

Stat 102C HW2: Answer Key

Muzhou Liang

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Problem 1

(1)

$$E(\hat{I}) = \frac{1}{n} \sum_{i=1}^n E[h(X_i)] = \frac{1}{n} n E[h(X)] = E[h(X)] = I$$

(2)

$$\begin{aligned} \text{Cov}[h(X_i), h(X_j)] &= E[h(X_i)h(X_j)] - E[h(X_i)]E[h(X_j)] \\ &= \int \int h(X_i)h(X_j)f(x_i)f(x_j)dx_i dx_j - I^2 \\ &= \int h(x_i)f(x_i)dx_i \int h(x_j)f(x_j)dx_j - I^2 \\ &= E[h(X_i)]E[h(X_j)] - I^2 = 0 \end{aligned}$$

$$\begin{aligned} \text{Var}(\hat{I}) &= \frac{1}{n^2} \text{Var}\left[\sum_{i=1}^n h(X_i)\right] \\ &= \frac{1}{n^2} \left[\sum_{i=1}^n \text{Var}[h(X_i)] + \sum_{i \neq j} \text{Cov}[h(X_i), h(X_j)] \right] \\ &= \frac{1}{n^2} n \text{Var}[h(X)] = \frac{1}{n} \text{Var}[h(X)] \end{aligned}$$

Let $V = \text{Var}[h(X)]$, $\hat{V} = \frac{1}{n^2} \sum_{i=1}^n [h(X_i) - \hat{I}]^2$ would be the estimator.

(3) \hat{I} follows $\mathcal{N}(I, V)$ when sample size is large, according to the Central Limit Theorem (CLT). Hence, the 95% confidence interval is given as $[\hat{I} - z_{.025} \sqrt{\hat{V}}, \hat{I} + z_{.025} \sqrt{\hat{V}}]$, where $z_{.025} = 1.96$.

(4) `> h = function(x) x^4`
`> E = function(n) {`

```

+   x = rnorm(n)
+   I = mean(h(x))
+   return(I)
+ }
> E(1e5)

[1] 3.010065

```

Problem 2

(1)

$$\begin{aligned}
 E_g[h(X)W(X)] &= \int h(x)w(x)g(x)dx \\
 &= \int h(x)\frac{f(x)}{g(x)}g(x)dx \\
 &= \int h(x)f(x)dx = E_f[g(X)]
 \end{aligned}$$

(2)

$$E(\hat{I}) = \frac{1}{n} \sum_{i=1}^n E[h(X_i)w_i] = E_g[h(X)W(X)] = I$$

Same as 1.2, $\text{Cov}[h(X_i)w_i, h(X_j)w_j] = 0$.

$$\begin{aligned}
 \text{Var}(\hat{I}) &= \frac{1}{n^2} \left[\sum_{i=1}^n \text{Var}[h(X_i)w_i] + \sum_{i \neq j} \text{Cov}[h(X_i)w_i, h(X_j)w_j] \right] \\
 &= \frac{1}{n} \text{Var}_g[h(X_i)w_i]
 \end{aligned}$$

(3) f is a truncated Normal density with $C = 0$; g is an Exponential density with $\lambda = 2$.

```

> f = function(x) sqrt(2/pi) * exp(-x^2/2)
> g = function(x) dexp(x, 2)
> E = function(n) {
+   x = rexp(n, 2)
+   w = f(x)/g(x)
+   I = mean(x * w)
+   return(I)
+ }
> E(1e5)

[1] 0.8018441

```

Problem 3

(1) We take g as the density of $\mathcal{N}(C, 1)$, $h(x) = \mathbf{1}_{\{X > C\}}$.

```
> f = function(x) dnorm(x)
> g = function(x, C) dnorm(x, mean = C)
> h = function(x, C) (x > C) # this function returns a vector
> # consists of TRUEs and FALSEs, which equal to 1 and 0 when
> # taken into computing
> E = function(n, C) {
+   x = rnorm(n, C)
+   w = f(x) / g(x, C)
+   I = mean(h(x, C) * w)
+   return(I)
+ }
> E(1e5, 2); 1-pnorm(2) #true value
```

```
[1] 0.02281004
```

```
[1] 0.02275013
```

