z_{1-\frac{\alpha}{2}}\right]\right) \approx 1-\alpha$$

$$\Rightarrow P\left(\frac{\theta-\hat{\theta}}{\sqrt{\frac{\theta(1-\theta)}{n}}} \in \left[z_{1-\frac{\alpha}{2}}, +z_{1-\frac{\alpha}{2}}\right]\right) \approx 1-\alpha$$

$$\Rightarrow P\left(\theta \in \left[\hat{\theta} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\theta(1-\theta)}{n}}, \hat{\theta} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\theta(1-\theta)}{n}}\right]\right) \approx 1-\alpha$$

$$\Rightarrow CI_{\theta,1-\alpha} \approx \left[\hat{\hat{\theta}} - z_{1-\frac{\alpha}{2}}\sqrt{\boxed{\frac{\theta(1-\theta)}{n}}}, \hat{\hat{\theta}} + z_{1-\frac{\alpha}{2}}\sqrt{\boxed{\frac{\theta(1-\theta)}{n}}}\right] \quad \times$$

this is a fail... I don't know theta!

$$\Rightarrow CI_{\theta,1-\alpha} \approx \left[\hat{\hat{\theta}} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\hat{\theta}}(1-\hat{\hat{\theta}})}{n}}, \hat{\hat{\theta}} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\hat{\theta}}(1-\hat{\hat{\theta}})}{n}}\right]$$

this is *a* CI for the binomial proportion. It is actually a bad approximation for low n and theta near 0 or 1. There are other CI's we won't study and it is actually an area of modern research.

DGP: $X_{11},\ldots,X_{1n_1} \stackrel{iid}{\sim} Bern(\theta_1)$ indep. of $X_{21},\ldots,X_{2n_2} \stackrel{iid}{\sim} Bern(\theta_2)$

the Richard maneuver

From Lec 11, $\frac{(\hat{\theta}_1-\hat{\theta}_2)-(\theta_1-\theta_2)}{\sqrt{\frac{\theta_1(1-\theta_1)}{n_1}+\frac{\theta_2(1-\theta_2)}{n_2}}} \xrightarrow{d} N(0,1) \xRightarrow[\text{Slutsky's}]{\text{Thm 5.5.4 \&}} \frac{(\hat{\theta}_1-\hat{\theta}_2)-(\theta_1-\theta_2)}{\sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1}+\frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}} \xrightarrow{d} N(0,1)$

$$\Rightarrow CI_{\theta_1-\theta_2,1-\alpha} \approx \left[(\hat{\hat{\theta}}_1-\hat{\hat{\theta}}_2) \pm z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{\hat{\theta}}_1(1-\hat{\hat{\theta}}_1)}{n_1}+\frac{\hat{\hat{\theta}}_2(1-\hat{\hat{\theta}}_2)}{n_2}}\right]$$

e.g. from the medical study,
$n_1 = 81$, $\hat{\hat{\theta}}_1 = 0.333$, $n_2 = 79$, $\hat{\hat{\theta}}_2 = 0.152$

$$CI_{\theta_1-\theta_2,95\%} \approx \left[(0.333-0.152) \pm 1.96\sqrt{\frac{0.333\cdot 0.667}{81}+\frac{0.152\cdot 0.848}{79}}\right]$$

$$= [.181 \pm 1.96 \cdot 0.066] = [0.051, 0.311]$$

"You're 95% confident that the true proportion difference is between 5.1% and 31.1%.

By CLT $\frac{\hat{\theta}-\theta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1) \Downarrow \frac{\hat{\theta}-\theta}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1)$.

DGP $\stackrel{iid}{\sim}$ some rv with mean θ, variance σ^2 unknown. $\hat{\theta} = \bar{X}$

$$CI_{\theta,1-\alpha} \approx \left[\hat{\hat{\theta}} \pm z_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}}\right]$$

if you use the t it won't be "so bad"

Prob 11 on midterm I: $\bar{x} = 2.57$, $s = 1.00$

$$CI_{\theta,95\%} \approx \left[2.57 \pm 1.96 \frac{1.00}{\sqrt{30}}\right] = [2.212, 2.928]$$

DGP $\stackrel{iid}{\sim} f(\theta)$ where $\hat{\theta} = \hat{\theta}^{MLE}$.

From lec 11, $\frac{\hat{\theta}^{MLE}-\theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} \xrightarrow{d} N(0,1) \xRightarrow{\text{Richard}} \frac{\hat{\theta}^{MLE}-\theta}{\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}} \xrightarrow{d} N(0,1)$

$$\Rightarrow CI_{\theta,1-\alpha} \approx \left[\hat{\hat{\theta}} \pm z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$$

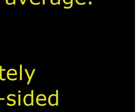
example from last class. DGP: iid Gumbel(theta, 1) and the data is <2.15, 1.91, 3.66, 4.85, 3.03, 1.03, 3.58> and n = 7. Find a 95% CI for theta:

$$\hat{\theta}^{MLE} = \ln\left(\frac{n}{\sum e^{-x_i}}\right), \quad \hat{\theta}^{MLE} = 2.26$$

$$\sqrt{I(\theta)^{-1}} = e^{\theta} \quad \Rightarrow \sqrt{I(\hat{\theta}^{MLE})^{-1}} = 9.57$$

$$CI_{\theta,95\%} \approx \left[2.26 \pm 1.96 \cdot \frac{9.57}{\sqrt{7}}\right] = [0.58, 3.93]$$

Now that we've been properly introduced to statistical inference (all three goals), let's talk about some big picture things.



For an unbiased estimator, MSE (being small) is KING. Why?

(1) Point Estimation
The lower the MSE, the closer thetahathat is to theta on average.

(2) Hypothesis Testing
Most estimators we discussed with exactly or approximately normally distributed. Thus the retainment region for a 2-sided test looks like:

$$RET = \left[\theta_0 \pm z_{1-\frac{\alpha}{2}} \underbrace{\sqrt{MSE}}_{=SE(\hat{\theta})}\right]$$

with a smaller MSE => smaller RET => higher power!

(3) Confidence Intervals

For exactly or approximately normally distributed estimators,

$$CI_{\theta,1-\alpha} \stackrel{=}{\underset{\approx}{or}} \left[\hat{\hat{\theta}} \pm z_{1-\frac{\alpha}{2}} \cdot \sqrt{MSE}\right]$$

or $\sqrt{\widehat{MSE}}$ via Richard maneuver

A lower MSE means a tighter / smaller CI which means you're more confidence about where theta lies e.g.

$$CI_{\theta,95\%} = [0.49, 5.1] \quad vs. \quad CI_{\theta,95\%} = [0.4999, 0.5001]$$

Let's picture all three goals: