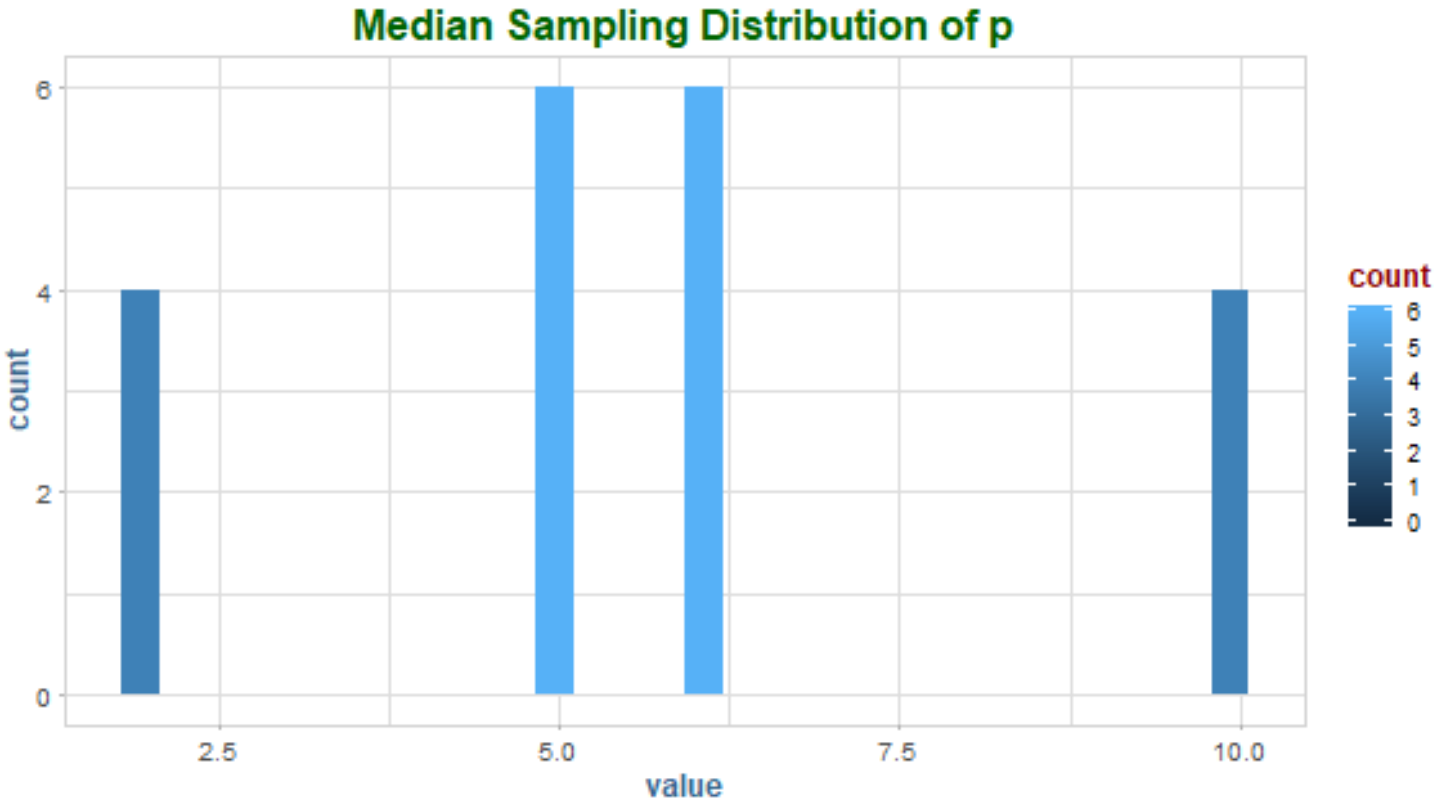
4.1

Consider the population {1, 2, 5, 6, 10, 12}.

Find (and plot) the sampling distribution of medians for samples of size 3 without replacement.

```
p <- c(1, 2, 5, 6, 10, 12)
c <- combinations(v = p, n = 6, r = 3)
t <- apply(c, 1, median)

ggplot(data.table(value = t), aes(value, fill = ..count..)) +
   geom_histogram(bins = 30) +
   labs(title = "Median Sampling Distribution of p")
```



Compare the median of the population to the mean of the medians.

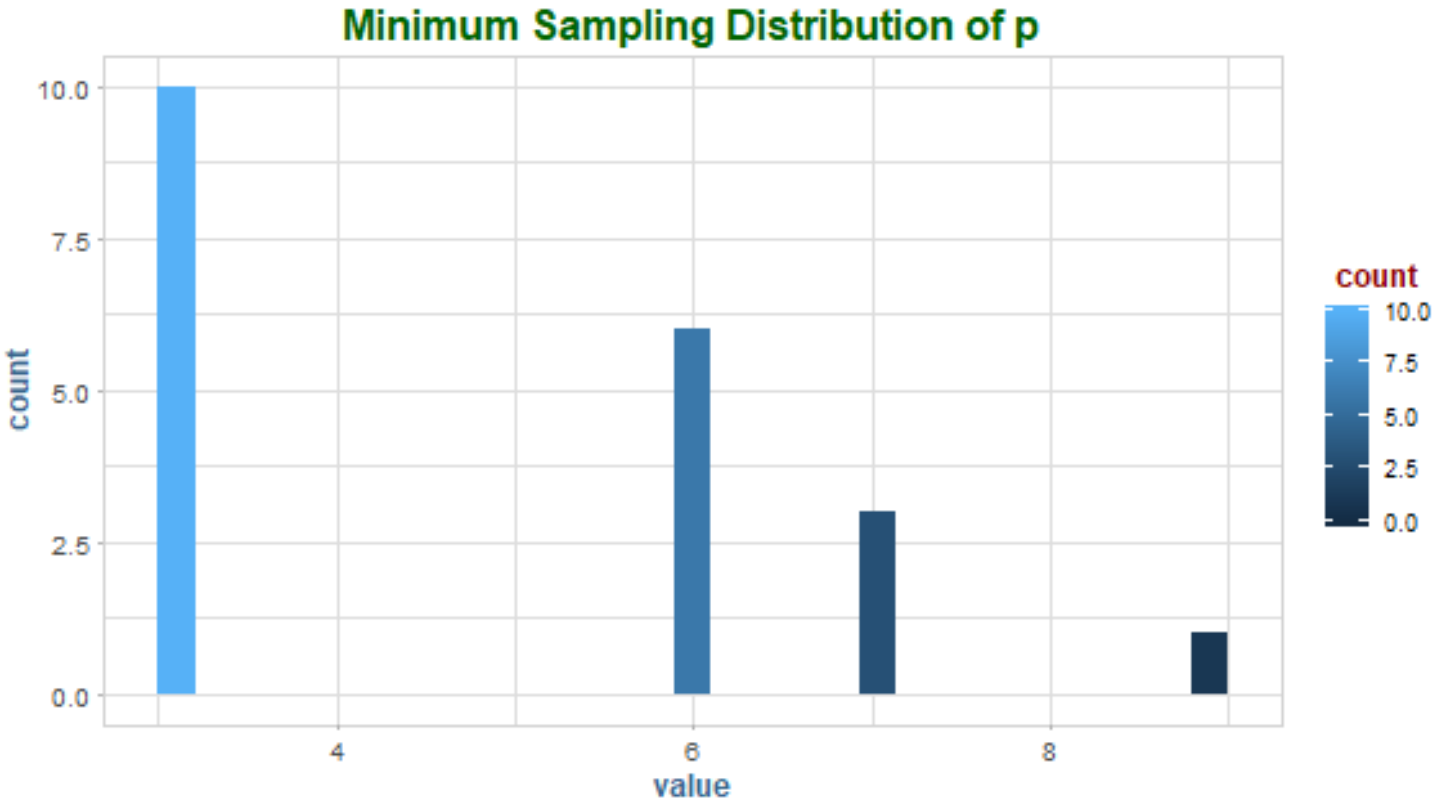
Median of p = **5.5**. Mean of Medians of p = **5.7**

4.2

Consider the population {3, 6, 7, 9, 11, 14}.

For samples of size 3 without replacement, find (and plot) the sampling distribution for the minimum.

```
p <- c(3, 6, 7, 9, 11, 14)
c <- combinations(v = p, n = 6, r = 3)
t <- apply(c, 1, min)

ggplot(data.table(value = t), aes(value, fill = ..count..)) +
   geom_histogram(bins = 30) +
   labs(title = "Minimum Sampling Distribution of p")
```



What is the mean of the sampling distribution? **4.8**

The statistic is an estimate of some parameter - what is the value of that parameter?

This is an estimation of the minimum, which is: **3**

4.3

Let *A* denote the population {1, 3, 4, 5} and *B* the population {5, 7, 9}.

```
A <- c(1, 3, 4, 5)
B <- c(5, 7, 9)
```

Let *X* be a random value from *A*, and *Y* and random value from *B*.

a.) Find the sampling distribution of *X* + *Y*.

```
result = numeric(12)
index <- 1
for(j in 1:length(A))
{
   for(k in 1:length(B))
   {
      result[index] <- A[j] + B[k]
      index <- index + 1
   }
}

sort(result)
```

```
 [1]  6  8  8  9 10 10 10 11 12 12 13 14
```

b.) In this example, does the sampling distribution depend on whether you sample with or without replacement?

No.

Why or why not?

Because 5 in is both sets.

c.) Compute the mean of the values for each of *A* and *B* and the values in the sampling distribution of *X* + *Y*.

Mean of *A*: **3.25**. Mean of *B*: **7**.

Mean of *A* + *B*: **10.25**

How are the means related?

mean(A) + mean(B) = mean(A + B).

d.) Suppose you draw a random value from *A* and a random value from *B*.

```
prob <- sum(result >= 13) / length(result)
```

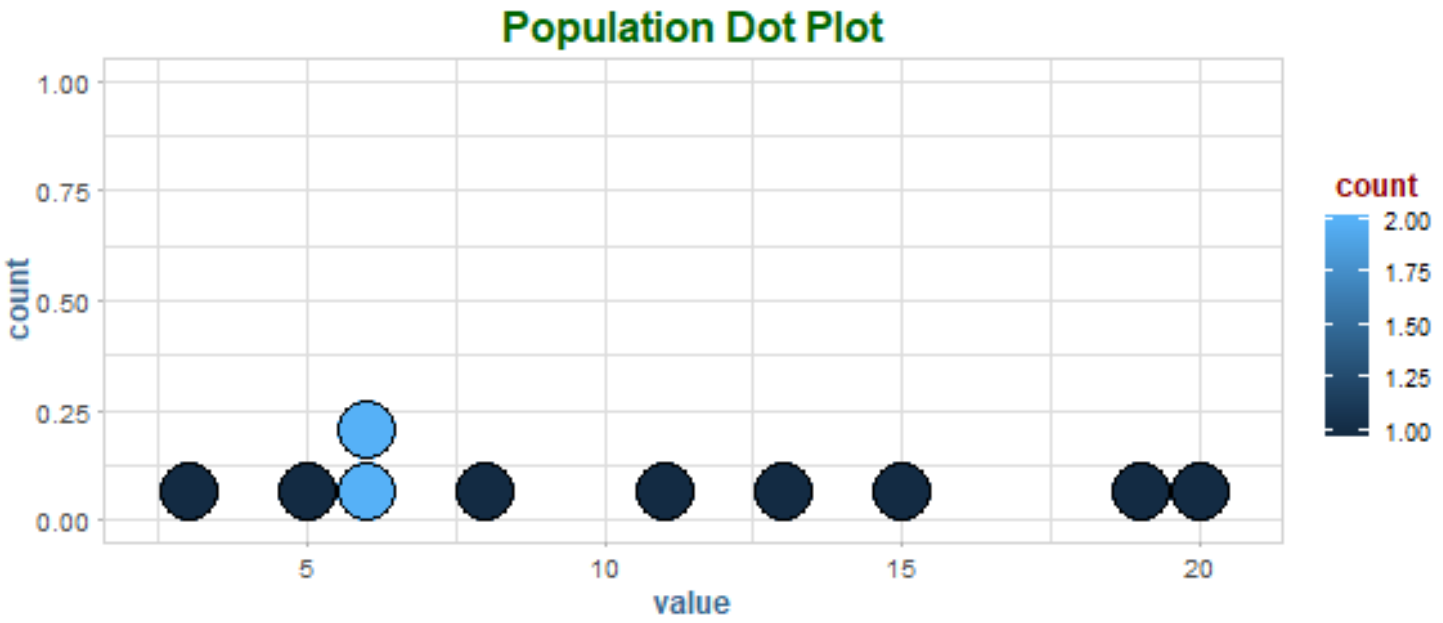
What is the probability that the sum is 13 or larger? **16.6666667%**

4.4

Consider the population {3, 5, 6, 6, 8, 11, 13, 15, 19, 20}.

a.) Compute the mean and standard deviation and create a dot plot of its distribution.

```
p <- c(3, 5, 6, 6, 8, 11, 13, 15, 19, 20)

mu <- mean(p)
sigma <- sd(p)

ggplot(data.table(value = p)) +
   geom_dotplot(aes(value, fill = ..count..), binwidth = 1) +
   labs(title = "Population Dot Plot")
```

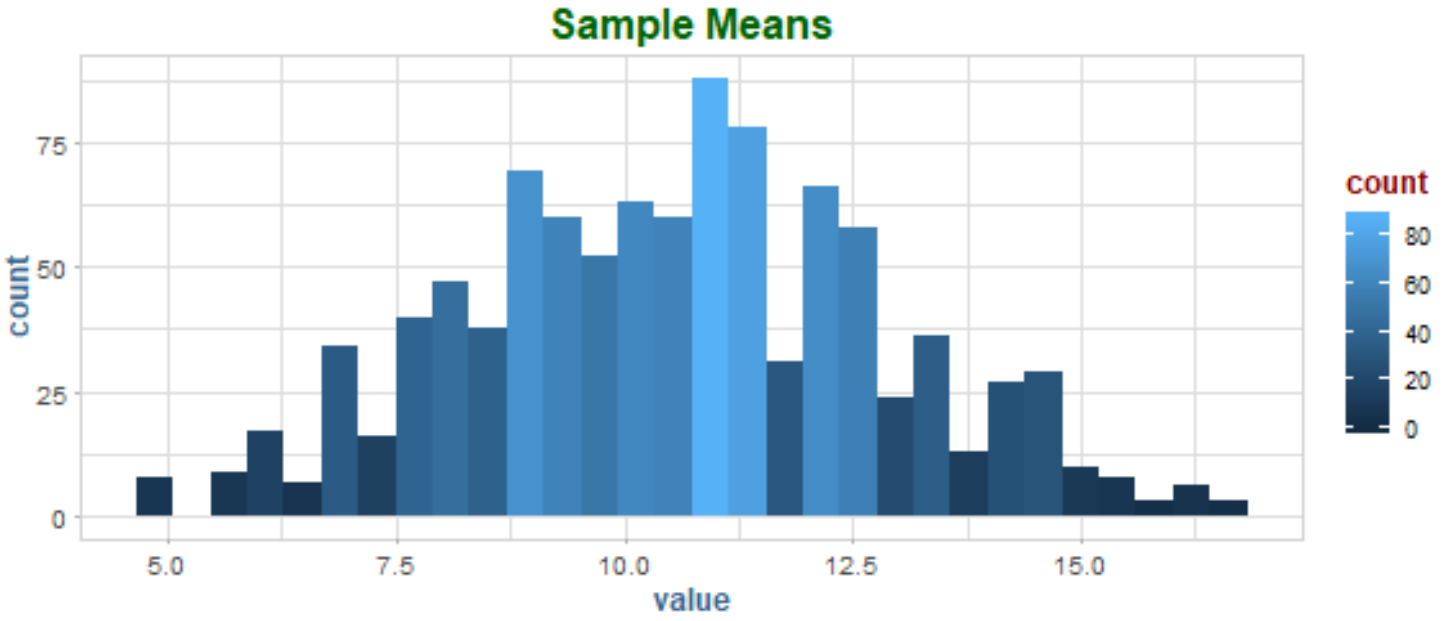


$\mu = 10.6, \sigma = 5.9851668$

b.) Simulate the sampling distribution of $\bar{X}$ by taking random samples of size 4 and plot your results.

```
N <- 10e2
results <- numeric(N)

for( i in 1:N)
{
   index <- sample(length(p), size = 4, replace = F)
   results[i] <- mean( p[index] )
}
```

```
ggplot(data.table(value = results)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Sample Means")
```




```
xbar <- mean(results)
se <- sd(results) / sqrt(N)
```

