Lecture 11 10/14/20

We want to prove the *asymptotic normality and asymptotic efficiency of the MLE thm*. This means we want to show:

$$\frac{\hat{\theta}^{MLE}-\theta}{\sqrt{CRLB}} \xrightarrow{d} N(0,1) \Rightarrow \hat{\theta}^{MLE} \,\dot{\sim}\, N\left(\theta, \sqrt{\frac{I(\theta)^{-1}}{n}}^{\,2}\right)$$

$$CRLB := \frac{I(\theta)^{-1}}{n}$$

The asymptotic normality of the MLE is very useful but the asymptotic efficiency is like a huge bonus. The MLE estimates with the approximately theoretically guaranteed minimum variance.

The proof mostly follows from p 472 of C&B. Recall the Taylor series formula for $f(y)$ "centered at" a.

$$f(y) = \overbrace{f(a) + (y-a)f'(a)}^{\text{1st order approximations}} + (y-a)^2\frac{f''(a)}{2} + \ldots$$

Let $f=\ell'$, $y=\hat{\theta}^{MLE}$, $a=\theta$, we obtain:

$$\ell'(\hat{\theta}^{MLE}; x_1,\ldots,x_n) = \ell'(\theta; x_1,\ldots,x_n) + (\hat{\theta}^{MLE}-\theta)\ell''(\theta; x_1,\ldots,x_n) + \frac{(\hat{\theta}^{MLE}-\theta)^2}{2}\ell'''(\ ) + \ldots$$

If you assume the technical conditions on p 516 C&B and a large enough sample size n, then the first order approximation can be employed:

$$\ell'(\hat{\theta}^{MLE}; x_1,\ldots,x_n) = \ell'(\theta; x_1,\ldots,x_n) + (\hat{\theta}^{MLE}-\theta)\ell''(\theta; x_1,\ldots,x_n)$$

$$\hat{\theta}^{MLE} := \operatorname{argmax}\{\mathcal{L}(\theta; x_1,\ldots,x_n)\} = \operatorname{argmax}\{\ell(\theta; x_1,\ldots,x_n)\}$$

$$\Rightarrow \text{Solve for } \theta \text{ in: } \ell''(\theta; x_1,\ldots,x_n) \overset{set}{=} 0$$

$$\Rightarrow 0 = \ell'(\theta; x_1, \dots, x_n) + (\hat\theta^{MLE} - \theta)\,\ell''(\theta; x_1, \dots, x_n)$$

$$\Rightarrow \hat\theta^{MLE} - \theta = \frac{-\ell'(\theta; x_1,\dots,x_n)}{\ell''(\theta; x_1,\dots,x_n)} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \frac{\frac{1}{n}\ell'(\theta; x_1,\dots,x_n)}{-\frac{1}{n}\ell''(\theta; x_1,\dots,x_n)}$$

Multiply both sides by $1\Big/\sqrt{\frac{I(\theta)^{-1}}{n}}$

$$\frac{\hat\theta^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} = \frac{\frac{1}{n}\ell'(\theta; x_1,\dots,x_n)}{-\frac{1}{n}\ell''(\theta; x_1,\dots,x_n)\sqrt{\frac{I(\theta)^{-1}}{n}}} \cdot \frac{I(\theta)}{I(\theta)}$$

$$= \underbrace{\frac{I(\theta)}{-\frac{1}{n}\ell''(\theta; x_1,\dots,x_n)}}_{\hat A} \cdot \underbrace{\frac{\frac{1}{n}\ell'(\theta; x_1,\dots,x_n)}{\sqrt{\frac{I(\theta)}{n}}}}_{\hat B}$$

If we can prove that $\hat A \xrightarrow{P} 1$, $\hat B \xrightarrow{d} N(0,1)$, then we're done by Slutsky's theorem.

$\boxed{\text{Proof } \hat A \xrightarrow{P} 1}$

Recall $\ell'(\theta; x_1,\dots,x_n) = \sum_{i=1}^n \ell'(\theta; x_i)$ Lec 9, def 7.8 of score function.

