IRT Model

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- The central issue of doing IRT model is to scale the distance between the latent positions of individuals. Consider the following problem:
- A legislator i has an ideal point θ_i. If he votes for bill j, he derives utility $u_i(\eta_j) = -(\theta_i - \eta_j)^2 + \epsilon_{ij}^{[\eta]}$. If he votes against the bill, his utility is $u_i(s_j) = -(\theta_i - s_j)^2 + \epsilon_{ij}^{[s]}$, where s_j is the status quo. The terms $\epsilon_{ij}^{[s]}$ and $\epsilon_{ij}^{[\eta]}$ are i.i.d. standard normal variables (this is a random utility model). The legislator votes for bill $j = 1, \ldots, J$ if and only if $u_i(\eta_j) > u_i(s_j)$.

IRT model

- The probability that a legislator i votes for proposition j is given by:

$$Pr(y_{i,j} = 1|\theta_i) = Pr(u_i(\eta_j) > u_i(s_j)).$$

Replacing the values of $u_i(\eta_j)$ and $u_i(s_j)$, we have the following expression:

$$Pr(y_{i,j} = 1|\theta_i) = Pr(-(\theta_i - \eta_j)^2 + \epsilon'_{i,j} > -(\theta_i - s_j)^2 + \epsilon_{i,j})$$

$$= Pr((\theta_i - s_j)^2 - (\theta_i - \eta_j)^2 > \epsilon_{i,j} - \epsilon'_{i,j})$$

$$= Pr(2\theta_i(s_j - \eta_j) + (\eta_j^2 - s_j^2) > \epsilon_{i,j} - \epsilon'_{i,j})$$

$$= Pr(\theta_i d_j - b_j > \epsilon_{i,j} - \epsilon'_{i,j}), \quad where \quad d_j = 2(s_j - \eta_j) \quad and \quad b_j = s_j^2 - \eta_j^2.$$

So we have: $Pr(y_{i,j} = 1|\theta_i) = \Phi(\theta_i d_j - b_j)$. d_j here is an item-discrimination parameter and b_j is an item-difficulty parameter.

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- To express the parameters of the voting model as functions of the estimated IRT parameters we use Bayes rules:

$$P(\boldsymbol{\theta}, d_j, b_j|y) \propto P(y|\boldsymbol{\theta}, d_j, b_j)P(\boldsymbol{\theta})P(b)P(d)$$

$$P(\boldsymbol{\theta}, d_j, b_j|y) \propto \left(\prod_{i=1}^{N} \prod_{i=1}^{J} \Phi(\theta_i d_j + b_j)^{y_{ij}} (1 - \Phi(\theta_i d_j + b_j))^{1-y_{ij}} \right) P(\boldsymbol{\theta})P(b)P(d).$$

IRT Model

- The key steps in Stan are to specify the prior distributions for d, b and θ, and also specify the model that links the choice and parameters.

```
model {
  alpha ~ normal(0, 25);
  beta ~ normal(0, 25);
  theta ~ normal(0, 1);
  for (n in 1:N)
    y[n] ~  bernoulli_logit(theta[j[n]] * beta[k[n]] - alpha[k[n]]);
}
```

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- One reasonable modeling assumption to make is to have a hierarchical structure for ideology:

$$\gamma_j \sim N(\beta_{party} * PartyID_j, \sigma^2),$$

$$\theta_j \sim N(\gamma_j, \tau^2)$$

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- Since we have one more parameter β_{party} in our model, we need to give a prior for this parameter as well.
- In Stan, we just have to add this:

```
beta_party ~ normal(0, 2);
for (i in 1:J){
  theta[i] ~ normal(gamma[i], 1);
  gamma[i] ~ normal(beta_party * party[i], 1); }
```