Bayes for beginners: **Bayes factors**

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Reprise: An approximate Bayes factor

A difference in BIC between models on the same data can be expressed as a likelihood ratio:

$$e.g., \Delta_{BIC} = BIC_{M0} - BIC_{M1}$$

$$LR_{BIC} = e^{1/2\Delta_{BIC}}$$

This quantity is a well-known approximation to a Bayes factor with a *unit-information prior*:

$$BF \approx LR_{BIC} = e^{1/2\Delta_{BIC}}$$

Here the prior is meant to carry as much information as a single observation (Kass & Wasserman, 1995)

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They are equal to likelihood ratios only for nested models with simple hypotheses and no nuisance parameters (Kass & Raftery, 1995)

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... thus the posterior probability is proportional to the likelihood times the prior

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- promising 'default' objective priors have been developed

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Unlike AIC or BIC there is no need to add a penalty for complex models or hypotheses

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(It can also be useful to use the logarithm $ln(BF)$ when the Bayes factor is large)

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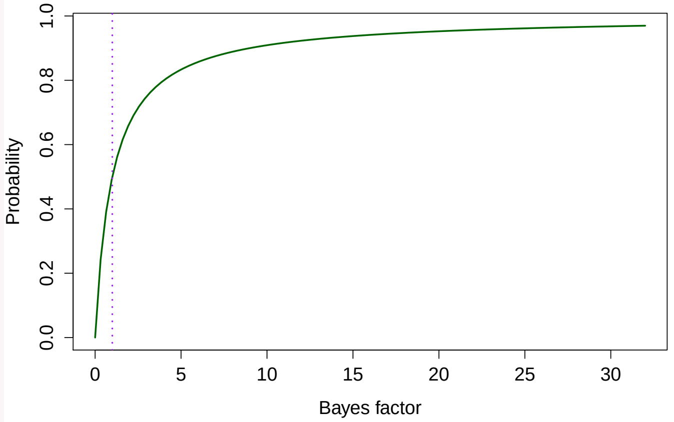
Transforming from an odds to a probability scale



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Jeffries (1961) proposed the following interpretation:

BF_{10}	Strength of evidence for H_1
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... but remember that it is a continuous measure!

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- unit-information prior
  (e.g., Kass & Raftery, 1995; Wagenmakers, 2007)
- Cauchy or JZS prior
  (e.g., Liang et al., 2008; Rouder et al., 2009)

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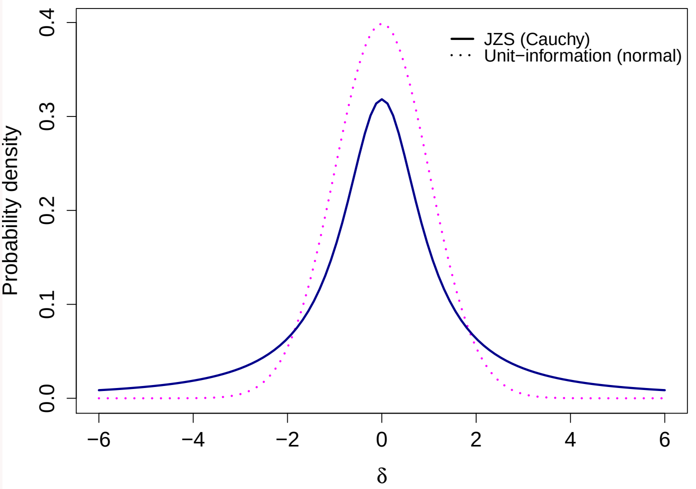
Standard Cauchy versus standard normal priors



Bayesian t test with unit-information prior

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However, when H_0 is that $\delta = 0$ it can be calculated by hand (e.g., for one sample t):

$$BF_{01} = \frac{\left(1 + \frac{t^2}{n-1}\right)^{-n/2}}{(1 + nr^2)^{-1/2}\left(1 + \frac{t^2}{(1+nr^2)(n-1)}\right)^{-n/2}}$$

... here r is the scale factor for the variance (typically set at 1)

Bayesian t test with a Jeffreys-Zellner-Siow (JZS) prior

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The analytic solution for the one sample case is:

$$BF_{01} = \frac{\left(1 + \frac{t^2}{n-1}\right)^{-n/2}}{\int_0^\infty (1 + ngr^2)^{-1/2} \left(1 + \frac{t^2}{(1+ngr^2)(n-1)}\right)^{-n/2} (2\pi)^{-1/2} g^{-3/2} e^{-1/(2g)} dg}$$

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... and here r is the scale factor for the variance typically set at $1/2$ for a one sample or $\sqrt{2}/2$ for two samples

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This is around $n > 5{,}000$ for the unit-information prior or and $n > 50{,}000$ for the JZS prior

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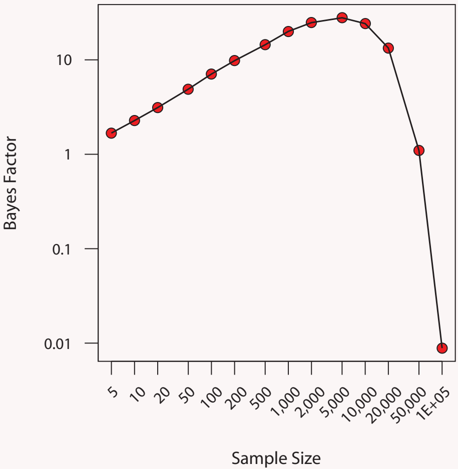
Tipping point for the unit-information prior when $\delta = 0.02$



Reprise: Voting intention example

Does a single exposure to an US flag influence voting intention?

Carter et al. (2011) report a priming study in which a single exposure to a US flag changes voting intentions

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How do our objective Bayes factors work in this case?

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```
> unit.prior.Bf.2s(t=2.02, n1=91.5, n2=91.5)
$`Bayes factor for H0`
[1] 0.9394962

$`Bayes factor for H1`
[1] 1.0644
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Baguley (2012) also provides R functions for these tests.

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... other constraints are possible