$\Rightarrow \ell''(\theta; x_1,\dots,x_n) = \sum_{i=1}^n \ell''(\theta; x_i)$

$$-\frac{1}{n}\ell''(\theta; x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^n -\ell''(\theta; x_i) = \frac{1}{n}\sum Y_i = \bar Y \xrightarrow{P} E[Y] = I(\theta)$$

let $Y_i := -\ell''(\theta; x_i)$

$E[Y_i] = E[-\ell''(\theta; x_i)] = \overset{HW}{\cdots} = I(\theta)$

By Thm 5.5.4, $\hat A \xrightarrow{P} 1$

$\boxed{\text{Proof } \hat B \xrightarrow{d} N(0,1)}$

$$\frac{1}{n}\ell'(\theta; x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^n \ell'(\theta; x_i) = \frac{1}{n}\sum W_i = \bar W$$

Let $W_i := \ell'(\theta; x_i)$

By the CLT, $\dfrac{\bar{w} - E[\bar{w}]}{SE[\bar{w}]} \xrightarrow{d} N(0,1)$

$$E[\bar{w}] = E[w] = E[\ell'(\theta; x_i)] = 0 \quad \text{by fact 1b, lec 9}$$

$$SE[\bar{w}] = \sqrt{\frac{Var[w]}{n}} = \sqrt{\frac{I(\theta)}{n}}$$

$$Var[w] = E[w^2] - E[\cancel{w}]^2 = E[\ell'(\theta; x_i)^2] = I(\theta)$$

$$\hat{B} = \frac{\bar{w}}{\sqrt{\frac{I(\theta)}{n}}} = \frac{\bar{w} - E[\bar{w}]}{SE[\bar{w}]} \xrightarrow{d} N(0,1)$$

Concludes the proof of the asymptotic normality and the asymptotic efficiency of the MLE.

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} \xrightarrow{d} N(0,1)$$

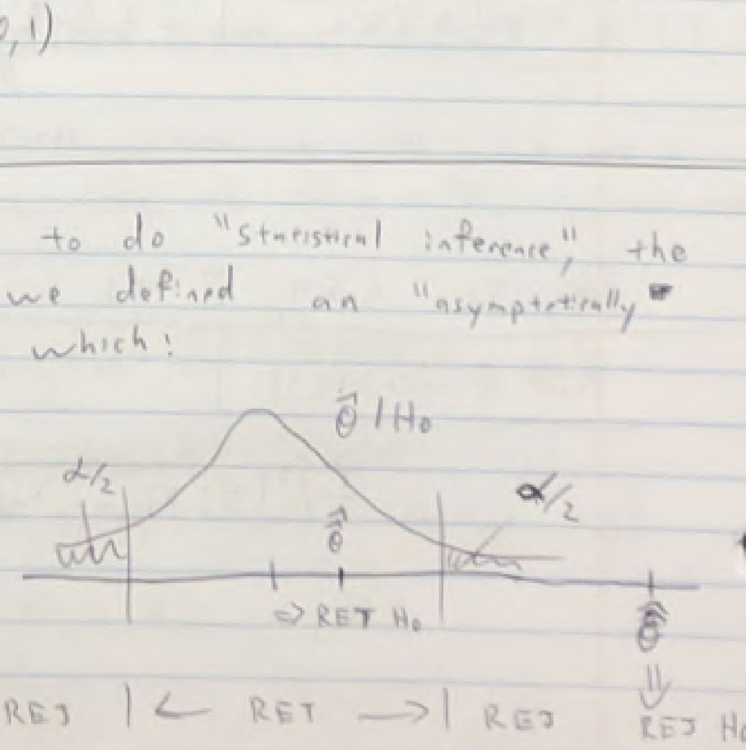
By one more use of Slutsky's, the above implies:

$$\frac{\hat{\theta} - \theta}{SE[\hat{\theta}]} \xrightarrow{d} N(0,1) \Rightarrow \frac{\hat{\theta} - \theta}{\hat{SE}[\hat{\theta}]} \xrightarrow{d} N(0,1)$$

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}} \xrightarrow{d} N(0,1)$$

Lets use these theorems to do "statistical inference", the name of the class. Recall we defined an "asymptotically normal estimator" as one which:

$$\frac{\hat{\theta} - \theta}{SE[\hat{\theta}]} \xrightarrow{d} N(0,1)$$



Using an asymptotically normal estimator (whether the normality comes from the CLT directly or from the fact that the MLE is ~~approximately~~ asymptotically normal) to create an approximate z test is called a "Wald Test" (p 153 AoS). We've seen a Wald test before: the 1 proportion z test. Let's review that.