Compute the mean and standard error, and compare to the population mean and standard deviation.

mean: 10.5285, standard error: 0.0730601

c.) Use the simulation to find $P(\bar{X} < 11)$.

```
prob <- mean(results < 11)
```

$P(\bar{X} < 11) = 55.7\%$

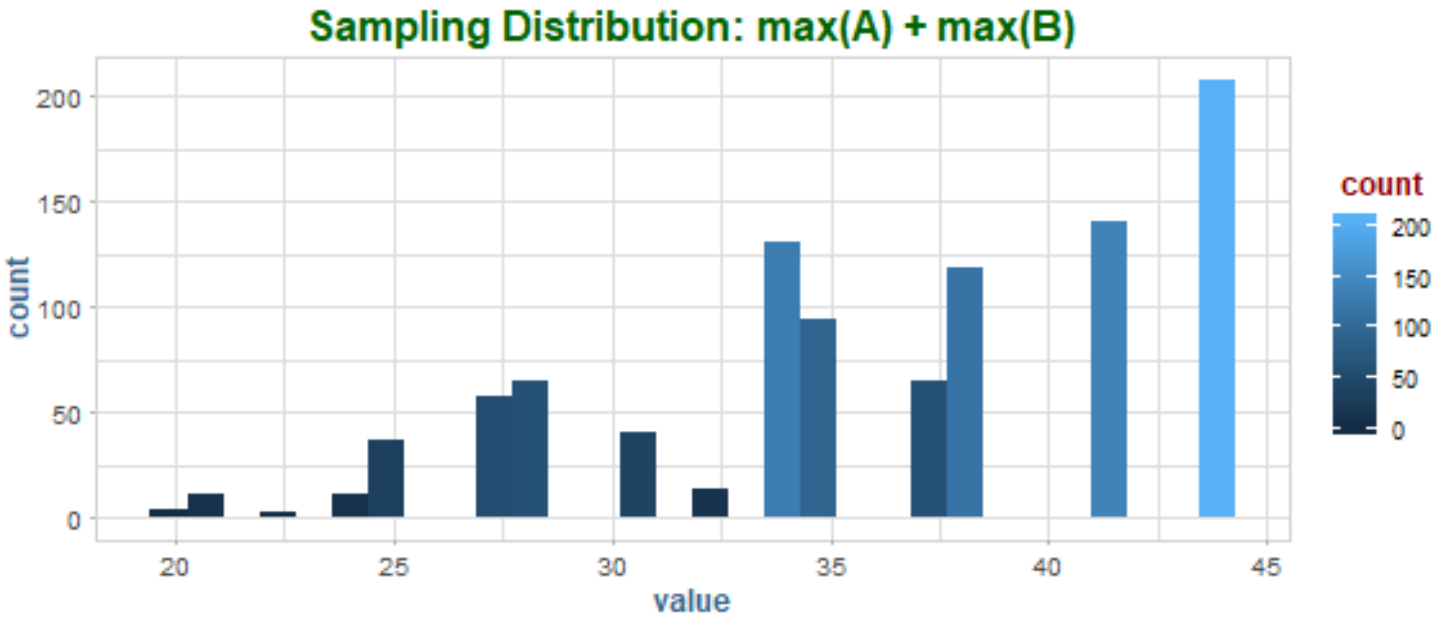
4.5

Consider two populations A = {3, 5, 7, 9, 10, 16}, B = {8, 10, 11, 15, 18, 25, 28}.

```
A <- c(3, 5, 7, 9, 10, 16)
B <- c(8, 10, 11, 15, 18, 25, 28)
```

a.) Using R, draw random samples (without replacement) of size 3 from each population, and simulate the sampling distribution of the sum of their maximums.

```
N <- 10e2
results <- numeric(N)

for(i in 1:N)
{
   samp.a <- sample(A, 3, replace = F)
   samp.b <- sample(B, 3, replace = F)

   results[i] <- max(samp.a) + max(samp.b)
}

ggplot(data.table(value = results)[, index := .I]) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Sampling Distribution: max(A) + max(B)")
```



b.) Use your simulation to estimate the probability that the sum of the maximums is less than 20.

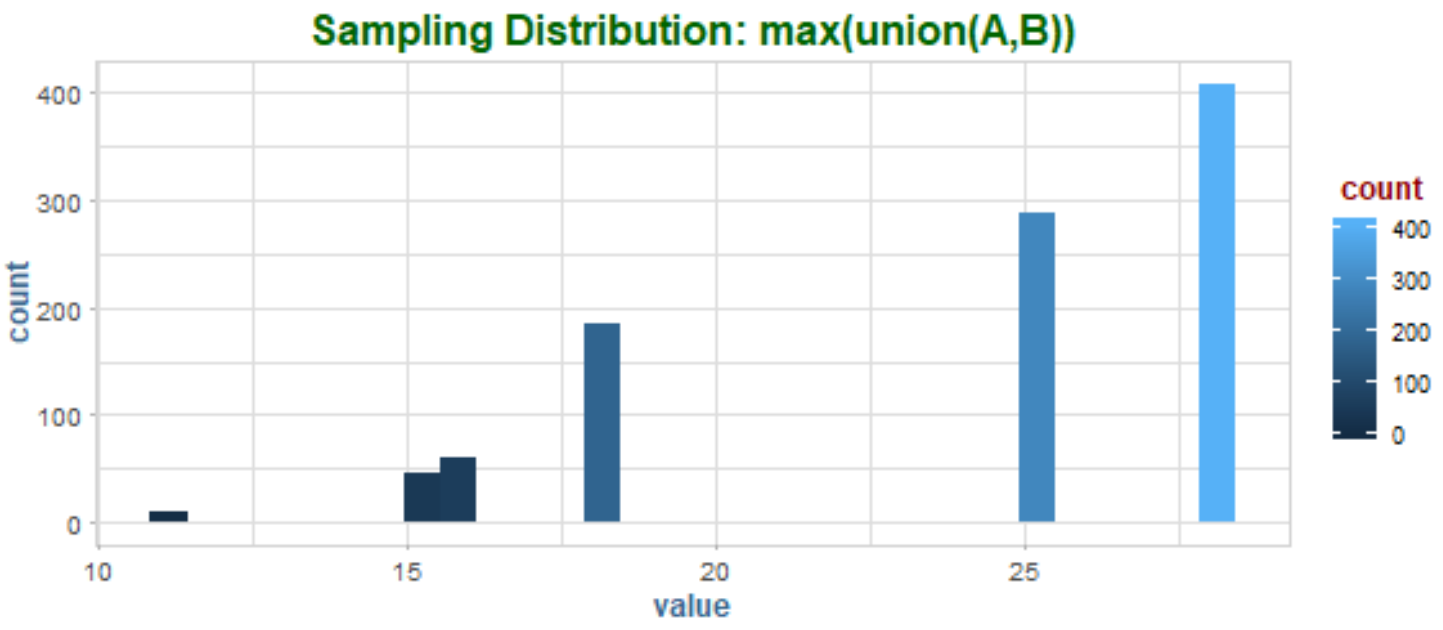
```
prob <- mean(results < 20)
```

Probability: 0%

c.) Draw random samples of size 3 from each population, and find the maximum of the union of these two sets.

Simulate the sampling distribution of the maximums of this union.

```
results <- numeric(N)

for(i in 1:N)
{
   samp.a <- sample(A, 3, replace = F)
   samp.b <- sample(B, 3, replace = F)

   results[i] <- max(union(samp.a, samp.b))
}

ggplot(data.table(value = results)[, index := .I]) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Sampling Distribution: max(union(A,B))")
```



d.) Use simulation to find the probability that the maximum of the union is less than 20.

```
prob <- mean(results < 20)
```

Probability: 30.2%

4.6

The data set *Recidivism* contains the poopulation of all Iowa offenders convicted of either a felony or misdemeanor who were released in 2010 (case study in Section 1.4).

```
Recidivism <- data.table(read.csv(paste0(data.dir, "Recidivism.csv"),
                                  header = T))
```

Of these, 31.6% recidivated and were sent back to prision.

Simulate the sampling distribution of $\hat{p}$, the sample proportion of offeneders who recidivated, for random samples of size 25.

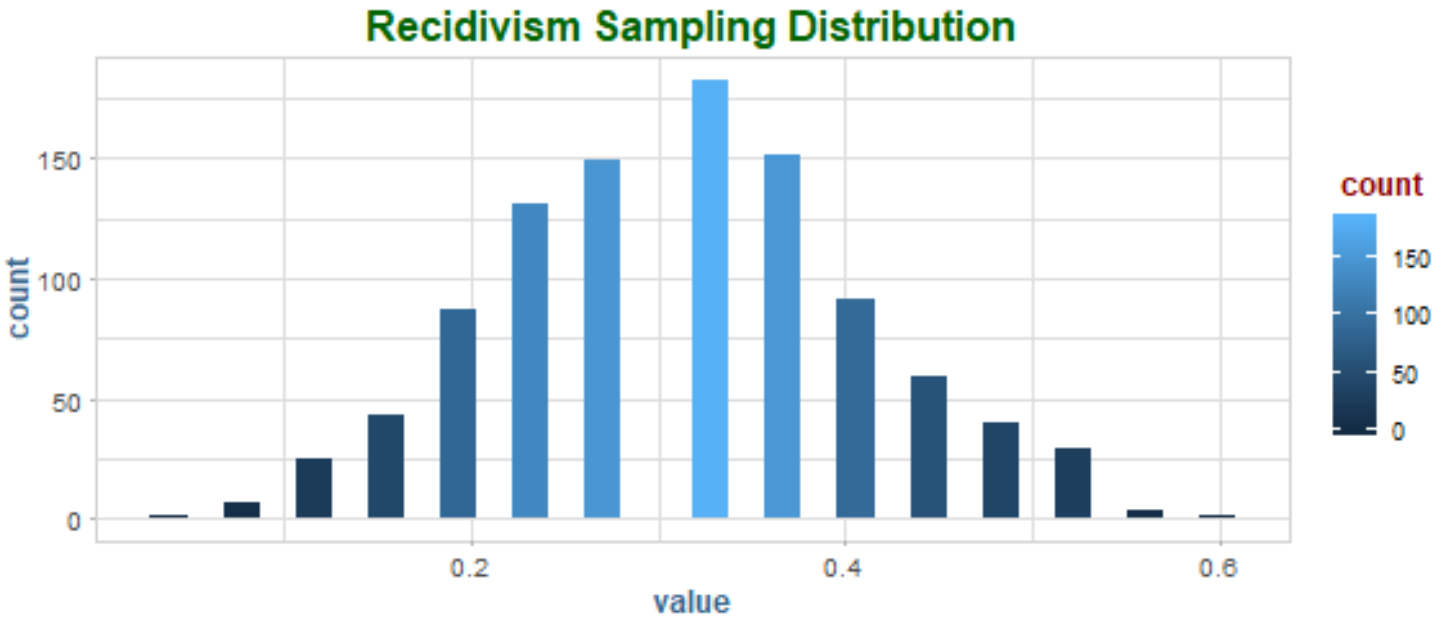
```
mean(Recidivism$Recid == "Yes")
```

```
[1] 0.3164141
```

```
N <- 10e2
results <- numeric(N)

for(i in 1:N)
{
   samp <- sample(Recidivism$Recid, 25)
   results[i] <- mean(samp == "Yes")
}
```

a.) Create a histogram and describe the simulated sampling distribution of $\hat{p}$.

```
ggplot(data.table(value = results)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Recidivism Sampling Distribution")
```



Estimate the mean and standard error.

```
mu <- mean(results)
se <- sd(results) / sqrt(25)
```

$\mu = 0.31288, \sigma = 0.0189516$

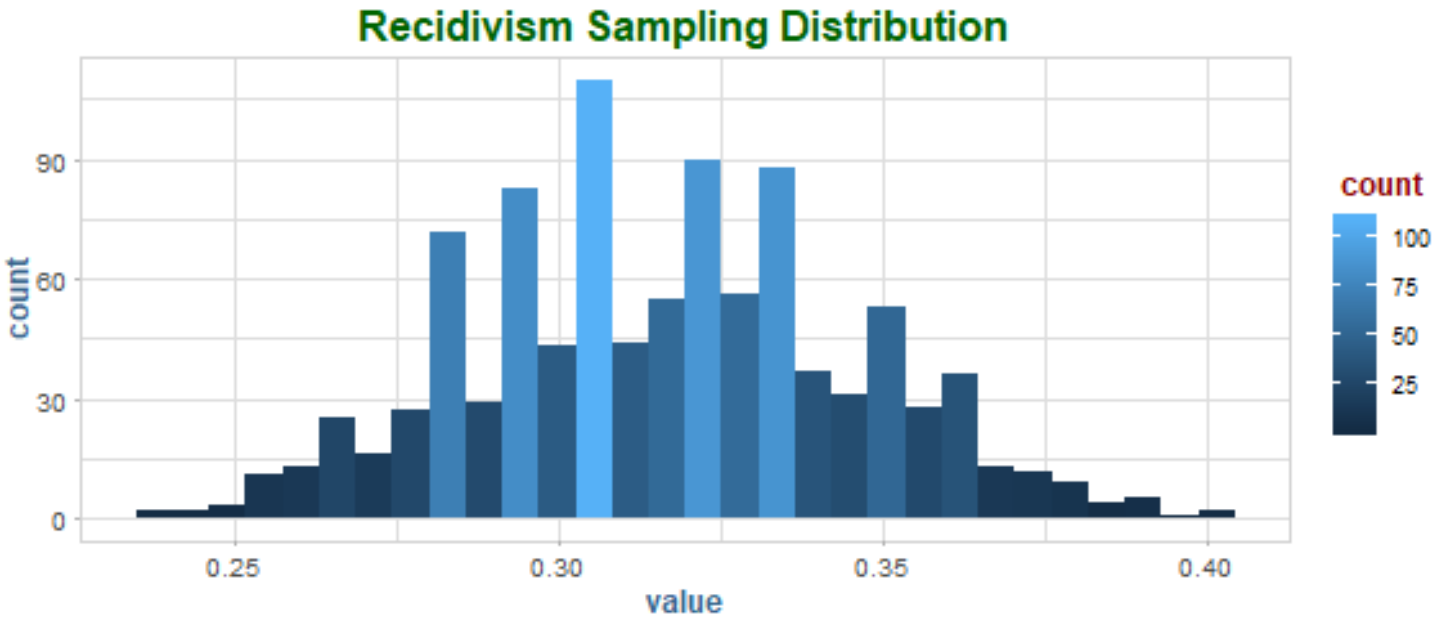
b.) Compare your estimate of the standard error with the theoretical standard error (*Corollary 4.3.2*).

```
tse <- mu * ( 1 - mu ) / sqrt(25)
```

