Statistical Methods for Insurance: Multilevel Models

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W6.C1

Overview of this class

- Fixed effects vs random effects
- Mixed effects models
- Diagnostics

What is a multilevel model?

- Observations are not independent, but belong to a hierarchy
- Example: individual level demographics (age, gender), and school level information (location, cours offerings, classroom resources)
- Multilevel model enables fitting different types of dependencies

Fixed vs random

- `Fixed effects` can be used when you know all the categories, e.g. age, gender, smoking status
- `Random effects` are used when not all groups are captured, and we have a random selection of the groups, e.g. individuals (if you have multiple measurements), schools, hospitals

Mixed effects models - a type of multilevel model

For data organized in g groups, consider a continuous response linear mixed-effects model (LME model) for each group i, $i = 1, \ldots, g$:

$$\underset{(n_i \times 1)}{\mathbf{y}_i} = \underset{(n_i \times p)(p \times 1)}{\mathbf{X}_i \boldsymbol{\beta}} + \underset{(n_i \times q)(q \times 1)}{\mathbf{Z}_i \mathbf{b}_i} + \underset{(n_i \times 1)}{\varepsilon_i}$$

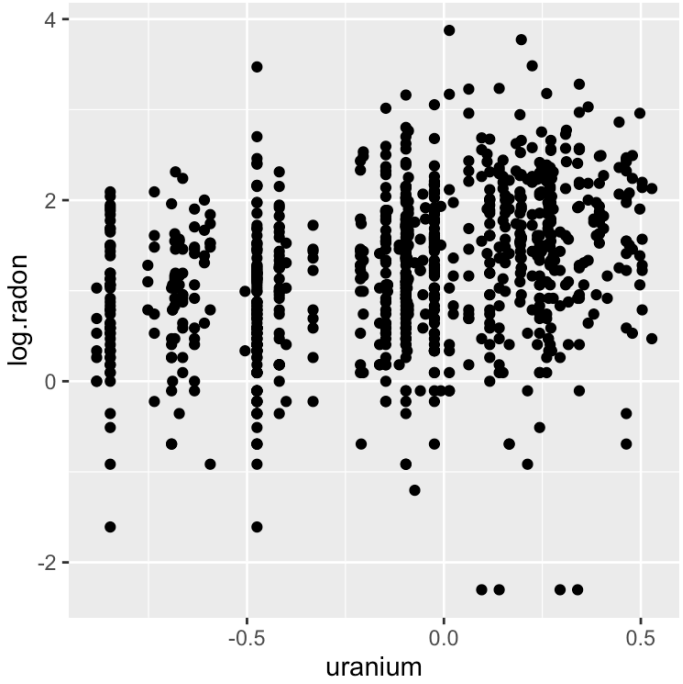
- $\mathbf{y}_i$ is the vector of outcomes for the n_i level-1 units in group i
- $\mathbf{X}_i$ and $\mathbf{Z}_i$ are design matrices for the fixed and random effects
- $\boldsymbol{\beta}$ is a vector of p fixed effects governing the global mean structure
- $\mathbf{b}_i$ is a vector of q random effects for between-group covariance
- ε_i is a vector of level-1 error terms for within-group covariance

Example

- Data: *radon*, 919 owner-occupied homes in 85 counties of Minnesota. Available in the `HLMdiag` package
- Response: $log.radon$
- Fixed: *storey* (categorical), *uranium* (quantitative)
- Random: *county* (house is a member of county)

```
## Observations: 919
## Variables: 5
## $ log.radon   <dbl> 0.7885, 0.7885, 1.0647, 0.0000, 1.1314, 0.9163, 0....
## $ storey      <int> 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,...
## $ uranium     <dbl> -0.689, -0.689, -0.689, -0.689, -0.847, -0.847, -0...
## $ county      <int> 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,...
## $ county.name <fctr> AITKIN, AITKIN, AITKIN, AITKIN, ANOKA, ANOKA, ANO...
```

Take a look



Plot of response vs covariate. What do you see?

