Copulas

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Week 4 - Predict 451

Why Copulas?

- We can have *tail dependence* without *correlations* and without univariate *fat tails*.
- But t-distribution mixes up all these effects.
- Want to model tail dependence separately from correlations and fat tails.
- Free us to combine “normal” univariate distributions with “abnormal” multivariate distributions.
  - Or vice versa!
- Construct multivariate distributions by factoring out the co-dependence: easier!

From correlation to copulas

- Correlation is a measure of how 2 or more random variables move together, without dealing with how much each move.

$$\sigma_{XY} = \rho_{XY}\sigma_X\sigma_Y$$

  where ρ_{XY} is the correlation and σ_{XY} the covariance, and σ_X and σ_Y the standard deviations.
- Note how correlation factors out the co-dependence, not the “marginal” univariate distributions.

From correlation to copulas

- But correlations only capture the second moment of distributions.
- More generally, if $f(Y_1, \ldots, Y_d)$ is a multivariate density, then we can similarly factor our co-dependence by writing

$$f(Y_1, \ldots, Y_d)=c_Y(F_{Y_1}(Y_1), \ldots, F_{Y_1}(Y_d))f(Y_1)\ldots f(Y_d)$$

where c_Y is called the copula density, and F's are the marginal CDF of Y's.

From correlation to copulas

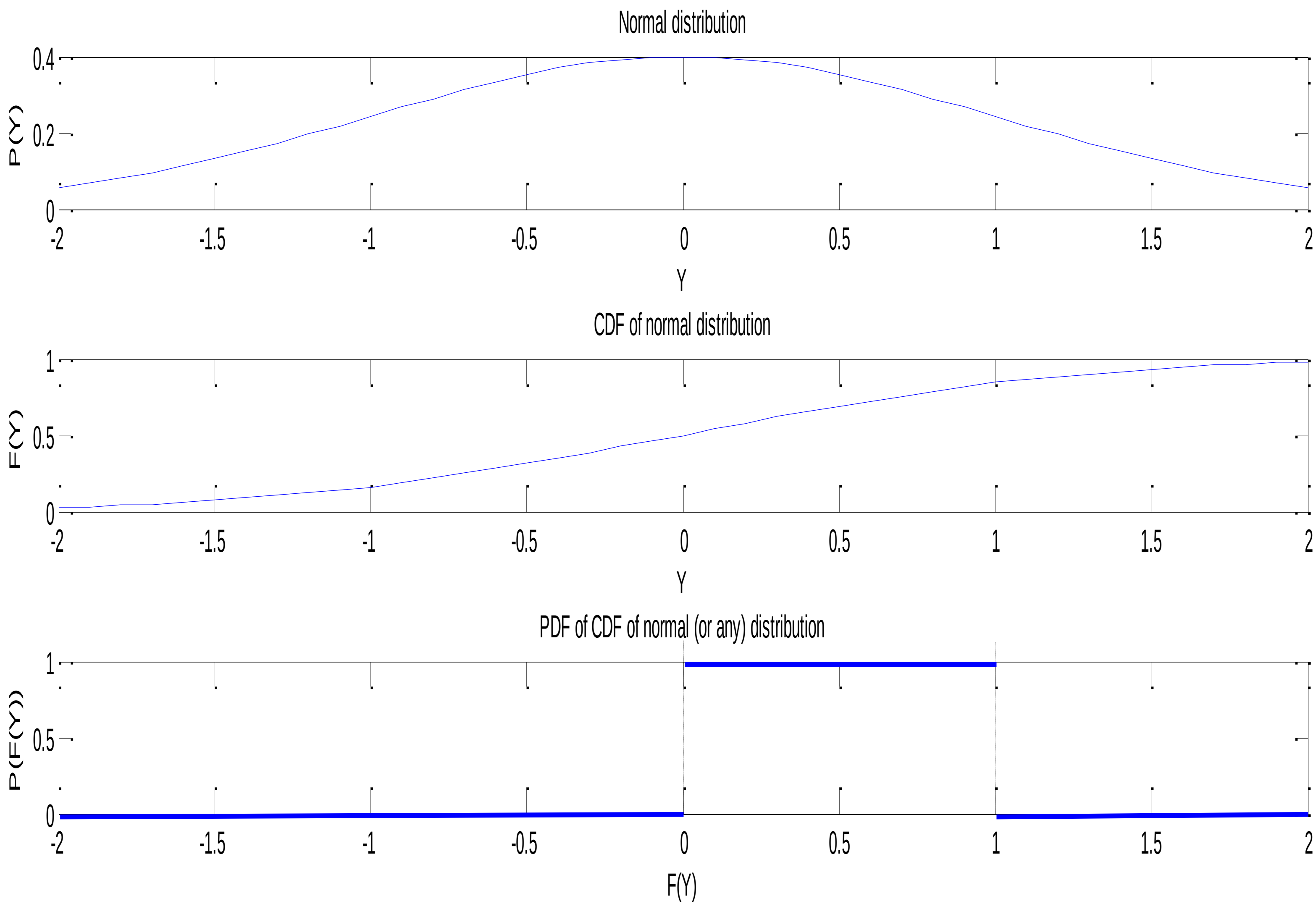
- This correspondence between correlation and copulas can be made precise because there is a 1-1 correspondence between them.
- I find this way of motivating copula is more intuitive than the usual definition, which states that the copula C of Y_1, …, Y_d is just the CDF of $F_{Y_1}(Y_1)$, …, $F_{Y_1}(Y_d)$, where "of course"* the marginal univariate distribution of $F_{Y_i}(Y_i)$ is Uniform(0, 1).

