Example 1 Posterior + Predictive

A | 1

Assume. BM

$$X_i|\theta \overset{IID}{\sim} Br(\cdot|\theta) \quad , \theta \in [0,1]$$

$$\theta \sim Be(\alpha, b)$$

Prior

$$\boxed{\begin{array}{l} Br(x|\theta) = \theta^x (1-\theta)^{1-x} \quad \theta \in (0,1) \\ Be(x|\alpha,b) = \frac{1}{B(\alpha,b)} x^{\alpha-1} (1-x)^{b-1} \\ \alpha > 0 \\ b > 0 \\ x \in (0,1) \end{array}}$$

① Find Posterior distr.

$$\pi(\theta|\underline{X}) = \frac{f(\underline{X}|\theta)\pi(\theta)}{\int f(\underline{X}|\theta)\pi(\theta)\,d\theta} \propto f(\underline{X}|\theta)\pi(\theta)$$

$$\propto \prod_{i=1}^{n} f(X_i|\theta)\,\pi(\theta)$$

$$\propto \prod_{i=1}^{n} Br(X_i|\theta)\,Be(\theta|\alpha,b)$$

$$\propto \prod_{i=1}^{n} \theta^{X_i}(1-\theta)^{1-X_i} * \frac{1}{B(\alpha,b)}\theta^{\alpha-1}(1-\theta)^{b-1}$$

$$\propto \theta^{(\alpha+n\bar{X}_i)-1}(1-\theta)^{(b+n-n\bar{X})-1}$$

I recognise that

$$\theta|\underline{X} \sim Be(\underbrace{\alpha+n\bar{x}}_{\alpha^*}, \underbrace{b+n-n\bar{x}}_{b^*})$$

② Find Prediction of future $y = x_{n+1}$ given $\underset{\sim}{x}_n$

$$p(y|\underset{\sim}{x}) = \int_R f(y|\theta)\,\pi(\theta|x) = \int_0^1 Br(y|\theta)\,Be(\theta|\alpha^*, b^*)\,d\theta$$

$$= \int_0^1 \theta^y (1-\theta)^{1-y} \frac{1}{B(\alpha^*, b^*)}\,\theta^{\alpha^*-1}(1-\theta)^{b^*-1}\,d\theta$$

$$= \left[\int \theta^{y+\alpha^*-1}(1-\theta)^{1-y+b^*-1}\,d\theta\right] \frac{1}{B(\alpha^*, b^*)} \quad 1_{\{0,1\}}(y)$$

$$= \frac{B(\alpha^*+y, 1+b^*-y)}{B(\alpha^*, b^*)} \quad 1_{\{0,1\}}(y)$$

$$= \frac{B(\alpha+n\bar{x}, 1-y+b+n-n\bar{x})}{B(\alpha+n\bar{x}, b+n-n\bar{x})} \quad 1_{\{0,1\}}(y)$$

Theorem 5.1

B

Sufficient statistic of EF

It is $f(X|\theta) = \prod_{i=1}^{n} u(x) g(\theta) \exp\left(\sum_{j=1}^{c} c_j \phi_j(\theta) h_j(x)\right)$

$$= \prod_{i=1}^{n} u(x)_i \exp\left(\sum_{j=1}^{k} c_j \sum_{i=1}^{n} h_j(x) \phi_j(\theta) + n \log(g(\theta))\right)$$

so $t_n = \left(n, \sum_{i=1}^{n} h_1(x_i), \dots \sum_{i=1}^{n} h_k(x_i)\right)$

