

# 1 Motivation

The ability of Bayesian methods to handle hierarchical models in an unusually tidy way is why they are becoming the first choice for complex problems in ecology and social science, problems with multiple unknowns, sources of data and sources of uncertainty. Recall that the posterior distribution of all of the unobserved quantities is proportionate to the joint distributions of the unobserved quantities and the data:

$$[\boldsymbol{\theta} \mid \mathbf{y}] \propto \underbrace{[\boldsymbol{\theta}, \mathbf{y}]}_{\text{Factor into sensible parts}}$$

It follows that the starting point for developing hierarchical models is to write a properly factored expression for the proportionality between the posterior and joint distribution of the observed and unobserved quantities. Properly means that the expression for the factored joint distribution obeys the chain rule of probability after assumptions about independence have been made. Bayesian networks, also called directed acyclic graphs (or, unattractively in my view, DAGs), offer a way to visually assure that your model does so. This will be true if there is one unknown and one data set or one hundred unknowns and ten data sets. This factored expression is all that is required to specify a “roll-your-own” MCMC algorithm or to write code in one of the current software packages that sample from the marginal posterior distributions, JAGS, STAN, OpenBUGS etc. The expression for posterior and joint is where you start discussions with statistical colleagues. It must be included in all papers and proposals using Bayesian methods because it communicates virtually everything about where your inferences come from.

Learning to write proper mathematical and statistical expressions for Bayesian models is 90 percent of the battle of learning how to do Bayesian analysis. We will

return to this battle time and time again during this course. In this exercise, we begin to learn the vital skill of model building. The problems increase in difficulty as we proceed, so it will be important to understand what you did right and wrong before we proceed to the next problem. In addition to practice drawing Bayesian networks and writing posterior and joint distributions, the problems will challenge you to:

- Choose distributions appropriate for the support of the random variable.
- Deftly use moment matching to convert means and standard deviations to parameters of distributions.
- Make inferences on derived quantities.

## 2 Instructions

For each problem below, draw the Bayesian network, write the posterior and joint distributions using generic bracket notation with appropriate products. Next, choose specific distributions. At this point, don't worry too much about the specific forms for prior distributions. We will learn more about composing these as the course proceeds. You may use uniform distributions with bounds that are vague for non-negative parameters. Use normal distributions centered on zero with large variances for real-valued parameters. Again, don't sweat this too much.

Work in groups to allow discussion and to teach each other.

## 3 Problems

### 3.1 The Kuznets effect

You are interested in modeling the relationship between per capita income and an index of air pollution for  $i = 1, \dots, 80$  nations around the world. You hypothesize that air pollution increases then declines as per capita income increases (i.e., the Kuznets effect). Choose a deterministic model to represent this humped relationship, bearing in mind of course that air pollution cannot be negative.

#### 3.1.1 A simple model

Start this problem by writing a model for the relationship between air quality for the  $i^{th}$  country measured as an air pollution index  $y_i$ , a continuous, non-negative response variable. The predictor variables is average per-capita income  $w_i$ . Assume both are measured perfectly.

#### 3.1.2 A hierarchical model with raw observations

Now imagine that you have a sample of  $n_i$  observations of the air pollution index in each country<sup>1</sup>. You also have a sample of annual incomes for 1000 individuals from each country. How would you model the effect of income on air pollution to include uncertainty in the response and the predictor?

#### 3.1.3 A hierarchical model with data summaries

Rewrite your model assuming that you didn't have the raw data, but rather the mean and the standard deviation of the mean for the air pollution index and annual income

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<sup>1</sup>For now, we will ignore the possibility of spatial structure in the data within each country.

for each country.

### 3.2 Effect of radon on cancer risk

You seek to understand how radon levels influence risk of cancer. You have data on the incidence of lung cancer in households (1 if cancer is present and 0 if no cancer) and radon levels (a continuous, non-negative number) for 1000 houses in each of 40 counties within a state. You also have data on the clay soil content at the county level. You heroically assume both clay content and radon levels are known without error. How would you model the effect of radon and soil type on the probability of lung cancer? Some hints–

1. What deterministic model would you use to predict the probability of cancer in a household as a function of radon level?
  - (a) What likelihood would you use for these 0 or 1 data?
  - (b) Assume that the intercept in your deterministic model of the effect of radon level on probability of cancer in a household is a linear function of county level clay soil content.

### 3.3 Controls on willow seedling establishment

You are interested in the way that herbaceous plant cover influences establishment of willow seedlings in riparian communities. You have data on the number of willow seedlings that establish on 100  $10 \times 10$  meter plots. Cover is estimated visually on each plot, creating a calibration problem – the visual estimate of cover is not the true cover. To deal with this problem, you obtained visual estimates of cover paired with the actual proportion of vegetated area (measured using many small sub-plots) on 15  $10 \times 10$  m plots. After days of sweaty labor, you regressed visual estimates

$(x_i)$  on the true cover  $(z_i)$  and developed a calibration equation  $h(\boldsymbol{\alpha}, z_i)$ :

$$h(\boldsymbol{\alpha}, z_i) = \frac{e^{\alpha_o + \alpha_1 z_i}}{1 + e^{\alpha_o + \alpha_1 z_i}} \quad (1)$$

$$x_i \sim \text{beta}(m(h(\boldsymbol{\alpha}, z_i), \zeta^2)) \quad (2)$$

$$\alpha_o \sim \text{normal}(.05, .006) \quad (3)$$

$$\alpha_1 \sim \text{normal}(1.07, .13) \quad (4)$$

$$\zeta^2 \sim \text{inverse gamma}(10.2, 630) \quad (5)$$

The function  $m()$  returns parameters of the beta distribution given moments as inputs.

Write a model of willow establishment as a function of cover that models the observation process and the ecological process separately. Hints—think about the predictor variable for herbaceous cover. Do you want to use the observed value of cover  $(x_i)$  or the true value  $(z_i)$  to model its effect on establishment? Use informed prior distributions on  $\alpha_o, \alpha_1$  and  $\zeta^2$  to relate the true, unobserved cover to the observations of cover.

Now presume that the 100 plots are arranged in 5 different stream reaches, 20 plots in each reach. You have data on peak runoff in each of the reaches, which you may assume is measured perfectly. Describe verbally how you might model variation at the catchment scale created by peak runoff.

### 3.4 Advanced: Diversity of a plant community

You have plot-level data on diversity of plant communities. The data consist of counts  $y_{ij}$  of the number of individuals of species  $i$  on  $j = 1, \dots, J$  same-sized plots, and the total number of individuals on plot  $j$  is reported as  $n_j$ . How would you model an

index ( $H$ ) of species diversity across the community, where  $H = -\sum_{i=1}^R \phi_i \log(\phi_i)$ ,  $\phi_i$  is the unobserved proportion of the  $i_{\text{th}}$  species in the community, and  $R$  is the total number of species present? Hints—

1. Model the observed count data as a random variable (a vector) arising from the unobserved vector  $\phi$  of proportions.
2. Take a look at the Dirichlet distribution as a way to form an prior on the vector  $\phi$ . The Dirichlet is to the multinomial likelihood as the beta distribution is to the binomial likelihood. A vague Dirichlet has parameters = 1 for all categories.
3. Calculate  $H$  as a derived quantity of the  $\phi_i$  and  $R$ , which will allow us to obtain a posterior distribution for  $H$  because any quantity that is a function of a random variable becomes a random variable in Bayesian analysis.