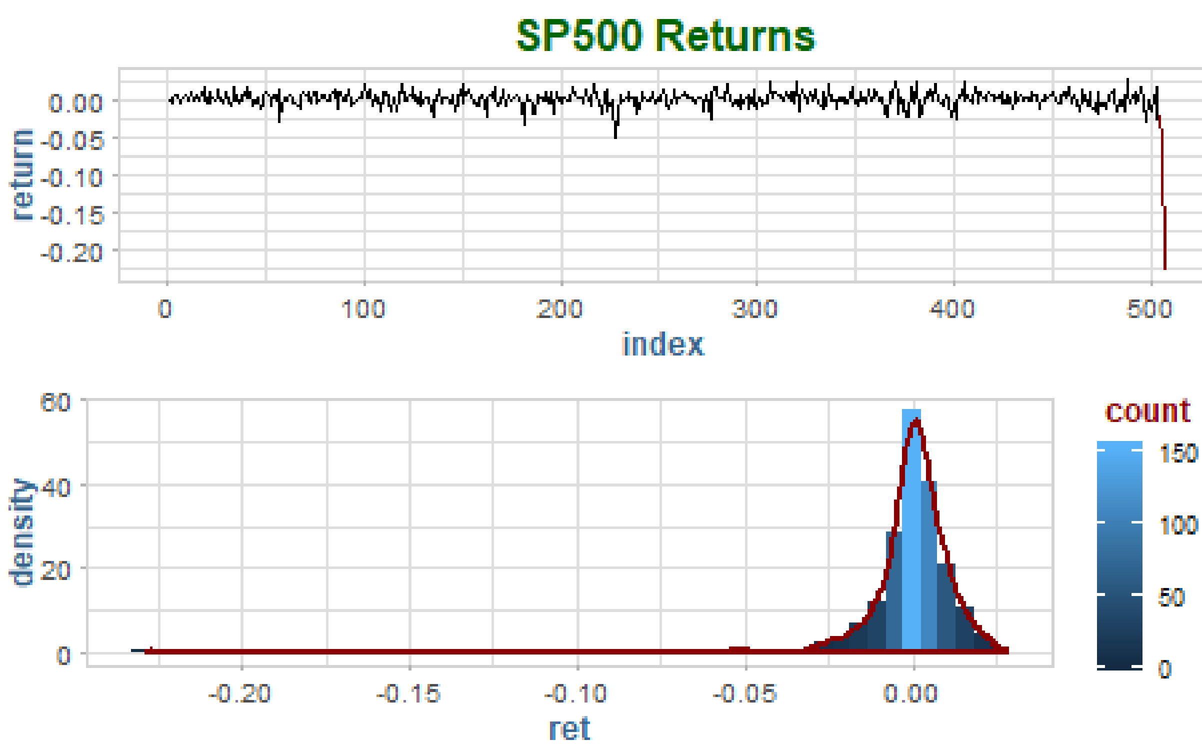
Problem Background

On Black Monday, the return on the S&P500 was *-22.8%*.

In this lab we are going to look at GARCH models and how they relate to predicting extreme events in financial markets.

```
data(SP500, package = "Ecdat")

ret.blm <- SP500$r500[1805] # Black Monday is the 1805 obs.
x <- SP500$r500[(1804 - 2*253+1 ): 1804]
returns <- data.table(ret = c(x, ret.blm))[, x := .I]

suppressWarnings({
   p1 <- ggplot(returns, aes( x = x)) +
      geom_line(aes(x = x, y = ifelse( x < 505, ret, NA))) +
      geom_line(aes(x = x, y = ifelse( x >= 505, ret, NA)), col = "darkred") +
      labs(title = "SP500 Returns", x = "index", y = "return")

   p2 <- ggplot(returns, aes(ret, y = ..density..)) +
      geom_histogram(aes(fill = ..count..), bins = 50) +
      geom_density(aes(y = ..density..), col = "darkred", lwd = 1)

   grid.arrange(p1, p2, nrow = 2)
})
```



Now, we fit the GARCH model.