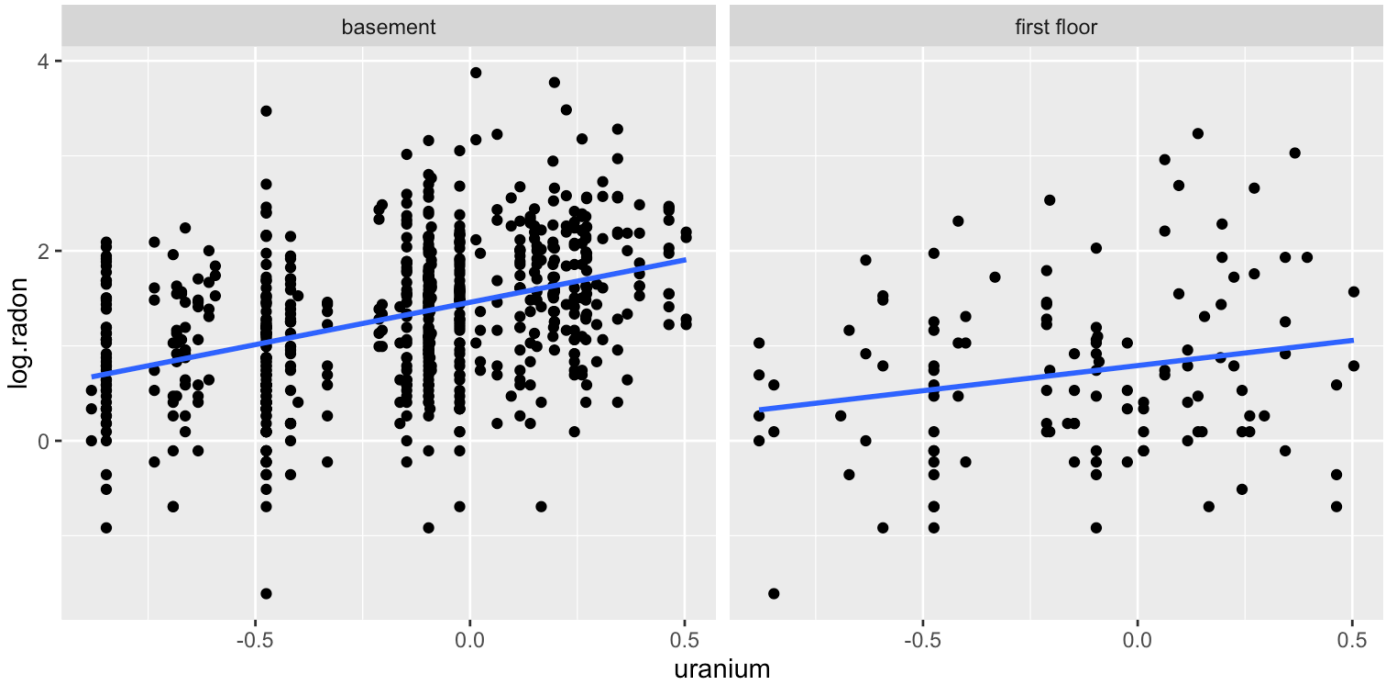
Here's what we see

- Vertical stripes: each county is represented by an average uranium value
- Weak linear association, lots of variation for houses within county
- Four points inline horizontally at the base (be suspicious)
- Some counties only have 2, 3 points
- Scales?

Pre-processing

- Counties with less than 4 observations removed
- Four flat-line observations should be removed, really suspect these were erroneously coded missing values

Look again



Fit a simple model

$$log.radon = \beta_0 + \beta_1 storey + \beta_2 uranium + \varepsilon$$

```
##
## Call:
## glm(formula = log.radon ~ storey + uranium, data = radon_sub)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.6610  -0.4928   0.0191   0.4745   2.4205
##
## Coefficients:
##                   Estimate Std. Error t value Pr(>|t|)
## (Intercept)         1.4483     0.0313   46.25  < 2e-16 ***
## storeyfirst floor  -0.6112     0.0733   -8.34  3.3e-16 ***
## uranium             0.8359     0.0742   11.26  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 0.547)
##
```

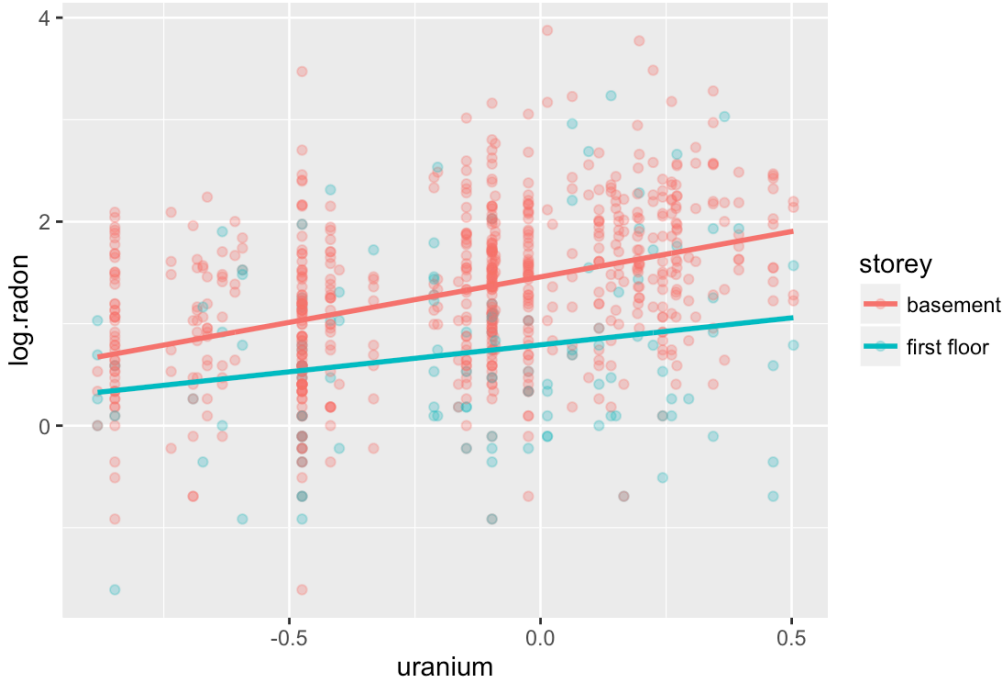
Your turn

Make a sketch of what this model looks like.

