Bayesian Dynamic Models

Models for Socio-Environmental Data

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SESYNC

Roadmap

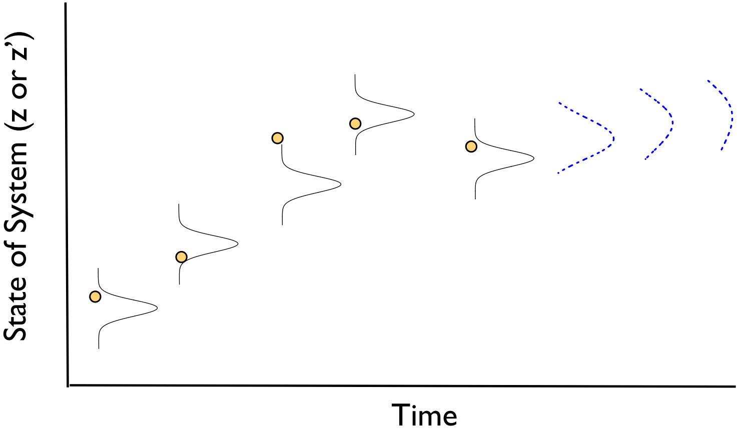
- Overview
- Model types with examples
  - discrete time, single state
  - continuous time (briefly)
- Forecasting
- Coding tips
- Discrete time, multiple states

Dynamic hierarchical models (aka state space models)

Also called “state space” models

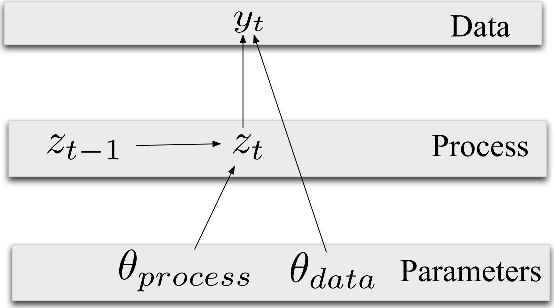
$$\left[y_t \middle| \boldsymbol{\theta}_d, z_t\right]$$
$$\left[z_t \middle| \boldsymbol{\theta}_p, z_{t-1}\right]$$

The idea is simple. We have a stochastic model of an unobserved, true state (z_t) and a stochastic model that relates our observations (y_t) to the true state.


State of System (z or z')
Time

A broadly applicable approach to modeling dynamic processes in ecology

$$[\mathbf{z},\theta_{process},\theta_{data}|\mathbf{y}] \propto \prod_{t=2}^{T}[y_t|\theta_{data},z_t]\,[z_t|\theta_{process},z_{t-1}]\,[\theta_{process},\theta_{data},z_1]$$



Sources of uncertainty in state space models

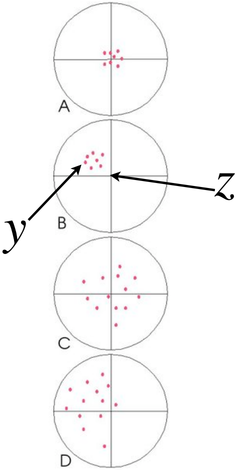
Process uncertainty

- Failure to perfectly represent process
- Propagates in time
- Decreases with model improvement
- Estimation allows forecasting

Observation uncertainty

- Failure to perfectly observe process
- Does not propagate
- Sampling uncertainty decreases with increased sampling effort.
- Measurement uncertainly decreases with improved instrumentation, calibration, etc.

Components of observation uncertainty



- Measurement $\left[y|h\left(z,\boldsymbol{\theta}_d\right),\boldsymbol{\sigma}^2_{measurement}\right]$
- Sampling $\left[y|z,\boldsymbol{\sigma}^2_{sampling}\right]$

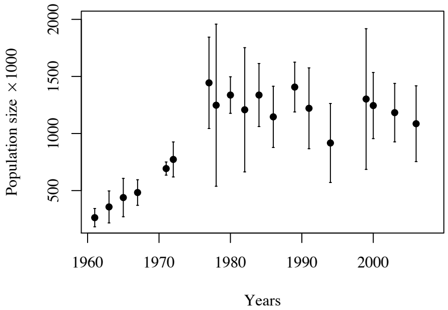
When can we separate process variance from observation variance?