Conjugate Prior

$$\pi(\theta|X_{1:n}) \propto f(X_{1:n}|\theta)\pi(\theta) = \prod_{i=1}^{n} f(X_i|\theta)\pi(\theta)$$

$$\propto g^{n}(\theta) \exp\left(\sum_{j=1}^{k} c_j \phi_j(\theta) \sum_{i=1}^{n} h_j(x_i)\right) *$$

$$* \, g^{\tau_0}(\theta) \exp\left(\sum_{j=1}^{k} c_j \phi_j(\theta) \tau_j\right)$$

$$\propto f(\theta)^{\tau_0 + n} \exp\left(\sum_{j=1}^{k} c_j \phi_j(\theta) \left[\tau_j + \sum_{i=1}^{n} h_j(x_i)\right]\right)$$

$$\propto \pi(\theta|\underset{\sim}{\tau}^*)$$

$$\underset{\sim}{\tau}^* = \left(\tau_0 + n, \tau_1 + \sum_{i=1}^{n} h_1(x_i), \dots, \tau_k + \sum_{i=1}^{n} h_k(x_i)\right)$$

Example 2

C 2

BM $\quad X_i|\theta \overset{iid}{\sim} Br(\cdot|\theta)$

$\theta \sim \pi(d\cdot)$

Find conjugate prior $\pi(d\cdot)$ for θ

Answer

- It is

$$Br(x|\theta) = \theta^{x}(1-\theta)^{1-x}$$
$$= \exp(x\log(\theta)) + (1-x)\log(1-\theta))$$
$$= (1-\theta)\exp\left(x\log\left(\frac{\theta}{1-\theta}\right)\right)$$

with $x \in \mathcal{X} = \{0,1\}$, $\theta \in [0,1]$

So $Br(\theta)$ is member of the $\mathcal{E}\mathcal{F}_1$

with

$u(x) = 1$

$g(\theta) = 1-\theta$

$h(x) = x$

$\phi(\theta) = \log\frac{\theta}{1-\theta}$

$c = 1$

(!) Find sufficient statistic.

It is $\quad t_n = (n, \sum x_i)$ or $(n, \bar{x})$

- So the conjugate prior is

$$\pi(\theta|\tau) \propto g(\theta)^{\tau_0} \exp\left(c_1 \phi_1(\theta)\, \tau_1\right)$$
$$= (1-\theta)^{\tau_0} \exp\left(1 \cdot \log \frac{\theta}{1-\theta} \cdot \tau_1\right)$$
$$= \theta^{\tau_1} (1-\theta)^{\tau_0-\tau_1}$$
$$\propto K(\tau)\, \theta^{(\tau_1+1)-1} (1-\theta)^{(\tau_0-\tau_1+1)-1}$$

with $K(\tau) = \int_0 \theta^{(\tau_1+1)-1} (1-\theta)^{(\tau_0-\tau_1+1)-1} d\theta$

I recognize

$$\theta|\alpha, b \sim Be(\alpha, b)$$

with $\alpha = \tau_1 + 1$

$b = \tau_0 - \tau_1 + 1$

obviously $K(\tau) = \frac{1}{Be(\alpha, b)}$

Example 3

BM $X_i|\vartheta \overset{\text{IID}}{\sim} Br(\cdot|\vartheta)$ $i = 1 \dots n$

$\vartheta \sim Be(\alpha, b)$

Find the $(1-a)100\%$ CI for ϑ
posterior

for $a = 0.05$, $\alpha = b = 2$, $n = 30$, $\sum_{i=1}^{30} x_i = 35$

Answer

- The posterior is

$$\vartheta | x_{1:n} \sim Be(\alpha^*, b^*) \equiv Be(17, 17)$$

- It is unimodal so I use the Thm 7.1

consider $C_\alpha = [L, U]$

so

$$\Rightarrow 1 - a = \int_L^U Be(\vartheta|17, 17)\, d\vartheta \quad (Q1)$$

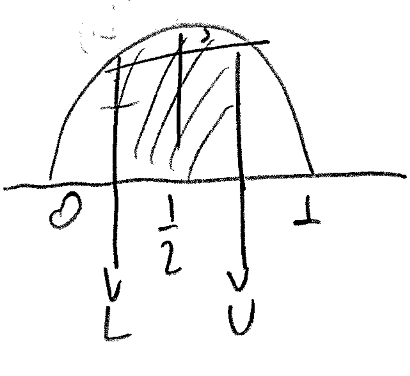
$$= Be(\vartheta \le U|17, 17) - Be(\vartheta < L|17, 17) \Rightarrow$$