Below, we will fit a AR(1) + GARCH(1, 1) model for the data 2 years prior to Black Monday (assuming 253 trading days/year).

```
spec <- ugarchspec( mean.model = list( armaOrder = c(1, 0) ),
                    variance.model = list( garchOrder = c(1, 1)),
                    distribution.model = "std")

fit <- ugarchfit(data = x, spec = spec)

dfhat <- coef(fit)[6]

forecast <- ugarchforecast(fit, data = x, n.ahead = 1)
forecast
```

```
*------------------------------------*
*       GARCH Model Forecast         *
*------------------------------------*
Model: sGARCH
Horizon: 1
Roll Steps: 0
Out of Sample: 0

0-roll forecast [T0=1971-05-21 20:00:00]:
       Series   Sigma
T+1 -0.003204 0.01208
```

The parameter estimates for the model are below:

Table 1: Parameter Estimates

mu	ar1	omega	alpha1	beta1	shape
0.001333	0.082048	0	1.2e-05	0.99898	3.292482

Problem 1.)

What is the conditional probability of a return less than or equal to -0.228 on Black Monday?

```
ret.pred <- fitted(forecast)
ret.sd <- sigma(forecast)

scale <- ret.sd * sqrt((dfhat-2)/dfhat)

z_score <- as.numeric( ( ret.blm - ret.pred) / scale )

bm.prob <- pt(z_score, dfhat, lower.tail = T)
```

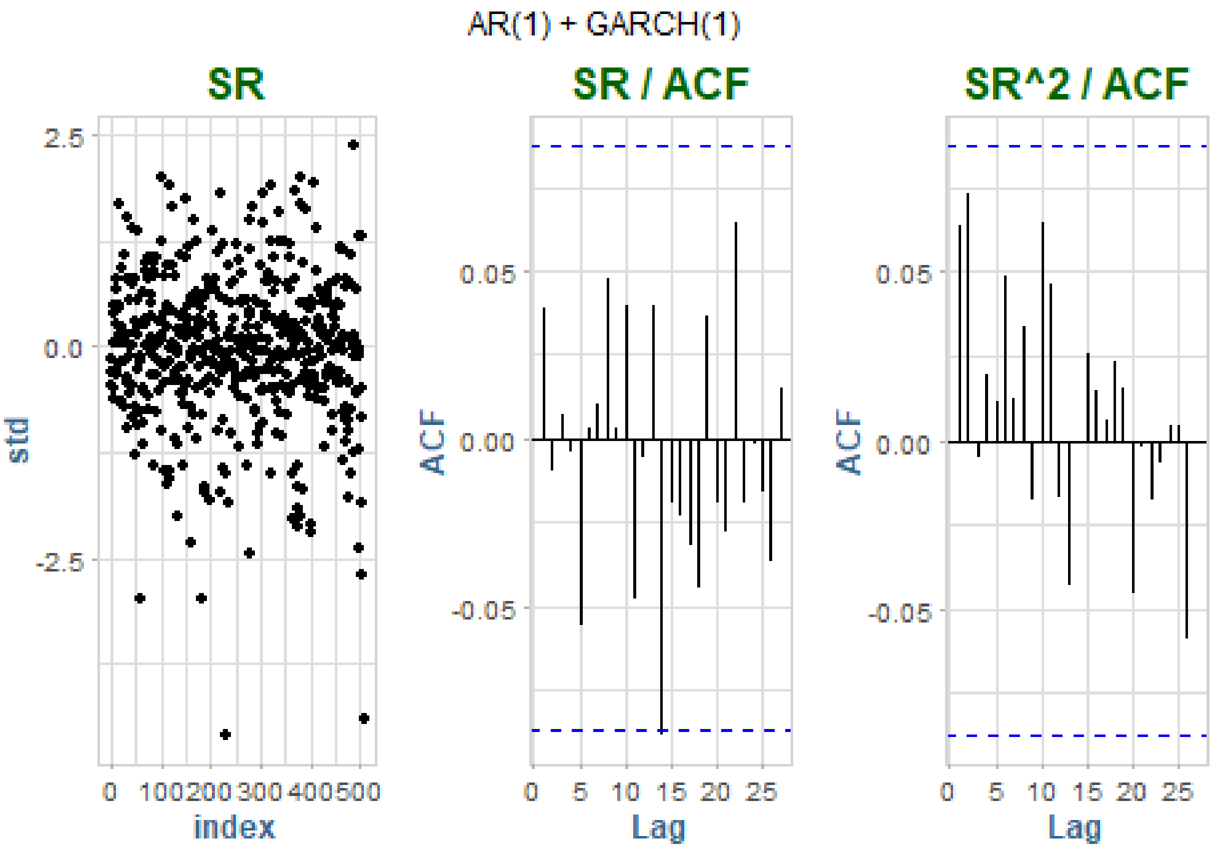
The probability of a return less than or equal to that of Black Monday is: 0.00002042755

Problem 2.)

