Scribbles on Gibbs distributions and Markov random fields

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1 Markov random fields

Let $y = (y_1, \ldots, y_n)$ be a random vector with elements y_i. Define $\textsc{Ne}(i) \subset \{1, \ldots, i-1, i+1, \ldots n\}$ to be the subset of indices such that, for all i, the conditional distribution for y_i may be written as

$$p(y_i \mid y_{(-i)} = p(y_i \mid y_j : j \neq i) = p(y_i \mid y_j : j \in \textsc{Ne}(i)) \,. \tag{1}$$

Clearly specifying $\textsc{Ne}(i)$ for all i species a network or graph. The image here is of each element of the random vector as a node in this network, and $\textsc{Ne}(i)$ is the set of neighboring nodes. The neighborhood set $\textsc{Ne}(i)$ is often called the Markov blanket of node i.

A joint probability distribution specified in this manner is called a *Markov random field*. Clearly not all forms for the conditionals in (1) lead to a valid joint probability distribution. The interesting question is: which ones do?

2 Gibbs distributions

An important idea in understanding Gibbs distributions is that of a clique. A clique is a set C of nodes in a network such that every node in C is a neighbor of every other node in C. A potential function ϕ_d of order d is a function of d arguments that is invariant to permutations of those arguments. For example, $\phi_2(y_1, y_2) = y_1 y_2$ is a potential function. (Note: in some presentations of these ideas potential functions are defined as being strictly positive. I will not adopt that restriction.)

A *Gibbs distribution* is a probability distribution over y that depends on the $y_1, \ldots, y_n$ only through potentials on cliques. Specifically,

$$p(y) = p(y_1, \ldots, y_n) \propto \exp\left\{ d \sum_{j=1}^{n} \sum_{a \in \mathcal{A}_j} \phi_k(y_{a_1}, \ldots, y_{a_j}) \right\},$$

where $\mathcal{A}_j$ is the set of all subsets of $\{1, \ldots, n\}$ of size j.

Loosely speaking, the Hammersley–Clifford theorem states that if Equation (1) defines a valid joint probability distribution, then this distribution must be a Gibbs distribution.