Rules of Probability

Bayesian Modeling for Socio-Environmental Data

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Road map

- Rules of probability
  - Conditional probability
  - Independence
  - The law of total probability
- Factoring joint probabilities

Why do we need to know this stuff?

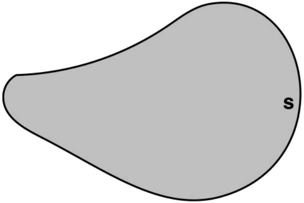
1. **Conditional probability** foundational for all the inferences that we make.
2. **The law of total probability** is the denominator of Bayes' Theorem.
3. **Factoring** joint distributions is how we deal with complexity, reducing high dimensional problems.
4. **Independence** allows us to simplify fully factored joint distributions.

Random variables

- are quantities governed by chance.
- have a specific value called an *events* or *outcomes*.
- are summarized by probability distributions.
- *Bayesians treat every unobserved quantity as a random variable*.

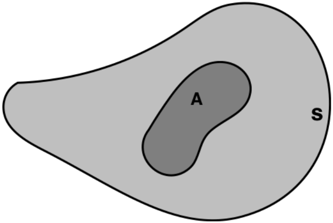
S=Sample Space

- The set of all possible events or outcomes of an experiment or survey.
- The sample space, S has a specific area.

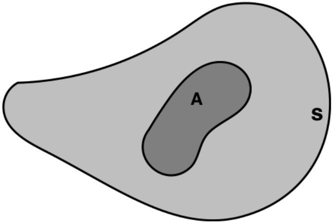


Events in S

- Can define and event, A.
- The area of event A is less than S.



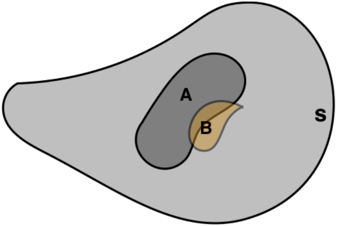
What is the probability of event A?



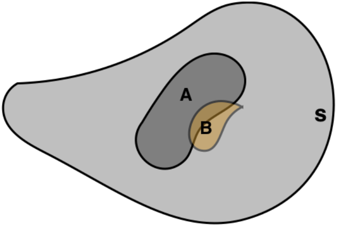
$\Pr(A) = \frac{\text{Area of } A}{\text{Area of } S}$

Conditional Probability

Conditional probability: the probability of an event given that *we know* another event has occurred.

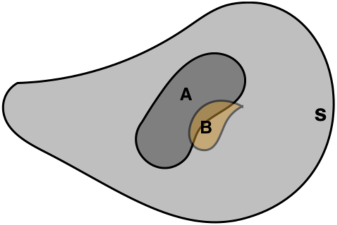


What is the probability of event B, given that event A has occurred?



$\Pr(B|A) =$ probability of B conditional on knowing A has occurred

What is the probability of event B, given that event A has occurred?



$$Pr(B|A) = \frac{\text{Joint Probability}}{\text{Probability of A}} = \frac{\Pr(A,B)}{\Pr(A)}$$

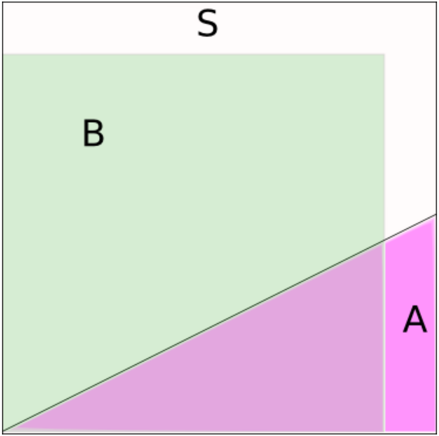
If the occurance of event A does not tell us anything about event B?

In this case, events A and B are said to be ***independent***

Events are independent if and only if...

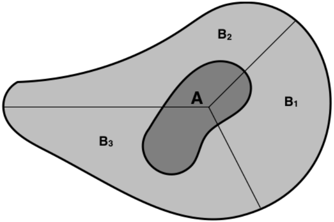
$\Pr(A|B) = \Pr(A)$

Assuming independence, the joint probability of event A and event B



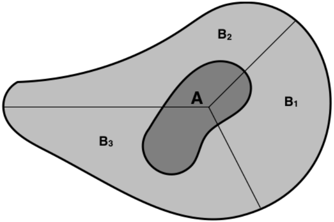
$\Pr(A, B) = \Pr(A)\Pr(B)$

The Law of Total Probability



We can define a set of events $\{B_n : n = 1, 2, 3, ...\}$, which taken together define the entire sample space, $\sum_n B_n = S$.

What is the probability of event A?



$\Pr(A) = \sum_n \Pr(A|B_n)\Pr(B_n)$ (discrete case)

$\Pr(A) = \int \Pr(A|B)\Pr(B)dB$ (continuous case)

The Chain Rule of Probability

The chain rule of probability allows us to calculate any number of joint distributions using only conditional probabilities.

$$\Pr(z_1, z_2, ..., z_n) = \Pr(z_n | z_{n-1}, ..., z_1) ... \Pr(z_3 | z_2, z_1) \Pr(z_2 | z_1) \Pr(z_1)$$

Notice the pattern here.

- z's can be scalars or vectors.
- Sequence of conditioning doesn't matter.
- When we build models, we choose a sequence that makes sense.

Chain rule of probability board work and independence

Factoring joint probabilities

Why is factoring useful?

- The rules of probability allow us to simplify complicated joint distributions, breaking them down into chunks.
- Chunks can be analyzed one at a time.
- Provide a usable graphical and mathematical foundation, *critical* for the model specification step.

Consider a Bayesian Network (represented by a directed acyclic graph or DAG)

- Bayesian networks specify how joint distributions are factored into conditional distributions using nodes to represent RV's and arrows to represent dependencies.
- Nodes at the heads of arrows *must* be on the left hand side of the conditioning symbols;
- Nodes at the tails of arrows are on the right hand side of the conditioning symbols.
- Any node at the tail of an arrow without and arrow leading into it must be expressed unconditionally.

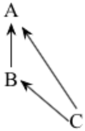
Factoring with DAGs

$\Pr(A, B) =$

Factoring with DAGs

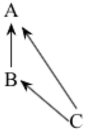
$\Pr(A, B) = \Pr(A|B)\Pr(B)$

Factoring with DAGs



$\Pr(A, B, C) =$

Factoring with DAGs



$\Pr(A, B, C) = \Pr(A|B, C)\Pr(B|C)\Pr(C)$

Work on lab

Complete parts I-VI