* See equation A.9 in Section A.9.2

Copulas as CDF of CDF of Y

- A copulas (“C”) is the multivariate CDF of the univariate marginal CDF (“F”$_i$) of Y_i.
- The univariate marginal distribution of F_i is Uniform(0,1).

Distribution of any CDF is Uniform



Highlights of copulas

- Independence copula $\leftrightarrow$ zero correlations.
- Co-monotonicity copula $\leftrightarrow$ perfect (1) correlations.
- Counter-monotonicity copula $\leftrightarrow$ perfect (-1) anti-correlations (only for 2 variables).
- Gaussian copulas $\leftrightarrow$ meta-Gaussian distribution
  $\neq$ Gaussian distribution
- t copulas $\leftrightarrow$ meta-t distribution
  $\neq$ t distribution.

Highlights of copulas

- Archimedean copulas:
  - Created by “generator” function.
  - Unchanged under permutation of variables (“exchangeable”)
  - Useful for modeling variables where all pairs have similar dependence.

Correlations

- Pearson's correlation = mean of differences between Y_1 ,Y_2 (assuming zero means and unit variances).
- Kendall's tau = mean of *sign* of differences between Y_1 ,Y_2 .
- Spearman's correlation = Pearson's correlation of *ranks* of Y_1 ,Y_2 = Pearson's correlation of $F(Y_1)$ and $F(Y_2)$

Correlations

- Pearson correlation still depends on univariate distributions of Y_1 ,Y_2 even though independent of variances.
- Kendall's tau and Spearman correlation depend only on the copula of Y_1 ,Y_2 .
- Kendall's tau of $F(Y_1)$ and $F(Y_2)$ gives the tau of the copula of Y_1 , Y_2 .

Highlights of copulas

- Kendall's tau is related to Pearson correlation for both meta-Gaussian and meta-t distributions through $\rho_\tau(Y_i, Y_j) = \frac{2}{\pi}\arcsin(\Omega_{i,j})$ where ρ_τ is Kendall's tau and $\Omega_{i,j}$ is the Pearson correlation.
- Tail-dependence as captured (or not!) by copulas.
  – Why is Gaussian copula "the formula that kills Wall Street"?

Formula that kills Wall Street

- Gaussian copula allows for fat tails of individual distributions (of mortgage-backed securities)
- Gaussian copula does *not* allow for tail dependence of individual distributions.
- Mortgages all went south at the same time during 2008 crisis!