Theoretical: 0.0429972

c.) Repeat the above using samples of size 250, and compare with the $n = 25$ case.

```
N <- 10e2
results <- numeric(N)

for(i in 1:N)
{
   samp <- sample(Recidivism$Recid, 250)
   results[i] <- mean(samp == "Yes")
}

ggplot(data.table(value = results)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Recidivism Sampling Distribution")
```




```
mu <- mean(results)
se <- sd(results) / sqrt(250)
```

$\mu = 0.31602, \sigma = 0.0018666$

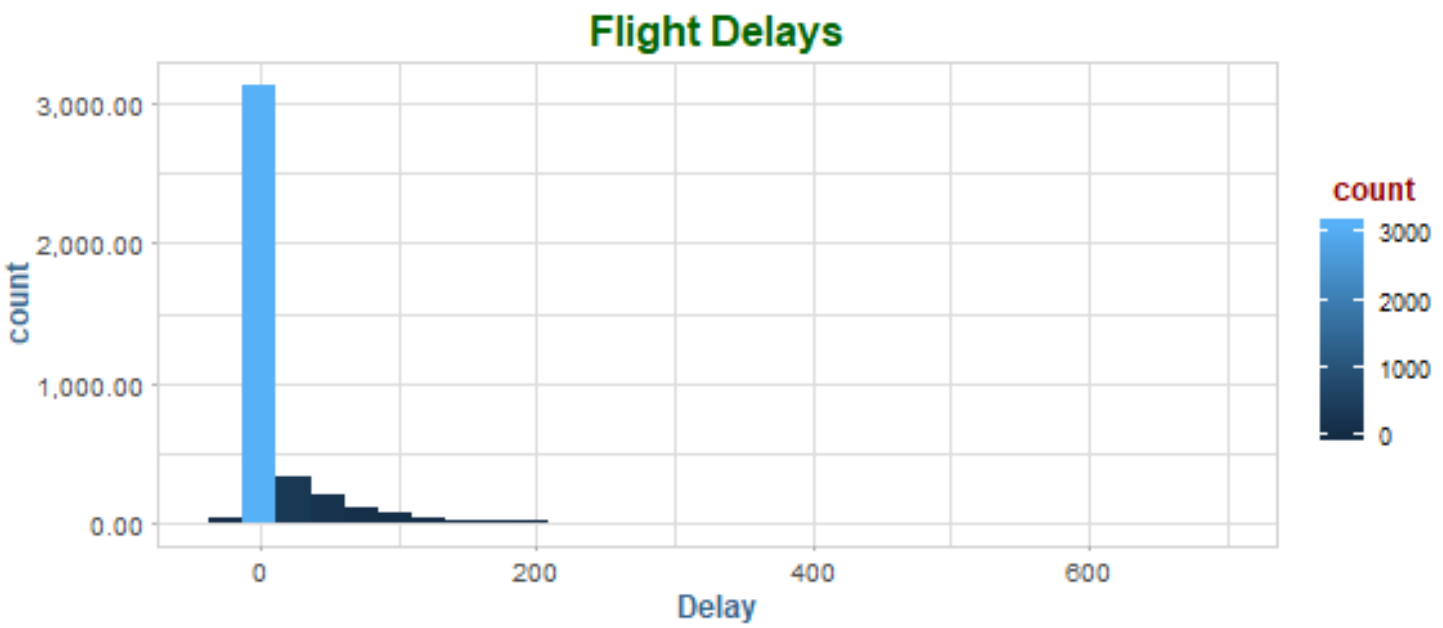
4.7

The data set *FlightDelays* contains the population of all flight departures by United Airlines and American Airlines out of LGA during May and June 2009 (case study in Section 1.1).

```
Flights <- data.table(read.csv(paste0(data.dir, "FlightDelays.csv"),
                               header = T))
```

a.) Create a histogram of *Delay* and describe the distribution.

```
ggplot(Flights, aes(Delay)) +
   geom_histogram(aes(fill = ..count..), bins = 30) +
   scale_y_continuous(labels = comma) +
   labs(title = "Flight Delays")
```



Compute the mean and standard deviation.

```
mu <- mean(Flights$Delay)
sigma <- sd(Flights$Delay)
```

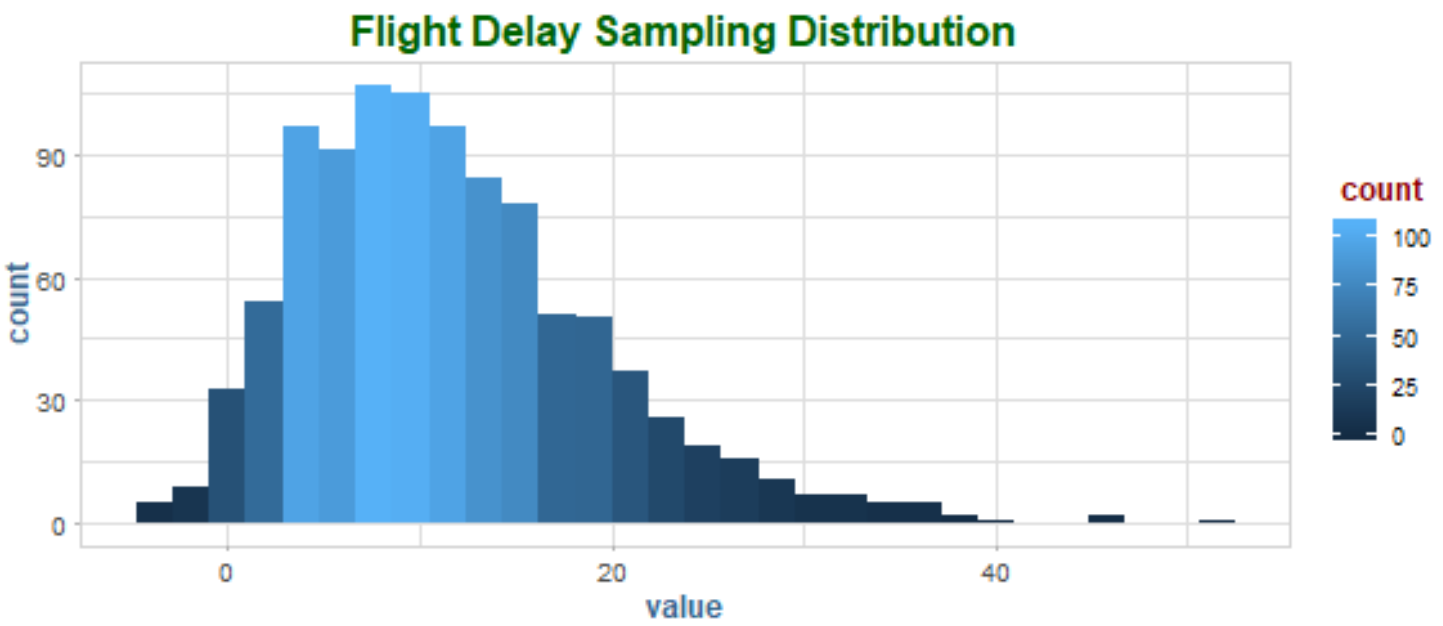
$\mu = 11.7379002, \sigma = 41.6304951$

b.) Simulate the sampling distribution of $\bar{x}$, the sample mean of the length of the flight delays (*Delay*), for sample size 25.

```
N <- 10e2
results <- numeric(N)

for(i in 1:N)
{
   samp <- sample(Flights$Delay, 25, replace = F)
   results[i] <- mean(samp)
}
```

Create a histogram and describe the simulated sampling distribution of $\bar{x}$.

```
ggplot(data.table(value = results)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Flight Delay Sampling Distribution")
```



Estimate the mean and standard error.

```
mu <- mean(results)
se <- sd(results) / sqrt(25)
```

$\mu = 11.67064, \Sigma = 1.5891353$

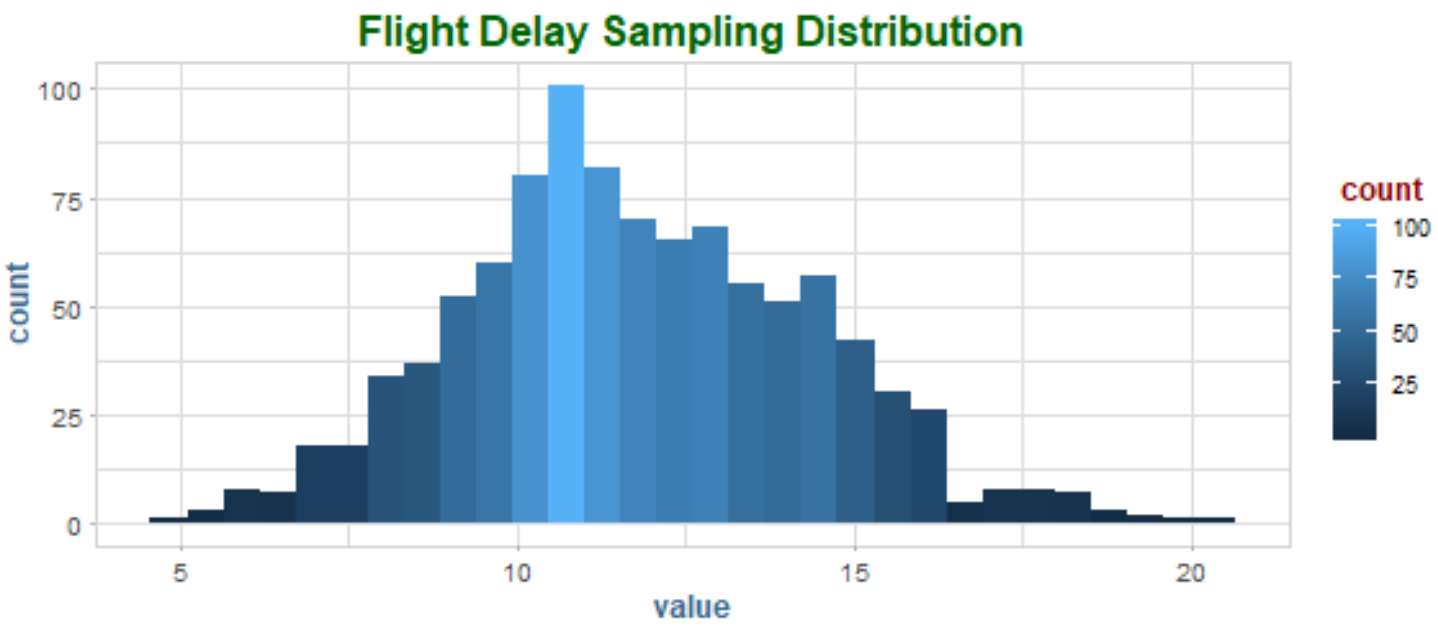
c.) Compare your estimate of the standard error with the theoretical standard error (*Corollary A.4.1*).

```
tse <- var(results) / 25
```

Theoretical: 2.5253509

d.) Repeat with sample size 250.

```
N <- 10e2
results <- numeric(N)

for(i in 1:N)
{
   samp <- sample(Flights$Delay, 250, replace = F)
   results[i] <- mean(samp)
}

ggplot(data.table(value = results)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs(title = "Flight Delay Sampling Distribution")
```




```
mu <- mean(results)
se <- sd(results) / sqrt(250)
tse <- var(results) / 250
```

$\mu = 11.758072, \Sigma = 0.1643619$

Theoretical: 0.0270148

4.8

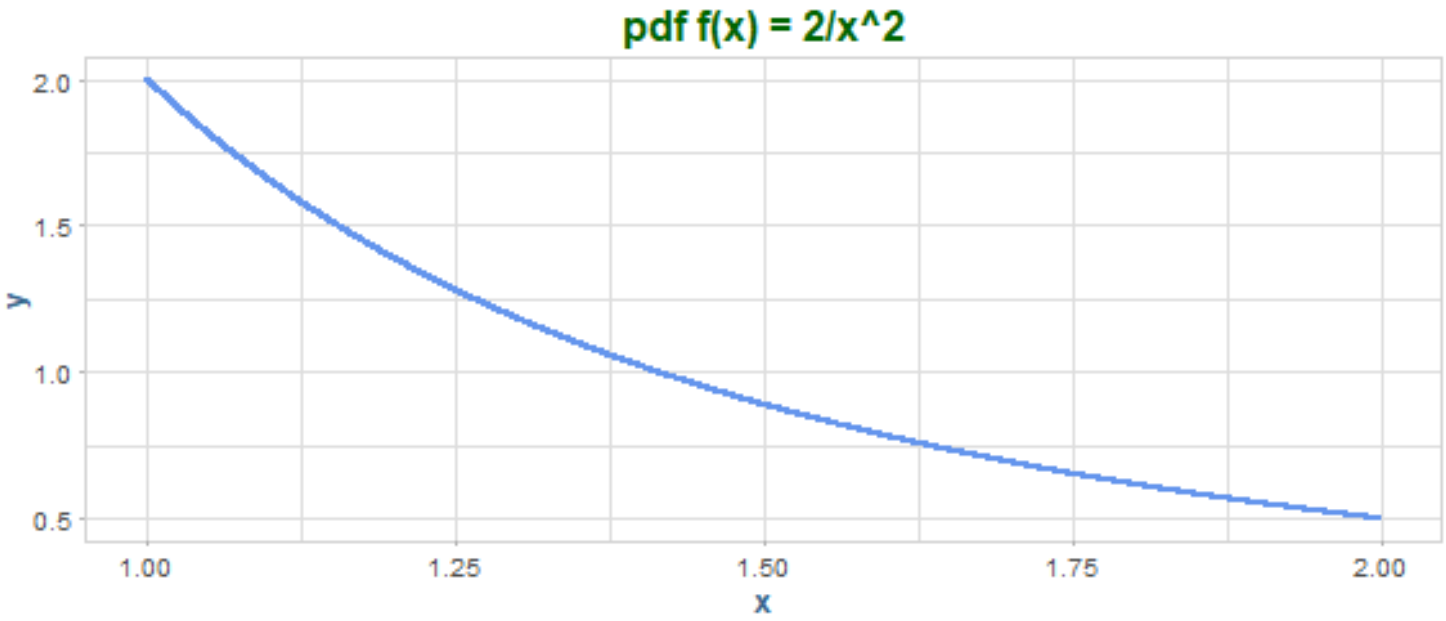
Let $X_1, X_2, \ldots, X_{25}$ be a random sample from some distribution and $W = T(X_1, X_2, \ldots, X_n)$ be a statistic.