Fit an interaction term

```
##
## Call:
## glm(formula = log.radon ~ storey * uranium, data = radon_sub)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.6445  -0.4898   0.0131   0.4653   2.4369
##
## Coefficients:
##                          Estimate Std. Error t value Pr(>|t|)
## (Intercept)                1.4580     0.0318   45.91  < 2e-16 ***
## storeyfirst floor         -0.6659     0.0796   -8.37  2.7e-16 ***
## uranium                    0.8909     0.0805   11.07  < 2e-16 ***
## storeyfirst floor:uranium -0.3620     0.2066   -1.75     0.08 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for gaussian family taken to be 0.546)
##
##     Null deviance: 538.51  on 795  degrees of freedom
```

What does this model look like?



Your turn

Write down the equation of the fitted model

Mixed effects model

$$log.radon_{ij} = \beta_0 + \beta_1 storey_{ij} + \beta_2 uranium_i + b_{0i} + b_{1i} storey_{ij} + \varepsilon_{ij}$$

$$i = 1, \ldots, \#counties; j = 1, \ldots, n_i$$

```
library(lme4)
radon_lmer <- lmer(log.radon ~ storey + uranium +
  (storey | county.name), data = radon_sub)
summary(radon_lmer)
radon_lmer_fit <- augment(radon_lmer)
```

Your turn

For the radon data:

- What is p, q, g?
- And hence $n_i, i = 1, \ldots, g$?

$$log.radon_{ij} = \beta_0 + \beta_1 storey_{ij} + \beta_2 uranium_i + b_{0i} + b_{1i} storey_{ij} + \varepsilon_{ij}$$

$$i = 1, \ldots, \#counties; j = 1, \ldots, n_i$$

Examining the model output: fixed effects

```
Fixed effects:
                  Estimate Std. Error t value
(Intercept)        1.48066    0.03856   38.40
storeyfirst floor -0.59011    0.11246   -5.25
uranium            0.84600    0.09532    8.88
```

How do these compare with the simple linear model estimates?

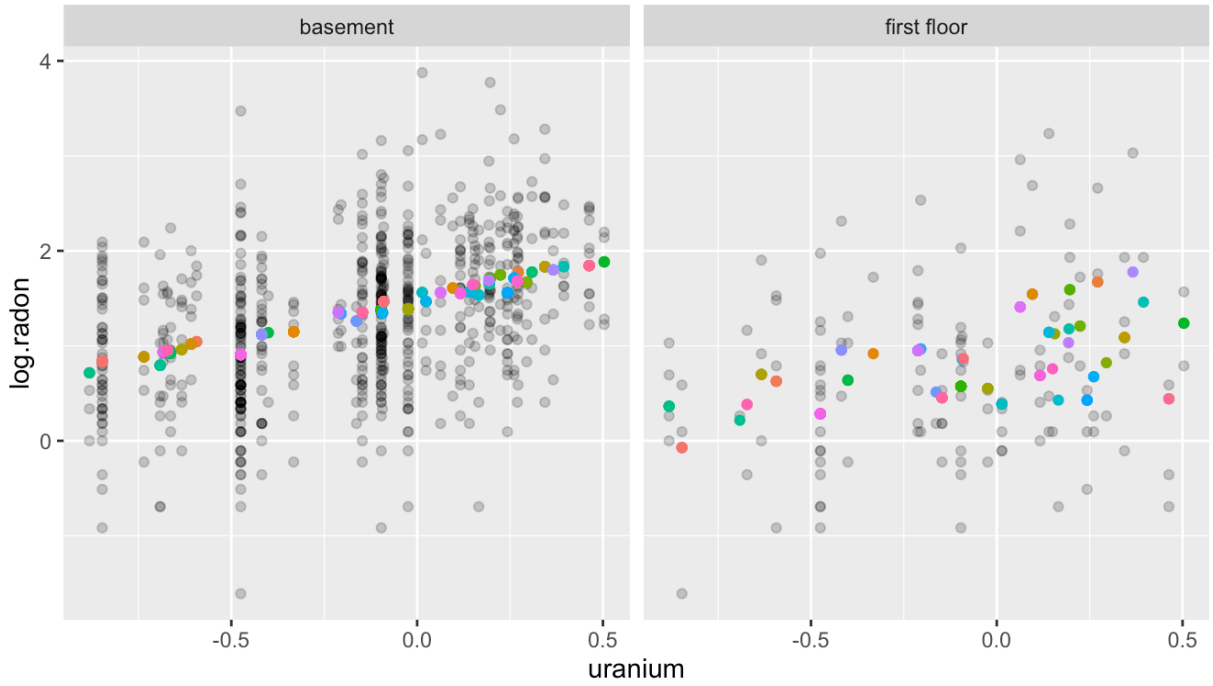
```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      1.44830    0.03131  46.254  < 2e-16 ***
storeyfirst floor -0.61125    0.07332  -8.337 3.35e-16 ***
uranium           0.83591    0.07422  11.262  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Examining the model output: random effects

