Coin Toss Problem

Yuk Tung Liu

2017-02-17

1 Introduction

What is the probability that at least k consecutive heads or tails occur in n coin flips? This problem can be solved analytically using a recurrence formula.

First, we will map this problem to a related problem. We will prove that the probability of getting at least k consecutive heads or tails in n coin flips is equal to the probability of getting at least $k-1$ tails in $n-1$ coin flips. Next we will derive a recurrence formula for the probability. Finally, we derive a formula for the expected value of the number of consecutive heads or tails in n coin flips.

2 Mapping

We first want to prove that the probability of getting at least k consecutive heads or tails in n coin flips is equal to the probability of getting at least $k-1$ consecutive tails in $n-1$ coin flips.

Let X_i be the random variable corresponding to the outcome of the ith coin flip. We set $X_i = 0$ if the ith flip is a tail and $X_i = 1$ is the ith flip is a head. As an example, consider the following outcome of 10 coin flips: 0100110001, where 0 denotes a tail and 1 denotes a head. For this example, we have

$X_1 = 0, X_2 = 1, X_3 = 0, X_4 = 0, X_5 = 1, X_6 = 1, X_7 = 0, X_8 = 0, X_9 = 0, X_{10} = 1.$

So the outcome can also be represented by the sequence $\{0, 1, 0, 0, 1, 1, 0, 0, 0, 1\}$. In general, the outcome of n coin flips can be represented by the sequence of 0s and 1s as $S_n = \{X_1, X_2, ..., X_n\}$.

Given any sequence $S_n = \{X_1, X_2, ..., X_n\}$, construct $n-1$ random variables $Y_1, Y_2, ..., Y_{n-1}$ according to the following rule:

$$Y_i = |X_{i+1} - X_i|, \qquad i = 1, 2, \cdots, n-1. \tag{1}$$

That is, set $Y_i = 0$ if $X_{i+1} = X_i$ and $Y_i = 1$ if $X_{i+1} \neq X_i$. In terms of coin flips, we set $Y_i = 0$ if the outcome of the $(i+1)$th flip is the same as the outcome of the ith flip (i.e. both heads or both tails), and set $Y_i = 1$ if the outcome of the $(i+1)$th flips is not the same as the outcome of the ith flip (i.e. one is head and one is tail).

Given any sequence $S_n = \{X_1, X_2, ..., X_n\}$, we can construct another unique sequence $S'_{n-1} = \{Y_1, Y_2, ..., Y_{n-1}\}$. For example, the outcome of the 10 coin flips in the above example can be represented by the sequence $S_{10} = \{0, 1, 0, 0, 1, 1, 0, 0, 0, 1\}$, and its associated S'_9 is $S'_9 = \{1, 1, 0, 1, 0, 1, 0, 0, 1\}$. It is easier to see the relationship between the two sequences if we display them in a staggered pattern:

S_{10}: 0 1 0 0 1 1 0 0 0 1

S'_9: 1 1 0 1 0 1 0 0 1

By construction, giving any S_n there is a unique S'_{n-1}. What about the inverse? Given any sequence S'_{n-1}, can we construct a sequence S_n that maps to S'_{n-1}? It turns out that we can always do that, but the sequence S_n is not unique. In fact, given any S'_{n-1}, we can construct exactly two sequences S_n that map to S'_{n-1}.

The construction can be done as follows. Given any $S'_{n-1} = \{Y_1, Y_2, ..., Y_{n-1}\}$, there is no way we can determine X_1. So we can have either $X_1 = 0$ or $X_1 = 1$. Once X_1 is fixed, the other X_i can be constructed uniquely by the values of Y_i by solving equation (1). If $Y_i = 0$, we have $|X_{i+1} - X_i| = 0$ and so $X_{i+1} = X_i$. If $Y_i = 1$, we have $|X_{i+1} - X_i| = 1$. Since X_i and X_{i+1} can only take values of 0 and 1, we must have $X_{i+1} = 1$ if $X_i = 0$ and $X_{i+1} = 1$ if $X_i = 0$. That is, $X_{i+1} = 1 - X_i$ if $Y_i = 1$. To summarize, we have

$$X_{i+1} = \begin{cases} X_i & \text{if } Y_i = 0, \\ 1 - X_i & \text{if } Y_i = 1, \end{cases} \qquad i = 1, 2, \cdots, n-1. \tag{2}$$

This equation can be combined to give

$$X_{i+1} = (1 - 2X_i)Y_i + X_i, \qquad i = 1, 2, \cdots, n-1. \tag{3}$$

When $Y_i = 0$, the above equation gives $X_{i+1} = X_i$; when $Y_i = 1$, the above equation gives $X_{i+1} = 1 - X_i$. This is exactly the same as equation (2).

