Moment Matching

Models for Socio-Environmental Data

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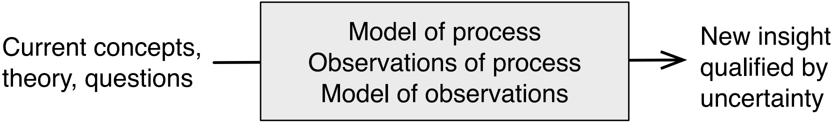
Roadmap

- The rules of probability
  - conditional probability
  - independence
  - the law of total probability
- Factoring joint probabilities
- Probability distributions for discrete and continuous random variables
- Marginal distributions
- Moment matching

Motivation: A general approach to scientific research



Motivation: Why do we need to know this stuff?

Concept to be taught	Why do you need to understand this concept?
Conditional probability	It is the foundation for Bayes' Theorem and all inferences we will make.
The law of total probability	Basis for the denominator of Bayes' Theorem $[y]$
Factoring joint distributions	This is the procedure we will use to build models.
Independence	Allows us to simplify fully factored joint distributions.
Probability distributions	Our toolbox for representing uncertainty
Moments	The way we summarize distributions.
Marginal distributions	Bayesian inference is based on marginal distributions of unobserved quantities.
Moment matching	Allows us to embed the predictions of models into any statistical distribution.

Motivation: flexibility in analysis

Deterministic models	Types of data
general linear	real numbers
nonlinear	non-negative real numbers
differential equations	counts
difference equations	0 to 1
auto-regressive	0 or 1
occupancy	counts in categories
state-transition	proportions in categories
integral-projection	

univariate and
multivariate

Motivation: flexibility in analysis

Probability model	Support for random variable
normal	all real numbers
lognormal	non-negative real numbers
gamma	non-negative real numbers
beta	0 to 1 real numbers
Bernoulli	0 or 1
binomial	counts in 2 categories
Poisson	counts
multinomial	counts in > 2 categories
negative binomial	counts
Dirichlet	proportions in ≥ 2 categories
Cauchy	real numbers

$$\mu_i = g(\boldsymbol{\theta}, x_i)$$

A familiar approach

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_i + \varepsilon_i \\ \varepsilon_i &\sim \text{normal}(0, \sigma^2) \end{aligned}$$

which is identical to

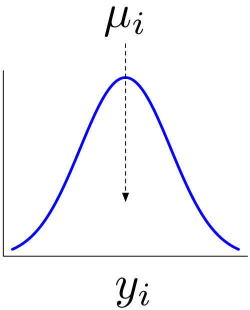
$$\begin{aligned} \mu_i &= \beta_0 + \beta_1 x_i \\ y_i &\sim \text{normal}(\mu_i, \sigma^2) \end{aligned}$$

A familiar approach

$$\boldsymbol{\theta} = (\beta_0, \beta_1)'$$

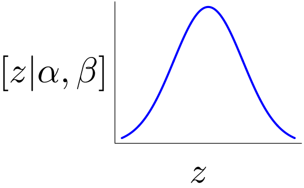
$$\mu_i = g(\boldsymbol{\theta}, x_i) = \beta_0 + \beta_1 x_i$$

$$\underbrace{[y_i|\mu_i, \sigma^2]}_{\text{normal}}$$



The problem

All distributions have parameters:



α and β are parameters of the distribution of the random variable z.

Types of parameters

Parameter name	Function
intensity, centrality, location	sets position on x axis
shape	controls dispersion and skew
scale, dispersion parameter	shrinks or expands width
rate	scale^{-1}

The problem

The normal and the Poisson are the only distributions for which the parameters of the distribution are the *same* as the moments. For all other distributions, the parameters are *functions* of the moments.

$$
\begin{aligned}
\alpha &= m_1(\mu, \sigma^2) \\
\beta &= m_2(\mu, \sigma^2)
\end{aligned}
$$

We can use these functions to “match”' the moments to the parameters.

Moment matching

$$
\begin{aligned}
\boldsymbol{\mu}_i &= g(\boldsymbol{\theta}, x_i) \\
\boldsymbol{\alpha} &= m_1(\boldsymbol{\mu}_i, \boldsymbol{\sigma}^2) \\
\boldsymbol{\beta} &= m_2(\boldsymbol{\mu}_i, \boldsymbol{\sigma}^2) \\
& [y_i | \boldsymbol{\alpha}, \boldsymbol{\beta}]
\end{aligned}
$$

Moment matching the gamma distribution

The gamma distribution: $[z|\alpha, \beta] = \frac{\beta^\alpha z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)}$
The mean of the gamma distribution is

$$\mu = \frac{\alpha}{\beta}$$

and the variance is

$$\sigma^2 = \frac{\alpha}{\beta^2}.$$

Discover functions for α and β in terms of μ and σ^2.

Note: $\Gamma(\alpha) = \int_0^\infty t^\alpha e^{-t} \frac{\mathrm{d}t}{t}$

Moment matching the beta distribution

The beta distribution gives the probability density of random variables with support on $0, ..., 1$.

$$[z|\alpha,\beta] = \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)}$$

$$B = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\mu = \frac{\alpha}{\alpha+\beta}$$

$$\alpha = \frac{\mu^2 - \mu^3 - \mu\sigma^2}{\sigma^2}$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\beta = \frac{\mu - 2\mu^2 + \mu^3 - \sigma^2 + \mu\sigma^2}{\sigma^2}$$

You need some functions...

```
#BetaMomentMatch.R
# Function for parameters from moments
shape_from_stats <- function(mu, sigma){
 a <-(mu^2-mu^3-mu*sigma^2)/sigma^2
 b <- (mu-2*mu^2+mu^3-sigma^2+mu*sigma^2)/sigma^2
shape_ps <- c(a,b)
return(shape_ps)
}
# Functions for moments from parameters
beta.mean=function(a,b)a/(a+b)
beta.var = function(a,b)a*b/((a+b)^2*(a+b+1))
```

Moment matching for a single parameter

We can solve for α in terms of μ and β,

$$\begin{aligned} \mu &= \frac{\alpha}{\alpha+\beta} \\ \alpha &= \frac{\mu\beta}{1-\mu}, \end{aligned}$$

which allows us to use

$$\begin{aligned} \mu &= g(\theta, x) \\ y &\sim \text{beta}\left(\frac{\mu\beta}{1-\mu}, \beta\right) \end{aligned}$$

to moment match the mean alone.

Moment matching for a single parameter

The first parameter of the lognormal $= \alpha$, the mean of the random variable on the log scale. The second parameter $= \sigma^2_{\log}$, the variance of the random variable on the log scale
We often moment match the median the lognormal distribution:

$$\begin{aligned} \text{median} = \mu &= g(\theta, x) \\ \mu &= e^{\alpha} \\ \alpha &= \log(\mu) \\ y &\sim \text{lognormal}(\log(\mu), \sigma^2_{\text{log}}) \end{aligned}$$

In this case, σ^2 remains on log scale.

Problems continued

Do section on Moment Matching