Model Diagnostics

Compute and plot the standardized residuals. Also, plot the ACF of the standardized residuals and their squares.

```
plot_residuals <- function(res, sd, mname) {

   residual <- data.table(
      std = res / sd)[,
        res_sq := std*std][,
                           index := .I]

   p1 <- ggplot(residual, aes(x = index, y = std)) +
      geom_point() +
      labs(title = "SR")

   p2 <- ggAcf(residual$std) +
      labs(title = "SR / ACF")

   p3 <- ggAcf(residual$res_sq) +
      labs(title = "SR^2 / ACF")

   grid.arrange(p1, p2, p3, nrow = 1, top = mname)
}

plot_residuals(forecast@model$modeldata$residuals,
               forecast@model$modeldata$sigma,
               "AR(1) + GARCH(1)")
```



Does the model fit adequately?

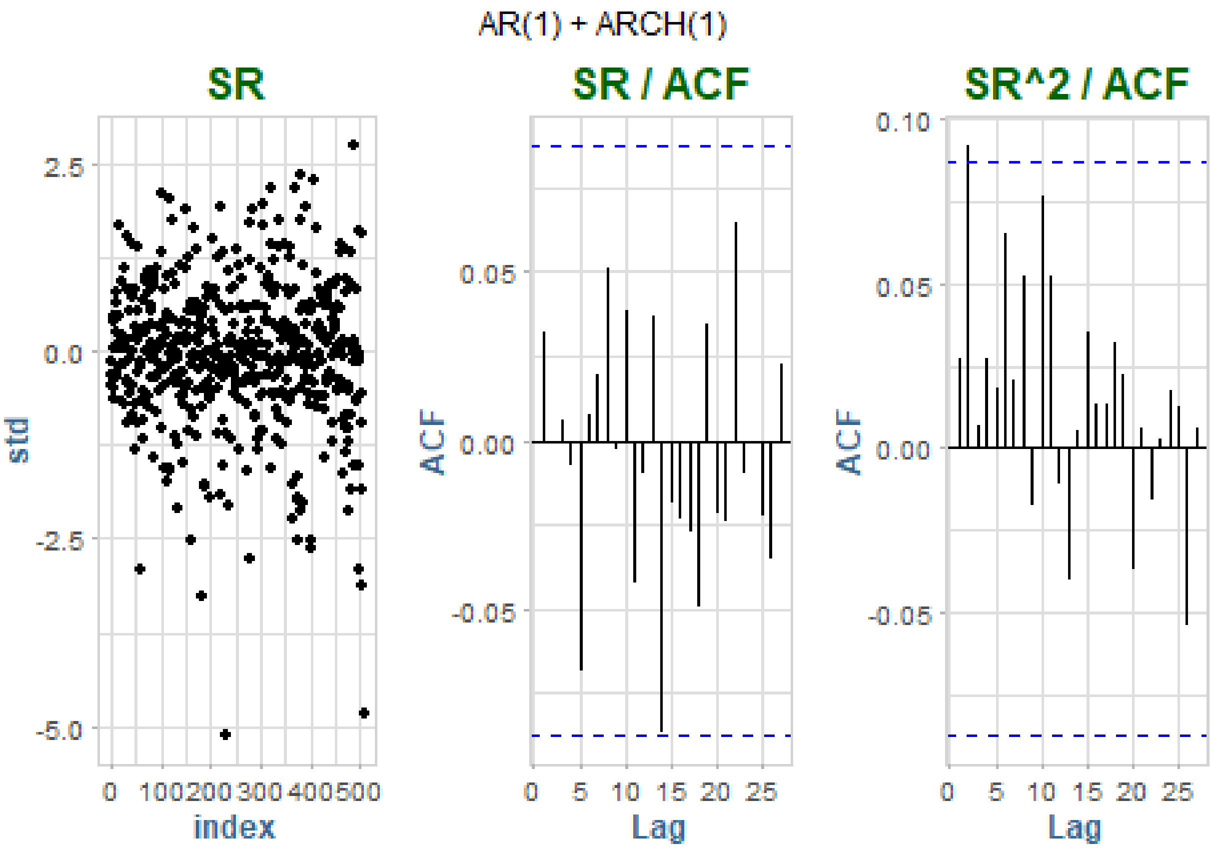
In the residual plot above, we see what appears to be white noise for the residual plots. Neither of the ACF plots display significant residual auto-correlation, so we would say this model fits reasonably well outside of a few points (specifically at lag 14 in the SR ACF). The SR^2 plot shows no significant autocorrelations, and we would conclude this model is of adequate fit.

