

## Problem 1

Assume that a stock's log returns at any time scale have normal distribution.

- Suppose that its average annual log return is 100%, and its annual standard deviation ("volatility") of log returns is 200%.

What are its average ( $\mu$ ) and standard deviation ( $\sigma$ ) of daily log returns, assuming a year has 250 trading days?

```
trading.days <- 250

annual.ret <- 1
annual.vol <- 2

mu <- annual.ret / trading.days
sigma <- annual.vol / sqrt(trading.days)

prob1 <- list(mu = round(mu, 5), sigma = round(sigma, 5))
```

Average: **0.004**

Standard Deviation: **0.12649**

## Problem 2

Simulate 250 instances of the daily log returns described in 1.) with random seed `set.seed(2015)`.

- Compute the net returns of these instances, and compute their average and standard deviation.

```
set.seed(2015)

logRet <- rnorm(250, mu, sigma)
netRet <- exp(logRet) - 1

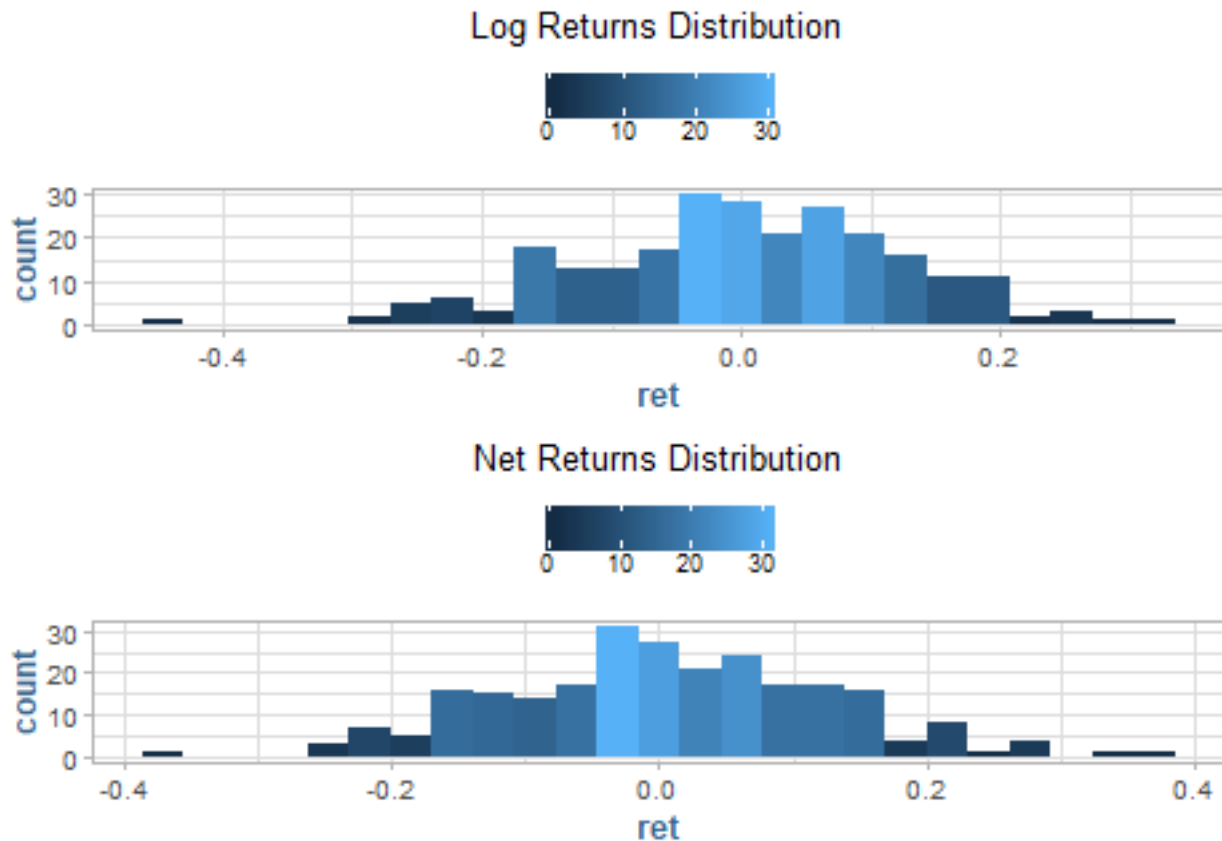
m <- round(mean(netRet), 5)
s <- round(sd(netRet), 5)
```

Average Net Return: **0.00695**

Standard Deviation of Net Returns: **0.12108**

Are the average ( $m$ ) and standard deviation ( $s$ ) of net returns same as the average and standard deviation of log returns computed in 1.)?

