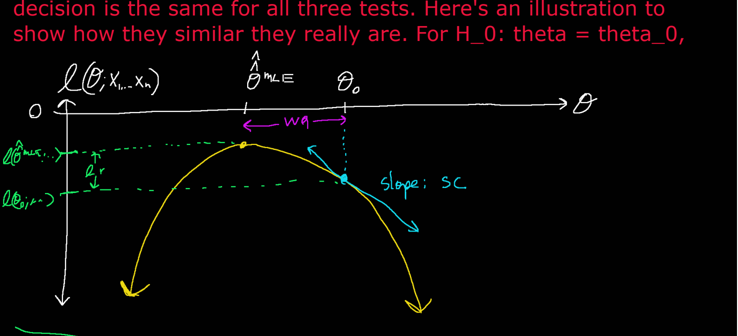
Thm: The Wald test, the score test and the LR test are all "asymptotically equivalent" which means as n gets large, the decision is the same for all three tests. Here's an illustration to show how they similar they really are. For H_0: theta = theta_0,



How about a general test for goodness of fit? Pearson chisquared required K categories. What if I have the following data:

x_1 = 1.73, x_2 = -0.49, x_3 = 0.93, x_4 = 2.16, x_5 = 0.03

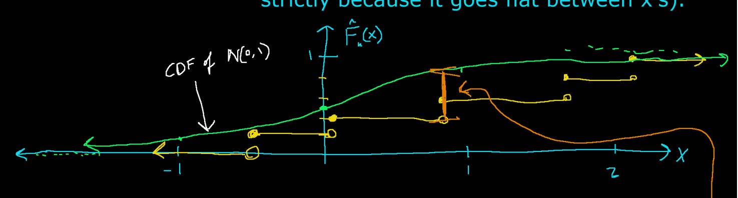
and I want to test against H_0: DGP is iid N(0, 1) vs
H_a: any other DGP

You can't use the LRT here because the LRT would force you to have an H_a: N(theta_1, theta_2). For continuous data (DGP), we can use the Kolmogorov-Smirnov (KS) test. Here's how it works.

We first compute the "empirical distribution function", Fhat_n:

$$\hat{F}_n(x) := \frac{\#\{x_i \leq x\}}{n}$$

which is monotonically increasing (but not strictly because it goes flat between x's).



Fhat_n is a "function estimator" for the "true" CDF, F(x).

If H_0 is assumed, then I assume a DGP explicitly which means I assume the CDF of the data explicitly, $F(x) = F_{H_0}(x)$.

So now to test, we need a test statistic that gauges the data's departure from H_0. What should that look like?

$$D_n = \text{difference}\left(\hat{F}_n(x), F_{H_0}(x)\right)$$

The KS test uses the "supremum norm difference" which means it measures the "largest absolute difference between the two over all x":

$$D_n := \sup_x \left\{ \left| \hat{F}_n(x) - F_{H_0}(x) \right| \right\} \leq 1 \qquad \text{e.g}$$

Advanced note: the Glivenko-Cantelli thm (1933) proves that D_n converges to zero under H_0. This means that the empirical distribution function converges to the real CDF at all x. They also prove that if H_0 is false then it converges to something >0 meaning that the power of the KS test converges to 100%.

We need the sampling distribution to see if our sample d_n value (thetahathat) is within tolerance limits / chance variation of H_0 in order to decide "retain H_0" or "reject H_0". Advanced note: Kolmogorov proved in 1933 that:

$$\sqrt{n}\, D_n \xrightarrow{d} K,$$ the "Kolmogorov distribution", an amazing "distribution-free" result kind of like the CLT.

$$\Downarrow$$

$$\sqrt{n}\, D_n \stackrel{\cdot}{\sim} K$$

Tables of critical values are precomputed. For example at alpha = 5%, the criticall cutoff is a K value of 1.359. However, these critical value are very approximate for n < 50, so there are better tables for finite n. We won't bother with that in 369.

There is an extension to the KS test for non-continuous DGP's which we won't cover in this class.

A major limitation to the KS test is you need a null hypothesis which is an explicit DGP i.e. parameter values specified e.g. H_0: iid N(0,1). You can't say H_0: normal! You need a different test for that situation. One example is the Shapiro-Wilk test which we don't study.

