Some info for autoregressive time series models

18 March 2020

1 AR(1): Introduction

Suppose X_t is an AR(1) process with $|\rho| < 1$, then:

$$
\begin{aligned}
X_t &= \rho X_{t-1} + \varepsilon_t, && (1)\\
\varepsilon_t &\sim N(0, \sigma^2), && (2)\\
\boldsymbol{X} &= (X_0, \dots, X_T)', && (3)\\
&\sim N_{T+1}(\mathbf{0}, \boldsymbol{\Sigma}), && (4)\\
\Sigma_{ij} &= \frac{\sigma^2}{1-\rho^2} \cdot \rho^{|i-j|}, && (5)\\
\boldsymbol{\Sigma} &= \frac{\sigma^2}{1-\rho^2} \cdot \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^T \\ \rho & 1 & \rho & \dots & \rho^{T-1} \\ \dots & \dots & \dots & \dots & \dots \\ \rho^T & \dots & \rho^2 & \rho & 1 \end{bmatrix}. && (6)
\end{aligned}
$$

2 Model specifications to skip over years without observations

$$
\begin{aligned}
X_0 &\sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right), && (7)\\
X_t|X_0 &= \rho^t X_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i} \cdot \rho^i, && (8)\\
\mathrm{Var}\{X_t|X_0\} &= \mathrm{Var}\{\sum_{i=0}^{t-1} \varepsilon_{t-i} \cdot \rho^i\}, && (9)\\
&= \sigma^2 \sum_{i=0}^{t-1} \rho^{2i}, && (10)\\
&= \sigma^2 \frac{1-\rho^{2t}}{1-\rho^2}, && (11)\\
X_t|X_0 &\sim N\left(\rho^t X_0, \frac{\sigma^2}{1-\rho^2}(1-\rho^{2t})\right), && (12)\\
X_{t+s}|X_t &\sim N\left(\rho^s X_t, \frac{\sigma^2}{1-\rho^2}(1-\rho^{2s})\right), && (13)
\end{aligned}
$$

where derivation of the conditional variance follows from

$$s = \sum_{k=0}^{n-1} r^k \quad = \quad \frac{1-r^n}{1-r}, \tag{14}$$

because

$$s - rs = s(1-r) = 1 - r^n.$$

3 Useful to know for next sections

If

$$X = (X_1, X_2) \quad \sim \quad N((\mu_1, \mu_2)', \Sigma), \tag{15}$$

then

$$X_1 | X_2 = x_2 \quad \sim \quad N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}). \tag{16}$$

Eq. 16 extended to vectors of length > 2, e.g., if

$$X = (X_1, \boldsymbol{X}_2) \quad \sim \quad N((\mu_1, \boldsymbol{\mu}_2)', \boldsymbol{\Sigma}), \tag{17}$$

then

$$X_1 | \boldsymbol{X}_2 = \boldsymbol{x}_2 \quad \sim \quad N(\mu_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}). \tag{18}$$

or

$$\boldsymbol{X}_2 | X_1 = x_1 \quad \sim \quad N(\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\Sigma_{11}^{-1}(x_1 - \mu_1), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\Sigma_{11}^{-1}\boldsymbol{\Sigma}_{12}). \tag{19}$$

4 Constructing AR(1)-trajectories given realizations for years $t = (t_0, t_1, \ldots, t_n)$

- At the start, for $t = t_0 - 1$ etc.:

$$X_{t_0 - 1} | X_{t_0} \quad \sim \quad N(\rho X_{t_0}, \sigma^2). \tag{20}$$

- At the end, for $t = t_n + 1$ etc.:

$$X_{t_n + 1} | X_{t_n} \quad \sim \quad N(\rho X_{t_n}, \sigma^2). \tag{21}$$

- For any missing years between t_i and t_{i+1}, e.g. to get X_t given X_{t-1} and X_s, where $s > t$:

$$E(X_t | X_{t-1}, X_s) \quad = \quad \frac{1}{1-A}\left(\rho \cdot (1 - \rho^{2(s-t)}) \cdot X_{t-1} + \rho^{s-t}(1-\rho^2) \cdot X_s\right), \tag{22}$$