- Replication of the observation for the latent state with sufficient n
- Calibration model with properly estimate prediction variance
- Strongly differing "structure" in process and observation models
- We may not need to separate them–sometimes the observed state and the true state are the same.

General joint and posterior distribution for single state model

$$
\begin{aligned}
\text{Deterministic model} &= g(\boldsymbol{\theta}_{process}, z_{t-1}, \mathbf{x}_{t-1}) \\
\left[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, \boldsymbol{\sigma}_p^2, \boldsymbol{\sigma}_d^2 | \mathbf{y}\right] &\propto \prod_{t=2}^{T} \left[y_t | \boldsymbol{\theta}_{data}, z_t, \boldsymbol{\sigma}_o^2\right] \\
&\times \left[z_t | g(\boldsymbol{\theta}_{process}, z_{t-1}, \mathbf{x}_{t-1}), \boldsymbol{\sigma}_p^2\right] \\
&\times \left[\boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, \boldsymbol{\sigma}_p^2, \boldsymbol{\sigma}_o^2, z_1\right]
\end{aligned}
$$

Modeling the Serengeti wildebeest population



- 48 year time series
- Annual means and standard deviations of population size for 19 years
- Spatially replicated census
- Annual data on dry season rainfall

How does rainfall influence density dependence?

$$g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right) = z_{t-1} e^{(\beta_0 + \beta_1 z_{t-1} + \beta_2 x_{t-1} + \beta_3 z_{t-1} x_{t-1})\Delta t}$$

- $z_t =$ true population size
- $x_{t-1} =$ standardized, annual dry season rainfall during time $t-1$ to t.
- $\beta_0 = r_{max} =$ intrinsic, per-capita rate of increase at average rainfall
- $\beta_1 =$ strength of density dependence, $\frac{r}{K}$ at average rainfall.
- $\beta_2 =$ change in rate of increase per standard deviation change in rainfall
- $\beta_3 =$ effect of rainfall on strength of density dependence

Process model predicting medians

$$z_t \sim \text{lognormal}\left(\log\left(g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right)\right), \boldsymbol{\sigma}_p^2\right)$$

- $\log\left(g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right)\right)$, the centrality parameter, the mean of z_t on the log scale
- $\boldsymbol{\sigma}_p^2$, the scale parameter, the variance of z_t on the log scale
- What does the deterministic model predict?
  - define centrality parameter $= \alpha_t$
  - median$(z_t) = e^{\alpha_t}$
  - $\alpha_t = \log(\text{median}(z_t))$
  - median$\left(z_t\right) = g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right)$

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Review of relationships between normal and lognormal

1. $z_t = g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right) \exp\left(\varepsilon_t\right),\ \varepsilon_t \sim \text{normal}\left(0, \sigma_p^2\right)$
2. $\log\left(z_t\right) = \log\left(g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right)\right) + \varepsilon_t,\ \varepsilon_t \sim \text{normal}\left(0, \sigma_p^2\right)$
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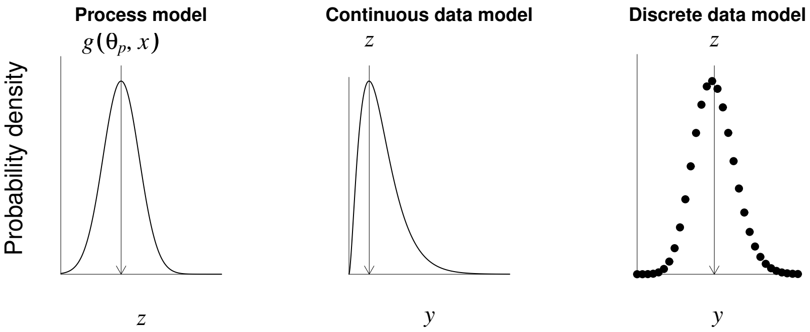
It is also possible to moment match the mean

$$
\begin{aligned}
\boldsymbol{\mu}_t &= g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right) && (1)\\
\boldsymbol{\alpha}_t &= \log(\boldsymbol{\mu}_t) - \frac{1}{2}\log\left(\frac{\boldsymbol{\mu}_t^2 + \boldsymbol{\sigma}^2}{\boldsymbol{\mu}_t^2}\right) && (2)\\
z_t &\sim \text{lognormal}(\boldsymbol{\alpha}_t, \boldsymbol{\sigma}^2) && (3)
\end{aligned}
$$

You should do it this way if you have derived quantities computed as sums of the z_t, for example when modeling a total population from subpopulations in different sites.