```
Random effects:
 Groups      Name                Variance Std.Dev. Corr
 county.name (Intercept)         0.01388  0.1178
             storeyfirst floor   0.22941  0.4790   0.02
 Residual                        0.50694  0.7120
Number of obs: 796, groups:  county.name, 46
```

This is saying that the variance of the estimates for first floor observations is larger than the storey.

What it looks like



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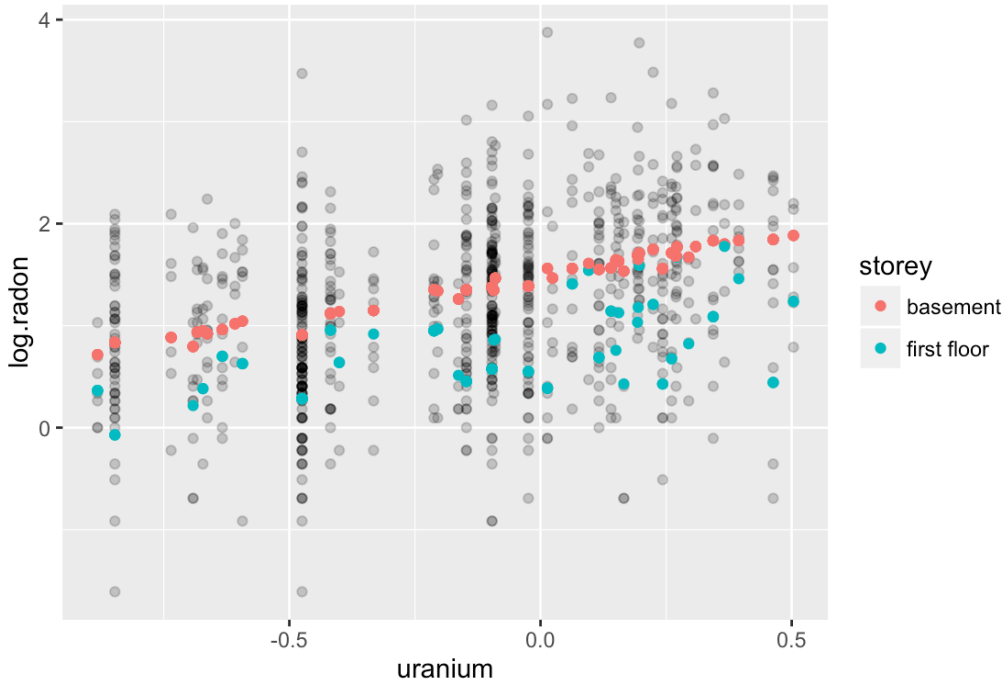
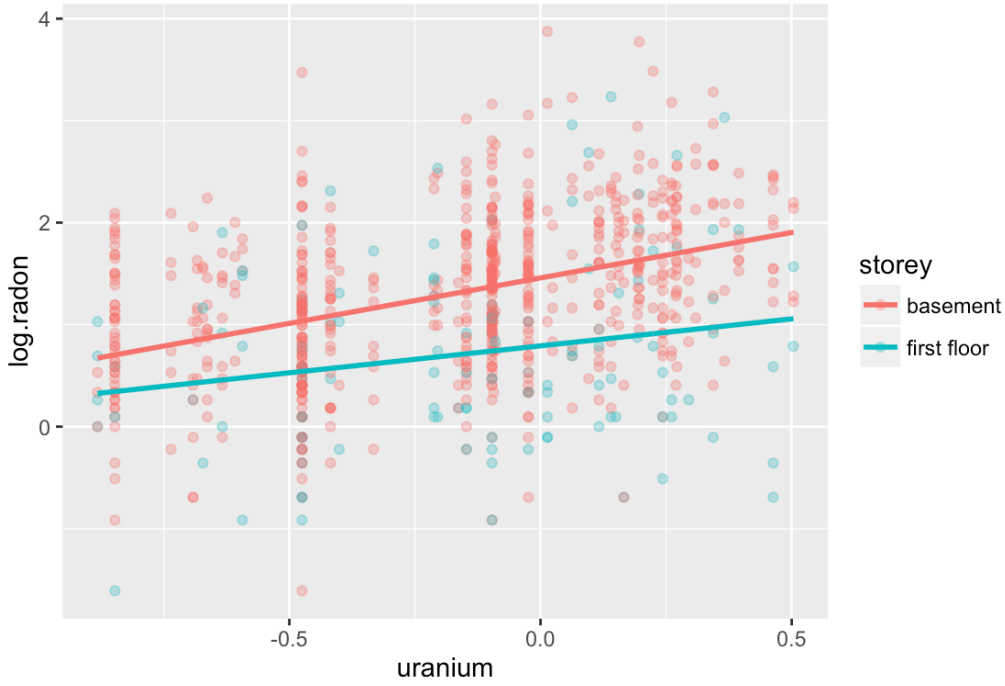
Or like this



Your turn

How does the mixed effects model differ from the simple linear model? (Hint: Think about the variance.)



Assumptions

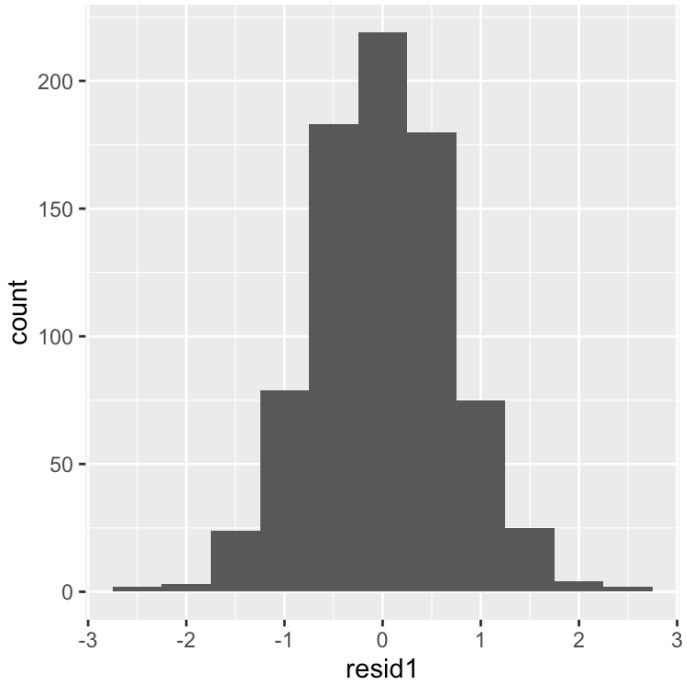
Recall:

$$\underset{(n_i \times 1)}{\mathbf{y}_i} = \underset{(n_i \times p)(p \times 1)}{\mathbf{X}_i \boldsymbol{\beta}} + \underset{(n_i \times q)(q \times 1)}{\mathbf{Z}_i \mathbf{b}_i} + \underset{(n_i \times 1)}{\boldsymbol{\varepsilon}_i}$$