Suppose the *sampling distribution* of W has a pdf given by $f(x) = \frac{2}{x^2}$, for 1 < x < 2.

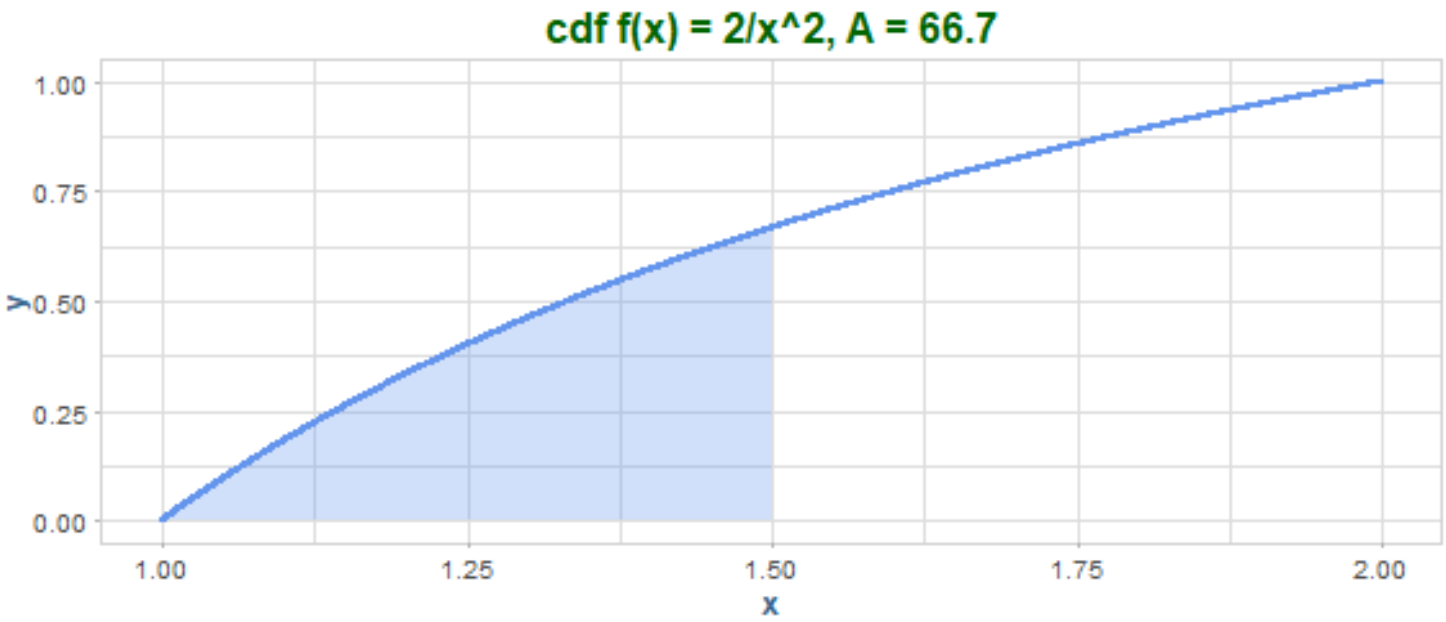
Find $P(w < 1.5)$

Solution:

```
f <- function(x) 2 / x^2

x <- seq( from = 1.0001, to = 1.999, by = 0.0001)

y <- f(x)

ggplot(data.table(x, y)) +
   geom_point(aes(x, y), color = "cornflowerblue", size = .6) +
   labs(title = "pdf f(x) = 2/x^2")
```




```
a <- cumsum(y) / sum(y)
p <- round( a[x == 1.5], 4 ) * 100

d <- data.table(x, y = a)
ggplot(d) +
   geom_point(aes(x, y), color = "cornflowerblue", size = .6) +
   geom_area(aes(x, y), data = d[x < 1.5], fill = "cornflowerblue", alpha = .3) +
   labs(title = paste("cdf f(x) = 2/x^2, A =", p ))
```



Numerical solution: 66.7%

Analytical Solution: $\int_1^{1.5} \frac{2}{x^2} = \frac{2}{3}$

4.9

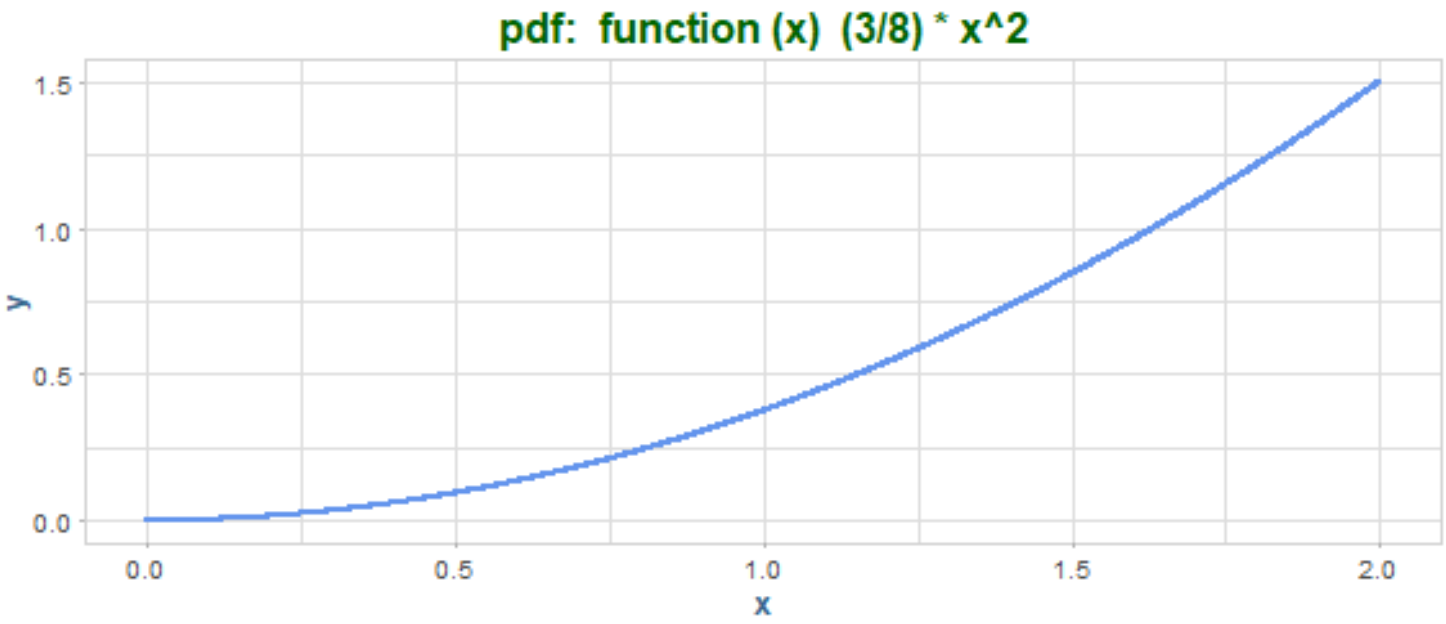
Let $X_1, X_2, \ldots, X_n$ be a random sample from some distribution and $Y = T(X_1, X_2, \ldots, X_n)$ be a statistic.

Suppose the *sampling distribution* of Y has pdf $f(y) = (3/8)y^2 \; for \; 0 \leq y \leq 2.$

Find $P(0 \leq Y \leq \frac{1}{5})$

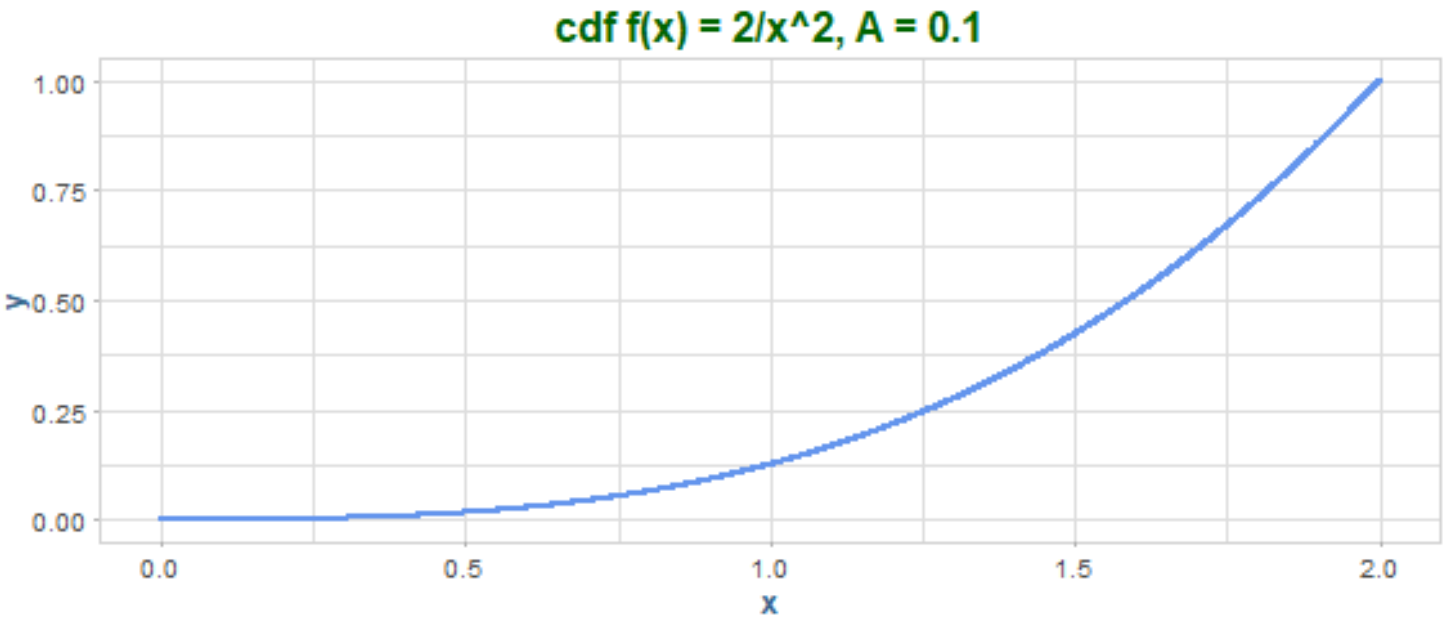
Solution:

```
f <- function(x) (3/8)*x**2

x <- seq( from = 0, to = 2, by = 0.001)

y <- f(x)

ggplot(data.table(x, y)) +
   geom_point(aes(x, y), color = "cornflowerblue", size = .6) +
   labs(title = paste("pdf: ", paste0(deparse(f), collapse = " ")))
```




```
a <- cumsum(y) / sum(y)
p <- round( a[x == 1/5], 4 ) * 100

d <- data.table(x, y = a)
```

```
ggplot(d) +
   geom_point(aes(x, y), color = "cornflowerblue", size = .6) +
   geom_area(aes(x, y), data = d[x < 1/5], fill = "cornflowerblue", alpha = .3) +
   labs(title = paste("cdf f(x) = 2/x^2, A =", p ))
```



Numerical Solution: 0.1%

Analytical Solution: $\int_0^{\frac{1}{5}} \frac{x^3}{8} = \frac{.008}{8} = .001 = .1\ \%$

4.10

Suppose the heights of boys in a certain large city follow a distribution with mean 48 in. and variance 9^2.

Use the CLT approximation to estimate the probability that in a random sample of 30 boys, the mean height is more than 51 in.

```
z <- (51 - 48) / (9^2 / sqrt(30))
p <- pnorm(z, lower.tail = F)
```

Probability: **41.96%**

4.11

Let $X_1, X_2, \ldots, X_{36} \sim Bern(.55)$ be independent, and let $\hat{p}$ denote the sample proportion.

Use the CLT approximation with continuity correction to find the probability that $\hat{p} \leq 0.5$.

```
z <- ( .5 - .55 ) / sqrt(.55 * (1 - .55) / 36)

p <- pnorm(z, lower.tail = T)
```

Probability: 27.32%

4.12

A random sample of size $n = 20$ is drawn from a distribution with mean 6 and variance 10.