Once the value of X_1 is known, equation (3) determines X_i uniquely for $i > 1$. There are two possible values for X_1, so the sequence S'_{n-1} has exactly two distinct sequences of S_n that map to it.

We are now ready to prove that the probability of getting at least k consecutive heads or tails in n coin flips is equal to the probability of getting at least $k-1$ consecutive tails in $n-1$ coin flips. Suppose S_n is the sequence corresponding to at least k consecutive heads or tails in n coin flips. Then S_n contains at least k consecutive 1s or 0s. It follows that the associated sequence S'_{n-1} contains at least $k-1$ consecutive 0s. Let $N_1(k, n)$ be the number of sequences with at least k consecutive 1s or 0s, and $N_2(k-1, n-1)$ be the number of sequences S'_{n-1} with at least $k-1$ consecutive 0s. Since there are exactly two distinct sequences S_n that map to the same S'_{n-1}, we have $N_1(k, n) = 2N_2(k-1, n-1)$. In n coin flips, each flip has two possible outcomes. So there are 2^n total possible outcomes in n coin flips. Hence,

$$P(\text{at least } k \text{ consecutive 1s or 0s in } n \text{ flips}) = \frac{N_1(k,n)}{2^n} = \frac{2N_2(k-1,n-1)}{2^n} = \frac{N_2(k-1,n-1)}{2^{n-1}}, \tag{4}$$

which is the probability of getting $k-1$ consecutive tails in $n-1$ coin flips. This completes the proof.

3 Recurrence Relations

Let $Q(k, n)$ be the probability of getting at least k consecutive heads or tails in n coin flips; and let $G(k, n)$ be the probability of getting at least k consecutive tails in n coin flips. In the previous section, we show that $Q(k, n) = G(k-1, n-1)$. We now want to find a general formula for $Q(k, n)$. It turns out that it is easier to analyze $G(k, n)$. We will find a formula for $G(k, n)$ and then use the relationship between Q and G to obtain $Q(k, n)$.

The value of $G(k, n)$ is easy for $k \geq n$. To have at least k tails, we must flip the coin at least k times. Hence, $G(k, n) = 0$ if $k > n$. For $k = n$, we must get all tails in the n flips. Since P(tail)=1/2 and all flips are independent, $G(n, n) = 1/2^n$. So we have

$$G(k, n) = 0 \quad \text{if} \quad k > n, \quad G(n, n) = \frac{1}{2^n}. \tag{5}$$

Now we want to derive a formula for $G(k, n)$ when $n > k$.

When $n > k$, we look at the outcome of the last k flips. For concreteness, we consider $k = 6$ but the analysis works for general k. We consider 2 cases: (1) the last flip is a head and (1') the last flip is a tail. The outcomes of the last 6 flips are as follows:

Case 1: xxxxx1

Case 1': xxxxx0

Here x stands for 0 (for tail) or 1 (for head). Case 1 and 1' are mutually exclusive. We further split Case 1' into two mutually exclusive cases:

Case 2: xxxx10

Case 2': xxxx00

In Case 2, the last 2 flips are head and tail. In Case 2', the last 2 flips are all tails. We further split Case 2' into two mutually exclusive cases:

Case 3: xxx100

Case 3': xxx000

So Case 3 and Case 3' differ from the last third flip. We can go on to split Case 3' into Cases 4 and 4', 4' into 5 and 5' and so on. The rule is that we keep splitting the cases with all tails until we get a head or all 6 outcomes are tails. In the end, we have the following 7 mutually exclusive cases:

Case 1: xxxxx1

Case 2: xxxx10

Case 3: xxx100

Case 4: xx1000

Case 5: x10000

Case 6: 100000

Case 7: 000000

These are all the possible and mutually exlusive cases we want to consider in the last 6 flips. Let "≥ 6" denote "getting at least 6 consecutive tails in n coin flips". Then we have

$$\begin{aligned} P(\geq 6) = G(6,n) &= P(\geq 6 \cap \text{Case 1}) + P(\geq 6 \cap \text{Case 2}) + P(\geq 6 \cap \text{Case 3}) \\ &\quad + P(\geq 6 \cap \text{Case 4}) + P(\geq 6 \cap \text{Case 5}) + P(\geq 6 \cap \text{Case 6}), \end{aligned} \tag{6}$$

where $P(\geq 6 \cap \text{Case i})$, $i = 1,2,3,4,5,6,7$, means the probability of getting at least 6 consecutive tails in n flips and the last 6 flips have the pattern described by Case i. We can rewrite the above equation in terms of conditional probabilities as

$$\begin{aligned} G(6,n) &= P(\geq 6|\text{Case 1})P(\text{Case 1}) + P(\geq 6|\text{Case 2})P(\text{Case 2}) + P(\geq 6|\text{Case 3})P(\text{Case 3}) \\ &\quad + P(\geq 6|\text{Case 4})P(\text{Case 4}) + P(\geq 6|\text{Case 5})P(\text{Case 5}) + P(\geq 6|\text{Case 6})P(\text{Case 6}) \\ &\quad + P(\geq 6|\text{Case 7})P(\text{Case 7}). \end{aligned} \tag{7}$$