Why a continuous distribution for a "discrete state"?



The data

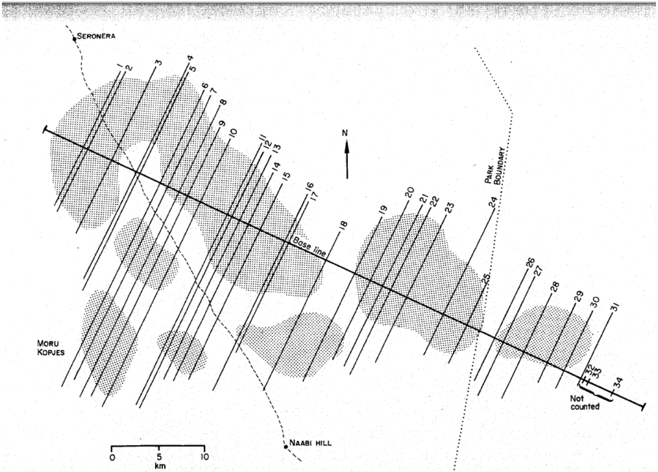


Fig. 2. The orientation of the base-line and of the random transects in the May 1971 sample count. Shading shows approximate positions of the main wildebeest herds.

Observation model

$$y_t \sim \text{normal}\left(z_t, y.sd_t\right)$$

- y_t is the observed mean number of animals across all transects
- $y.sd_t$ is the observed standard deviation across transects
- z_t is the unobserved, true state, the *mean of the data distribution*

We choose a normal distribution for the likelihood because the y_t are the annual mean of means of densities of wildebeest on many transects. For now, we ignore the potential for spatial autocorrelation among transects.

Posterior and joint distributions

$$\left[\mathbf{z},\boldsymbol{\beta},\boldsymbol{\sigma}_p^2|\mathbf{y}\right] \propto \underbrace{\prod_{\forall t\in\mathbf{y}.i}\left[y_t \,\middle|\, z_t, y.sd_t\right]}_{\text{data model}}$$

$$\times \underbrace{\prod_{t=2}^{48}\left[z_t|g\left(\boldsymbol{\beta},z_{t-1},x_{t-1}\right),\boldsymbol{\sigma}_p^2\right]}_{\text{process model}} \times \underbrace{\left[\beta_0\right]\left[\beta_1\right]\left[\beta_2\right]\left[\beta_3\right]\left[\boldsymbol{\sigma}_p^2\right]\left[z_1\right]}_{\text{parameter models}}$$

- $\mathbf{y}.i$ is a vector of years with non-missing census data
- $y_t \sim$ normal$(z_t, y.sd_t)$
- $z_t \sim$ lognormal$\left(\log\left(g\left(\boldsymbol{\beta},z_{t-1},x_{t-1}\right)\right),\boldsymbol{\sigma}_p^2\right)$
- $\beta_0 \sim$ normal$\left(.234,.136^2\right)$ informative prior
- $\beta_{i\in 1,2,3} \sim$ normal$(0,1000)$
- $\boldsymbol{\sigma}_p^2 \sim$ gamma$\left(.01,.01\right)$
- $z_1 \sim$ normal$(y_1, y.sd_1)$

General joint and posterior distribution for multi-state model

$$\boldsymbol{\mu}_t = \mathbf{A}\mathbf{z}_t, \text{ process parameters are elements of matrix } \mathbf{A}$$

$$[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data} | \mathbf{Y}] \propto$$
$$\prod_{t=2}^{T} [\mathbf{y}_t | \boldsymbol{\theta}_{data}, \mathbf{z}_t] [\mathbf{z}_t | \boldsymbol{\mu}_t] [\boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, \mathbf{z}_1]$$

- Problem: Evaluate management alternatives for managing overabundant deer in national parks.