Use the CLT approximation to estimate $P(\bar{X} \leq 4.6)$.

```
z <- ( 4.6 - 6 ) / ( 10 * sqrt(20) )

p <- pnorm(z, lower.tail = T)
```

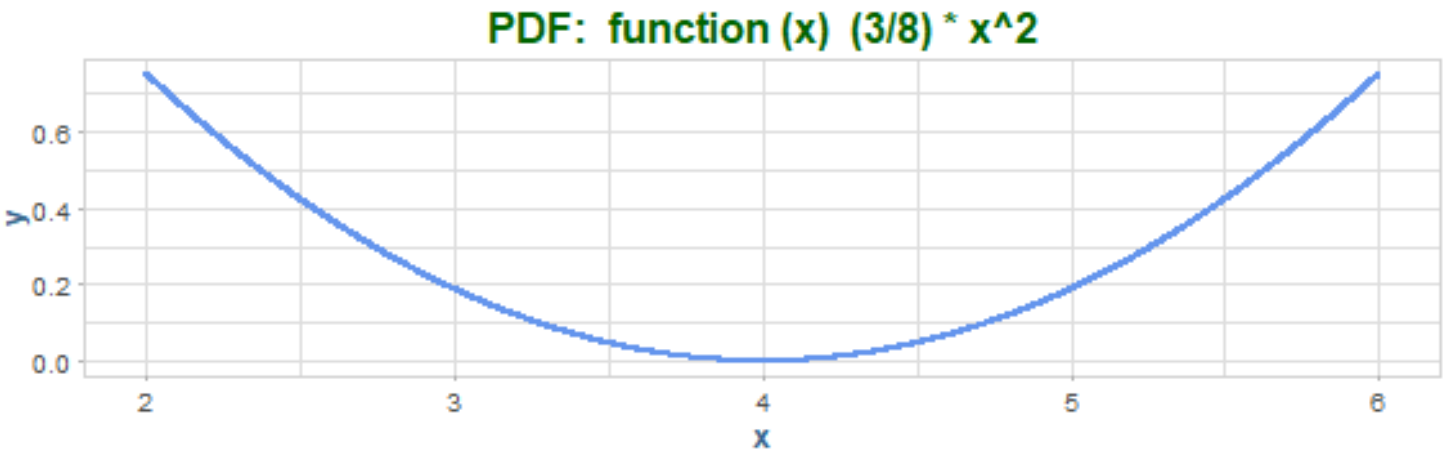
Probability: 48.75%

4.13

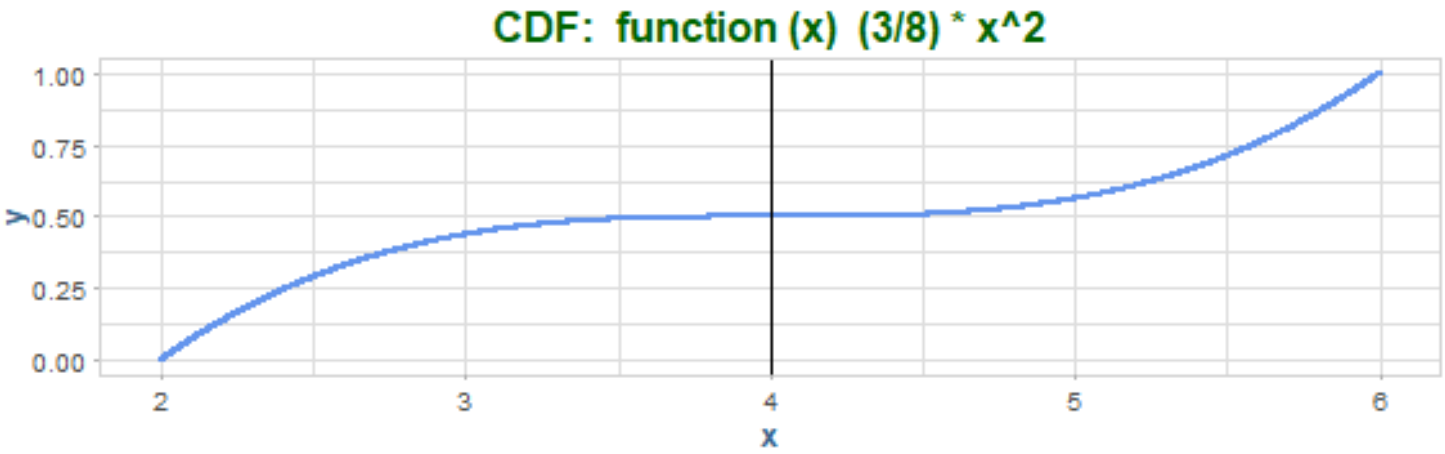
A random sample of size $n = 244$ is drawn from a distribution with pdf $f(x) = (3/16)(x-4)^2, 2 \leq x \leq 6$.

Use the CLT approximation to estimate $P(X \geq 4.2)$.

```
pdf <- function(x) (3/16)*(x - 4)^2
x <- seq(from = 2, to = 6, by = 0.001)
y <- pdf(x)

ggplot(data.table(x,y)) +
   geom_point(aes(x, y), col = "cornflowerblue", lwd = .8) +
   labs(title = paste("PDF: ", paste0(deparse(f), collapse = " ")))
```



```
cdf <- function(x) (3/8)*(x - 4)

y <- cumsum(y) / sum(y)

ev <- x[min(which(y > .5))]

ggplot(data.table(x,y)) +
   geom_point(aes(x, y), col = "cornflowerblue", lwd = .8) +
   geom_vline(xintercept = ev) +
   labs(title = paste("CDF: ", paste0(deparse(f), collapse = " ")))
```




```
z <- ( 4.2 - ev ) / sqrt(244)
pnorm(z, lower.tail = F)
```

```
[1] 0.4949177
```

4.14

According to the 2000 census, 28.6% of the US adult population recieved a high school diploma.

In a random sample of 800 US adults, what is the probability that between 220 and 230 (inclusive) people have a high school deploma?

Use the CLT approximation with continuity correction, and compare with the exact probability.

Solution:

The sampling distribution of $\hat{p}$ is approximately normal with:

```
n <- 800
mu <- .286
```

```
ev <- 800 * mu
sigma <- sqrt(n*mu*(1-mu))
```

$\mathbb{E}[X] = 228.8$ and $\sigma = \sqrt{800(.286)(1-.286)} = 12.7814$

```
l <- pnorm((ev - 219.5) / sigma)
h <- pnorm((ev - 230.5) / sigma)

p <- l - h
```

Probability: 0.3195

4.15

If $X_1, \ldots, X_n$ are i.i.d. from Unif[0, 1], how large should n be so that $P(\bar{X} - \frac{1}{2} < 0.05) \geq 0.90$,

that is, is there at least a 90% chance that the sample mean is within 0.05 of $\frac{1}{2}$? Use the CLT approximation.

4.16

Maria claims that she has drawn a random sample of size 30 from the exponential distribution with $\lambda = 1/10$.

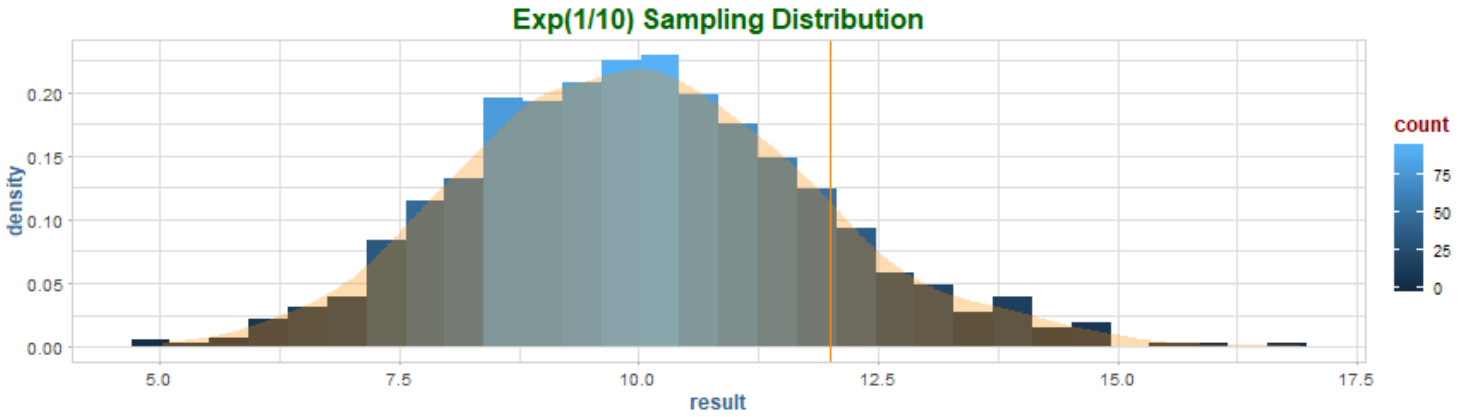
The mean of her sample is 12.

a.) What is the expected value of a sample mean?

$X \sim Exp(\frac{1}{10}), \mathbb{E}(x) = 10$

b.) Run a simulation by drawing 1000 random samples, each of size 30, from Exp(1/10), and compute the mean for each sample.

```
N <- 1000
result <- numeric(N)

for( i in 1:N)
{
   samp <- rexp( n = 30, rate = 1/10)
   result[i] <- mean(samp)
}

ggplot(data.table(result), aes(result)) +
   geom_histogram(aes(y = ..density.., fill = ..count..), bins = 30) +
   geom_vline(xintercept = 12, col = "darkorange") +
   stat_density( kernel = "gaussian", fill = "darkorange", alpha = .3) +
   labs(title = "Exp(1/10) Sampling Distribution")
```




```
p <- mean(result > 12)
```

What proportion of the sample means is as large or larger than 12? **13.2%**

c.) Is a mean of 12 unusual for a sample of size 30 from Exp(1/10)?

Yes, only ~13% of the sample means have a value of 12 or higher.

4.17

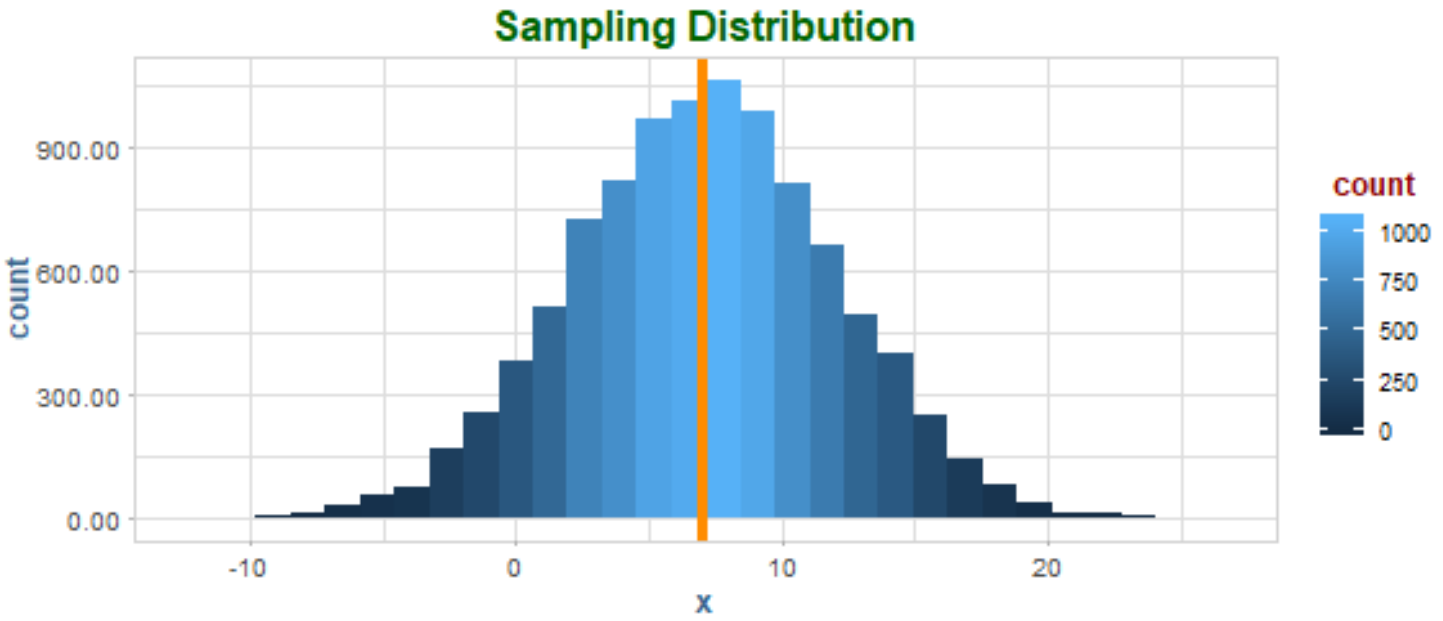
Let $X \sim N(15, 3^2)$ and $Y \sim N(4, 2^2)$ be independent random variables.

a.) What is the exact sampling distribution of $W = X - 2Y$?

$W \sim N(7, 5^2)$

b.) Use R to simulate the sampling distribution of W and plot your results.

```
X <- rnorm(10e3, 15, 3)
Y <- rnorm(10e3, 4, 2)

W <- X - 2*Y

ggplot(data.table(x = W)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   geom_vline(xintercept = 7, col = "darkorange", lwd = 1.5) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```



Check that the simulated mean and standard error are close to the theoretical mean and standard error.

```
mu <- mean(W)
sigma <- sd(W)
```

$\mu = 7.070161, \sigma = 4.9699342$

c.) Use the simulated sampling to estimate $P(W \leq 10)$, and then check your estimate with an exact calculation.

```
phat <- mean(W <= 10)

p <- pnorm(10, mean = 7, sd = 5)
```

$\hat{p} = 72.3\%$

$P(W \leq 10) = 72.57\%$

4.18

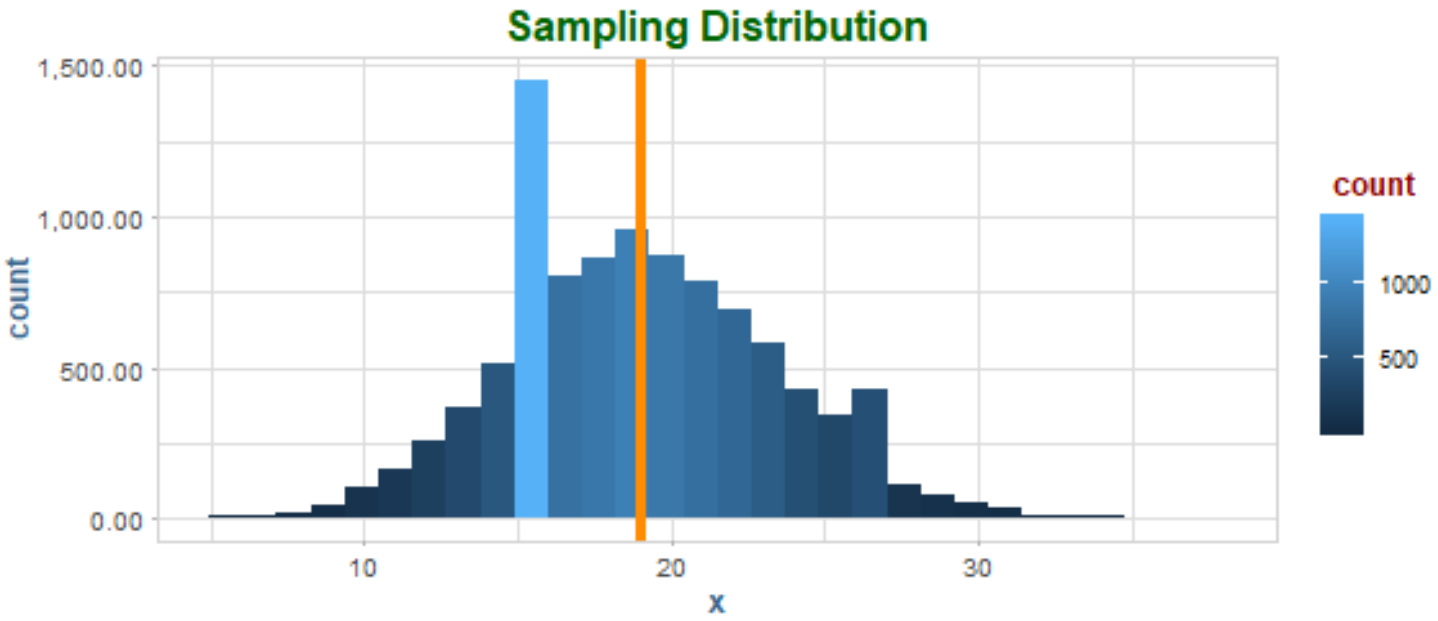
Let $X \sim Pois(4), Y \sim Pois(12), U \sim Pois(3)$ be independent random variables.

a.) What is the exact sampling distribution of $W = X + Y + U$?

$W \sim Pois(19)$

b.) Use R to simulate the sampling distribution of W and plot your results.