Problem 3.)

AR(1) + ARCH(1)

Would an AR(1) + ARCH(1) model provide an adequate fit?

```
spec2 <- ugarchspec( mean.model = list( armaOrder = c(1, 0) ),
                     variance.model = list( garchOrder = c(1, 0)),
                     distribution.model = "std")

fit2 <- ugarchfit(data = x, spec = spec2)

plot_residuals(as.vector(residuals(fit2)),
               as.vector(sigma(fit2)),
               "AR(1) + ARCH(1)")
```



The residual diagnostics for the AR(1) + ARCH(1) are close to the AR(1) + GARCH(1), however, in the SR^2 plot we start to see pronounced autocorrelations at lag 2, which is concerning.

```
ic.data <- as.data.table(cbind(infocriteria(fit), infocriteria(fit2)),
                         keep.rownames = T)[,
                                            D := V1 - V2]
colnames(ic.data) <- c("Information Criterion", "fit1", "fit2", "Delta")

pretty_kable(ic.data, "Model Fit Comparison", dig = 4)
```

Table 2: Model Fit Comparison

Information Criterion	fit1	fit2	Delta
Akaike	-6.5205	-6.5177	-0.0028
Bayes	-6.4704	-6.4759	0.0055
Shibata	-6.5208	-6.5179	-0.0029
Hannan-Quinn	-6.5008	-6.5013	0.0005

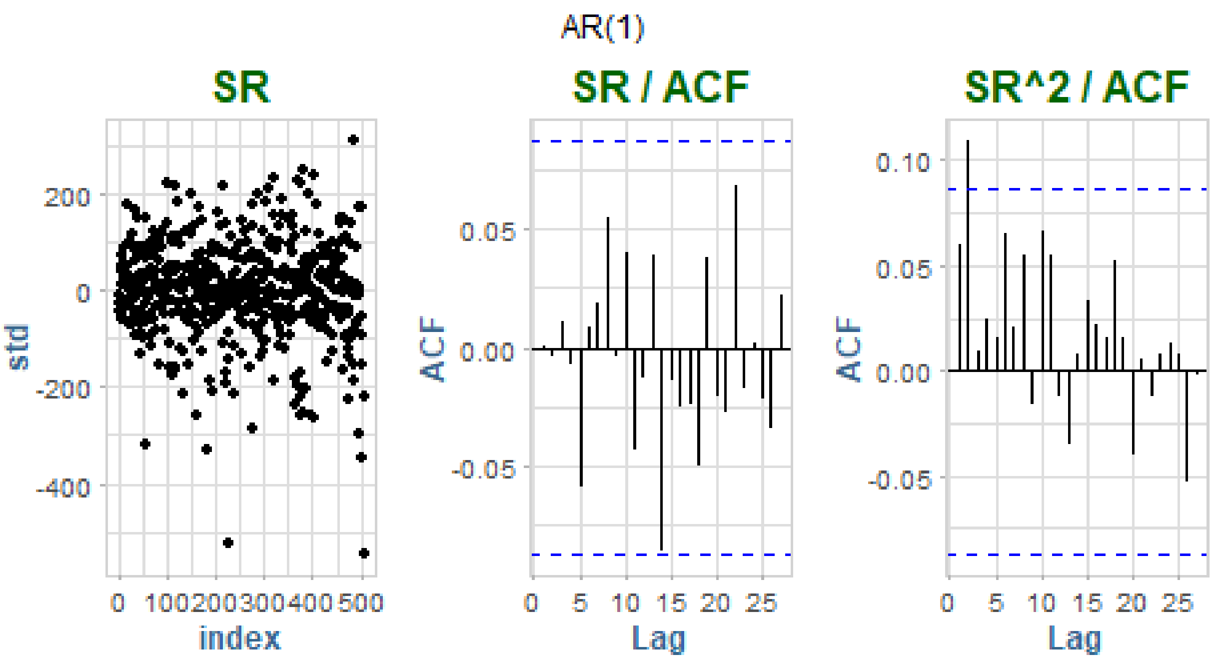
If we compare the information criterion for the model fits, we see that the AR(1) + GARCH(1) has lower AIC, but the AR(1) + ARCH(1) has a lower BIC.

