GLM, FGLS, econometric application: RE estimator

Consider the panel LRM

$$y_{it} = \boldsymbol{x}'_{it}\boldsymbol{\beta} + \mu_i + \nu_{it}.$$

y_{it}: an observation for individual i at time t.

$\boldsymbol{x}'_{it}$: K independent variables for individual i at time t. $1 \times K$. A variable can be time-varying or time-invariant.

$\boldsymbol{\beta}$: vector of true coefficients. $K \times 1$.

GLM, FGLS, econometric application: RE estimator

$$y_{it} = \boldsymbol{x}'_{it}\boldsymbol{\beta} + \mu_i + \nu_{it}.$$

μ_i: time-invariant error, specific to i. $\mu_i \sim N(0, \sigma^2_\mu)$, and hence it is a random effect. Hence, the model is called the random effects model. μ_i captures individual heterogeneity. μ_i is assumed to be uncorrelated with $\boldsymbol{x}_{it}$ which is a strong assumption. μ_i are uncorrelated among individuals.

ν_{it}: time-variant error. $\nu_{it} \sim N(0, \sigma^2_\nu)$. ν_{it} is uncorrelated with X_{it}. ν_{it} are uncorrelated among individuals, but also within individuals.

ε_{it}: define it as $\mu_i + \nu_{it}$. It is the composite error of the model. μ_i and ν_{it} are independent.

GLM, FGLS, econometric application: RE estimator

Derive the variance-covariance matrix of ε_{it}. That is, derive $\mathrm{E}\left[\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}' \mid \boldsymbol{X}\right]$.

GLM, FGLS, econometric application: RE estimator

The variance for i, at a given t:

$$\mathrm{E}\left[\varepsilon_{it}\varepsilon_{it} \mid \boldsymbol{x}'_{it}\right] = \sigma_\mu^2 + \sigma_\nu^2.$$

The covariance for i, across t: ε_{it} are correlated within an individual, due to the time-invariant μ_i. Serial correlation! That is, for every i, and $t \neq s$,

$$\mathrm{E}\left[\varepsilon_{it}\varepsilon_{is} \mid \boldsymbol{x}'_{it}\right] = \sigma_\mu^2.$$

The covariance for i and j, across t: ε_{it} are not correlated across individuals. That is, for every $i \neq j$,

$$\mathrm{E}\left[\varepsilon_{it}\varepsilon_{js} \mid \boldsymbol{x}'_{it}\right] = 0.$$

GLM, FGLS, econometric application: RE estimator

How does $\mathrm{E}\left[\varepsilon\varepsilon' \mid \boldsymbol{X}\right]$ look like?

GLM, FGLS, econometric application: RE estimator

For T errors of individual i, stored in $\boldsymbol{\varepsilon}_i$,

$$\mathrm{E}\left[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i' \mid \boldsymbol{X}_i'\right] = \begin{bmatrix} \sigma_\mu^2 + \sigma_\nu^2 & \sigma_\mu^2 & \dots & \sigma_\mu^2 & \dots & \sigma_\mu^2 \\ \sigma_\mu^2 & \sigma_\nu^2 + \sigma_\nu^2 & \dots & \sigma_\mu^2 & \dots & \sigma_\mu^2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sigma_\mu^2 & \sigma_\mu^2 & \dots & \sigma_\mu^2 + \sigma_\nu^2 & \dots & \sigma_\mu^2 \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ \sigma_\mu^2 & \sigma_\mu^2 & \dots & \sigma_\mu^2 & \dots & \sigma_\mu^2 + \sigma_\nu^2 \end{bmatrix}_{T \times T}$$

$$\equiv \boldsymbol{\omega}_i$$

ε_{it} are serially correlated within an individual!