```
W <- rpois(10e3, lambda = 19)

ggplot(data.table(x = W)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   geom_vline(xintercept = 19, col = "darkorange", lwd = 1.5) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```



Check that the simulated mean and standard error are close to the theoretical mean and standard error.

```
mu <- mean(W)
sigma <- sd(W)
```

$\mu = 19.0308, \sigma = 4.3647401$

c.) Use the simulated sampling distribution to estimate $P(W \leq 14)$ and then check your estimate with an exact calculation.

```
phat <- mean(W <= 14)

p <- ppois(14, lambda = 19)
```

$\hat{p} = 14.88\%$

$P(W \leq 14) = 14.97\%$

4.19

Let $X_1, X_2, \ldots, X_{10} \sim^{i.i.d} N(20, 8^2)$ and $Y_1, Y_2, \ldots, Y_{15} \sim^{i.i.d} N(16, 7^2)$.

Let $W = \bar{X} + \bar{Y}$

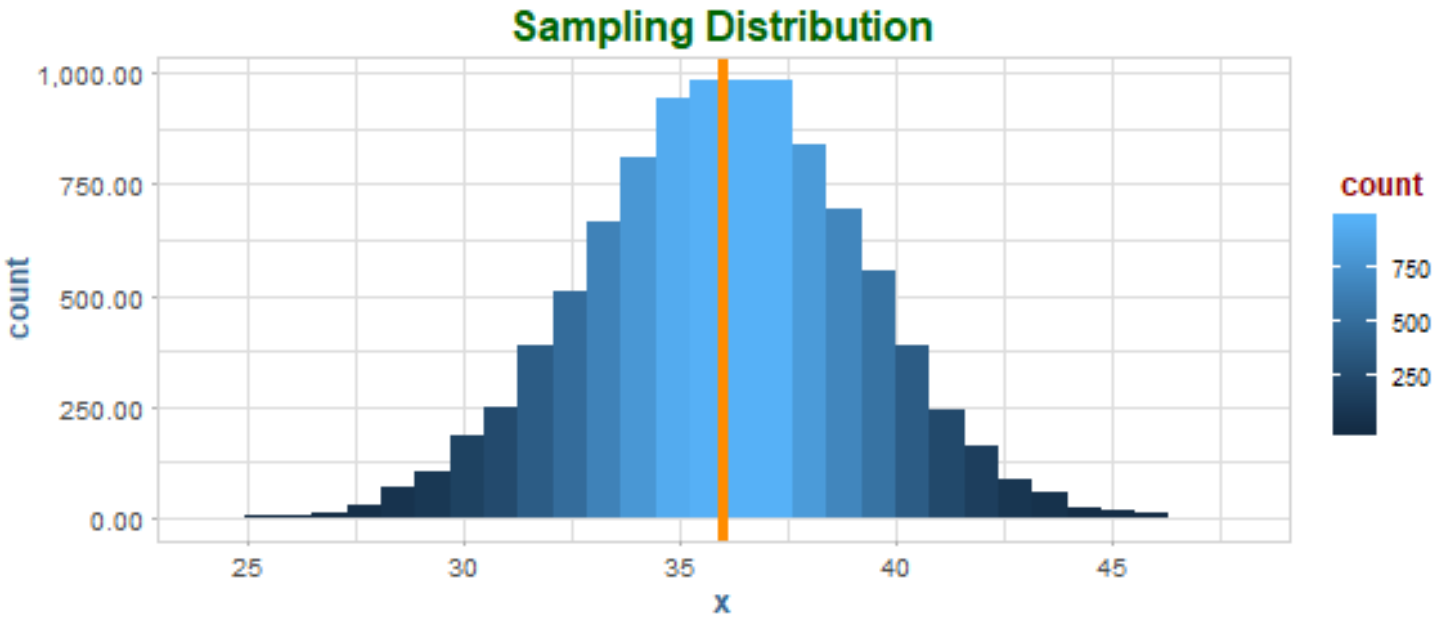
a.) Give the exact sampling distribution of W.

$\sigma = (10 + 15)/sqrt(10 + 15 - 1) = 3.1$

$W \sim N(36, 3.1^2)$

b.) Simulate the sampling distribution in R and plot your results.

```
N <- 10e3
result <- numeric(N)

for( i in 1:N)
{
   X <- rnorm(10, 20, 8)
   Y <- rnorm(15, 16, 7)

   result[i] <- mean(X) + mean(Y)
}

ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   geom_vline(xintercept = 36, col = "darkorange", lwd = 1.5) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```



Check that the simulated mean and standard error are close to the exact mean and standard error.

```
mu <- mean(result)
sigma <- sd(result)
```

$\mu = 36.0314146, \sigma = 3.1268221$

c.) Use your simulation to find $P(W < 40)$. Calculate an exact answer and compare.

```
phat <- mean(result <= 40)

p <- pnorm(40, 36, 3)
```

$\hat{p} = 90.08\%$

$P(W < 40) = 90.88\%$

4.20

Let $X_1, X_2, \ldots, X_9 \sim^{i.i.d.} N(7, 3^2)$, and $Y_1, Y_2, \ldots, X_{12} \sim^{i.i.d.} N(10, 5^2)$.

Let $W = \bar{X} - \bar{Y}$.

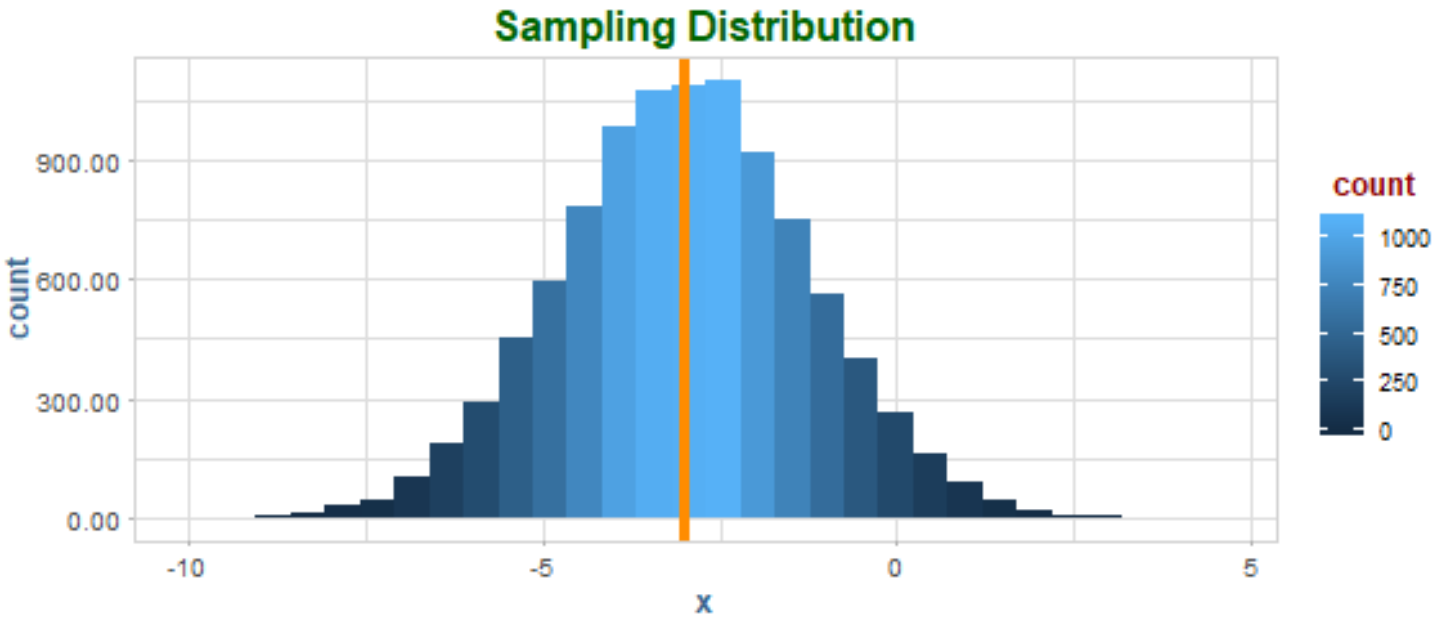
a.) Give the sampling distribution of W.

$\sigma = (3 + 5)/sqrt(9 + 12 - 1) = 1.79$

$W = N(-3, 1.79^2)$

b.) Simulate the sampling distribution of W in R, and plot your results.

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   X <- rnorm(9, 7, 3)
   Y <- rnorm(12, 10, 5)
   result[i] <- mean(X) - mean(Y)
}

ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   geom_vline(xintercept = -3, col = "darkorange", lwd = 1.5) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```



Check that the simulated mean and standard error are close to the theoretical mean and standard error.

```
mu <- mean(result)
sigma <- sd(result)
```

$\mu = -2.9932759, \sigma = 1.7650974$

c.) Use your simulation to find $P(W < -1.5)$.

```
phat <- mean(result <= -1.5)

p <- pnorm(-1.5, -3, 1.79)
```

$\hat{p} = 80.28\%$

Calculate an exact answer and compare.

$P(W < 1.5) = 79.9\%$

4.21

Let $X_1, X_2, \ldots, X_N$ be a random sample from $N(0, 1)$. Let $W = X_1^2 + X_2^2 + \ldots + X_n^2$

What is the mean and variance of the sampling distribution of W?

$\mu = 0, \sigma = 1$

Repeat using N = 4, N = 5.

$N = 4, \sigma = 4/sqrt(4 - 1) = 2.3$

$N = 5, \sigma = 5/sqrt(5-1) = 2.5$

What observations or conjectures do you have for general **n**?

4.22

Let X be a uniform random variable on the interval $[40, 60]$ and Y be a uniform random variable on $[45, 80]$.

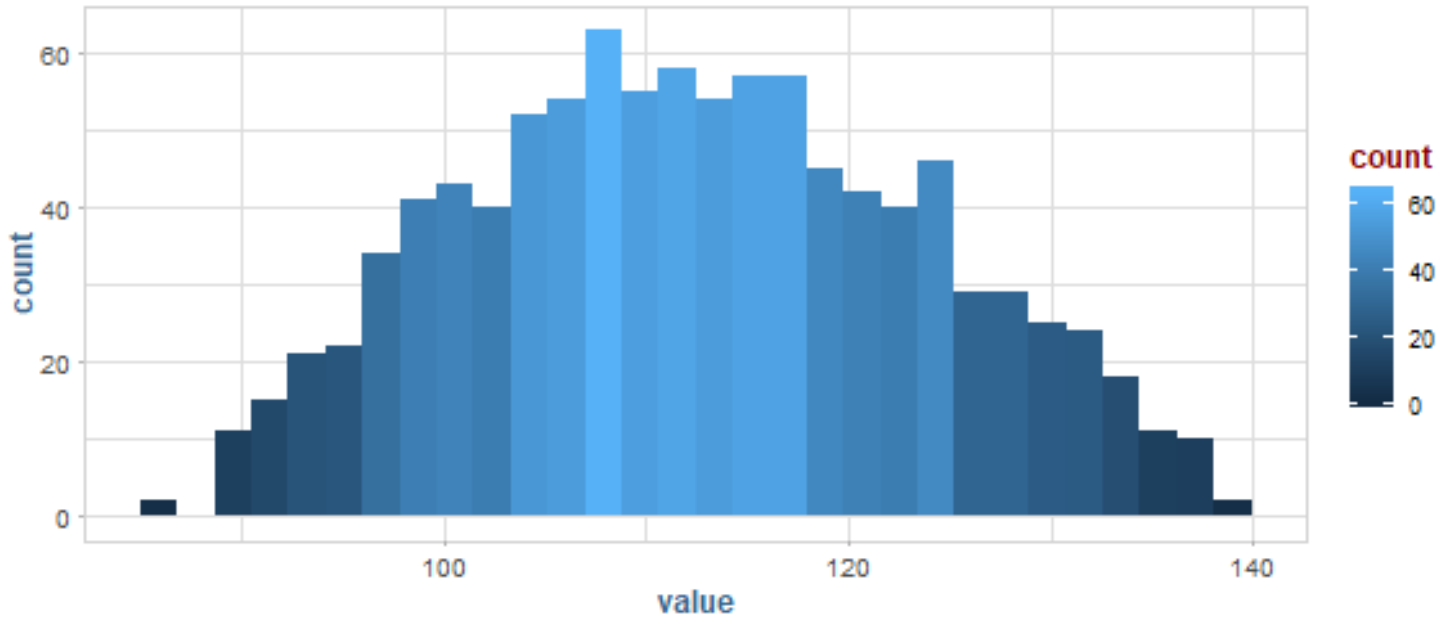
Assume that X and Y are independent.

a.) Compute the expected value and variance of $X + Y$.

$\mu = 112.5, Var = 1/24 * (140 - 85)^2 = 126.04$

b.) Simulate a sampling distribution of $X + Y$.

```
X <- runif(1000, 40, 60)
Y <- runif(1000, 45, 80)

total <- X + Y

ggplot(data.table(value = total)) +
   geom_histogram(aes(value, fill = ..count..), bins = 30) +
   labs("X + Y Sampling Distribution")
```



Describe the sampling distribution of $X + Y$. **Approximately Normal**

Compute the mean and variance of the sampling distribution and compare this with the theoretical mean and variance.

```
mu <- mean(total)
var <- var(total)
```

$\mu = 112.5068443, Var = 128.3520317$

c.) Suppose the time (in minutes) Andy takes to complete his statistics homework is $Unif[40, 60]$ and the time Adam takes is $Unif[45, 80]$.

Assume they work independently.

One day they announce that their total time to finish an assignment was less than 90 minutes.

How likely is this?

```
p <- punif(90, 85, 140)
```

Probability: **9.09%**

4.23

Let $X_1, X_2, \ldots, X_{20} \sim^{i.i.d.} Exp(2)$. Let $X = \sum_{i=1}^{20} X_i$.

a.) Simulate the sampling distribution of X in R.

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   samp <- rexp(n = 20, rate = 2)
   result[i] <- sum(samp)
}
```

b.) From your simulation, find $\mathbb{E}[X]$ and $Var[X]$.

```
mu <- mean(result)
var <- var(result)
```

$\mu = 10.0109526, Var = 4.9987997$

c.) From your simulation, find $P(X \leq 10)$.

```
phat <- mean(result <= 10)
```

Probablity: *52.41%*

4.24

Let $X_1, X_2, \ldots, X_{30} \sim^{i.i.d.} Exp(1/3)$ and let $\bar{X}$ denote the sample mean.

a.) Simulate the sampling distribution of $\bar{X}$ in R.

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   samp <- rexp(n = 30, rate = 1/3)
   result[i] <- mean(samp)
}
```

b.) Find the mean and standard error of the sampling distribution, and compare with the theoretical results.

```
mu <- mean(result)
sigma <- sd(result)
```

Sample: $\mu = 2.9971571, \sigma = 0.5534277$

Theoretical: $\mu = \frac{\lambda}{1} = 3, sigma = 3/sqrt(30-1) = .56$

c.) From your simulation, find $P(\bar{X} \leq 3.5)$.

