# Three period lifecycle

## Problem Set 1: The Lifecycle Model

Agents are assumed to live for three periods in this model. In periods 1 and 2 they work, in period 3 they retire. Wages are given by $w_1, w_2$ in periods 1 and 2. All agents are identical conditional on age. There exists a perfect capital market with constant interest rate $r$ and the price of consumption acts as the numeraire in each period, i.e. it is normalized to one. Let's call assets at the start of period 1 $A_1$, and we assume that after period 3 all individuals die, and they must have non-negative assets at that point. There is no bequest motive, so everything needs to be consumed by the end of period 3. We assume the following period perferences:

$$U(c_t, l_t) = \alpha \ln c_t + (1-\alpha) \ln l_t$$

and point out that $L - h_t = l_t$, i.e. $h$ is hours worked and $L$ is total time endowment.

### 1. Write down the consumers lifecycle maximization problem at age 1.

$$\begin{aligned}
&\max_{\substack{\{c_t\}_{t=1}^{3} > 0 \\ l_t \in (0, L]}} \quad U(c_t, l_t) \\
&\text{subject to } A_1 + \sum_{t=1}^{2} \left(\frac{1}{1+r}\right)^{t-1} w_t (L - l_t) = \sum_{t=1}^{3} \left(\frac{1}{1+r}\right)^{t-1} c_t \\
&\qquad\qquad A_1 \text{ given.}
\end{aligned}$$

### 2. Call $\lambda$ the Lagrange Multiplier on the budget constraint

and solve this problem. Provide an expression for $\lambda$. Show and provide intuition for $\frac{\partial \lambda}{\partial A_1} < 0$, $\frac{\partial \lambda}{\partial w_t} < 0$.

We disregard the zero lower bound on both $c_t, l_t$ because of the utility functional form. However, we write explicitly the non-negativity constraint on hours, i.e. the **upper** bound on $l_t$. Then the Lagrangian writes as

$$\begin{aligned}
\mathcal{L} = &\ \alpha \ln c_1 + (1-\alpha) \ln l_1 + \beta \left(\alpha \ln c_2 + (1-\alpha) \ln l_2\right) + \beta^2 \alpha \ln c_3 \\
&+ \lambda \left[ A_1 + \sum_{t=1}^{2} \left(\frac{1}{1+r}\right)^{t-1} w_t \left(L - l_t\right) - \sum_{t=1}^{3} \left(\frac{1}{1+r}\right)^{t-1} c_t \right] \\
&+ \mu_1 \left[L - l_1\right] + \mu_2 \left[L - l_2\right]
\end{aligned}$$

This has the following FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 : \lambda = \frac{\alpha}{c_1} = \frac{\beta(1+r)\alpha}{c_2} = \frac{\left(\beta(1+r)\right)^2 \alpha}{c_3} \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial l_1} = 0 : \frac{1-\alpha}{l_1} = \lambda w_1 + \mu_1 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial l_2} = 0 : \frac{\beta(1-\alpha)}{l_2} = \lambda \frac{w_2}{1+r} + \mu_2 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 : A_1 + \sum_{t=1}^{2} \left(\frac{1}{1+r}\right)^{t-1} w_t \left(L - l_t\right) = \sum_{t=1}^{3} \left(\frac{1}{1+r}\right)^{t-1} c_t \tag{4}$$

We assume an **interior solution** for this part of the problem, hence $\mu_i = 0$. Now plug in for $c_t$ and $l_t$ from (1), (2) and (3) into (4):

$$\begin{aligned}
A_1 + w_1 L - w_1 l_1 + \frac{w_2}{1+r} - \frac{w_2}{1+r} l_2 &= c_1 + \frac{c_2}{1+r} + \frac{c_2}{(1+r)^2} \\
A_1 + w_1 L - \frac{1-\alpha}{\lambda} + \frac{w_2}{1+r} - \frac{\beta(1-\alpha)}{\lambda} &= \frac{\alpha(1+\beta+\beta^2)}{\lambda} \\
A_1 + w_1 L + \frac{w_2}{1+r} L &= \frac{1+\beta+\beta^2\alpha}{\lambda} \\
\lambda &= \frac{1+\beta+\beta^2\alpha}{A_1 + w_1 L + \frac{w_2}{1+r} L} \qquad (5)
\end{aligned}$$

We find the required derivatives

$$\begin{aligned}
\frac{\partial \lambda}{\partial A_1} &= -\frac{1+\beta+\beta^2\alpha}{\left(A_1 + w_1 L + \frac{w_2}{1+r} L\right)^2} < 0 \\
\frac{\partial \lambda}{\partial w_t} &= -\frac{1+\beta+\beta^2\alpha}{\left(A_1 + w_1 L + \frac{w_2}{1+r} L\right)^2} \left(\frac{1}{1+r}\right)^{t-1} L < 0
\end{aligned}$$

This means that higher initial wealth $A_1$ and higher period wage $w_t$ causes higher consumption throughout the lifecycle and therefore lower marginal utility of consumption (and, hence, wealth).