States

state	definition
n_1	The number of juvenile deer, aged 6 months on their first census
n_2	The number of adult female deer, aged 18 months and older
n_3	The number of adult male deer, aged 18 months and older

Deterministic Model

f	number of recruits per female surviving to census
ϕ_j	probability that a juvenile (aged 6 months) survives to 18 months
ϕ_d	annual survival probabilty of adult females
ϕ_b	annual survival probability of adult males
m	proportion of juveniles surviving to adults that are female

$$
\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 0 & \phi_d^{\frac{1}{2}} f & 0 \\ m\phi_j & \phi_d & 0 \\ (1-m)\,\phi_j & 0 & \phi_b \end{pmatrix} \\
\mathbf{n}_t &= \mathbf{A}\mathbf{n}_{t-1}.
\end{aligned}
$$

The posterior and joint distribution

$$\left[\boldsymbol{\phi}, m, f, \mathbf{N}, \underbrace{\boldsymbol{\sigma}_p, \boldsymbol{\rho}}_{\text{elements of } \boldsymbol{\Sigma}} \,|\mathbf{y}^{\text{census}}, \mathbf{y}^{\text{census.sd}}, \mathbf{Y}^{\text{class}}\right] \propto$$

$$\underbrace{\prod_{t=2}^{T} \text{multivariate normal}\left(\log(\mathbf{n}_t) | \log\left(\mathbf{A}_t \mathbf{n}_{t-1}\right), \boldsymbol{\Sigma}\right)}_{\text{process model}}$$

$$\times \prod_{t=2}^{T} \underbrace{\text{normal}\left(y_t^{\text{census}} | \sum_{i=1}^{3} n_{i,t}, y_t^{\text{census.sd}}\right)}_{\text{data model 1}}$$

$$\underbrace{\times \text{multinomial}\left(\mathbf{y}_t^{\text{class}} \,|\, \left(\frac{n_{1,t}}{\sum_{i=1}^{3} n_{i,t}}, \frac{n_{2,t}}{\sum_{i=1}^{3} n_{i,t}}, \frac{n_{3,t}}{\sum_{i=1}^{3} n_{i,t}}\right)'\right)}_{\text{data model 2}}$$

$$\times \text{ priors}$$

Systems of differential equations

$$\frac{dz_1}{dt} = k_1 z_1 - k_2 z_1 z_2 \tag{4}$$

$$\frac{dz_2}{dt} = - k_3 z_1 + \alpha k_2 z_1 z_2 \tag{5}$$

$$\frac{dz_3}{dt} = \frac{k_4 z_3}{k_5 + z_3} \tag{6}$$

$\left[\mathbf{z_t} | g\left(\left(\mathbf{k}, \mathbf{z}_{t-1}, x_t\right), \boldsymbol{\sigma}_p^2\right]\right.$

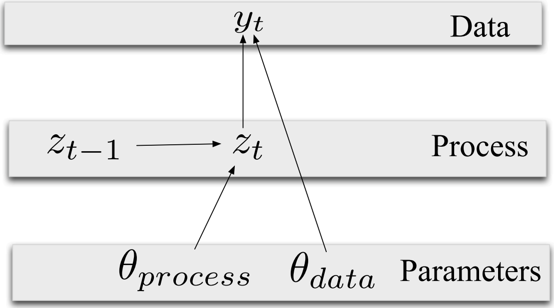
Implementing the process model needs a numerical solver to align the states with the data.

Continuous time models

- Must deterministically update states at discrete intervals to match with data
- To estimate states:
  - Use analytical solutions to ODE system if available.
  - For models without analytical solutions:
    - OpenBUGS and STAN have ODE solvers.
    - Euler's or Runge-Kutta IV can be embedded in JAGS or OpenBUGS for simple models.
    - Best: Write your own MCMC sampler with embedded numerical solver (e.g. `lsoda()` in R).
    - See: Campbell, E. E., W. J. Parton, J. L. Soong, K. Paustian, N. T. Hobbs, and M. F. Cotrufo. 2016. Using litter chemistry controls on microbial processes to partition litter carbon fluxes with the Litter Decomposition and Leaching (LIDEL) model. Soil Biology & Biochemistry 100:160-174.