```
phat <- mean(result <= 3.5)
```

$\hat{p} = 0.8216$

d.) Estimate $P(\bar{X} \leq 3.5)$ by assuming the CLT approximation holds.

Compare this result with the one in part **(c)**.

```
z <- (3.5 - 3) / sigma
p <- pnorm(z)
```

$p = 0.8169$

4.25

Consider the exponential distribution with density $f(x)\frac{1}{20}e^{-x/20}$, with mean and standard deviation of 20.

a.) Calculate the median of this distribution.

$0.5 = \int_m^{\inf} f(x)dx = -e^{-x/20}$

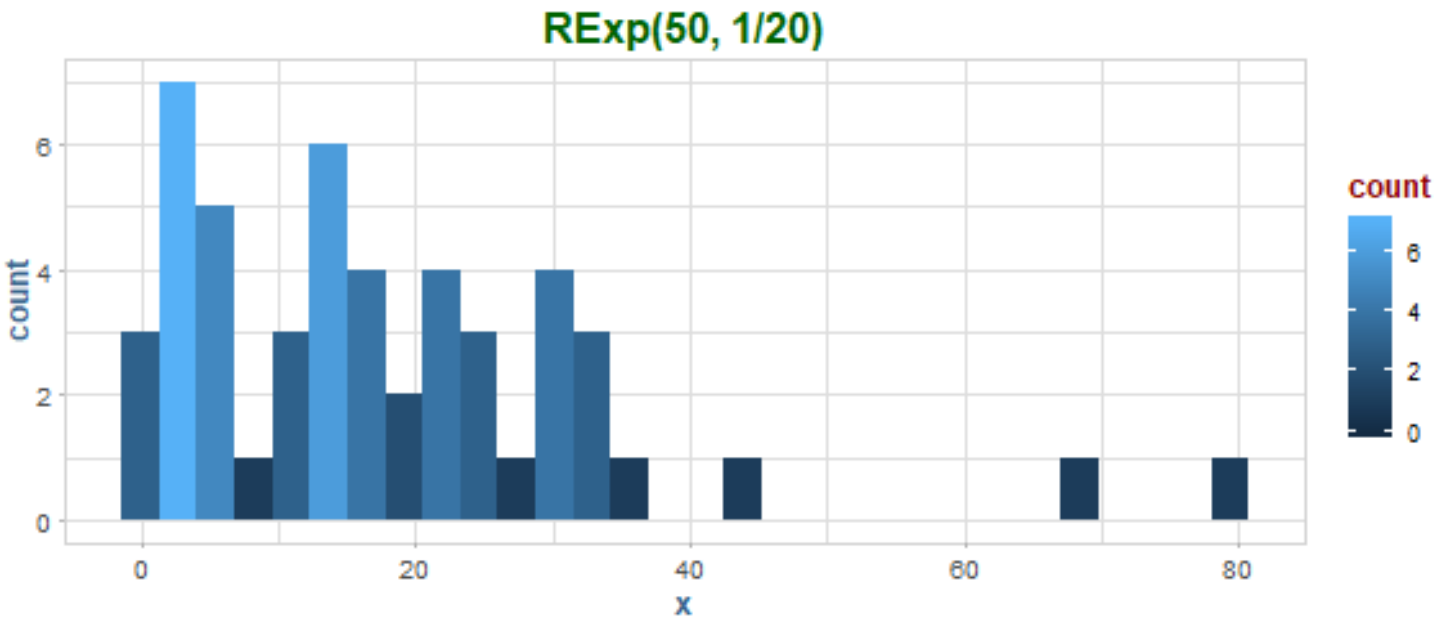
$\ldots = M = 20 * log(2)$

$\ldots = 13.8629436$

b.) Using R, draw a random sample of size 50 and graph the histogram.

```
samp <- rexp( n = 50, rate = 1/20)

ggplot(data.table(x = samp)) +
   geom_histogram(aes(x, fill = ..count..)) +
   labs(title = "RExp(50, 1/20)")
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



What are the mean and standard deviation of your sample?

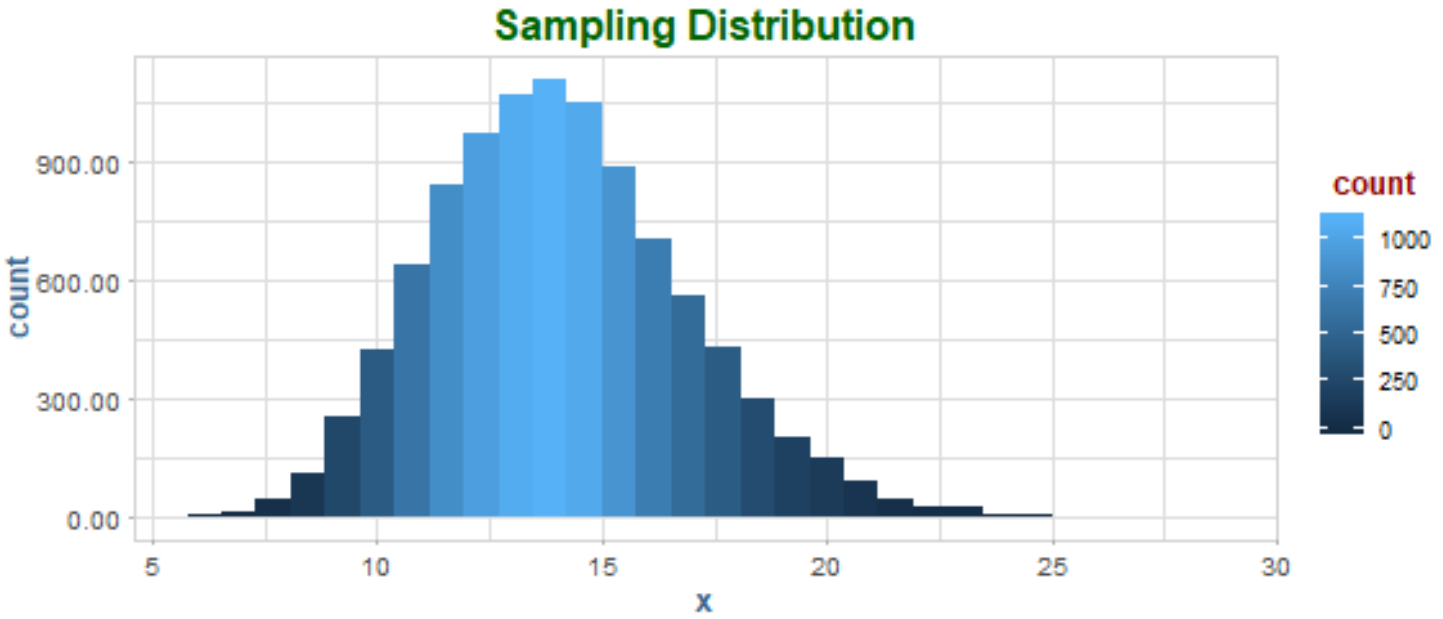
```
mu <- mean(samp)
sigma <- sd(samp)
```

$\mu = 18.3487021, \sigma = 15.9548065$

c.) Run a simulation to find the (approximate) sampling distribution for the median of sample size 50 from the exponential distribution and describe it.

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
```

```
{
   samp <- rexp( n = 50, rate = 1/20)
   result[i] <- median(samp)
}

ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```



What is the mean and the standard error of this sampling distribution?

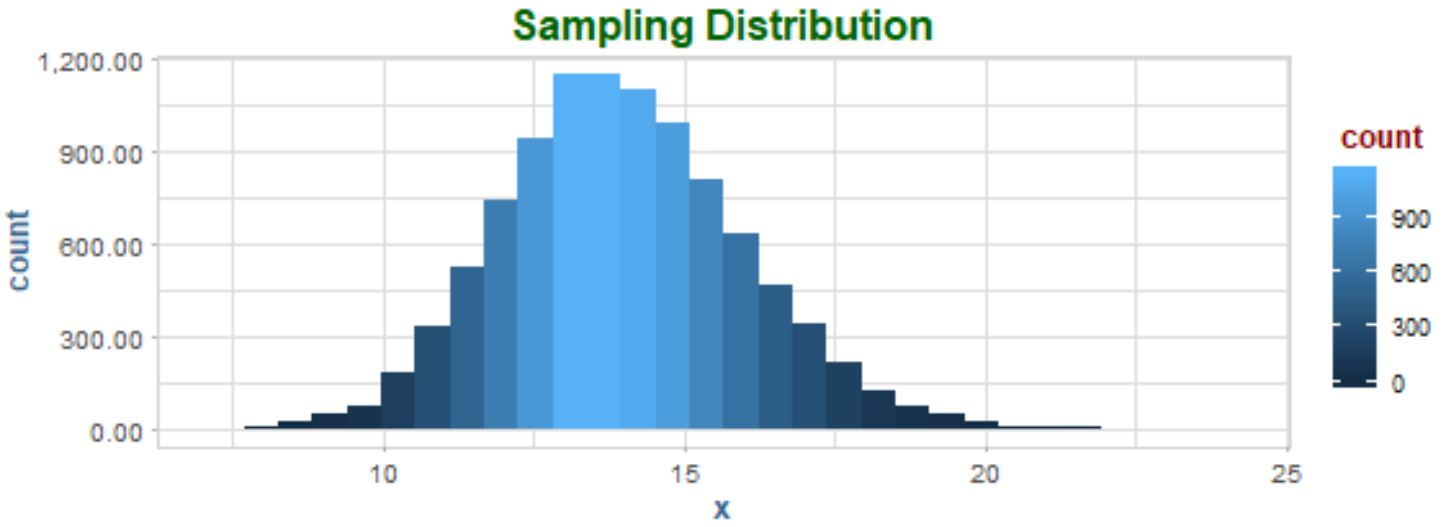
```
mu <- mean(result)
sigma <- sd(result) / sqrt(50)
```

$\mu = 14.0699157, \sigma = 0.4002735$

d.) Repeat the above but use sample sizes $n = 100, 500\ and\ 1,000.$

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   samp <- rexp( n = 100, rate = 1/20)
   result[i] <- median(samp)
}
```

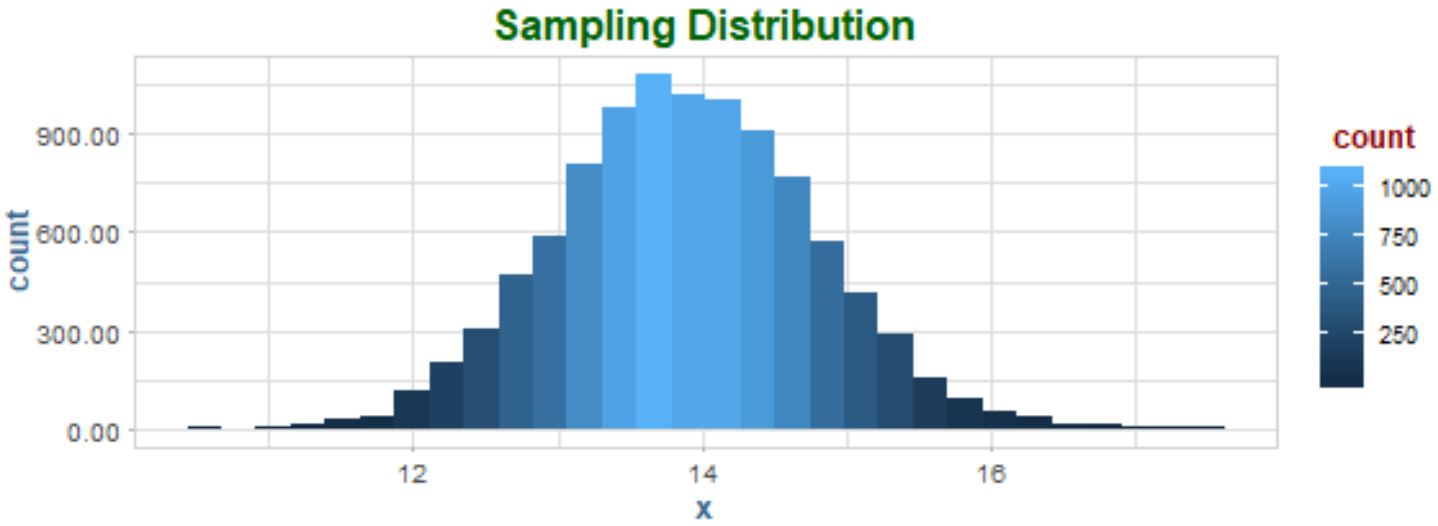
```
ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```




```
mu <- mean(result)
sigma <- sd(result) / sqrt(50)
```

$\mu = 13.9708534, \sigma = 0.2820773$

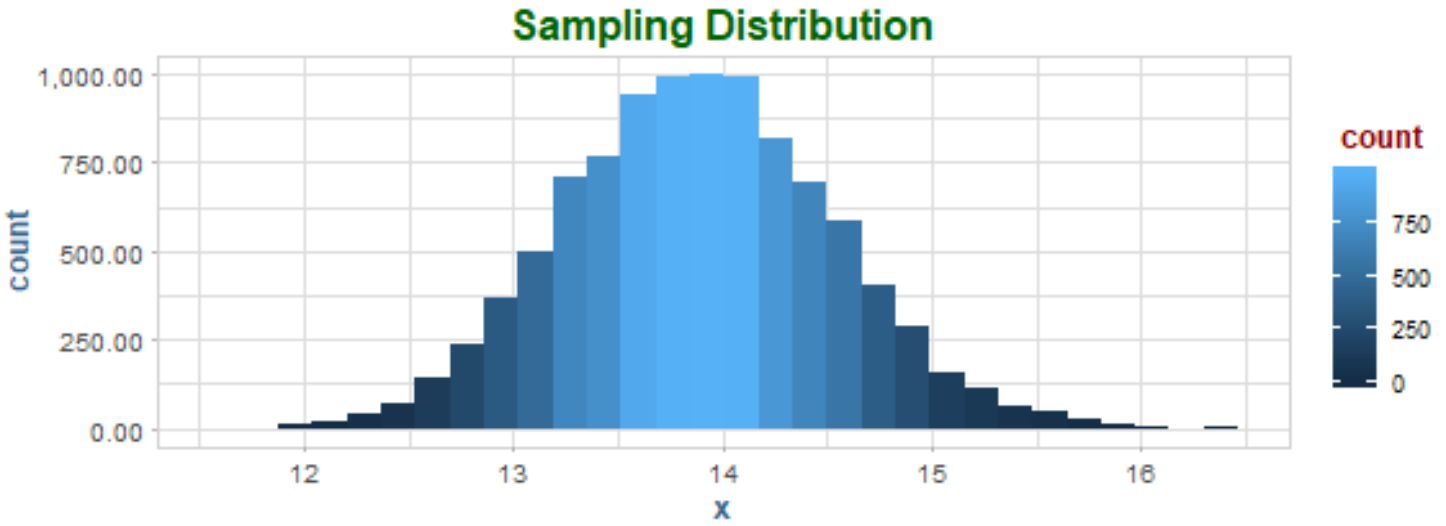
```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   samp <- rexp( n = 500, rate = 1/20)
   result[i] <- median(samp)
}

ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```