**Answer:** While not identical, they are close enough for practical purposes.



### Problem 3

- Compute the quantity  $m - \frac{s^2}{2}$ .

```
prob3.a <- round(m - s^2/2, 5)
```

- $m - \frac{s^2}{2} = -0.00038$

How does this compare with the average log return  $\mu$  computed in part 1.)?

**Answer:** This is a close approximation to the original value of  $\mu$ . We should note the extremely small sample size (250), and that a larger sample size (2.5m) produces a closer approximation:

```
set.seed(2015)

logRet <- rnorm(2500000, mu, sigma)
netRet <- exp(logRet) - 1

prob3.b <- round(mean(netRet) - sd(netRet)^2/2, 5)
```

**2.5m samples = 0.00399**

Their equality can be proven analytically through a mathematical theorem called Ito's Lemma that lies at the foundation of Black-Scholes options pricing formula. Their numerical equality is not that good here because Ito's Lemma assumes we can divide a period into infinitesimally small sub-periods.

So divide a year into 25,000 sub-periods (think of these sub-periods roughly as minutes) instead.

```
set.seed(2015)

trading.periods <- 25e3

min_from_days <- function(days) {
  60 * 6.5 * days
}

min.avg <- annual.ret / trading.periods
min.sd <- annual.vol / sqrt(trading.periods)

min.logRet <- rnorm( min_from_days(250), min.avg, min.sd)
min.netRet <- exp(min.logRet) - 1

new.m <- mean(min.netRet)
new.s <- sd(min.netRet)

new.mu <- mean(min.logRet)
new.sigma <- sd(min.logRet)

prob3.c <- round(new.m - new.s^2/2, 5)
```

Compare the new mu (average log return per minute) with the new  $m - \frac{s^2}{2}$  (m is now the average net return per minute).

- New mu = 0.00009
- New  $m - \frac{s^2}{2} = 0.00009$

Also, compare the new sigma (standard deviation of log returns per minute) with the new s (standard deviation of net return per minute).

- New sigma: **0.01267**
- New s: **0.01268**

**Problem 4**

- If we assume that the stock's initial price is \$1, what is the expected value of its log price  $\log(P(t))$  after  $t$  minutes expressed in terms of  $\mu$ ?

**Assuming:**

$\mu$  is the average daily log return,

$\sigma$  is the standard deviation of the daily log return

There are  $x$  minutes in a day, then 1 day =  $x$  minutes and 1 minute =  $\frac{1}{x}$  day

For some minute  $i$ ,

$$\mathbb{E}(r/\text{minute}) = \mathbb{E}(r_i) = \frac{\mu}{\sigma}$$

$$\mathbb{V}(r/\text{minute}) = \mathbb{V}(r_i) = \frac{\sigma^2}{x^2}$$

$$\sigma(r/\text{minute}) = \frac{\sigma}{x}$$

The expected return after  $t$  minutes is,

$$P_0 = 1$$

$$P_t = P_0(1 + R_1) \dots (1 + R_t)$$

$$\log(P_t) = (\log P_0) + \log(1 + R_1) + \log(1 + R_t)$$

$$P_t = \log(P_0) + r_1 + r_2 + r_t$$

$$\text{Since } \log(1) = 0$$

$$P_t = \sum_{i=1}^t r_i$$

$$\mathbb{E}(P_t) = \mathbb{E}(\sum_{i=1}^t r_i),$$

$$\mathbb{E}(r_i) = \frac{\mu}{x}$$

- And what is the expected value of its price  $P(t)$  expressed in terms of  $\mu$  and  $\sigma$ ?

$$\mathbb{V}(P_t) = \mathbb{V}(\sum_{i=1}^t r_i) = t \frac{\sigma^2}{x^2}$$

$$\mathbb{E}(P_t) = t \frac{\sigma}{x}$$

- Finally, express these expected values in terms of  $m$  and  $s$  instead.

$$mP_t = \frac{s}{x} \sqrt{t}$$

**Problem 5**

The continuously compounded rate of growth of a stock is  $\log(P(t))/t$ .

- What is the expected continuously compounded rate of growth of the stock in part 4?

$$\begin{aligned}\frac{P_t}{t} &= \sum_{i=1}^t \frac{r_i}{t} \\ \mathbb{E}\left[\frac{P_t}{t}\right] &= \mathbb{E}\left[\sum_{i=1}^t \frac{r_i}{t}\right] = t \frac{\mu}{t} \\ \mathbb{E}\left[\frac{P_t}{t}\right] &= \frac{\mu}{x}\end{aligned}$$