$$\mathrm{Var}(X_t | X_{t-1}, X_s) \quad = \quad \frac{\sigma^2}{1-\rho^2}\left(1 - \frac{\rho^2 - 2A + \rho^{2(s-t)}}{1-A}\right), \tag{23}$$

$$A \quad = \quad \rho^{2(s-t+1)}. \tag{24}$$

- For $X_t|X_{t-1}, X_s$:

$$
\begin{aligned}
\boldsymbol{X} &= (X_t, X_{t-1}, X_s)', && (25)\\
&\sim N_3(\mathbf{0}, \boldsymbol{\Sigma}), && (26)\\
\boldsymbol{\Sigma} &= \frac{\sigma^2}{1-\rho^2}\cdot\begin{bmatrix} 1 & \rho & \rho^{s-t} \\ \rho & 1 & \rho^{s-t+1} \\ \rho^{s-t} & \rho^{s-t+1} & 1 \end{bmatrix}, && (27)\\
\boldsymbol{\Sigma}_{12} &= \frac{\sigma^2}{1-\rho^2}(\rho, \rho^{s-t}), && (28)\\
\boldsymbol{\Sigma}_{22} &= \frac{\sigma^2}{1-\rho^2}\cdot\begin{bmatrix} 1 & \rho^{s-t+1} \\ \rho^{s-t+1} & 1 \end{bmatrix}, && (29)\\
\boldsymbol{\Sigma}_{22}^{-1} &= \frac{\sigma^2}{1-\rho^2}\cdot\begin{bmatrix} 1 & -\rho^{s-t+1} \\ -\rho^{s-t+1} & 1 \end{bmatrix}\cdot\frac{1}{\left(\frac{\sigma^2}{1-\rho^2}\right)^2\cdot(1-A)}, && (30)\\
&= \frac{1}{\frac{\sigma^2}{1-\rho^2}}\frac{1}{1-A}\begin{bmatrix} 1 & -\rho^{s-t+1} \\ -\rho^{s-t+1} & 1 \end{bmatrix}, && (31)
\end{aligned}
$$

- Then the conditional mean is given by:

$$
\begin{aligned}
E(X_t|X_{t-1}, X_s) &= \frac{1}{1-A}(\rho, \rho^{s-t})\begin{bmatrix} 1 & -\rho^{s-t+1} \\ -\rho^{s-t+1} & 1 \end{bmatrix}(X_{t-1}, X_s)', && (32)\\
&= \frac{1}{1-A}(\rho - \rho^{2(s-t)+1}, -\rho^{s-t+2} + \rho^{s-t})(X_{t-1}, X_s)', && (33)\\
&= \frac{1}{1-A}\left(\rho\cdot(1-\rho^{2(s-t)})\cdot X_{t-1} + \rho^{s-t}(1-\rho^2)\cdot X_s\right), && (34)
\end{aligned}
$$

- And the conditional variance is given by:

$$
\begin{aligned}
\mathrm{Var}(X_t|X_{t-1}, X_s) &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, && (35)\\
&= \frac{\sigma^2}{1-\rho^2}\left(1 - \frac{1}{1-A}(\rho, \rho^{s-t})\begin{bmatrix} 1 & -\rho^{s-t+1} \\ -\rho^{s-t+1} & 1 \end{bmatrix}(\rho, \rho^{s-t})'\right), && (36)\\
&= \frac{\sigma^2}{1-\rho^2}\left(1 - \frac{1}{1-A}(\rho - \rho^{2(s-t)+1}, -\rho^{s-t+2} + \rho^{s-t})(\rho, \rho^{s-t})'\right), && (37)\\
&= \frac{\sigma^2}{1-\rho^2}\left(1 - \frac{1}{1-A}(\rho^2 - \rho^{2(s-t)+2} - \rho^{2(s-t)+2} + \rho^{2(s-t)})\right), && (38)\\
&= \frac{\sigma^2}{1-\rho^2}\left(1 - \frac{\rho^2 - 2A + \rho^{2(s-t)}}{1-A}\right). && (39)
\end{aligned}
$$