- $\mathbf{b}_i$ is a random sample from $\mathcal{N}(\mathbf{0}, \mathbf{D})$ and independent from the level-1 error terms,
- $\boldsymbol{\varepsilon}_i$ follow a $\mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{R}_i)$ distribution
- $\mathbf{D}$ is a positive-definite $q \times q$ covariance matrix and $\mathbf{R}_i$ is a positive-definite $n_i \times n_i$ covariance matrix

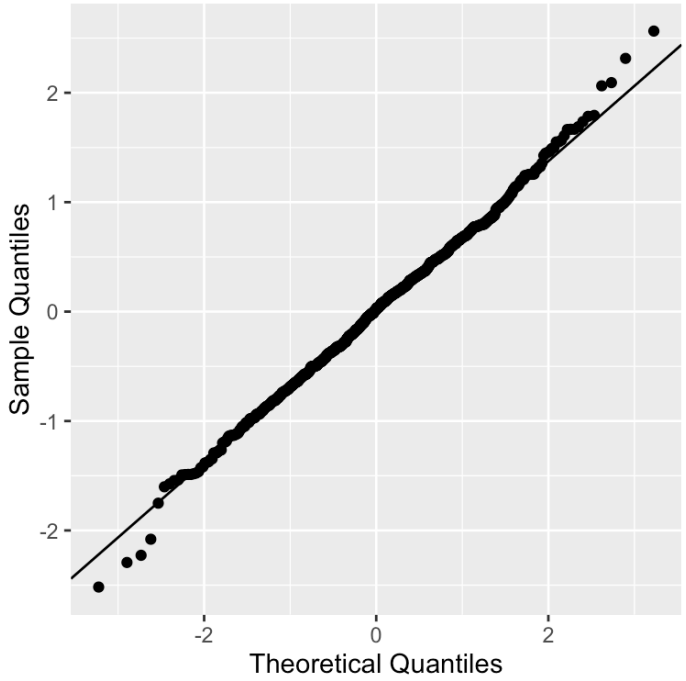
Extract and examine level-1 residuals

$$\boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{R}_i)$$



Level-1 (observation level) look normal.

qqplot



Level-1 (observation level) do look nearly normal.

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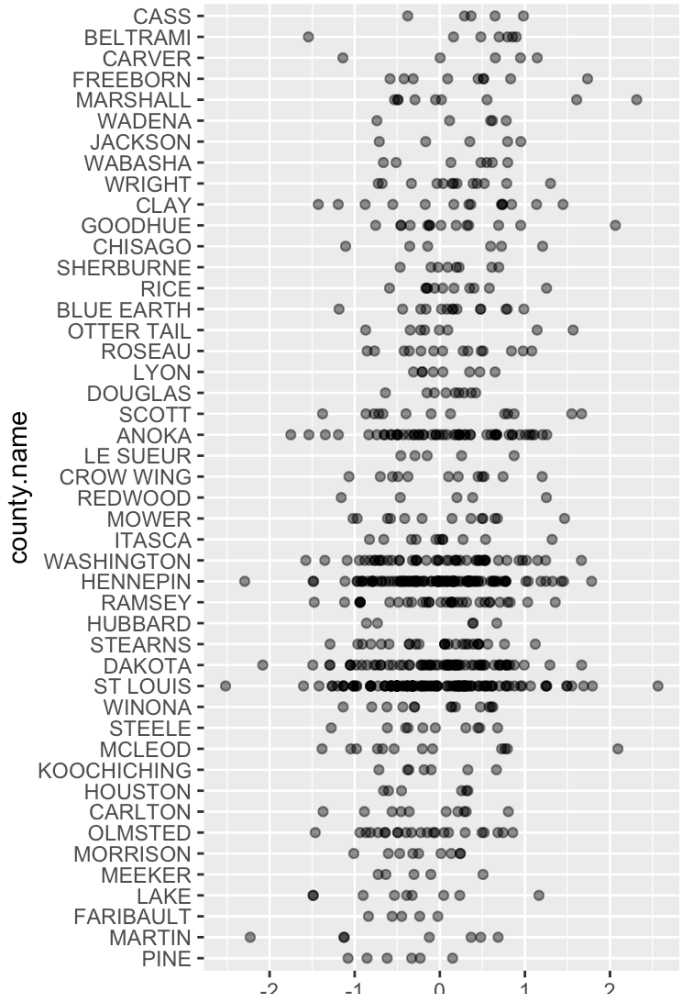
Examine within group

Summary statistics

