Exercises 7: Latent-feature models

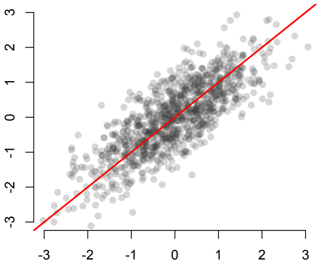
Projecting downward

A question one often encounters in statistics is: given a p-dimensional vector, how do we project it down into a smaller k-dimensional space in a way that preserves as much of the information in the original data as possible? The point is to represent a large amount of information in a tractable, more parsimonious way—in other words, to cut through the clutter.

You already know one way of doing this, namely linear regression. Given an n-dimensional outcome vector y and a matrix of covariates X, the fitted values $\hat{y} = X(X^T X)^{-1} X^T y$ (or their equivalents arising from a Bayesian model) involve a projection from $\mathcal{R}^n$ to $\mathcal{R}^p$. But what if you don't have regressors X, only the outcomes y?

A simple example: projection to $\mathcal{R}$

Say we have n observations of a p-dimensional outcome vector y_i. By Y, I mean the matrix whose ith row is the ith observation $y_i^T = (y_{i1}, \ldots, y_{ip})^T$. (Remember our convention that vectors are column vectors.) Suppose for the moment that every column of Y is standardized to have mean zero and unit variance.



Imagine projecting every observation y_i into a one-dimensional subspace—that is, defining a new scalar outcome $z_i = y_i^T w_1$ for some vector w_1. Presumably w_1 should be maximally information-preserving. The question is how to operationalize this rather loose idea.

Here's one way: choose w_1 so as to *maximize the variance* of the projected values z_i. The intuition for this is straightforward. In the picture at right, the points can be described fairly well by projecting each one onto the diagonal line and reporting the single number z_i. (Or equivalently, its actual position in p-dimensional space, which is $z_i w_1$—though this requires p numbers.) It's also easy to see that the projected points will have greater variance in this subspace than they would in any other choice of subspace. Try drawing some other line through the point cloud; you'll see that the projections of the points onto this line would be more scrunched up than along the line I've drawn.

Mathematically, this means choosing w_1 such that the projected variance

$$V_w = \frac{1}{n} \sum_{i=1}^{n} (z_i - \bar{z})^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i^T w_1 - \bar{z})^2$$

is as large as possible. Of course, we can blow up the variance to be as large as we want by choosing w_1 itself to be huge, so we must constrain it somehow. A natural constraint is that w_1 is a unit vector: $w_1^T w_1 = 1$.

1. Characterize the relationship between the singular value decomposition of Y and the eigenvalue decomposition of $\frac{1}{n} Y^T Y$.

2. Prove[1] that the unit-length w_1 which maximizes the projection variance is a right-singular vector of the data matrix Y corresponding to the largest singular value d_1. What is the relationship between V_w and d_1?

3. Load the data in "congress109.csv." The rows are members of the 109th U.S. Congress; the columns are phrases uttered during floor speeches. Entry (i, j) in the matrix is the number of times member i uttered phrase j. Find the variance-maximizing one-dimensional projection, and compute the location of each member in this one-dimensional space. (Meet R's built-in routines `svd` and `eigen`.) You've now moved from 1000 pieces of information about each member, to 1. Consult the information in "congress109members.csv." (You might find R's `merge` command helpful.) Does location in the subspace you've defined seem to correlate with relevant political facts about each member?

4. Since each projected value is $z_i = y_i^T w_1$, we can write the whole column vector of z_i's as $Z = Y w_1$, and the residuals from this projection as $R = Y - Z w_1^T$. Each row of R is the residual vector for the ith case, after the projection.

   Now imagine applying the same procedure as above to the residuals: that is, finding the maximum-variance projection of each residual vector r_i, defined by some new vector w_2. Prove that w_2 is the right-singular vector of the original data matrix Y corresponding to the second largest singular value d_2.

   Prove (by induction) that the kth principal component is the kth right-singular vector. Here the kth principal component is defined to be the variance-maximizing projection of the residuals after subtracting the contributions to Y of the first $k-1$ principal components.

[1] Remember that the method of Lagrange multipliers is useful for optimizing under constraints.

Factor models

The principal-components representation of a d-dimensional data vector y_i is

$$y_i = \sum_{k=1}^{d} f_{i,k} w_k \, ,$$

where w_k is the kth principal component of the data matrix Y, and where $f_{i,k}$ is the projection of y_i onto w_k.

The equation is of course exact if one uses all d principal components. But this involves keeping around the same amount of information as is required by the original data set—there is no summary or compression of the data, merely a re-expression of the data in the new coordinate system defined by the right-singular vectors of Y.

If we instead drop all but the first $c < d$ principal components, we can reconstruct the data approximately:

$$y_i \approx \sum_{k=1}^{c} f_{i,k} w_k \, ,$$

and by adding an error term, we restore precise equality:

$$y_i = \sum_{k=1}^{c} f_{i,k} w_k + \epsilon_i = W f_i + \epsilon_i \, .$$

where f_i is the c-vector of scores. Presto: we have changed principal-component analysis into a *factor model*, which involves explicit statistical assumptions about the errors ϵ_i. Each f_i is the vector of *factor scores* for observation i, while W is called the loadings matrix, and is common to all observations.

Stacking up the whole system into a single matrix equation, we have

$$Y^T = W F^T + E^T \, ,$$

where Y is the $n \times d$ matrix of observations, whose ith row is y_i; W is the loadings matrix, and F is the matrix whose ith row is f_i. (If you find the transposes messy, simply switch the dimensions of Y and F and re-interprets rows as columns.)

If this looks like a family of related regressions, that's because it is—just one where both the responses and predictors are unknown! We will require some identifying restrictions in order to estimate them both.

1. We'll start with a Gaussian factor model. Suppose that $f_i \sim N(0, I)$, and that $\epsilon_i \sim N(0, \Psi)$ for some diagonal matrix $\Psi = \text{diag}(\psi_1, \ldots, \psi_d)$. What does this imply about the distribution of y_i?

2. Suppose that each element of the factor-loadings matrix w_{jk} is assigned a Gaussian prior, whose variance is τ_k^2 (that is, a different variance for each column of the loadings matrix). What is the conditional posterior distribution of W_j, the jth row of W?

3. What is the conditional posterior distribution for f_i, given the data and the loadings matrix W?

4. Suppose that $\psi_j \sim IG(1/2, 1/2)$ a priori. What is its conditional posterior?

5. Code up a Gibbs sampler fot fitting a Gaussian factor model, for now assuming that τ_k^2, the hyperparameter governing the variance of the elements in column k of W, is large (effectively infinite). Use your Gibbs sampler to estimate a three-factor model to the country-level monthly stock returns in "CountryReturns.csv." Can you interpret the factors from your model?

6. If

   $$y_i = W f_i + \epsilon_i \, ,$$

   then we also have that

   $$y_i = W \Gamma^T \Gamma f_i + \epsilon_i$$

   as long as $\Gamma^T \Gamma = I$. This suggests that the factors in your model are not uniquely identified: we could right-multiply the loadings matrix by Γ^T, so long as we right-multiply the factor scores by Γ. This makes it hard to interpret the factors, since we could rotate them by an arbitrary orthogonal matrix of appropriate dimension.

   Ponder this issue, and suggest a solution (or way to sidestep the problem) if one occurs to you. We'll discuss in class!