Quant III

Lab 5: Beyesian: Basics

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October 09, 2020

Outline

- Homework Question
- Bayes Basics
- Estimates and Inference

Homework Question:

- Uniform distribution $U(0, \theta)$
- If θ is known:

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- Therefore: $f(y) = \frac{1}{\theta}\mathbb{I}(y \leq \theta)$
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- $\hat{\theta}_{MLE}$ is the smallest possible value, i.e. $max\{y_i\}$

Negative Binomial Model

- Remind me later.

Basic Bayes

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Basic Bayes

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- Frequentist v.s.Bayesian
  - Natural, fixed, unknown parameter θ v.s. distribution of θ
  - Infinite sampling (hypothetical)/ Asymptotics v.s. Limited observations
- Take-away: different way of thinking.

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- Kernel: $P(Y|\theta)P(\theta)$. Once you know this, you know the distribution.

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- Choose prior is important
  - Conjugate prior: get an analytic solution
  - Otherwise: Numerically approximate the posterior (MCMC, VI).
  - Computational convenience, and/or subjective knowledge
  - Sometimes: uninformative prior

Conjugate prior

- Beta is conjugate prior for Bernoulli distribution.
- What does that mean?
- Prior Beta, likelihood derived from Bernoulli $\rightarrow$ Posterior Beta.

Conjugate prior ctd.

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- Consider $\theta \sim N(\mu_0, \sigma_0^2)$
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$$\propto e^{-\frac{1}{2}(\frac{1}{\sigma^2}\sum(\theta-y_i)^2+\frac{1}{\sigma_0^2}(\theta-\mu_0)^2)}$$

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$$\theta|Y \sim N(\frac{\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}}{\sigma_0^2+\frac{n}{\sigma^2}}, (\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2})^{-1})$$

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- We know the whole posterior distribution.
  - Expected value of θ?
  - What is the most likely value of θ? (MAP)
  - The probability of $\theta > 0$?

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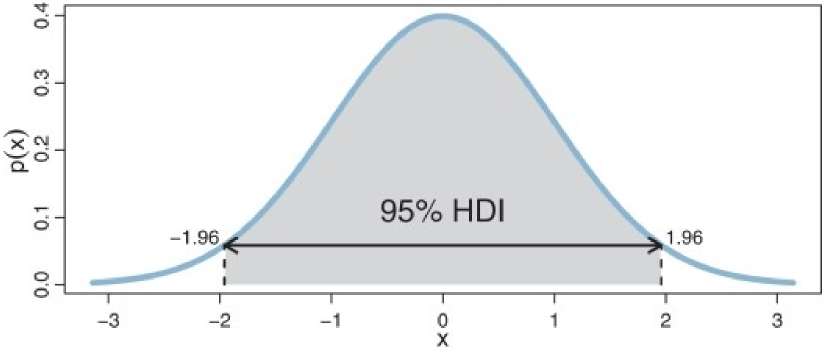
- Bayesian: We know the whole posterior distribution, and therefore:
  - Credible Interval
  - We call set A is 95% Credible interval of θ if $Pr(\theta \in A) = \int_A P(\theta|y)d\theta = 0.95$
- Difference?

Inference ctd.

- Symmetric credible interval: bounded by α and $1 - \alpha$ quantile.
- Highest posterior density interval (HPD): the density inside the region has to be higher than at any point outside the interval.

HDI



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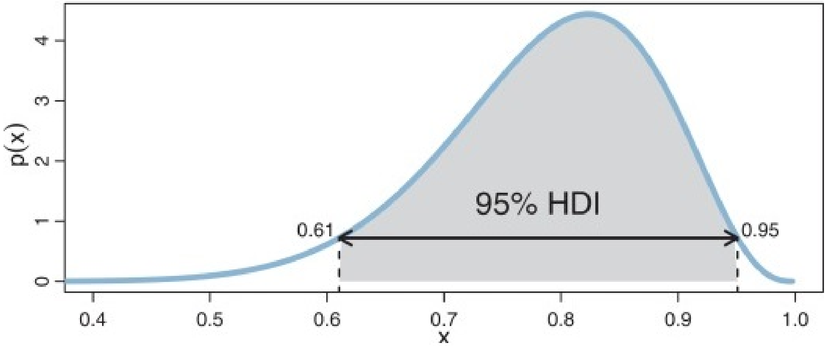

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