Depending on the IC, we would choose different models, however, the differences are rather small.

Problem 4.)

Does an AR(1) model with a Gaussian conditional distribution prove an adequate fit?

```
fit3 <- arima( x, order = c(1, 0, 0))

suppressMessages({
   plot_residuals(fit3$residuals,
             fit3$sigma2,
             "AR(1)")
})
```

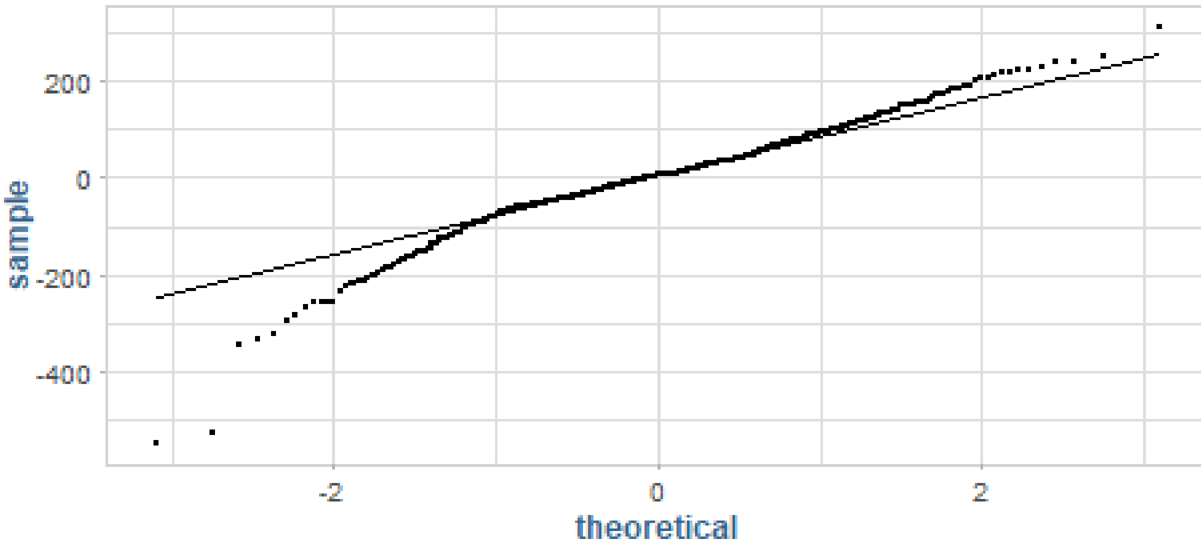



```
# normalize AIC/BIC to compare to previous models
pretty_kable(data.table(AIC = AIC(fit3) / fit3$nobs,
                        BIC = BIC(fit3) / fit3$nobs), "Information Criteron AR(1)", dig = 4)
```

Table 3: Information Criteron AR(1)

AIC	BIC
-6.4171	-6.392

Looking at the residual plots, we see similar characteristics of the previous models. There is some volatility clustering and the pronouced lags at 14 in the SR and 2 in the SR^2, and in general the lags are more pronounced in the Gaussian model. The information criterion is quite a bit higher in both AIC and BIC. I would say this model is not fully adequate, and would strongly prefer either of the two previous models.



Moreover, since the errors are assumed Gaussian, we can use the normal plot of the residuals to show the extremely poor fit of the standardized residuals at the tails.

Problem 5.)

