Time Series Analysis

Fractional Integration and Granger Causality

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Outline

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Fractional Integration

- A series of models where shocks decay much more slowly than ARIMA models would predict, but do not have infinite memory like random walk models. ARFIMA. These have linear decay, not exponential.
    - $(1 - B)^d y_t = \epsilon_t$
    - $d = 0$ gives us pure white noise.
    - $d = 1$ gives us fully integrated differencing
    - $0 < d < .5$, stationary, but non-geometric decay (stationary with loooong memory).
    - $.5 \geq |d| < 1$, finite mean, infinite variance (non stationary)
    - $-.5 < d < 0$, "anti-persistence" – strong negative memory (see example).

d-uh

What is d exactly?

- $(1-B)^d y_t = \epsilon_t; \forall 0 > d > .5$
- Take the binomial expansion: $(1-B)^d y_t = \sum_{j=0}^{\infty} \Pi_j B^j x_t = \sum_{j=0}^{\infty} \Pi_j x_{t-j}$ where, $Pi = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$ and Γ is the gamma function (generalized factorial).

The intuition: these are infinite weights Π_j that define very slow decay back to equilibrium

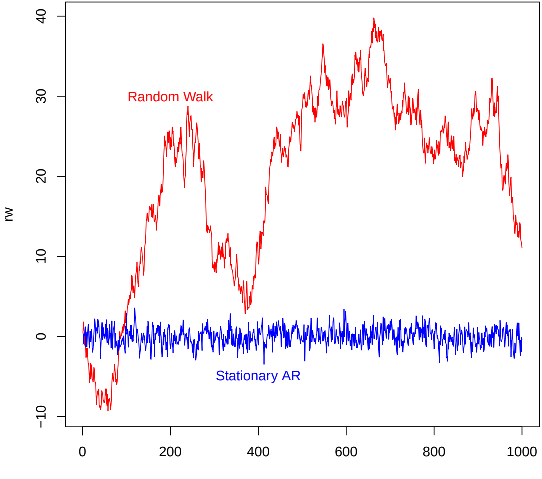
In D House

More intuition in Political Science:

- Come about when you have aggregated a population where some members have perfect memory and other do not.
  - Voting in the House and Senate: Some reps policy motivated and do not switch votes, other strategic and can switch
  - Presidential approval: Pure partisans and switchers.
  - International conflict: rivalries (continuous hostility) and alliance partners (reciprocate cooperation)
- These fractonally integrated models are described by the parameter d, which instead of taking on values of 0 or 1, can be a fraction (ARFIMA(p,d,q)). Most interesting cases are covered where $-.5 < d < .5$.
  - $0 < d < .5$, stationary, but non-geometric decay (stationary with loooong memory).
  - $.5 \geq d < 1$, finite mean, infinite variance (non-stationary)
  - $-.5 < d < 0$, "anti-persistence" – strong negative memory (see example).

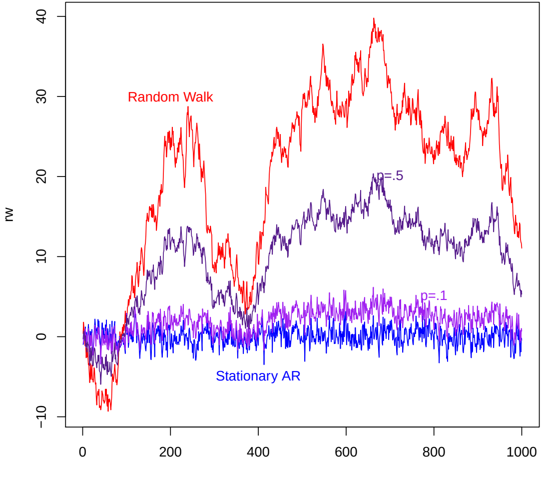
"D"-GP for Fractional Integration I

Aggregation of stationary with non-stationary.