```
mu <- mean(result)
sigma <- sd(result) / sqrt(50)
```

$\mu = 13.8879017, \sigma = 0.1261444$

```
N <- 10e3
result <- numeric(N)

for(i in 1:N)
{
   samp <- rexp( n = 1000, rate = 1/20)
   result[i] <- median(samp)
}

ggplot(data.table(x = result)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   scale_y_continuous(labels = comma) +
   labs(title = "Sampling Distribution")
```




```
mu <- mean(result)
sigma <- sd(result) / sqrt(50)
```

$\mu = 13.8853986, \sigma = 0.0905196$

How does sample size affect the sampling distribution?

The sample mean and standard error converge to the analytical solution with increased sample size.

4.26

Prove theorem 4.2.1.

4.27

Let $X_1, X_2 \sim^{i.i.d.} F$ with corresponding pdf $f(x) = \frac{2}{x^2}, 1 \leq x \leq 2.$

a.) Find the pdf of X_{max}.

$F_{max}(x) = 8(\frac{1}{x^2} - \frac{1}{x^3})$

b.) Find the expected value of X_{max}.

$\mathbb{E}[F_{max}] = 1.545$

4.28

Let $X_1, X_2, \ldots, X_N \sim^{i.i.d.}$ with corresponding pdf $f(x) = 3x^2, 0 \leq x \leq 1.$

a.) Find the pdf for X_{min}.

b.) Find the pdf for X_{max}.

c.) If $n = 10$, find the probability that the largest value, X_{max}, is greater than 0.92.

4.29

Compute the pdf of the sampling distribution of the maximum samples of size 10 from a population with an exponential distribtuion with $\lambda = 12$.

4.30

Let $X_1, X_2, \ldots, X_N \sim^{i.i.d.} Exp(\lambda)$ with pdf $f(x) = \lambda e^{-\lambda x}, \lambda > 0, x > 0$.

a.) Find the pdf $f_{min}(x)$ for the sample minimum X_{min}. Recognize this as the pdf of a known distribution.

b.) Simulate in R the sampling distribution of X_{min} of samples of size $n = 25$ from the exponential distribution with $\lambda = 7$.

Compare the theoretical expected value of X_{min} with the simulated expected value.

4.31

Let $X_1, X_2, \ldots, X_n \sim^{i.i.d.} Pois(3)$. Let $X = \sum_{i=1}^{10} X_i$.

Find the pdf for the sampling distribution of X.

4.32

Let X_1 and X_2 be independent exponential random variables, both with parameter $\lambda > 0$.

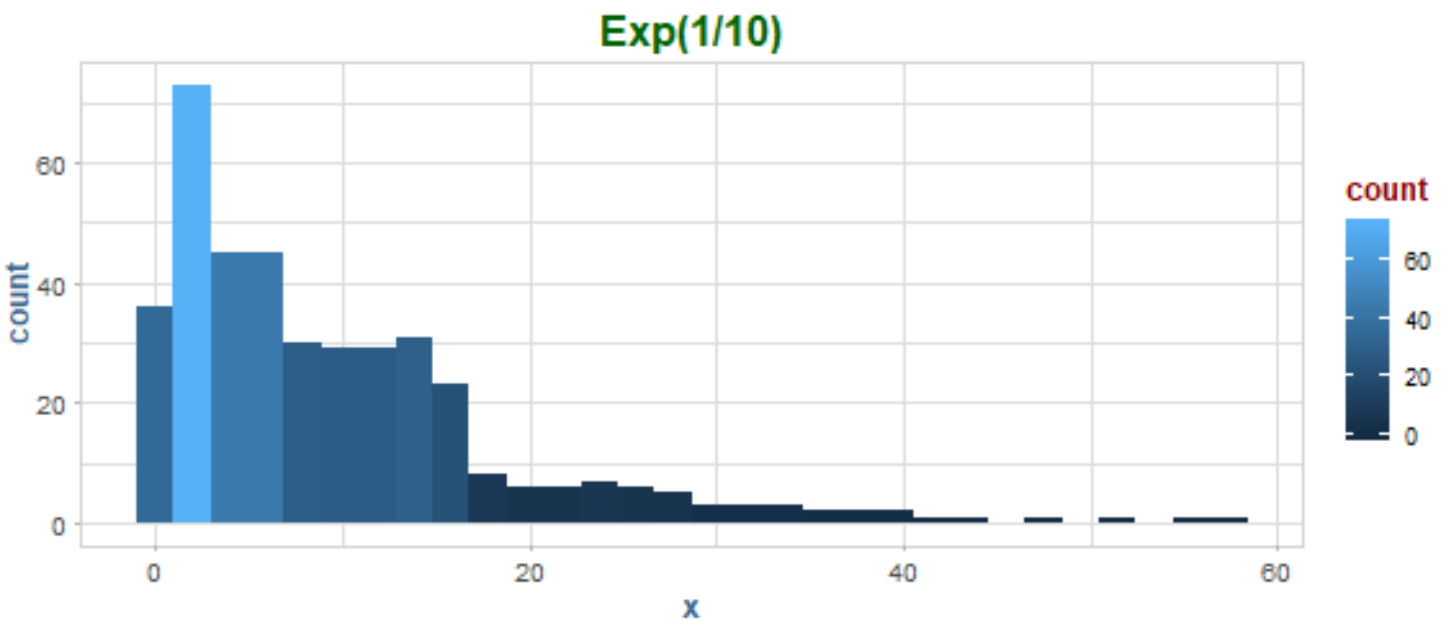
Find the cumulative distribution function for the sampling distribution of $X = X_1 + X_2$.

4.33

This simulation illustrates the CLT for a finite population.

```
N <- 400
n <- 5

finpop <- rexp(N, 1/10)

ggplot(data.table(x = finpop)) +
   geom_histogram(aes(x, fill = ..count..)) +
   labs(title = "Exp(1/10)")
```

```
`stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```




```
mean(finpop) #mean (mu) of your pop.
```

```
[1] 9.751862
```

```
sd(finpop) # stdev (sigma) of your pop.
```

```
[1] 9.551042
```

```
sd(finpop)/sqrt(n) # theoretical standard error of sampling distribution
```

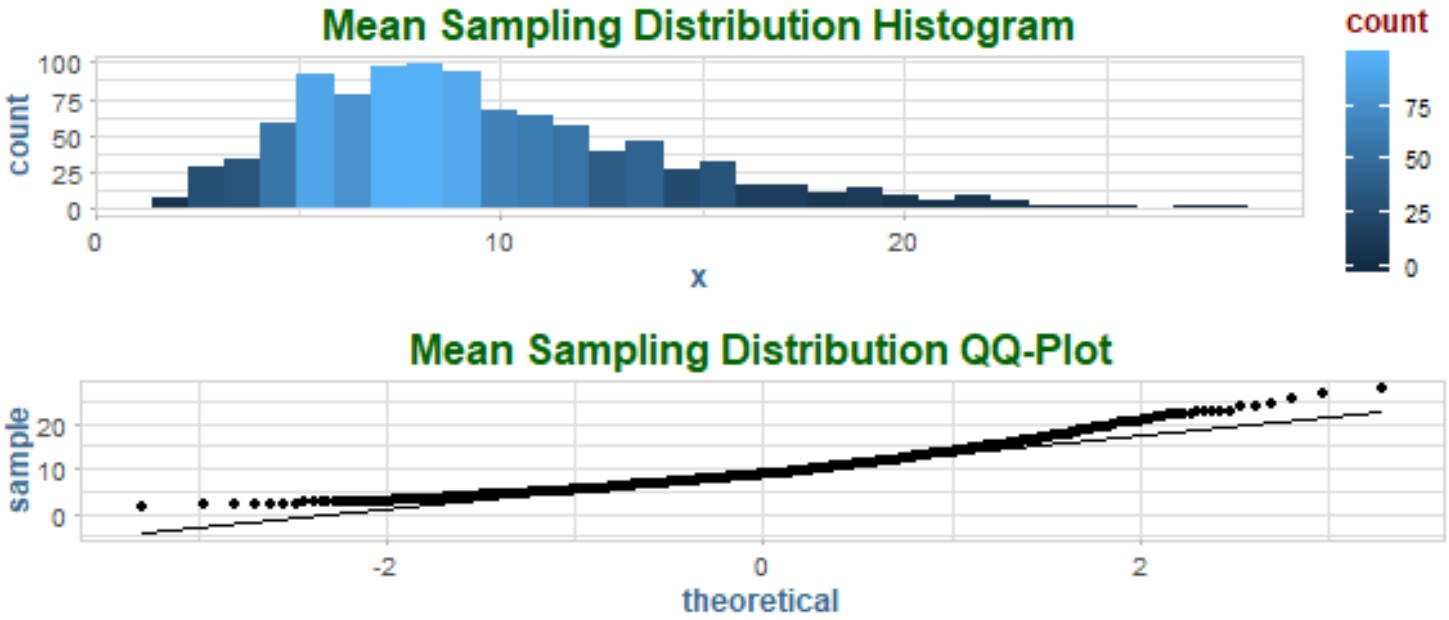
```
[1] 4.271356
```

```
sd(finpop)/sqrt(n) * sqrt((N-n)/(N-1)) # without replacement
```

```
[1] 4.249892
```

```
Xbar <- numeric(1000)
for(i in 1:1000)
{
   x <- sample(finpop, n) # Random sample of size n
                          # (w/o replacement)
   Xbar[i] <- mean(x)
}
```

```
p1 <- ggplot(data.table(x = Xbar)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   labs(title = "Mean Sampling Distribution Histogram")

p2 <- ggplot(data.table(x = Xbar), aes(sample = x)) +
   stat_qq() +
   stat_qq_line() +
   labs(title = "Mean Sampling Distribution QQ-Plot")

grid.arrange(p1, p2, nrow = 2)
```




```
mean(Xbar)
```

```
[1] 9.471362
```

```
sd(Xbar) # estimated standard error of sampling distribution
```

```
[1] 4.382019
```

a.) Does the sampling distribution of sample means appear approximately normal?

b.) Compare the mean and standard error of your simulated sampling distribution with the theoretical ones.

c.) Calculate $(\sigma/\sqrt{(n)})(\sqrt{(N-n)/(N-1)})$, where σ is the standard deviation of the finite population and compare with the (estimated) standard error of the sampling distribution.

d.) Repeat for larger **n**, say **n** = 20 and **n** = 100.

4.34

Let $X_1, X_2, \ldots, X_n$ be independent random variables from $N(\mu, \sigma)$.

We are interested in the sampling distribution of the variance.

Run a simulation to draw random samples of size 20 from $N(25, 7^2)$ and calculate the variance for each sample.

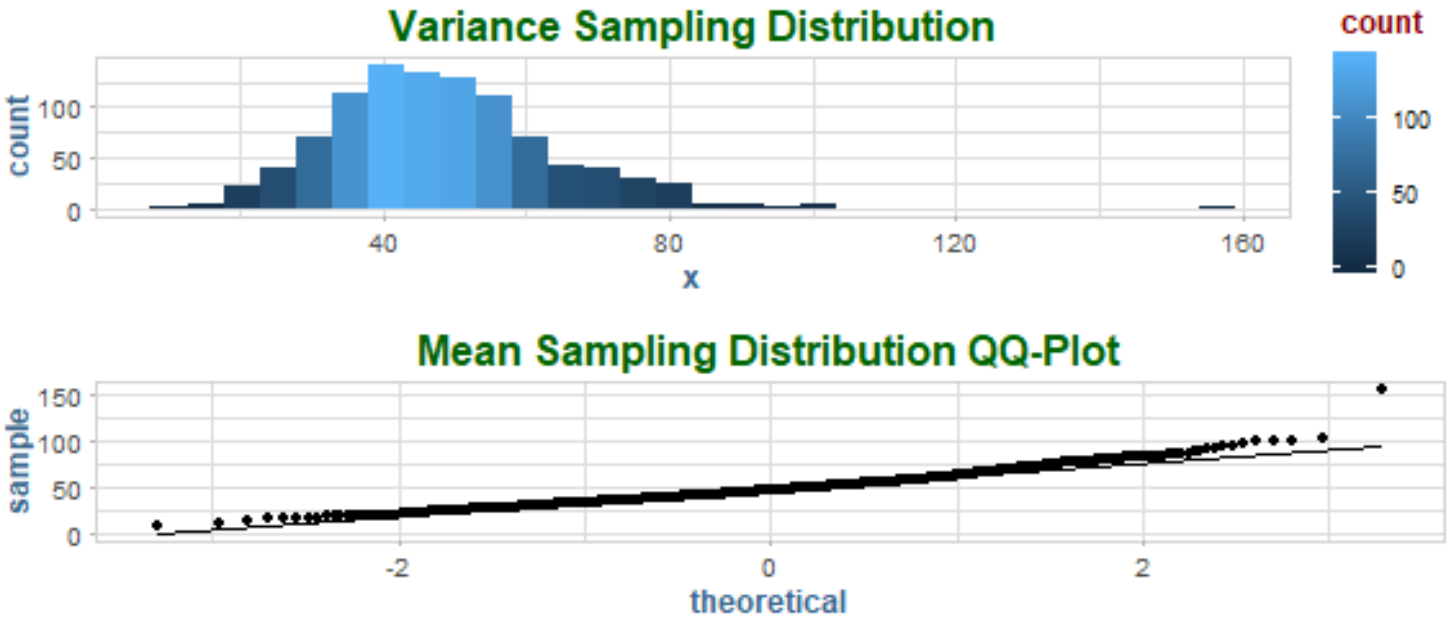
```
W <- numeric(1000)

for(i in 1:1000)
{
   x <- rnorm(20, 25, 7)
   W[i] <- var(x)
}

mean(W)
```

```
[1] 48.30215
```

```
var(W)
```

```
[1] 239.5773
```

```
p1 <- ggplot(data.table(x = W)) +
   geom_histogram(aes(x, fill = ..count..), bins = 30) +
   labs(title = "Variance Sampling Distribution")

p2 <-
   ggplot(data.table(value = W), aes(sample = value)) +
   stat_qq() +
   stat_qq_line() +
   labs(title = "Mean Sampling Distribution QQ-Plot")


grid.arrange(p1, p2, nrow = 2)
```



Does the sampling distribution appear to be normally distributed?

Repeat with n = 50 and n = 200.

4.35

A random sample of size $n = 100$ is drawn from a distribution with pdf $f(x) = 3(1-x)^2, 0 \leq x \leq 1$.

a.) Use the CLT approximation to estimate $P(\bar{X} \leq 0.27)$.

b.) Use the expanded CLT to estimate the same probability (dnorm).

c.) If $X_1, X_2, X_3 \sim^{i.i.d.} Unif[0,1]$, then the minimum has density **f** given above.

Use simulation to estimate the probability.