```
## # A tibble: 6 x 4
##   county.name      m     s     n
##         <fctr>  <dbl> <dbl> <int>
## 1       ANOKA  0.051 0.719    52
## 2    BELTRAMI  0.335 0.867     7
## 3  BLUE EARTH  0.152 0.562    14
## 4     CARLTON -0.194 0.651    10
## 5      CARVER  0.322 0.924     5
## 6        CASS  0.383 0.504     5
```


county.name
CASS
BELTRAMI
CARVER
FREEBORN
MARSHALL
WADENA
JACKSON
WABASHA
WRIGHT
CLAY
GOODHUE
CHISAGO
SHERBURNE
RICE
BLUE EARTH
OTTER TAIL
ROSEAU
LYON
DOUGLAS
SCOTT
ANOKA
LE SUEUR
CROW WING
REDWOOD
MOWER
ITASCA
WASHINGTON
HENNEPIN
RAMSEY
HUBBARD
STEARNS
DAKOTA
ST LOUIS
WINONA
STEELE
MCLEOD
KOOCHICHING
HOUSTON
CARLTON
OLMSTED
MORRISON
MEEKER
LAKE
FARIBAULT
MARTIN
PINE
-2
-1
0
1
2

resid1

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Learn

There is some difference on average between counties, which means that residuals still have some structure related to the county location.

Normality tests

Anderson-Darling, Cramer-von Mises, Lilliefors (Kolmogorov-Smirnov)

