Diagonalizability

- Which matrices are diagonalizable?

Proposition. If an $n \times n$ matrix A has n linearly independent eigenvectors, then it is diagonalizable. The matrix S (such that $S^{-1}AS$ is a diagonal matrix Λ) is the matrix whose columns are the n eigenvectors.

Proposition. If the eigenvectors $x_1, \ldots, x_k$ correspond to *different* eigenvalues $\lambda_1, \ldots, \lambda_n$, then they are linearly independent.

Remark 1. If the matrix A has no repeated eigenvalues, then it can be diagonalized.

Remark 2. The diagonalizing matrix S is not unique.

Remark 3. $AS = S\Lambda$ holds only if the columns of S are eigenvectors of A.

Example of a non-diagonalizable matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 0$, $1D$ eigenspace.

Other examples (with non-zero e-values):

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Example of a diagonalizable matrix with a zero eigenvalue:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We have

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- diagonalizability - n independent eigenvectors
- invertibility - no zero eigenvalues

Remark. Diagonalizability *can* fail only if there are repeated eigenvalues. However, matrices with repeated eigenvalues may still be diagonalizable (ex: identity matrix, situations when we have a p dimensional eigenspace corresponding to an eigenvalue with multiplicity p) .

Imaginary eigenvalues.

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_1 = i$, $\lambda_2 = -i$; the eigenvectors are complex as well:

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Powers and Products.

Proposition.
(i) If λ is an eigenvalue for A, then λ^k is an eigenvalue for A^k.
(ii) Each eigenvector of A is an eigenvector for A^k.
(iii) If A is diagonalizable and S diagonalizes A, S also diagonalizes A^k.
Remark. If A is invertible, the proposition holds for $k = -1$ as well.

No such general result for products of two matrices! Only when A and B share the *same* eigenvector, we can say that AB has the eigenvalue $\lambda\mu$ if λ is an eigenvalue for A and μ is an eigenvalue is an eigenvalue for B.

Proposition. If A and B are diagonalizable, they share the same eigenvector matrix if and only if $AB = BA$.