Practical Computing for Economists

Notes on MCMC

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Model

Individuals are $i = 1, \ldots, N$, and $j = 1, \ldots, M$ measures of them (eg. repeated high school/college tests, or different interest rate measures). The DGP is

$$\begin{aligned} Y_{ij} &= \alpha_j \theta_i + \beta_1 X_i + \epsilon_{ij} \\ &\vdots \\ Y_{mj} &= \alpha_m \theta_i + \beta_m X_i + \epsilon_{im} \end{aligned}$$

The thing we want to estimate here is (the distribution of) θ_i. That's the hidden state, such as how IQ determines exam performance.

Only the Y_s and X_s are observed. We assume that θ is independent of ϵ and X. So we define $\tilde{Y} = Y - X\beta$, and use the notation $\sum_{\tilde{Y}} = \alpha \sum_\theta \alpha' + \sum_\epsilon$. This $\tilde{Y}$ is the residualized version of Y and follows because of the uncorrelation of θ and X. We need 1) some normalizations of the factor (i.e. its units) and 2) some restrictions for identification.

The MCMC procedure

We assume that the errors are diagonal (for identification):

$$\epsilon_{i,j} \sim N(0, \sigma_j^2)$$

So the unknowns are $\beta, \theta, \alpha, \sigma$

And then the priors and posteriors are conjugate:

$$p(\sigma_j^2) \simeq IG(a, b) \qquad p(\sigma_j^2 | X, Y) \sim IG(a^*, b^*)$$

Where the posterior parameters are given by:

$$a^* = a + \frac{N}{2} \qquad b^* = b + \frac{1}{2} \sum \left[(Y_{ij} - X_i \beta_j - f_i \alpha_j)^2 \right]$$

We assume that all the other other parameters are normally distributed. The priors and posteriors are:

$$p(\beta_j) \sim N(\mu, B) \qquad p(\beta_j | \sigma^2, X, Y) \sim N(\mu^*, B^*)$$

$$B^* = \left[\frac{1}{\sigma^2} \left(X'X + B \right) \right]^{-1} \qquad \mu^* = B^* \left(\frac{1}{\sigma^2} X'Y_j + \frac{Bb}{\sigma^2} \right)$$

And for α:

$$p(\alpha) \sim N(c, A) \qquad p(\alpha | X, Y, \sigma) \sim N(c^*, A^*)$$

With analogous definitions of (c^*, A^*). Then the distribution of the hidden factor is:

$$\theta_i \sim N(\gamma, F) \qquad \theta_i^* \sim N(\gamma^*, F^*)$$

$$F^* = \left[\alpha'(\sigma^2)^{-1} \alpha \right]^{-1} \qquad \gamma^* = F^* \left(\alpha'(\sigma^2)^{-1} Y_i \right)$$

An example

Imagine we have data on test scores, GPAs in maths and science GPA_s and GPA_m, and another test score TS. Then if this is driven by IQ and class size according to the following model:

$$
\begin{aligned}
GPA_m &= \alpha_1 IQ_i + \beta_1 CS_i + \epsilon_{i1} \\
GPA_s &= \alpha_2 IQ_i + \beta_2 CS_i + \epsilon_{i2} \\
TS &= \alpha_3 IQ_i + \beta_3 CS_i + \epsilon_{i3}
\end{aligned}
$$

For $i = 1, \ldots, N$ individuals. f course, IQ is hidden, so we want to infer it from the data. The normalization is that $\alpha_1 = 1$ and the identifying restrictions on the error process are:

$$
\Sigma_\epsilon = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}
$$

Then the econometric problem is to compute the joint distribution of $\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, $IQ_1, \ldots, IQ_N$ and $\sigma_1^2, \sigma_2^2, \sigma_3^2$.

The aim of the exercise

Ultimately, we want to be able to compute the *joint* distribution of the parameters. MCMC lets us do this by holding all parameters but one fixed and then drawing correctly from the remaining conditional & marginal distributions. Doing this enough eventually produces draws from the joint distribution, which is what we want.