### 3. Find both the Marshallian and Frischian Labor Supply

i.e. functions $h_t^*(w_1, w_2, A_1)$ and $h_t^F(w_t, \lambda)$.

From the focs for leisure (2) and (3), we have

$$\begin{aligned}
l_1 &= \frac{1-\alpha}{\lambda w_1} \\
l_2 &= \frac{\beta(1-\alpha)(1+r)}{\lambda w_2}
\end{aligned}$$

Hence the frischian labor supplies are

$$\begin{aligned}
h_1^F(w_1, \lambda) &= L - l_1 = L - \frac{1-\alpha}{\lambda w_1} \\
h_2^F(w_2, \lambda) &= L - l_2 = L - \frac{\beta(1-\alpha)(1+r)}{\lambda w_2} \qquad (6)
\end{aligned}$$

and we obtain the marshallian ones by substituting our expression for $\lambda$ from (5)

$$\begin{aligned}
h_1^*(w_1, w_2, A_1) =& L - \frac{(1-\alpha)}{w_1} \left(\frac{1+\beta+\beta^2\alpha}{A_1 + w_1 L + \frac{w_2}{1+r} L}\right)^{-1} \\
h_2^*(w_1, w_2, A_1) =& L - \frac{\beta(1-\alpha)(1+r)}{w_2} \left(\frac{1+\beta+\beta^2\alpha}{A_1 + w_1 L + \frac{w_2}{1+r} L}\right)^{-1} \qquad (7)
\end{aligned}$$

### 4. Take parameters and evaluate optimal policies

Take the following parameter values and evaluate your optimal policy functions for consumption, leisure and assets:

$$\alpha = 0.3, \beta = 0.9, L = 8700, A_1 = 1000, r = 0.05, w_1 = 5, w_2 = 10$$

```r
# define a model
m1 <- list(A1=1000,
              r=0.05,
              w=c(5,10),
              L=8700,
              alpha=0.3,
              beta=0.9)

# define our model solution
lambda <- function(m){
  r = (1+m$beta+m$alpha*m$beta^2)/(m$A1+m$w[1]*m$L+m$w[2]*m$L/(1+m$r))
  return(r)
}

c1 <- function(m,lamb){
  m$alpha / lamb
}
c2 <- function(m,lamb){
  ((1+m$r)*m$beta*m$alpha)/lamb
}
c3 <- function(m,lamb){
    (((1+m$r)^2)*(m$beta^2)*m$alpha)/lamb
}
l1 <- function(m,lamb){(1-m$alpha)/(m$w[1] * lamb)}
l2 <- function(m,lamb){m$beta*(1+m$r)*(1-m$alpha)/(m$w[2] * lamb)}
hours <- function(m,leisure){
  m$L - leisure
}

# print to a table

lambda1 = lambda(m1)
df = data.frame(period = 1:3,cons = c(c1(m1,lambda1),
                                      c2(m1,lambda1),
                                      c3(m1,lambda1)),
                leisure = c(l1(m1,lambda1),
                            l2(m1,lambda1),
                            m1$L))
df$hours = m1$L - df$leisure
df
```

```
##   period     cons  leisure    hours
## 1      1 17828.81 8320.112  379.888
## 2      2 16848.23 3931.253 4768.747
## 3      3 15921.57 8700.000    0.000
```

### 5. Your friend estimates the regression

equation

$$\Delta \ln h_2 = \sigma \Delta \ln w_2 + u_2$$

using OLS and he claims to be estimating the Frisch elasticity of labor supply. What's the value of the estimate $\hat{\sigma}$? What's the estimate's standard error? (Hint: no statistics software needed to answer this question.)

There is no variation in this model as everybody is the same. Hence, $u_2 = 0$. Then,

$\sigma = \frac{\Delta \ln h_2}{\Delta \ln w_2} = 3.6499642$

### 6. Evalute the Frisch elasticity

under the numerical values from question 4. How would those results change if $A_1 = 20000$? Why? For the rest of the problem, use $A_1 = 1000$. Then calculate the Hicksian elasticity of labor supply in period 1 (i.e. keep discounted lifetime utility constant).

