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Problem 1

Consider the following integral

$$\theta = \int_0^{0.5} e^{-x} dx$$

(a) Write an algorithm to generate a Monte Carlo estimate of θ generating from the $\mathcal{U}(0, 0.5)$ distribution.

(b) Write down the variance of the MC estimator in (a) and calculate it numerically for a sample of 100.

(c) Write down another MC estimator θ^* by sampling from the exponential distribution with density $f(x) = \lambda e^{-\lambda x}$ for $\lambda > 0$ and $x > 0$.

(d) Write down the variance for (c) and evaluate it for 100 samples.

(e) Which estimator has smaller variance? Show all calculations. Does this make intuitive sense? Explain.

`In R:`

(f) Generate 100 samples of both algortihms in `R`, compute the MC variances and compare to the exact answers.

(g) Repeat (f) 100 times, produce violin plots of the variances and standard errors of variances for both algorithms using `geom_violin` of `ggplot2`. Compare the median to the exact answers.

Problem 2

(a) Let $\hat{\theta}_f^{IS}$ be an importance sampling estimator of $\theta = \int g(x)dx$ with $f(x)$ being the importance function. Prove that if $\frac{g(x)}{f(x)}$ is bounded, then the variance of the importance sampling estimator $\hat{\theta}_f^{IS}$ is finite.

(b) Consider the following integral

$$\int_1^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Write down the algorithm to calculate an estimate $\hat{\theta}$ of this integral by using the importance sampling method with the re-scaled Exp(1) density $f(x) = e^{-x}/e^{-1}, x > 1$ as importance function.

`In R:`

(c) Implement the algorithm from (b) in `R` and calculate an estimate $\hat{\theta}$ of the integral by using 1000 samples. Compute the standard deviation of $\frac{g(x)}{f(x)}$

Problem 3

(a) Revisit Problem 1 (a). Suppose we want to use $\mathcal{U}(0, 0.5)$ as a generating density, but also make use of antithetic variables. If $X \sim \mathcal{U}(0, 0.5)$ find a random variable Y based on the same distribution as X, but respectively correlated to X. Derive the density of Y and the covariance with X.

(b) Write down an algorithm to calculate the integral θ of Problem 1 (a) using only $\frac{m}{2}$ independent realizations of $\mathcal{U}(0, 0.5)$, with the other $\frac{m}{2}$ based on the transformation in (a).

(c) Calculate the variance of $\hat{\theta}$ in (b) and compare to Problem 1 (b) for a sample of $\frac{m}{2} = \frac{100}{2}$ iterates.

`In R:`

(d) Implement the antithetic algorithm in (b) for $\frac{m}{2} = 50$ iterates and compare the results with the theoretical values.

Problem 4

Consider following integral

$$\theta = \mathrm{E}\left[e^U\right] = \int_0^1 e^u du,$$

where $U \sim \mathcal{U}(0,1)$.

(a) Derive the variance of a simple Monte Carlo estimator $\hat{\theta}_{MC}$ with m replicates.

(b) Apply the control variate approach. Choose a control variate, derive the variance and the covariance of e^U and $f(U)$, write down the controlled estimator and calculate the variance reduction factor.

(c) Write down the algorithm for the control variate approach to calculate the integral θ using m independent realizations of $\mathcal{U}(0,1)$.

`In R:`

(d) Implement the algorithm of (c) in `R` for $m = 1000$ as well as the algorithm for the simple Monte Carlo estimator. Calculate the estimates and the variance reduction factor and compare to the theoretical results.