GLM, FGLS, econometric application: RE estimator

For T errors of N individuals, stored in ε,

$$\mathrm{E}\left[\varepsilon\varepsilon' \mid \boldsymbol{X}\right] = \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\omega}_1 & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_2 & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\omega}_n & \dots & \mathbf{0} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \boldsymbol{\omega}_N \end{bmatrix}_{NT \times NT}$$

GLM, FGLS, econometric application: RE estimator

Example $\mathbf{\Omega}$ for $N = 2$, $T = 2$,

$$\mathrm{E}\left[\varepsilon\varepsilon' \mid \boldsymbol{X}\right] = \mathbf{\Omega} = \begin{bmatrix} \sigma_\mu^2 + \sigma_\nu^2 & \sigma_\mu^2 & 0 & 0 \\ \sigma_\mu^2 & \sigma_\mu^2 + \sigma_\nu^2 & 0 & 0 \\ 0 & 0 & \sigma_\mu^2 + \sigma_\nu^2 & \sigma_\mu^2 \\ 0 & 0 & \sigma_\mu^2 & \sigma_\mu^2 + \sigma_\nu^2 \end{bmatrix}_{4\times 4}$$

GLM, FGLS, econometric application: RE estimator

ε_{it} are serially correlated within individuals. We cannot estimate $\boldsymbol{\beta}$ efficiently using OLS! What can we do?

GLM, FGLS, econometric application: RE estimator

We have constructed

$$\mathbf{\Omega}.$$

It includes the unknown σ_μ^2 and σ_ν^2. There are different ways to estimate them. We do not cover this. Use these estimates to obtain

$$\hat{\mathbf{\Omega}}.$$

Now GLS estimation is feasible. Then obtain

$$\hat{\mathbf{\Omega}}^{-1} = \hat{\boldsymbol{P}}'\hat{\boldsymbol{P}}.$$

GLM, FGLS, econometric application: RE estimator

$\hat{\boldsymbol{P}}\varepsilon$ is serially uncorrelated within individuals!

GLM, FGLS, econometric application: RE estimator

Obtain $\hat{\boldsymbol{P}}\boldsymbol{y}$ and $\hat{\boldsymbol{P}}\boldsymbol{X}$. Applying the OLS on them,

$$\begin{aligned}\boldsymbol{b}_{FGLS} &= (\underbrace{(\hat{\boldsymbol{P}}\boldsymbol{X})'}_{\boldsymbol{X}^{*\prime}}\underbrace{(\hat{\boldsymbol{P}}\boldsymbol{X})}_{\boldsymbol{X}^{*}})^{-1}\underbrace{(\hat{\boldsymbol{P}}\boldsymbol{X})'}_{\boldsymbol{X}^{*\prime}}\underbrace{\hat{\boldsymbol{P}}\boldsymbol{y}}_{\boldsymbol{y}^{*\prime}} \\ &= \boldsymbol{b}_{RE}\end{aligned}$$

The feasible generalised least squares estimator is the random effects estimator!

GLM, RE estimator, example

Example in MATLAB, where we check how the $\hat{\boldsymbol{P}}$ matrix looks like.

GLM, RE estimator, example

Editor – /Users/Tunga/Library/Mobile Documents/com~apple~CloudDocs/Academic/Te

exercisepartwo.m | exerciseparttwofunctionbgs.m | exerciseparttwofunctionfes.m

```
% The RE estimator as the FGLS estimator

% 1. Load the data
clear;
filename = 'C:\Users\username\Desktop\exercise.csv';
delimiterIn = ',';
headerlinesIn = 1;
exercisedata = importdata(filename,delimiterIn,headerlinesIn);

% 2. Create the systematic component of the regression
y = exercisedata.data(:,1);
N = length(y);
X = [ones(N,1) exercisedata.data(:,2:end)];

% 3. Create additional variables
T = 8;
M = N/T;
P_D = kron(eye(M),ones(T,T).*1/T);

% 4. Obtain the RE coefficient estimates as FGLS coefficient estimates
fes = exerciseparttwofunctionfes(y,X,M,P_D,N);
bgs = exerciseparttwofunctionbgs(y,X,M,P_D,T);
s2_m = fes.s2_hat;
s2_v = bgs.s2_hat-s2_m/T;
L = 1-(T*s2_v/s2_m+1)^(-1/2);
P_hat = eye(N)-L*P_D;
y_t = P_hat*y;
X_t = P_hat*X;
B_hat = (X_t'*X_t)\(X_t'*y_t);
```