The Frisch elasticities are given by

$$\frac{\partial h_1^F(w_1,\lambda)}{\partial w_1}\frac{w_1}{h_1^F} = \frac{1-\alpha}{L\lambda w_1 - 1 + \alpha}$$
$$\frac{\partial h_2^F(w_2,\lambda)}{\partial w_2}\frac{w_2}{h_2^F} = \frac{\beta(1-\alpha)(1+r)}{L\lambda w_2 - \beta(1-\alpha)(1+r)}$$

and evaluates to

```
frisch_e1 <- function(m,lamb){
  (1-m$alpha)/(lamb*m$w[1]*m$L - 1 + m$alpha)
  }
frisch_e2 <- function(m,lamb){
  m$beta*(1+m$r)*(1-m$alpha) / (lamb*m$w[2]*m$L -(m$beta*(1+m$r)*(1-m$alpha)))
  }
```

$\varepsilon_{f,1} = 21.9014863$ and $\varepsilon_{f,2} = 0.8243786$

**Change Model to $A_1 = 20000$**

Let's see what happens to our current leisure function (2) when we plug in this new model:

```
m2 <- m1
m2$A1 <- 20000
lambda1_1 = lambda(m2)
```

we get a first period leisure of 9561.36, which is larger than total time available: TRUE. So this is not an admissible solution.

We have a corner solution where *at least* $l_1 = L$. Doing the same calculation for $l_2$ we get 4517.74, which is fine. Although this is not entirely correct, because $\lambda$ is different if we don't work in the first period. The budget constraint becomes

$$A_1 + \frac{w_2}{1+r}L = \frac{\alpha+\beta+\alpha\beta^2}{\lambda}$$
$$\lambda = \frac{\alpha+\beta+\alpha\beta^2}{A_1 + \frac{w_2}{1+r}L}$$

hence we define a new $\lambda$ as

```
lambda_2 <- function(m){
  r = (m$alpha+m$beta+m$alpha*(m$beta^2))/(m$A1+m$w[2]*m$L/(1+m$r))
  return(r)
```

```
}

lambda2 = lambda_2(m2)
df2 = data.frame(period = 1:3,cons = c(c1(m2,lambda2),
                                       c2(m2,lambda2),
                                       c3(m2,lambda2)),
                 leisure = c(m2$L,
                             l2(m2,lambda2),
                             m2$L))
df2$hours = m2$L - df2$leisure
df2
```

```
##   period     cons  leisure    hours
## 1      1 21384.02 8700.000    0.000
## 2      2 20207.90 4715.177 3984.823
## 3      3 19096.47 8700.000    0.000
```

And with that we get a new Frisch elasticity for period 2 of 1.18.

### Change Model to $A_1 = 1000$ and get Hicksian Elasticity

Going back to $A_1 = 1000$, the Hicksian elasticity is derived either from the slutzky equation, or from a complete solution of the dual of the above maximization problem. The Slutzky equation tells us in this case that the compensated response (holding lifetime utility V fixed) is equal to the substitution effect minus the income effect:

$$\frac{\partial h_1^H(w_1, V)}{\partial w_1} = \frac{\partial h_1^*(w_1, w_2, A)}{\partial w_1} - \frac{\partial h_1^*(w_1, w_2, A)}{\partial A_1} h_1^*(w_1, w_2, A)$$

and then the hicksian elasticity is defined as

$$\varepsilon_{1,H} = \frac{\partial h_1^H(w_1, V)}{\partial w_1} \frac{w_1}{h_1^*(w_1, w_2)}$$

We get the required partial derivatives

$$\frac{\partial h_1^*(w_1, w_2, A)}{\partial w_1} = \frac{(1-\alpha)A_1}{w_1^2(1+\beta+\beta^2\alpha)} + \frac{(1-\alpha)w_2 L}{w_1^2(1+\beta+\beta^2\alpha)(1+r)}$$
$$\frac{\partial h_1^*(w_1, w_2, A)}{\partial A_1} = -\frac{(1-\alpha)}{w_1(1+\beta+\beta^2\alpha)}$$

and compute the elasticity as

```
h1_w1 = ((1-m1$alpha)*m1$A1 ) / (m1$w[1]^2 *(1+m1$beta+(m1$beta^2)*m1$alpha)) +
    ((1-m1$alpha)*m1$w[2]*m1$L ) / (m1$w[1]^2 *(1+m1$beta+(m1$beta^2)*m1$alpha)*(1+m1$r))
h1_A1 = - (1-m1$alpha ) / (m1$w[1] *(1+m1$beta+(m1$beta^2)*m1$alpha))
deriva = h1_w1 - h1_A1 * df[1,]$hours
hicks = deriva * m1$w[1] / df[1,]$hours
```

which yields result $\varepsilon_{1,H} = 14.747$.