A more efficient alternative: We take g as the density of the exponential distribution $Exp(1)$ truncated at C , $h(x) = \mathbf{1}_{\{X > C\}}$ still. But note that $h(x)$ will always equal to 1 since the generated sample is always bigger than C . Therefore, we can just ignore $h(x)$ here.

$$g(y) = e^{-y} / \int_C^{\infty} e^{-x} dx = e^{-(y-C)}$$

$$\hat{I} = \frac{1}{n} \sum_{i=1}^n \frac{f(X_i)}{g(X_i)}$$

```
> f = function(x) dnorm(x)
> g = function(x, C) dexp(x-C)
> E = function(n, C) {
+   x = rexp(n) + C
+   w = f(x)/g(x, C)
+   I = mean(w)
+   return(I)
+ }
> E(1e5, 2); 1-pnorm(2) #true value
```

```
[1] 0.02269973
```

```
[1] 0.02275013
```

- (2) We can generate (X_1, X_2, X_3) from $\mathcal{N}(\alpha, 1)$, where α is a positive number slightly smaller to C . Thus, $P(M > C)$ would be much larger. So g would be the joint density of three normal distributions $\mathcal{N}(\alpha, 1)$, and f is the joint density of three $\mathcal{N}(\mu, 1)$. $h(X) = \mathbf{1}_{\{M > C\}}$.

```
> f = function(x1, x2, x3, mu) {
+   dnorm(x1, mu, 1) * dnorm(x2, mu, 1) * dnorm(x3, mu, 1)
+ }
> g = function(x1, x2, x3, alpha) {
+   dnorm(x1, alpha, 1) * dnorm(x2, alpha, 1) * dnorm(x3, alpha, 1)
+ }
> h = function(M, C) (M > C)
> E = function(N, mu, alpha, C) {
+   X = cbind(
+     x1 = rnorm(N, alpha, 1),
+     x2 = rnorm(N, alpha, 1),
+     x3 = rnorm(N, alpha, 1)
+   )
+   M = apply(X, 1, FUN = function(x) {max(x[1], x[1]+x[2],
+     x[1]+x[2]+x[3])})
+   w = apply(X, 1, FUN = function(x) {f(x[1], x[2], x[3], mu) /
+     g(x[1], x[2], x[3], alpha)})
+   I = mean(w * h(M, C))
+   return(I)
+ }
> E(1e5, -1, 0.5, 1)

[1] 0.03687905
```

Problem 4

(1)

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(r \cos \theta, r \sin \theta) \left| \begin{vmatrix} \frac{\partial r \cos \theta}{\partial r} & \frac{\partial r \cos \theta}{\partial \theta} \\ \frac{\partial r \sin \theta}{\partial r} & \frac{\partial r \sin \theta}{\partial \theta} \end{vmatrix} \right| = \frac{r}{2\pi} e^{-r^2/2}$$

(2)

$$f_{R,\Theta}(r, \theta) = \frac{1}{2\pi} d\theta r e^{-r^2/2} dr$$

Since R and Θ are independent, we can get the two densities as below

$$g_R(r) = r e^{-r^2/2} dr = e^{-r^2/2} d\frac{r^2}{2} = e^{-t} dt \quad (\text{let } T = \frac{R^2}{2})$$

$$h_\Theta(\theta) = \frac{1}{2\pi} d\theta$$

Thus T follows $\mathcal{Exp}(1)$. Remember that an Exponential sample can be obtained by a uniform generator using inverse sampling method. Hence

$$T = -\log U$$

$$R = \sqrt{2T}$$

$$\Theta = 2\pi V$$

```
> normsamp = function(n) {
+   u = runif(n)
+   v = runif(n)
+   t = -log(u)
+   r = sqrt(2*t)
+   theta = 2*pi*v
+   x = r*cos(theta)
+   y = r*sin(theta)
+   res = list("X"=x, "Y"=y)
+   return(res)
+ }
> par(mfrow=c(1,2))
> sample = normsamp(1e4)
> hist(sample$X, freq=FALSE)
> curve(dnorm(x), col = 'red', lwd=2,add = T)
> hist(sample$Y, freq=FALSE)
> curve(dnorm(x), col = 'red', lwd=2,add = T)
```