A broadly applicable approach to modeling dynamic processes in ecology

$$[\mathbf{z}, \theta_{process}, \theta_{data} | \mathbf{y}] \propto \prod_{t=2}^{T} [y_t | \theta_{data}, z_t] \, [z_t | \theta_{process}, z_{t-1}] \, [\theta_{process}, \theta_{data}, z_1]$$



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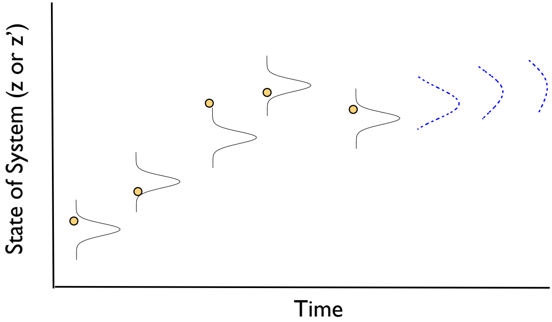
Bayesian forecasting future states z'

$$\underbrace{[z'_{T+1}|\mathbf{y}]}_{\text{predictive process distribution}} =$$

$$\int_{\theta_1 \ldots \theta_P} \int_{z_1 \ldots} \int_{z_T} [z'_{T+1}|\mathbf{z}, \boldsymbol{\theta}_{process}, \mathbf{y}] \underbrace{[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}|\mathbf{y}]}_{\text{posterior distribution}} dz \ldots dz_t \, d\theta_1 \ldots d\theta_P$$



Predictive process distribution

The MCMC output:

$$
\begin{array}{cccccccccccc}
i & \theta_1 & \theta_1 & \theta_3 \\
1 & .42 & 3.3 & 20.3 & z_{1,1} & z_{1,2} & \cdots & z_{1,T} & z'_{1,T+1} & z'_{1,T+2} & \cdots & z'_{1,T+F} \\
2 & .41 & 2.3 & 18.5 & z_{2,1} & z_{2,2} & \cdots & z_{2,T} & z'_{2,T+1} & z'_{,2T+2} & \cdots & z'_{2,T+F} \\
3 & .46 & 3.1 & 16.6 & z_{3,1} & z_{3,2} & \cdots & z_{3,T} & z'_{3,T+1} & z'_{3,T+2} & \cdots & z'_{3T+,F} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & .39 & 3.4 & 22.1 & z_{n,1} & z_{n,2} & \cdots & z_{n,T} & z'_{n,T+1} & z'_{n,T+2} & \cdots & z'_{n,T+F}
\end{array}
$$

$$
\begin{array}{rcl}
n & = & \text{number of iterations} \\
T & = & \text{final time with data} \\
F & = & \text{number of forecasts beyond data}
\end{array}
$$

Posterior and joint distribution with forecasts

$$\mu_t = g(\boldsymbol{\theta}_{process}, z_{t-1}, \mathbf{x}_{t-1})$$

$$[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data} | \mathbf{y}] \propto$$

$$\prod_{t=2}^{T} [y_t | \boldsymbol{\theta}_{data}, z_t] \prod_{t=2}^{T+F} [z_t | \mu_t] [\boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, z_1]$$

Posterior and joint distribution with missing data

$$\mu_t = g(\boldsymbol{\theta}_{process}, z_{t-1}, \mathbf{x}_{t-1})$$

$$[\mathbf{z}, \boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data} | \mathbf{y}] \propto$$

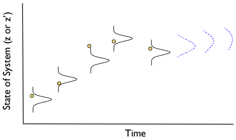
$$\prod_{\forall t \in \mathbf{y}.i}^{T} [y_t | \boldsymbol{\theta}_{data}, z_t] \prod_{t=2}^{T} [z_t | \mu_t] [\boldsymbol{\theta}_{process}, \boldsymbol{\theta}_{data}, z_1]$$