The conditional variance of an AR(1) + GARCH(1, 1) process is:

$Y_t - \mu = \phi(Y_{t-1} - \mu) + a_t$

$a_t = \sigma_t \epsilon_t$

is

$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$

Where $\mu, \phi, \omega, \alpha, \beta$ are constant parameters.

a.) What is $\mathbb{E}[Y_t - \mu]$?

$\mathbb{E}[Y_t - \mu] = \mathbb{E}[\phi(Y_{t-1} - \mu) + a_t],$

$\ldots = \phi\mathbb{E}[Y_{t-1} - \mu] + \mathbb{E}[a_t],$

From **12.6**, we know that $\mathbb{E}[Y_t] = \mu \;\; \forall t$ and from **12.3**, $\mu = 0 \;\; \forall t$, so that:

$\ldots = \phi(0) + \mathbb{E}[\sigma_t \epsilon_t],$

$\ldots = \sigma_t \mathbb{E}[\epsilon_t],$

From **14.3** we know ϵ_t is i.i.d, so that $\mathbb{E}[\epsilon_t] = 0$

$\ldots = 0$

b.) What is $\mathbb{E}[\epsilon_t]$?

From **14.3** we know ϵ_t is i.i.d, or $\mathbb{E}(\epsilon_t | \epsilon_{t-1}, \ldots) = 0$, so that $\mathbb{E}[\epsilon_t] = 0$

c.) What is $\mathbb{E}[\epsilon_t^2]$?

From **14.3** we know $\mathbb{V}(\epsilon_t | \epsilon_{t-1}, \ldots) = 1$, or $\mathbb{V}(\epsilon_t) = 1$, because ϵ is i.i.d.

Hence, $\mathbb{V}[X] = \mathbb{E}[X - \mu]^2 = \mathbb{E}[X^2] - \mu^2$

$\mathbb{V}[\epsilon_t] = \mathbb{E}[\epsilon_t^2] - \{\mathbb{E}[\epsilon_t]\}^2$

$\mathbb{E}[\epsilon_t^2] = \mathbb{V}(\epsilon_t) + \{\mathbb{E}[\epsilon_t]\}^2$

$\{\mathbb{E}[\epsilon_t]\}^2$ = 0,

So,

$\mathbb{E}(\epsilon_t^2) = \mathbb{V}(\epsilon_t) = 1$

d.) What is $\mathbb{E}[\epsilon_t \epsilon_{t-1}]$?

$Cov(\epsilon_t, \epsilon_{t-1}) = \mathbb{E}[\epsilon_t \epsilon_{t-1}] - \mathbb{E}[\epsilon_t]\mathbb{E}[\epsilon_{t-1}]$

$Cov(\epsilon_t, \epsilon_{t-1}) = 0$, by independence of ϵ.

$\mathbb{E}[\epsilon_t] = \mathbb{E}[\epsilon_{t-1}] = 0$ from **b.**,

$$\mathbb{E}[\epsilon_t \epsilon_{t-1}] = Cov(\epsilon_t, \epsilon_{t-1}) + \mathbb{E}[\epsilon_t]\mathbb{E}[\epsilon_{t-1}] = 0$$

e.) What is the *unconditional* variance of the process?

$$\mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha_{t-1}^2 + \beta_{t-1}^2]$$

$$\gamma_a(0) = \omega + \alpha\gamma_0 + \beta\gamma_0$$

$$\gamma_a(0)[1 - \alpha - \beta] = \omega$$

$$\gamma_a(0) = \frac{\omega}{1-\alpha-\beta}$$

f.) Show that this correctly reduces to Ruppert Eq. 12.8 for unconditional variance of an AR(1) process.

$$Cov(Y_y, Y_{t+h}) = \phi^{|h|}\frac{\sigma_\epsilon^2}{1-\phi^2}$$

$$\ldots = Cov(Y_y, Y_{t+h}) = \frac{\phi^{|h|}\gamma_\alpha(0)}{1-\phi^2}$$

From **e**, $\gamma_a(0) = \frac{\omega}{1-\alpha-\beta}$

$$\ldots = Cov(Y_y, Y_{t+h}) = \frac{\phi^{|h|}[\frac{\omega}{1-\alpha-\beta}]}{1-\phi^2}$$