The conditional probability $P(\geq 6|\text{Case 1})$ is the probability of getting at least 6 consecutive tails given that the last flip is a head. Since the last flip is a head, the ≥ 6 consecutive tails must occur in the first $n-1$ flips. Hence $P(\geq 6|\text{Case 1}) = G(6, n-1)$. Similarly, in case 2, the last two flips are head and tail, so if there are ≥ 6 consecutive tails, they must occur in the first $n-2$ flips. Hence $P(\geq 6|\text{Case 2}) = G(6, n-2)$. In general, you can convince yourself that $P(\geq 6|\text{Case i}) = G(6, n-i)$ for i =1, 2, 3, 4, 5 and 6. What about Case 7, in which the last 6 flips are all tails? Obviously, in Case 7, we already have 6 consecutive tails, so $P(\geq 6|\text{Case 7}) = 1$. We can now write equation (7) as

$$\begin{aligned} G(6,n) &= G(6,n-1)P(\text{Case 1}) + G(6,n-2)P(\text{Case 2}) + G(6,n-3)P(\text{Case 3}) \\ &\quad + G(6,n-4)P(\text{Case 4}) + G(6,n-5)P(\text{Case 5}) + G(6,n-6)P(\text{Case 6}) \\ &\quad + P(\text{Case 7}). \end{aligned} \tag{8}$$

We now need to calculate $P(\text{Case i})$ for i =1, 2, 3, 4, 5, 6 and 7. In Case 1, we want the last flip to be a head, so $P(\text{Case 1}) = 1/2$. In Case 2, we want the last flip to be a tail and the second last flip to be a head, so $P(\text{Case 2}) = (1/2) \times (1/2) = 1/4$. You can easily see that $P(\text{Case i}) = 1/2^i$ for all i =1, 2, 3, 4, 5, and 6. In Case 7, we want all the last 6 flips to be tails, so $P(\text{Case 7}) = 1/2^6$. Equation (8) can now be written as

$$G(6,n) = \frac{1}{2}G(6,n-1) + \frac{1}{2^2}G(6,n-2) + \frac{1}{2^3}G(6,n-3)$$

$$+\frac{1}{2^4}G(6,n-4)+\frac{1}{2^5}G(6,n-5)+\frac{1}{2^6}G(6,n-6)+\frac{1}{2^6}. \tag{9}$$

Note that (9) is a recurrence formula for $G(6,n)$. If we know the values of $G(6,n-1)$, $G(6,n-2)$, $G(6,n-3)$, ..., $G(6,n-6)$, we can use (9) to calculate $G(6,n)$. From (5), we know $G(6,1) = G(6,2) = G(6,3) = G(6,4) = G(6,5) = 0$ and $G(6,6) = 1/2^6$. It follows from (9) that

$$G(6,7) = \frac{1}{2}G(6,6) + \frac{1}{2^6} = \frac{1}{2^7} + \frac{1}{2^6} = \frac{3}{128}. \tag{10}$$

Using (9) with $n = 8$, we have

$$G(6,8) = \frac{1}{2}G(6,7) + \frac{1}{4}G(6,6) + \frac{1}{2^6} = \frac{3}{256} + \frac{1}{256} + \frac{1}{64} = \frac{1}{32}. \tag{11}$$

Similarly,

$$G(6,9) = \frac{1}{2}G(6,8) + \frac{1}{4}G(6,7) + \frac{1}{8}G(6,6) + \frac{1}{2^6} = \frac{5}{128}. \tag{12}$$

We can go on and on to compute $G(6,n)$ for other values of n.

Following the same analysis, one can derive a recurrence formula for $G(k,n)$ for a general k. The result is

$$\boxed{G(k,n) = \frac{1}{2^k} + \sum_{j=1}^{k}\frac{1}{2^j}G(k,n-j) \quad , \quad G(k,i) = 0 \text{ for } i < k \text{ and } G(k,k) = \frac{1}{2^k}} \tag{13}$$

Having otained $G(k,n)$, we can use the relation $Q(k,n) = G(k-1,n-1)$ to obtain a recurrence formula for $Q(k,n)$:

$$\boxed{Q(k,n) = \frac{1}{2^{k-1}} + \sum_{j=1}^{k-1}\frac{1}{2^j}Q(k,n-j) \quad , \quad Q(k,n) = 0 \text{ for } k < n} \tag{14}$$

This is the recurrence formula that can be used to calculate the probability of getting at least k consecutive heads or tails in n coin flips.