```
##
##  Anderson-Darling normality test
##
## data:  radon_lmer_fit$resid1
## A = 0.4, p-value = 0.4
```

all believe that the residuals are consistent with normality.

Conclusion about level-1 residuals

The assumption:

$$\boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{R}_i)$$

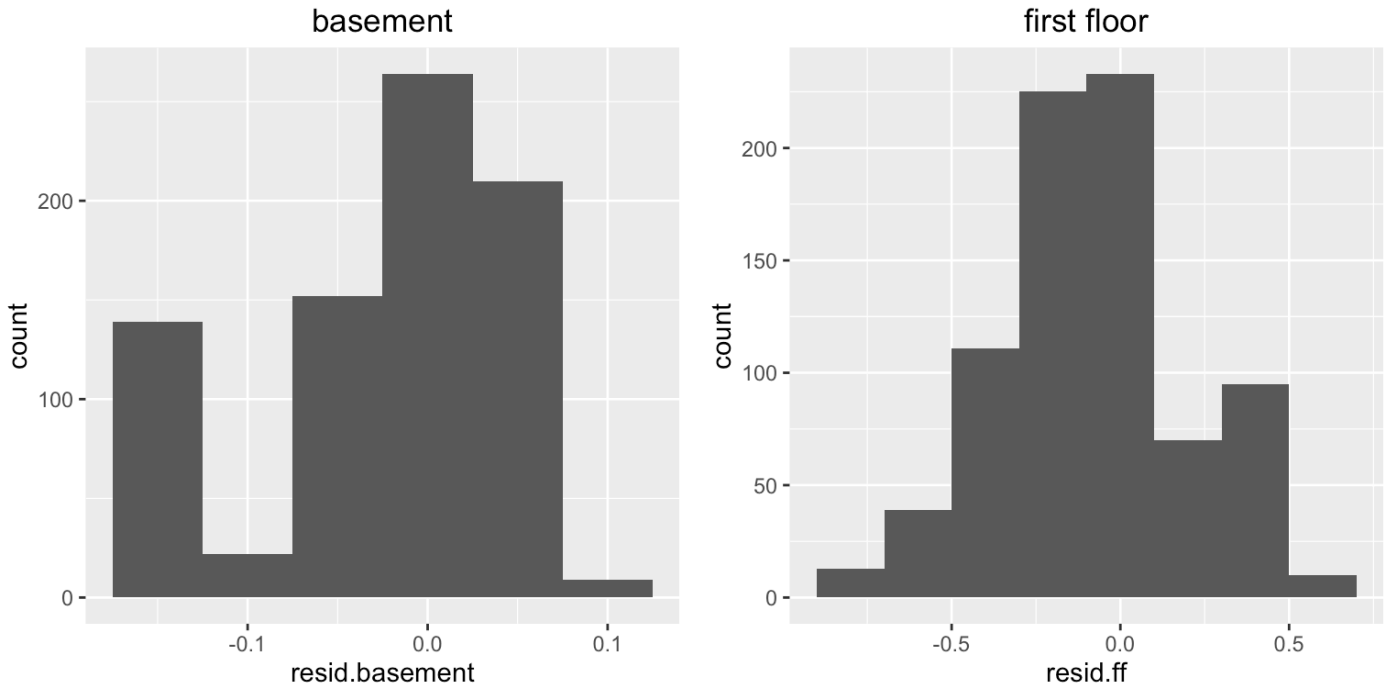
is probably ok, at the worst it is not badly violated.

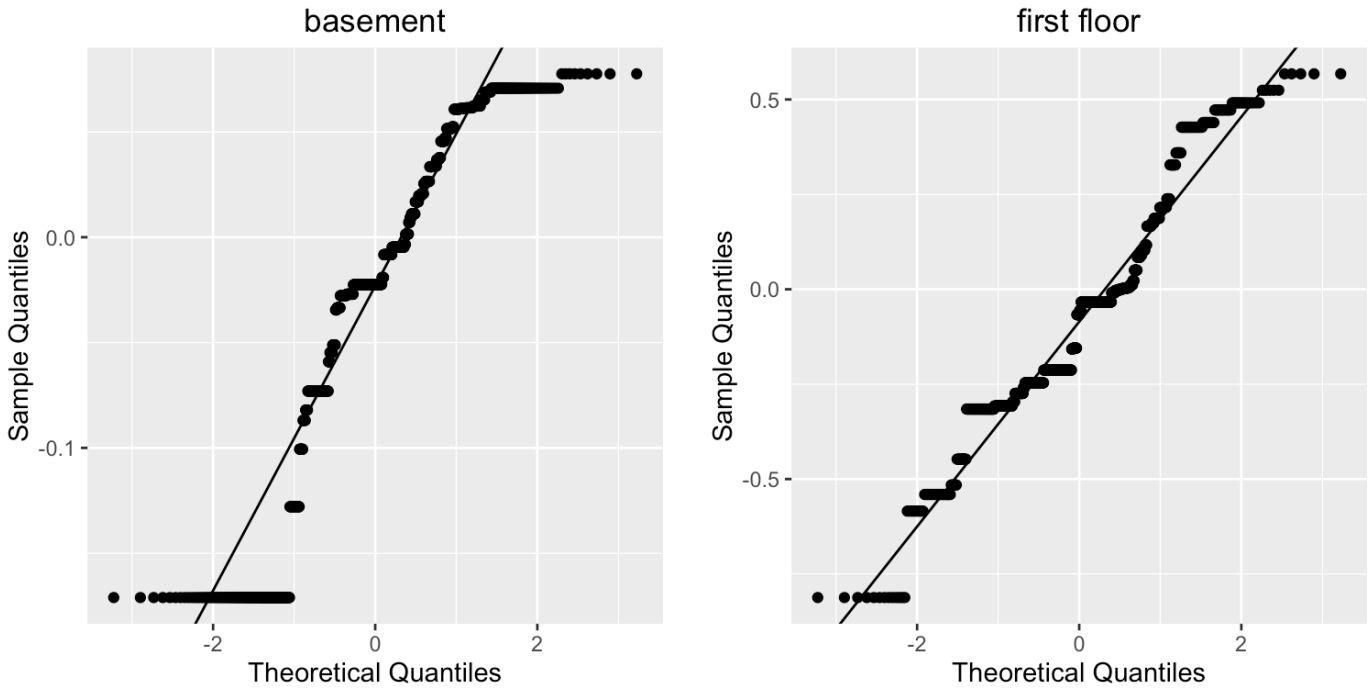
Random effects

$$\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{D}), \quad i = 1, \dots g$$

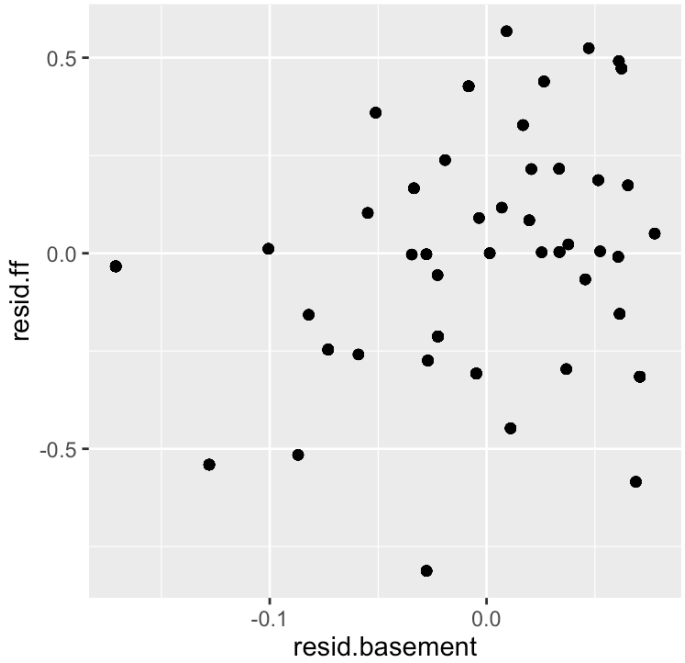
where $\mathbf{D}$ allows for correlation between random effects within group, and these should be independent from the level-1 error

We have both intercepts (basement) and slopes (first floor)


basement
count
200
100
0
-0.1
0.0
0.1
resid.basement
first floor
count
200
150
100
50
0
-0.5
0.0
0.5
resid.ff

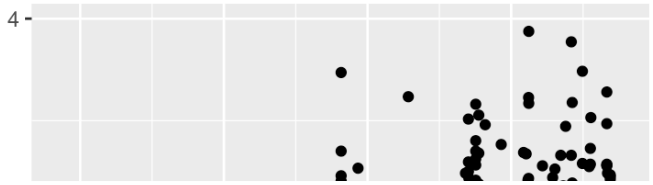

basement
first floor
Sample Quantiles
Theoretical Quantiles
0.0
-0.1
0.5
-0.5
-2
0
2

Should be no correlation



Fitted vs Observed

Plotting the observed vs fitted values, gives a sense for how much of the response is explained by the model. Here we can see that there is still a lot of unexplained variation.



Goodness of fit

From the linear model

