Bayes for beginners: **Introducing Bayesian data analysis**

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Reprise: Bayesian data analysis in three simple steps

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... this last step uses Bayes' theorem

Bayes' theorem

simple probability

$$\overbrace{Pr\left(A|B\right)}^{\text{Posterior}} = \frac{\overbrace{Pr\left(B|A\right)}^{\text{Evidence}} \quad \overbrace{Pr\left(A\right)}^{\text{Base rate (Prior)}}}{\underbrace{Pr\left(B\right)}_{\text{Probability of evidential event}}}$$

Bayes' theorem

probability distributions

… the same equation can be generalised to probability distributions (here Ω is the full set of parameters in the model including nuisance parameters)

$$\overbrace{\mathrm{P}(\theta|\mathcal{D},\Omega)}^{\text{Posterior}} = \frac{\overbrace{\mathrm{P}(\mathcal{D}|\theta)}^{\text{Likelihood}}\;\overbrace{\mathrm{P}(\theta|\Omega)}^{\text{Prior}}}{\underbrace{\int \mathrm{P}(\mathcal{D}|\theta)\mathrm{P}(\theta|\Omega)d\theta}_{\text{Marginal probability of data}}}$$

Marginal probability

... getting a posterior distribution for θ requires integrating out other parameters in the model

$$\int \mathrm{P}(\mathcal{D}|\theta)\mathrm{P}(\theta|\Omega)d\theta$$

Getting an analytic solution may be difficult or impossible

... but can also be very simple for some well-known problems

Normal distributions with known variance

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... together imply:

$$\overbrace{\mathrm{P}(\mu|\mathcal{D}, \sigma^2)}^{\text{Posterior}} \sim N\left(\mu_{post}, \sigma^2_{post}\right)$$

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... this weighting is sometimes termed a *fully automatic Occam's razor* (Spegelhalter & Smith, 1980)

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e.g., a 95% probability interval takes the form:

$$\mu_{post} \pm z_{.975}\sigma_{post} \approx \mu_{post} \pm 1.96\sigma_{post}$$

Selecting a prior

Subjective versus objective priors

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A complete alternative is the fully subjectivist position, which compels one to elicit priors on all parameters based on the personal judgement of appropriate individuals.

Spiegelhalter & Rice (2009)

Subjective Bayes

Eliciting a prior

A key task for a subjective Bayesian is to elicit a prior distribution:

> *... successful elicitation faithfully represents the opinion of the person being elicited. It is not necessarily "true" in some objectivistic sense, and cannot be judged that way. We see elicitation as simply part of the process of statistical modeling.*

Garthwaite et al. (2005)

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Objective Bayes

What is objectivity in Bayesian data analysis

... I take objectivity to mean that given the same data and the same assumptions regarding the model, different researchers will arrive at the same conclusions.

Wagenmakers (2007, Appendix)

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A viewpoint from subjective Bayes

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A viewpoint from objective Bayes

... it [objective Bayes] simultaneously achieves what should be a major goal of Bayesianism - ensuring that answers are conditional on the data actually obtained - while at the same time respecting the frequentist notion that the methodology must ensure success in repeated usage by scientists.

Berger (2006)

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- greater emphasis on transparency in modelling
- more flexibility in modelling complex, messy (real world) data sets