Derivation of Normal Distribution Conjugate

Junlong Aaron Zhou

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Here we show the normal distribution is the conjugate prior for the mean of normal distribution.

The setup is like follows:

$$
\begin{aligned}
\mu &\sim N(\mu_0, \sigma_0^2) \\
y_i &\sim N(\mu, \sigma^2)
\end{aligned}
$$

The derivation is as follows:

$$
\begin{aligned}
P(\mu|Y) &= \frac{\text{Likelihood} \times \text{Prior}}{P(Y)} \\
&\propto \prod \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}} * \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}} \\
&\propto exp(-\frac{1}{2}(\frac{\sum(y_i-\mu)^2}{\sigma^2} + \frac{(\mu-\mu_0)^2}{\sigma_0^2})) \\
&= exp(-\frac{1}{2}(\frac{\sum(y_i^2 - 2\mu y_i + \mu^2)}{\sigma^2} + \frac{\mu^2 - 2\mu\mu_0 + \mu_0^2}{\sigma_0^2})) \\
&\propto exp(-\frac{1}{2}(\frac{\mu * 2n\bar{y_i} + n\mu^2}{\sigma^2} + \frac{\mu^2 - 2\mu\mu_0}{\sigma_0^2})) \\
&= exp(-\frac{1}{2}(\mu^2 * (\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}) - \mu * (\frac{2n\bar{y_i}}{\sigma^2} + \frac{2\mu_0}{\sigma_0^2}))) \\
&= exp(-\frac{1}{2})(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2})(\mu^2 - 2\mu * ((\frac{n\bar{y_i}}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}) * \frac{1}{(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2})}) \\
&= exp(-\frac{1}{2})(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2})(\mu^2 - 2\mu * \frac{\sigma_0^2 n\bar{y_i} + \sigma^2\mu_0}{\sigma_0^2 n + \sigma^2}) \\
&= exp(-\frac{1}{2})(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2})(\mu^2 - 2\mu * \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}) \\
&\propto exp(-\frac{1}{2})(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2})(\mu - \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}})^2
\end{aligned}
$$

where for all the "proportional to" part, note we can disgard any term unrelated to μ, because it would only serve as a constant term in the posterior distribution. Similarly, in the last line, we just add a constant to make it a quadratic form.

So we arrive at a normal distribution with mean being $\frac{\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}$ and variance being $\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}$.