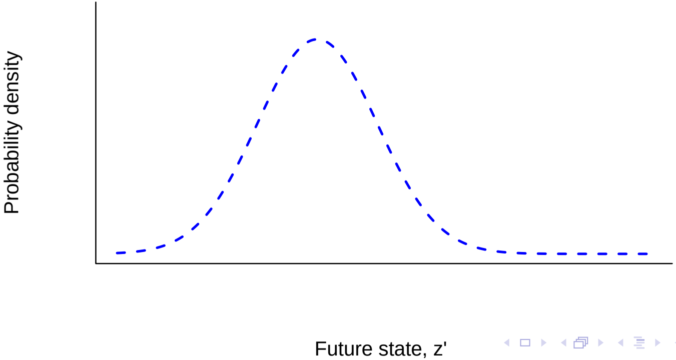
Forecasting

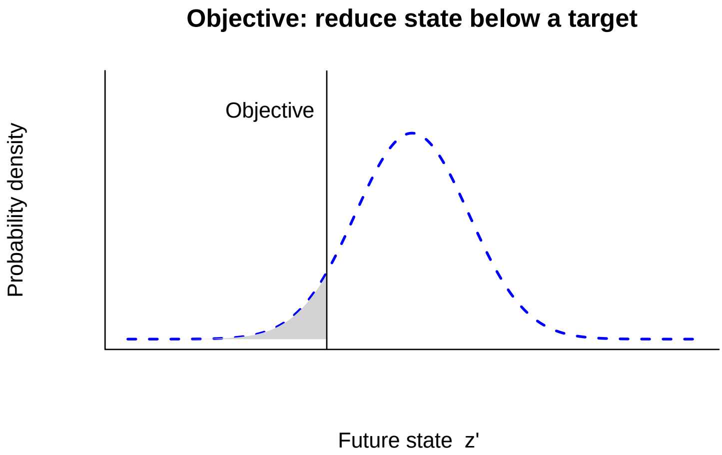
The fundamental problem of management:

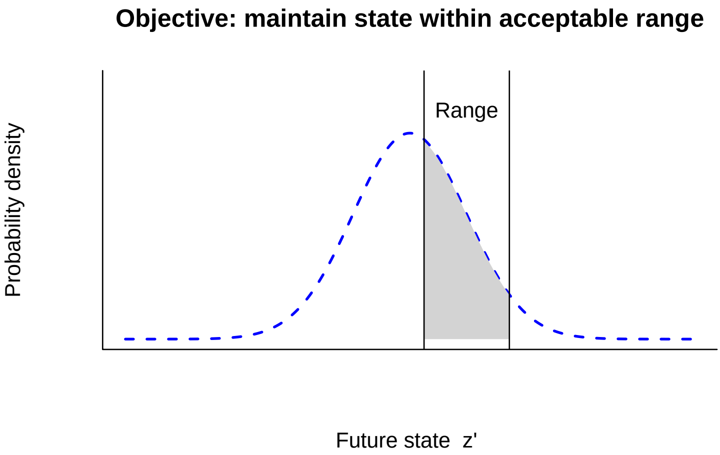
What actions can we take today that will allow us to meet goals for the future?

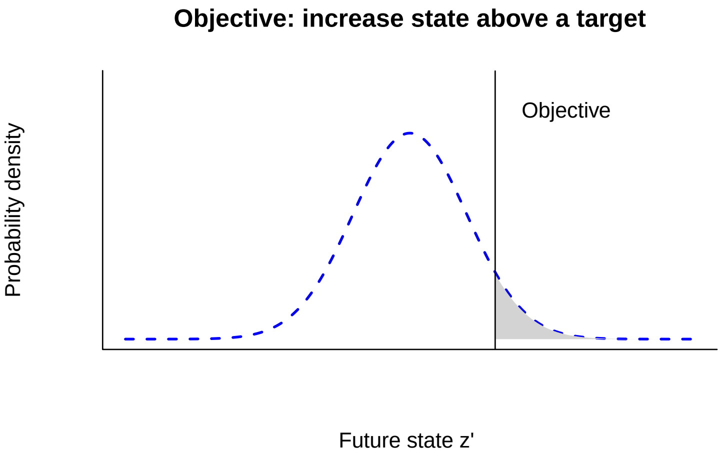
Predictive process distribution of z'