GLM, RE estimator, example

Editor – exercisepartwo.m | Variables – X_t

X_t

4360x10 double

	1	2	3	4	5	6	7	8	9
1	0.3634	0	-0.0796	0	0	5.0875	-1.8647	-15.2334	0
2	0.3634	0	0.9204	0	0	5.0875	-0.8647	-12.2334	0
3	0.3634	0	-0.0796	0	0	5.0875	0.1353	-7.2334	0
4	0.3634	0	-0.0796	0	0	5.0875	1.1353	-0.2334	0
5	0.3634	0	-0.0796	0	0	5.0875	2.1353	8.7666	0
6	0.3634	0	-0.0796	0	0	5.0875	3.1353	19.7666	0
7	0.3634	0	-0.0796	0	0	5.0875	4.1353	32.7666	0
8	0.3634	0	-0.0796	0	0	5.0875	5.1353	47.7666	0
9	0.3634	0	0	0	0	4.7241	-0.7745	-23.1512	0
10	0.3634	0	0	0	0	4.7241	0.2255	-14.1512	0
11	0.3634	0	0	0	0	4.7241	1.2255	-3.1512	0
12	0.3634	0	0	0	0	4.7241	2.2255	9.8488	0
13	0.3634	0	0	0	0	4.7241	3.2255	24.8488	0
14	0.3634	0	0	0	0	4.7241	4.2255	41.8488	0
15	0.3634	0	0	0	0	4.7241	5.2255	60.8488	0
16	0.3634	0	0	0	0	4.7241	6.2255	81.8488	0
17	0.3634	0	0	0.3634	0	4.3607	-0.7745	-23.1512	0
18	0.3634	0	0	0.3634	0	4.3607	0.2255	-14.1512	0
19	0.3634	0	0	0.3634	0	4.3607	1.2255	-3.1512	0
20	0.3634	0	0	0.3634	0	4.3607	2.2255	9.8488	0
21	0.3634	0	0	0.3634	0	4.3607	3.2255	24.8488	0
22	0.3634	0	0	0.3634	0	4.3607	4.2255	41.8488	0
23	0.3634	0	0	0.3634	0	4.3607	5.2255	60.8488	0
24	0.3634	0	0	0.3634	0	4.3607	6.2255	81.8488	0

GLM, RE estimator, example

Editor – exercisepartwo.m | Variables – P_hat

P_hat

4360x4360 double

	1	2	3	4	5	6	7	8	9
1	0.9204	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0
2	-0.0796	0.9204	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0
3	-0.0796	-0.0796	0.9204	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0
4	-0.0796	-0.0796	-0.0796	0.9204	-0.0796	-0.0796	-0.0796	-0.0796	0
5	-0.0796	-0.0796	-0.0796	-0.0796	0.9204	-0.0796	-0.0796	-0.0796	0
6	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0.9204	-0.0796	-0.0796	0
7	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0.9204	-0.0796	0
8	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	-0.0796	0.9204	0
9	0	0	0	0	0	0	0	0	0.9204
10	0	0	0	0	0	0	0	0	-0.0796
11	0	0	0	0	0	0	0	0	-0.0796
12	0	0	0	0	0	0	0	0	-0.0796
13	0	0	0	0	0	0	0	0	-0.0796
14	0	0	0	0	0	0	0	0	-0.0796
15	0	0	0	0	0	0	0	0	-0.0796
16	0	0	0	0	0	0	0	0	-0.0796
17	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0