What if you have the following setup: you are sampled from two different populations independently:

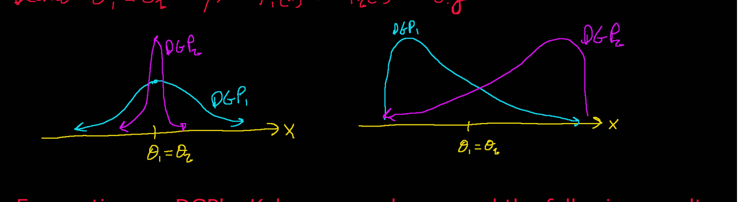
$$X_{11}, X_{12}, \ldots, X_{1n_1} \stackrel{iid}{\sim} DGP_1 \quad \text{independent of} \quad X_{21}, X_{22}, \ldots, X_{2n_2} \stackrel{iid}{\sim} DGP_2.$$

and you want to test against
H_0: DGP_1 = DGP_2 i.e. F_1(x) = F_2(x) vs
H_a: DGP_1 is not the same as DGP_2 i.e. F_1(x) not= F_2(x).

let $\theta_1 := E[X_1]$ ie DGP_1, $\theta_2 := E[X_2]$ ie DGP_2

If $\theta_1 \neq \theta_2 \Rightarrow F_1(x) \neq F_2(x)$ so why not just test via Wald H_a: $\theta_1 \neq \theta_2$?

Because $\theta_1 = \theta_2 \not\Rightarrow F_1(x) = F_2(x)$ e.g



For continuous DGP's, Kolmogorov also proved the following result:

$$\sqrt{\frac{n_1 n_2}{n_1 + n_2}}\, D_{n,m} \xrightarrow{d} K \quad \text{where} \quad D_{n,m} := \sup_x \left\{ \left| \hat{F}_1(x) - \hat{F}_2(x) \right| \right\}$$

The Anderson-Darling (AD) test is very similar (same setup for both one-sample and two-sample goodness of fit tests) so we won't study it. For non-continuous you can use the Mann-Whitney U test, but we won't study that either.

The 2-sample KS, AD, U tests are examples of "nonparametric tests" which means we make no explicit assumptions on the functional forms of the DGP's. They're also called "distribution-free" tests.

There is a completely different way of doing this type of testing which are called "resampling methods". And there are many different ones:

group invariance tests

permutation tests

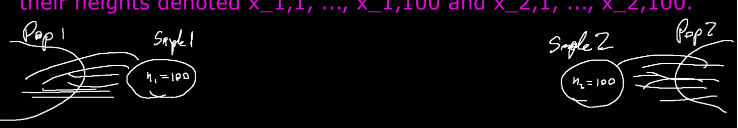
randomization tests

we will now study permutation tests.

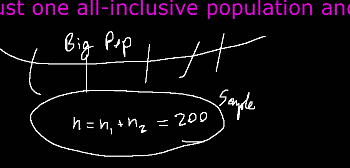
Assume the same setup as the 2-sample KS test. We have iid samples from DGP_1 independent of iid sample form DGP_2 (the two populations) and the same test objection:

$$H_a: DGP_1 \neq DGP_2, \; F_1(x) \neq F_2(x) \quad \text{vs} \quad H_0: DGP_1 = DGP_2, \; F_1(x) = F_2(x)$$

Fisher in 1936 had the following thought experiment. Imagine n_1 = 100 Englishmen and n_2 = 100 Frenchmen and you measure their heights denoted x_1,1, ..., x_1,100 and x_2,1, ..., x_2,100.



Under H_0, the DGP's are the same so there's no distinction between population 1 (sample 1) and population 2 (sample 2). So we can imagine just one all-inclusive population and one sample:



You've seen this idea before. Remember the 2-prop z-test? We estimated theta with $\hat{\hat{\theta}}_{pooled} := \frac{\sum x_{1i} + \sum x_{2i}}{n_1 + n_2}$.

Thus, we can imagine arbitrarily dividing this "master sample" into two 100-sized pieces. To draw the first piece, take a random sample of the master sample of size 100 and then second piece is the data left over:

1st fake sample 1: some subset of size n_1 of $\{x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots x_{2,n_2}\}$

1st fake sample 2: $\{x_{11}, \ldots x_{1n_1}, x_{21}, \ldots x_{2n_2}\} \setminus$ 1st fake sample 1.

Draw B of these fake samples where B is a large number.

Let $I_{b,1} \subset \{1, 2, \ldots, n\}$ and $I_{b,2} \subset \{1, 2, \ldots, n\}$, $|I_{b,1}| = n_1$, $|I_{b,2}| = n_2$, $I_{b,1} \cup I_{b,2} = \{1, 2, \ldots, n\}$ where I represents a set of indices in bth fake sample 1 and the bth fake sample 2.

Now calculate some statistic thetahathat_b which measures departure from H_0 and there are lots of choices:

(a) $\hat{\hat{\theta}}_b = \bar{X}_{b,1} - \bar{X}_{b,2} = \frac{1}{n_1} \sum_{i \in I_{b,1}} x_i - \frac{1}{n_2} \sum_{i \in I_{b,2}} x_i$

(b) $\hat{\hat{\theta}}_b = Med[\{x_i : i \in I_{b,1}\}] - Med[\{x_i : i \in I_{b,2}\}]$

(c) $\hat{\hat{\theta}}_b = d_{n_1,n_2}$ from the 2-sample KS test

(d) $\hat{\hat{\theta}}_b = \frac{\bar{X}_{b,1}}{\bar{X}_{b,2}}$