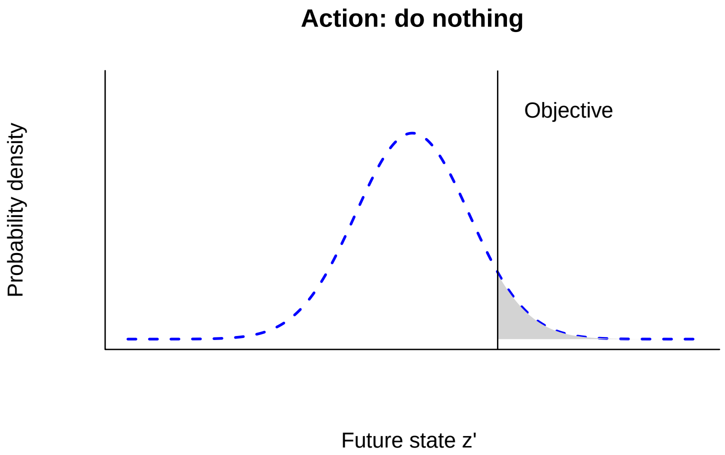




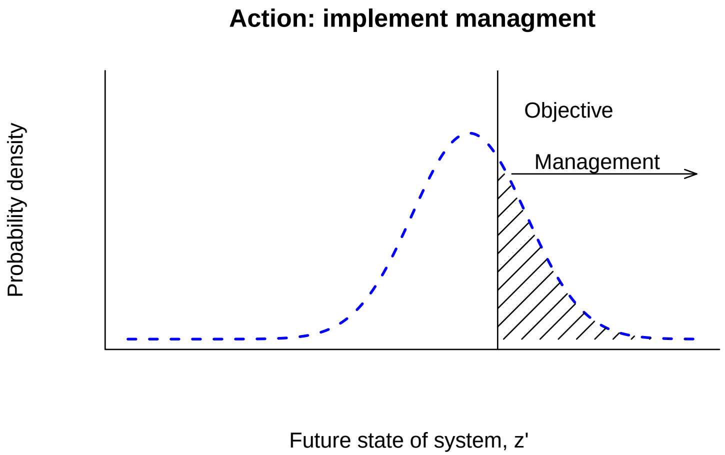

Objective: reduce state below a target
Objective
Probability density
Future state z'

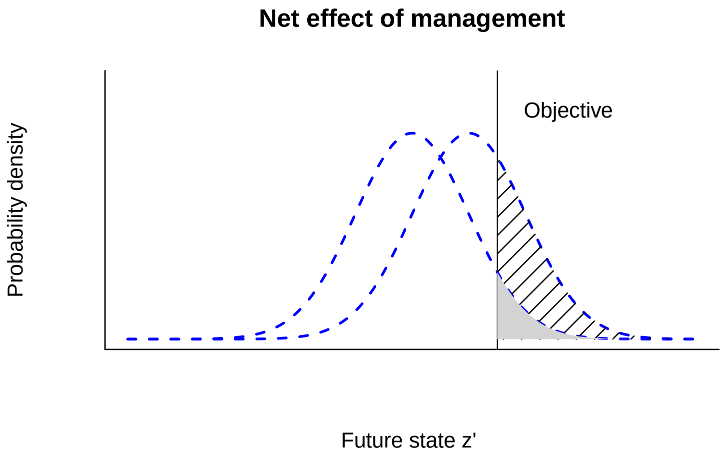

Objective: maintain state within acceptable range
Range
Probability density
Future state z'


Objective: increase state above a target
Objective
Probability density
Future state z'



Action: do nothing
Objective
Probability density
Future state z'


Action: implement managment
Probability density
Objective
Management
Future state of system, z'