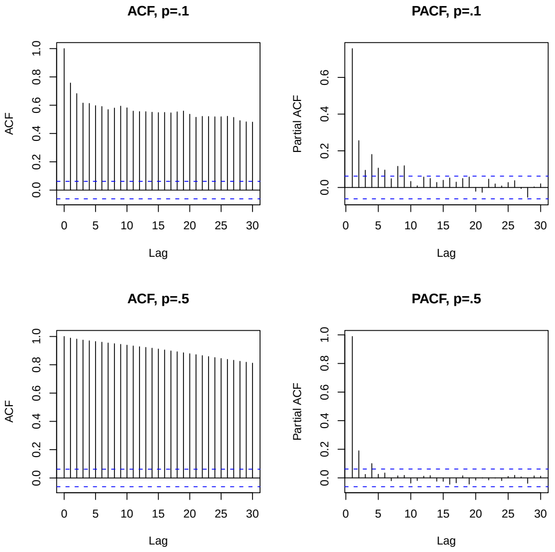


"D"-GP for Fractional Integration II

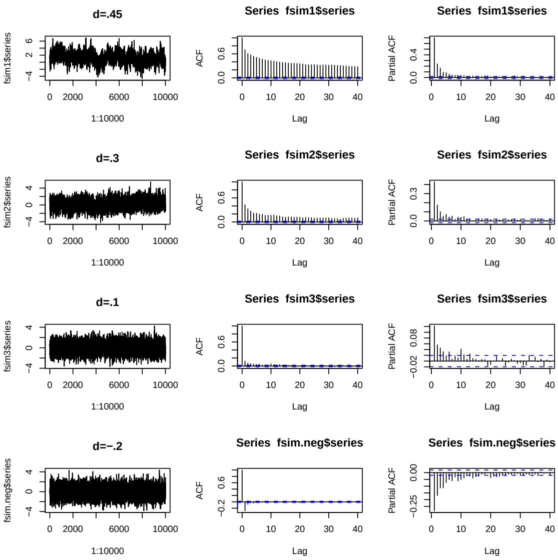
Aggregation of stationary with non-stationary.



I"D"-ing Frac. Integration I



I"D"-ing Frac. Integration II



Est-d-mation

You can estimate d along with any ar or ma components with either an approximation to the MLE or spectral methods (which are outside the domain of this class).

- Ox-metrics has the best software for this
- RATS, S-PLUS are pretty good
- R is ok, fracdiff, afmtools, and arfima packages
- Stata now has ARFIMA, so is in the game.

Fight d Power

Practical Advice

- Tests for fractional integration have very low power against close alternatives
- Note, that if you have a series that is fract. integrated, you can difference it. This will result in inducing dynamics, but these can be modelled away. This is inefficient, but most algorithms for estimating d are not great anyway.
- In small samples, it is impossible to tell ARIMA, from I(1) and in some cases I(0). Also, ARFIMA means you do not have to pick an order of integration that is 0 or 1 only, BUT you then have to pick d exactly. We never know d. Estimates are just that and the asymptotic properties of d-estimates remain unkown, especially when the true DGP is NOT ARFIMA.
- Hendry vs. Sims again.

Granger Causality I

Consider two variables (Y_t, Z_t):

$$
\begin{aligned}
Y_t &= \alpha_0 + \sum_{i=1}^{s} \alpha_i Y_{t-i} + \sum^{s} \beta_i Z_{t-i} + \epsilon_{1t} \\
Z_t &= \alpha_0 + \sum_{i=1}^{s} a_i Y_{t-i} + \sum^{s} b_i Z_{t-i} + \epsilon_{2t}
\end{aligned}
$$

Granger Causality Y_t Granger causes Z_t if the behavior of Y_t (past) can better predict the behavior of Z_t than $Z_t's$ past alone.

The reverse is true as well.

Granger Causality II

Issue: Granger causality tests only aid in "prediction." They are not helpful in drawing "inference" about specific parameters.
Consider the two variables, Y_t, Z_t.

$$
\begin{aligned}
Y_t &= \alpha Z_t + \gamma_{11} Y_{t-1} + \gamma_{12} Z_{t-1} + u_{1t} && (1)\\
Z_t &= \theta Y_t + \gamma_{21} Y_{t-1} + \gamma_{22} Z_{t-1} + u_{2t} && (2)
\end{aligned}
$$

and in reduced form we have:

$$
\begin{aligned}
Y_t &= \Pi_{11} Y_{t-1} + \Pi_{12} Z_{t-1} + N_{1t} && (3)\\
Z_t &= \Pi_{21} Y_{t-1} + \Pi_{22} Z_{t-1} + N_{2t} && (4)
\end{aligned}
$$

Using this reduced form representation, we can say Z "Granger causes" Y *when* $\Pi_{12} \neq 0$.