```
##   null.deviance df.null logLik  AIC  BIC deviance df.residual
## 1           539     795   -887 1783 1806      432         792
```

From the random effects model

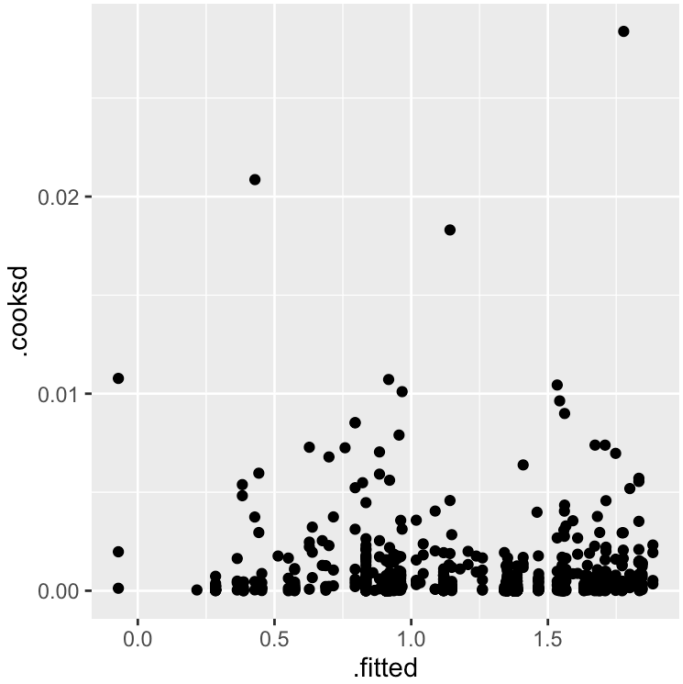
```
##   sigma logLik  AIC  BIC deviance df.residual
## 1 0.712   -885 1784 1817     1760         789
```

Hmmm… deviance looks strange! Compute sum of squares of residuals instead:

```
## [1] 387
```

Which model is best?

Influence



No overly influential observations

Resources

- HLMDiag package explanation
- HLM package

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