Net effect of management
Objective
Probability density
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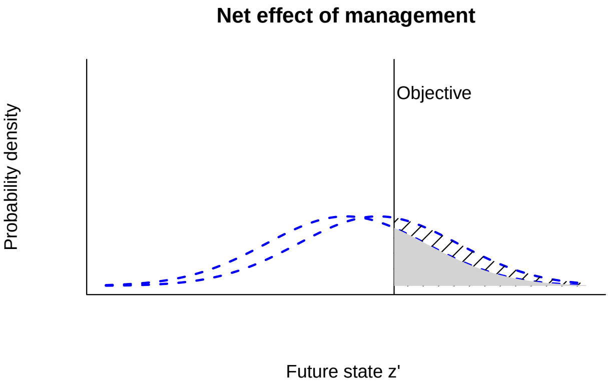
Net effect of management
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Future state z'

Papers using forecasting relative to goals

- Ketz, A. C., T. L. Johnson, R. J. Monello, and N. T. Hobbs. 2016. Informing management with monitoring data: the value of Bayesian forecasting. Ecosphere 7:e01587-n/a.

- Raiho, A. M., M. B. Hooten, S. Bates, and N. T. Hobbs. 2015. Forecasting the Effects of fertility control on overabundant ungulates: white-tailed deer in the National Capital Region. PLoS ONE 10.

- Hobbs, N. T., C. Geremia, J. Treanor, R. Wallen, P. J. White, M. B. Hooten, and J. C. Rhyan. 2015. State-space modeling to support management of brucellosis in the Yellowstone bison population. Ecological Monographs 85:3-28.

JAGS code for posterior and joint distributions

$$\left[\mathbf{z},\boldsymbol{\beta},\sigma_p^2|\mathbf{y}\right] \propto \underbrace{\prod_{\forall t \in y.i}\left[y_t \,\middle|\, z_t, y.sd_t\right]}_{\text{data model}}$$

$$\times \underbrace{\prod_{t=2}^{48}\left[z_t|g\left(\boldsymbol{\beta}, z_{t-1}, x_{t-1}\right), \sigma_p^2\right]}_{\text{process model}} \times \underbrace{[\beta_0][\beta_1][\beta_2][\beta_3]\left[\sigma_p^2\right][z_1]}_{\text{parameter models}}$$

```
model{
#Priors
b[1] ~ dnorm(.234,1/.136^2)
for(j in 2:n.coef){
b[j] ~ dnorm(0,.0001)
}
tau.p ~ dgamma(.01,.01)
sigma.p <- 1/sqrt(tau.p)
z[1]  ~ dnorm(N.obs[1],tau.obs[1])  #this must be treated as prior so that you have z[t-
1]
##Process model
for(t in 2:(T+F)){
mu[t] <- log(z[t-1]*exp(b[1] + b[2]*z[t-1] + b[3]*x[t] +b[4]*x[t]*z[t-1]))
z[t] ~ dlnorm(mu[t], tau.p)
}

#Data model
for(j in 2:n.obs){
N.obs[j] ~ dnorm(z[index[j]],tau.obs[j]) #index to match z[t] with data
}
}#end of model
```

Posterior predictive checks for time series data

Test statistic:

$$\frac{1}{T-1}\sum_{t=2}^{T} \mid y_t - y_{t-1} \mid \tag{7}$$

Conventional statistics are also used (mean, CV, discrepancy statistic for the y_t.

Reilly, C., A. Gelman, and J. Katz, 2001. Poststratification without Population Level Information 731 on the Poststratifying Variable, with Application to Political Polling. Journal of the American 732 Statistical Association 96:1–11.

Posterior predictive checks and test for autocorrelation

```
#Derived quantities for model evaluation

for(i in 1:n.obs){

    #for autocorrelation test

epsilon.obs[i] <- N.obs[i] - z[index[i]]

 # simulate new data

        N.new[i] ~ dnorm(z[index[i]],tau.obs[i])

sq[i] <- (N.obs[i] - z[index[i]] )^2
sq.new[i] <-(N.new[i] - z[index[i]]) ^2
}
fit  <- sum(sq[])
fit.new <- sum(sq.new[])
pvalue <-step(fit.new-fit)
```