Granger III

Calculate the reduced form for (3):

$$\begin{aligned}
Y_t &= \alpha\left[\theta Y_t + \gamma_{21} Y_{t-1} + \gamma_{22} Z_{t-1} + u_{2t}\right] + \gamma_{11} Y_{t-1} + \gamma_{12} Z_{t-1} + u_{1t} \\
&= \alpha\theta Y_t + \alpha\gamma_{21} Y_{t-1} + \alpha\gamma_{22} Z_{t-1} + \alpha u_{2t} + \gamma_{11} Y_{t-1} + \gamma_{12} Z_{t-1} + u_{1t} \\
Y_t(1-\alpha\theta) &= \alpha\gamma_{21} Y_{t-1} + \alpha\gamma_{22} Z_{t-1} + \alpha u_{2t} + \gamma_{11} Y_{t-1} + \gamma_{12} Z_{t-1} + u_{1t} \qquad (5) \\
Y_t &= \frac{\alpha\gamma_{21} + \gamma_{11}}{1-\alpha\theta} Y_{t-1} + \frac{\alpha\gamma_{22} + \gamma_{12}}{1-\alpha\theta} Z_{t-1} + \frac{\alpha u_{2t} + u_{1t}}{1-\alpha\theta} \\
\textit{where } \Pi_{12} &= \frac{\alpha\gamma_{22} + \gamma_{12}}{1-\alpha\theta}
\end{aligned}$$

Then, Z G-C Y if $\frac{\alpha\gamma_{22}+\gamma_{12}}{1-\alpha\theta} \neq 0$.

Conversely, if we wanted to determine if Y G-C Z it would require that $\Pi_{21} \neq 0$, which by the same substitution method above equals $\frac{\theta\gamma_{11}+\gamma_{21}}{1-\alpha\theta}$.

Granger Causality: What is it?

Now, say we wanted to obtain a *consistent* estimate of α in equation (1). This requires $\theta = 0$ in equation (2) which assures $E\left[Z_t u_{1t}\right] = 0$. Does a G-C test of $\Pi_{21} = 0$ (Y does not G-C Z) help give assurance that α is a consistent estimate for purposes of hypothesis testing?
No, at least not without additional assumptions. (More on this later.)
θ may or may not equal zero. And γ_{21} cannot tell us the value of θ. Consider that

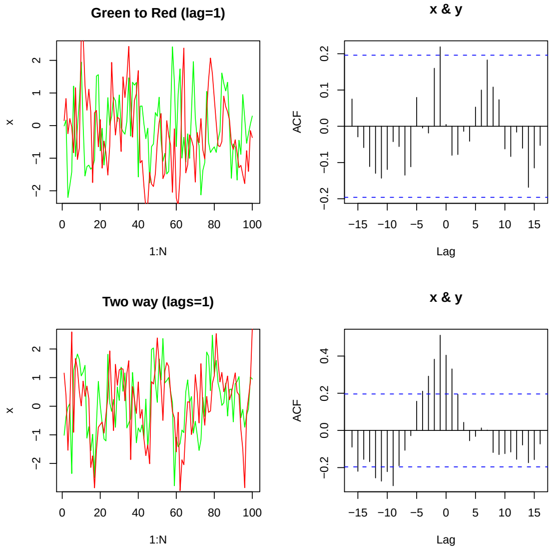
1. $\gamma_{21} \neq 0$ but $\theta = 0$.
2. $\gamma_{21} = -\theta\gamma_{11} \Rightarrow \Pi_{21} = 0$ but $\theta \neq 0$.
3. $\gamma_{21} = 0$ and $\gamma_{11} = 0$ but $\theta \neq 0$.

Ok, if Granger causality cannot assist in drawing valid statistical inferences on structural form, what can it do for us? It can assure valid prediction of Z or $\hat{Y}$.

Uses of G-C

- Prediction, forecasting (reduced form exercise)
- Under certain conditions (one-way causality), you can recover structural parameters.
- Basis for a VAR - more than 2 variables.
- Sets the “playing field”- can use reduced form and limit what structural parameters may or may not be.
- Can test for instantaneous G-C.

Granger Example



Implementation:

1. OLS is consistent as long as hypothesized restrictions on system of equations are block triangular. Classic G-C tests leave the system this way.
2. There are many different ways to test G-C, but the easiest and most straightforward is to use the F-statistic.
3. The lag-length should be sufficient that the errors are white noise. This is very important. Analysts should check whether tests are sensitive to lag-length.
4. Granger Causality can be estimated in levels when variables are non-stationary. One extra lag can be included in each equation but excluded from the F-test.