Special Cases

(1) $k = 1$. Clearly, the probability of getting at least one head or tail is 1, so $Q(1,n) = 1$ for all $n \geq 1$.

(2) $k = 2$. It follows from (14) that

$$Q(2,n) = \frac{1}{2} + \frac{1}{2}Q(2,n-1). \tag{15}$$

Stating from $Q(2,1) = 0$, we have

$$Q(2,2) = \frac{1}{2} + \frac{1}{2}\cdot 0 = \frac{1}{2} \tag{16}$$

$$Q(2,3) = \frac{1}{2} + \frac{1}{2}Q(2,2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \tag{17}$$

$$Q(2,4) = \frac{1}{2} + \frac{1}{2}Q(2,3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \tag{18}$$

It is easy to show that

$$Q(2,n) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}. \tag{19}$$

The last equality follows from the formula for the sum of a geometric series. It is easy to interpret the result. $Q(2,n)$ is the probability of getting at least two consecutive heads or tails in n coin flips. The only times that won't happen is if the outcome of the flips show alternate pattern, e.g. 0101010101... or 1010101010.... To get this pattern, the first flip can be 0 or 1, but the next $n-1$ flips are determined. So the probability of getting 01010101... or 101010101... is $1/2^{n-1}$. Another way to obtain the result is to notice that in the alternate pattern $Y_i = |X_{i+1} - X_i| = 1$ for all $i = 1, 2, \cdots, n-1$. Following the same analysis as the previous section, the probability of getting the alternate pattern in n flips is the same as the probability of getting $n-1$ heads in $n-1$ coin flips, which is $1/2^{n-1}$. Getting at least two consecutive heads or tails in n coin flips is the complement of getting the alternate pattern in n flips. So $G(2,n) = 1 - 1/2^{n-1}$.

Exercise 1

(a) Write a function to compute $Q(k,n)$ for any positive integers k and n.

(b) Verify that $Q(7,100) = 0.5423369$ and $Q(6,100) = 0.8068205$ as claimed in the Lon Capa HW problem on simulating coin flips.

(c) Verify that $1 - Q(5,100) = 0.02831033$. This is the probability that no more than 4 consecutive heads or tails occur in 100 coin flips.

4 Expected Value

For a fixed n, the probability mass function $f(k|n)$ is the probability that exactly k consecutive heads or tails occur in n coin flips. So we have

$$f(k|n) = Q(k,n) - Q(k+1,n). \tag{20}$$

The expected value of consecutive heads or tails in n coin flips is given by the formula

$$E(n) = \sum_{k=0}^{n} k f(k|n) = \sum_{k=1}^{n} k[Q(k,n) - Q(k+1,n)]. \tag{21}$$

The sum can be carried out by an index-shifting trick:

$$\begin{aligned}
E(n) &= \sum_{k=1}^{n} kQ(k,n) - \sum_{k=1}^{n} kQ(k+1,n) \\
&= \sum_{k=1}^{n} kQ(k,n) - \sum_{k=2}^{n+1} (k-1)Q(k,n) \\
&= Q(k,n) + \sum_{k=2}^{n} [k-(k-1)]Q(k,n) - nQ(n+1,n) \\
&= \sum_{k=1}^{n} Q(k,n),
\end{aligned} \tag{22}$$

where we have used the fact that $Q(n+1,n) = 0$. If you have trouble understanding the trick, write out the terms:

$$\begin{aligned}
E(n) = \sum_{k=1}^{n} kQ(k,n) \qquad & - \sum_{k=1}^{n} kQ(k+1,n) \\
= Q(1,n) \qquad & + 2Q(2,n) + 3Q(3,n) + 4Q(4,n) + \cdots + nQ(n,n)
\end{aligned}$$

$$
\begin{aligned}
& -Q(2,n) - 2Q(3,n) - 3Q(4,n) - \cdots - (n-1)Q(n,n) \\
= \quad & Q(1,n) + Q(2,n) + \cdots Q(n,n) \\
= \quad & \sum_{k=1}^{n} Q(k,n). \qquad (23)
\end{aligned}
$$

So we have

$$
\boxed{E(n) = \sum_{k=1}^{n} Q(k,n)} \qquad (24)
$$

Exercise 2: Write a function to calculate $E(n)$ and then verify that $E(100) = 6.977432$ as claimed in the Lon Capa problem on simulating coin flips.