Chapter 3

1

Compute the mean and the median of the following series of returns:

```
dat <- data.table(Returns = c(.12, .5, -.08, .2, .04, .1, .02))

pretty_kable(dat, "Returns")
```

Table 1: Returns

Returns
0.12
0.50
-0.08
0.20
0.04
0.10
0.02

```
mean(dat$Returns)
```

```
[1] 0.1285714
```

```
median(dat$Returns)
```

```
[1] 0.1
```

2

Compute the sample mean and the standard deviation of the following returns:

```
dat <- data.table(Returns = c(.12, .5, -.08, .2, .04, .1, .02))

pretty_kable(dat, "Returns")
```

Table 2: Returns

Returns
0.12
0.50
-0.08
0.20
0.04
0.10
0.02

3

Prove that Equation 3.2 is an unbiased estimator of the mean. That is, show that $\mathbb{E}[\hat{\mu}] = \mu$

$\mathbb{E}(\bar{x}) = \mathbb{E}[\frac{x_1+x_2+x_3+...+x_n}{n}]$

$... = \mathbb{E}\frac{1}{n}(x_1 + x_2 + x_3 + ... + x_n)$

$... = \frac{1}{n}\mathbb{E}(x_1 + x_2 + x_3 + ... + x_n)$

$... = \frac{1}{n}(\mathbb{E}[x_1] + \mathbb{E}[x_2] + \mathbb{E}[x_3] + ... + \mathbb{E}[x_n])$

$... = \frac{1}{n}(\mu_1 + \mu_2 + \mu_3 + ... + \mu_n)$

$... = \frac{1}{n}n(u)$

$... = \mu$

4

What is the standard deviation of the estimator in Eq. 3.2? Assume the various data points are i.i.d.

$\mu = \frac{1}{n}\sum_{i=1}^{n} r_i$

$\frac{1}{n}\sum_{i=1}^{n} r_i = \frac{1}{n}(r_1 + r_2 + r_3 ... + r_n)$

$\sigma^2 = \frac{1}{n}\sum(x_i - \mu)^2$

$\sigma_\mu = \frac{\sigma}{\sqrt{n}}$

5

Calculate the population covariance and correlation of the following series:

```
dat <- data.table(S1 = c(.21, .53, .83, .19), S2 = c(.2, .32, .8, .4))

pretty_kable(dat, "Returns")
```

Table 3: Returns

S1	S2
0.21	0.20
0.53	0.32
0.83	0.80
0.19	0.40

```
cov(dat$S1, dat$S2)
```

```
[1] 0.06493333
```

```r
cor(dat$S1, dat$S2)
```

```
[1] 0.8239775
```

6

Calculate the population mean, standard deviation and skewness of each of the following two series:

```r
dat <- data.table(S1 = c(-51, -21, 21, 51), S2 = c(-61, -7, 33, 35))

pretty_kable(dat, "Returns")
```

Table 4: Returns

S1	S2
-51	-61
-21	-7
21	33
51	35

```r
skew <- function(x) {
   mean(x - mean(x))^3 / sd(x)^3
}

apply(dat, 2, mean)
```

```
S1 S2
 0  0
```

```r
apply(dat, 2, sd)
```

```
      S1       S2
45.03332 45.03332
```

```r
apply(dat, 2, skew)
```

```
S1 S2
 0  0
```

7

Calculate the population mean, standard deviation and skewness of each of the following two series:

```r
dat <- data.table(S1 = c(-23, -7, 7, 23), S2 = c(-17, -17, 17, 17))

pretty_kable(dat, "Returns")
```

Table 5: Returns

S1	S2
-23	-17
-7	-17
7	17
23	17

```r
skew <- function(x) {
  mean(x - mean(x))^3 / sd(x)^3
}

apply(dat, 2, mean)
```

```
S1 S2
 0  0
```

```r
apply(dat, 2, sd)
```

```
      S1       S2
19.62991 19.62991
```

```r
apply(dat, 2, skew)
```

```
S1 S2
 0  0
```

8

Given the probability density function for a random variable, X

$f(x) = \frac{x}{18}$ for $0 \leq x \leq 6$

$\mu = \int_0^6 \frac{x^3}{18*3} = \frac{6^3}{3*18} - \frac{0^3}{3*18}$

$\frac{6^2}{3^2} = 4$

$\sigma^2 = \int_0^6 (x-4)^2 \frac{x}{18} dx$

$\frac{1}{18}(x^3 - 8x^2 + 16x)dx$

$\sigma^2 = 2(9 - 16 + 8) = 2$

9

Prove that Equation 3.19, reproduced here, is an unbiased estimator of the variance.

$\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu}_x)$

$\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n} = \frac{1}{n-1}\sum_{i=1}^{n}(\frac{n-1}{n}x_i - \frac{1}{n}\sum_{j \neq x} x_j)^2$

$\hat{\sigma}_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n(n-1)}\sum i = 1^n \sum_{i \neq j} x_i x_j$

$\mathbb{E}[x_i^2] = \sigma^2 + \mu^2$

$\mathbb{E}[x_i x_j] = \mu_i \mu_j + iff\, \sigma_{ij} = 0\, \forall i \neq j$

10

Given two random variables, X_A and X_B, with the corresponding means μ_A and μ_B and standard deviations σ_A and σ_B, prove that the variance of X_A plus X_B is:

$\mathbb{Var}[X_A + X_B] = \sigma_A^2 + \sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B$

$\mathbb{E}[X_A + X_B] = \mathbb{E}[X_A] + \mathbb{E}[X_B] = \mu_A + \mu_B$

$\mathbb{Var}[X_A + X_B] = \mathbb{E}[(X_A + X_B - \mathbb{E}[X_A + X_b])^2]$

$\mathbb{Var}[X_A + X_B] = \sigma_A^2 + \sigma_B^2 + 2Cov[X_A, X_B]$

11

A \$100 notional, zero coupon bond has one year to expiry. The probability of default is 10%. In the event of default, assume that the recovery rate is 40%.

The continiously compounded discount rate is 5%. What is the present value of this bond?

```r
EV <- .9 * 100 + 0.10 * 40
```

$\mathbb{E}(V) = 94$

```r
PV <- exp(-.05) * EV
```

$\mathbb{P}(V)$ = \$89.42