Lec 1: $n=20$, 12 had iphones $\Rightarrow \hat{\theta} = \bar{X}$, $\hat{\theta} = 0.6$

Lec 4: ~~Ha~~ $H_a: \theta \neq 0.524$, $H_0: \underset{\sim}{\theta = 0.524}$ DGP $\overset{iid}{\sim} \text{Bern}(\theta)$

Generally $\dfrac{\hat{\theta}-\theta}{SE[\hat{\theta}]} = \dfrac{\hat{\theta}-\theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \xrightarrow{d} N(0,1)$

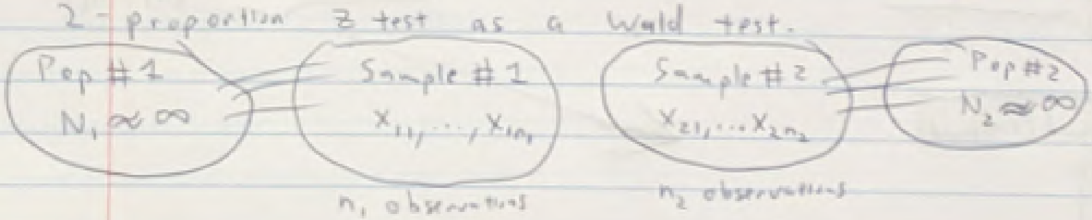
Under H_0, $\dfrac{\hat{\theta} - 0.524}{\underbrace{\sqrt{\frac{0.524(1-0.524)}{20}}}_{0.112}} \overset{\cdot}{\sim} N(0,1)$

$$\hat{\hat{\theta}}_{std} = \frac{\hat{\theta}-\theta}{SE[\theta]} = \frac{0.6-.524}{.112} = 0.678 \in RET = [-1.96, 1.96]$$

at $\alpha = 5\%$

$\Rightarrow$ Retain H_0.

We never saw a 2-proportion z test. We will now derive the approximate 2-proportion z test as a Wald test.



DGP: $X_{11}, \ldots, X_{1n_1} \overset{iid}{\sim} \text{Bern}(\theta_1)$ independent of $X_{21}, \ldots, X_{2n_2} \overset{iid}{\sim} \text{Bern}(\theta_2)$

$H_a: \theta_1 \neq \theta_2 \Leftrightarrow \theta_1 - \theta_2 \neq 0$, $H_0: \theta_1 = \theta_2 \Leftrightarrow \theta_1 - \theta_2 = 0$

Now we pick an estimator that can reflect a departure from H_0

Why not $\hat{\hat{\theta}}_1 - \hat{\hat{\theta}}_2$?

We need another fact from probability theory. HW in 368...
$X_1, \dots, X_{n_1} \overset{iid}{\sim}$ with mean μ_1, variance σ_1^2 independent of
$Y_1, \dots, Y_{n_2} \overset{iid}{\sim}$ with mean μ_2, variance σ_2^2 then...

$$\frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \xrightarrow{d} N(0,1)$$

implies

$$\frac{(\hat{\theta}_1 - \hat{\theta}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}}} \xrightarrow{d} N(0,1)$$

Under $H_0: \underbrace{\theta_1 = \theta_2}_{= \theta_{shared}} \iff \theta_1 - \theta_2 = 0$

$$\Rightarrow \frac{(\hat{\theta}_1 - \hat{\theta}_2) - (0)}{\sqrt{\frac{\theta_{shared}(1-\theta_{shared})}{n_1} + \frac{\theta_{shared}(1-\theta_{shared})}{n_2}}} = \frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\theta_{shared}(1-\theta_{shared})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$