$$\boxed{\frac{IB(U|17,17)}{B(17,17)} - \frac{IB(L|17,17)}{B(17,17)} - 1 + a = 0} \quad (Q2)$$

$$\Rightarrow \boxed{Be(U|17,17) - Be(L|17,17) = 0} \quad (Q3)$$

Solve system Q1 and Q2

(?) I observe that $Be(\alpha^*, b^*)$ with $\alpha^* = b^* = 17$ is symmetric around $\vartheta = 1/2$

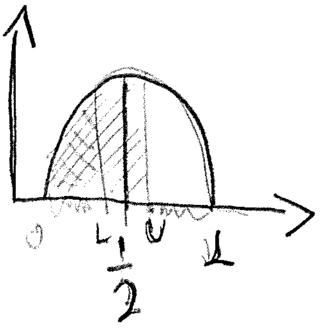


2

Q4

So (Q3) $\Rightarrow \frac{1}{2} - L = U - \frac{1}{2} \Rightarrow \boxed{L = 1 - U}$

(Q3) $1-\alpha = Be(\theta < U \mid 17, 17) - Be(\theta > U \mid 17, 17)$



$= -1 + 2 Be(\theta < U \mid 17, 17)$

$\boxed{U : Be(\theta < U \mid 17, 17) = 1 - \frac{\alpha}{2}}$

$1-\frac{\alpha}{2}$ -th quantile of $Be(17,17)$

and $\boxed{L = 1 - U}$

Example 4

E (1)

Consider observables $X_{1:n}$ s.t. $X_n|\theta \overset{IID}{\sim} Br(\theta)$

where $n=5$, $X_* = \sum_{i=1}^{5} X_i = 3$, $n=5$

Test if $\theta = 1/2$ or not. Based on the 1-1 loss function (8.6)

Answer

Usually $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

(2) Here $H_0: p_0(\underset{\sim}{X}) = Br(X_{1:n}|\theta_0)$

$H_1: p_1(\underset{\sim}{X}) = \int_0^1 Br(X_{1:n}|\theta) U(\theta|0,1) d\theta$

$$\boxed{\theta \sim U(0,1) \text{ if } U(\theta|0,1) = 1\, I_{(0,1)}(\theta)}$$

(1) Set prior $\Pi(\theta) = \Pi_0 1_{\theta_0}(\theta) + (1-\Pi_0) U(\theta|0,1)$

with $\theta_0 = 1/2$ and $\Pi_0 = 1/2$.

(3) Bayes factor

$$B_{01} = \frac{\prod_{i=1}^{n} Br(X_i|\theta_0)}{\int_0^1 \prod_{i=1}^{n} Br(X_i|\theta) U(\theta|0,1) d\theta} = \frac{\theta_0^{n\bar{x}} (1-\theta_0)^{n-n\bar{x}}}{\int_0^1 \theta^{n\bar{x}} (1-\theta)^{n-n\bar{x}} d\theta}$$

$$= \frac{\theta_0^{n\bar{x}} (1-\theta_0)^{n-n\bar{x}}}{B(n\bar{x}+1, n-n\bar{x}+1)} = \frac{(1/2)^5}{B(4,3)} = \frac{15}{8} \simeq 2$$

Also $\frac{C_{II}}{C_I}\frac{\Pi_1}{\Pi_0} = \frac{1}{1}\frac{1}{1} = 1$

Because $B_{01} > \frac{C_{II}}{C_I}\frac{\Pi_1}{\Pi_0}$ as $2 > 1$

I accept H_0