GR5065 Assignment 1

Due on Thursday, February 7, 2019 by 4PM

1 Bowling Revisted

Let the Fibonacci numbers be given by $\mathcal{F}_0 = 1$, $\mathcal{F}_1 = 1$, and else $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$. In this question, we are going to consider a *slightly different* Probability Mass Function as the data-generating process for bowling than the one we used the first week, namely

$$\Pr\left(x \middle| n\right) = \frac{\left(\mathcal{F}_x\right)^2}{\mathcal{F}_n \times \mathcal{F}_{n+1}}$$

where $x \geq 0$ is the number of pins knocked down and $n <= 10$ is the number of pins available to be knocked down, which can be implemented in R with

```r
# computes the x-th Fibonacci number without recursion and with vectorization
F <- function(x) {
  stopifnot(is.numeric(x), all(x == as.integer(x)))
  sqrt_5 <- sqrt(5) # defined once, used twice
  golden_ratio <- (1 + sqrt_5) / 2
  return(round(golden_ratio ^ (x + 1) / sqrt_5))
}

# probability of knocking down x out of n pins
Pr <- function(x, n = 10) return(ifelse(x > n, 0, (F(x) ^ 2) / (F(n) * F(n + 1))))

Omega <- 0:10 # 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is the sample space
```

1.1 Admissibility

How do you know that this is an admissible Probability Mass Function for bowling?

1.2 Three Cases

If the joint probability of two rolls in the same frame of bowling is given by the interior cells of

```r
joint_Pr <- matrix(0, nrow = 11, ncol = 11)
rownames(joint_Pr) <- colnames(joint_Pr) <- as.character(Omega)
for (x1 in Omega) {
  Pr_x1 <- Pr(x1, n = 10)
  for (x2 in 0:(10 - x1)) {
    joint_Pr[x1 + 1, x2 + 1] <- Pr_x1 * Pr(x2, n = 10 - x1)
  }
}
```

1. What number is the probability of a strike (knocking down all 10 pins) on the first roll of a frame of bowling?
2. What number is the probability of a spare (but not a strike) on a frame of bowling, i.e. knocking down all 10 pins on two rolls but not all 10 on the first roll?
3. What number is the probability of an "open frame", which is neither a strike nor a spare?

1.3 Conditional Expectation

What number is the expected number of pins knocked down in a frame of bowling, given that it is an "open frame"?

1.4 Tenth Frame

The tenth (final) frame of bowling works a bit differently than the previous nine frames:

1. If you get a strike on your first roll, then you get two additional rolls. In that case, your second roll will have all 10 pins available again. If you get another strike, then on your third roll, there will be 10 pins available again. If you do not get a strike on your second roll, then the remaining pins are available to be knocked down on the third roll.
2. If you do not get a strike on your first roll of the tenth frame but do get a spare, then you are given one additional roll with all 10 pins set back upright.
3. If you do not knock down all 10 pins by your second roll, then you do not get any additional rolls.

Compute (without random number simulation, although you could use it to check) the number that is the expectation of the sum of the number of pins knocked down in the tenth frame of bowling.

Hint: Use iterated expectations to obtain a marginal expectation like we did in lecture, where the three marginal probabilities are for a strike, a spare, and an open frame.

1.5 Perfect Game

What number is the probability of achieving a "perfect game", which is twelve consecutive strikes without any spares or open frames?

1.6 Scoring

Although it is plausible to assume that pins knocked down are independent across frames of bowling, the scoring rules for bowling are not independent across frames. The "traditional" scoring rules are described at

https://en.wikipedia.org/wiki/Ten-pin_bowling#Traditional_scoring

but basically they say that if you get a strike on frame j, then your score for frame j is 10 plus the number of pins knocked down on your next two rolls (for a maximum of 30). And if you get a spare on frame j, then your score for frame j is 10 plus the number of pins knocked down on the first roll of frame $j+1$ (for a maximum of 20). The tenth frame is scored as described in the previous subquestion in order to make it consistent with the previous nine frames. So, a perfect game would achieve the maximum possible score of 300.

What is the expectation of a person's score for an entire game of bowling? Again, do not use random number generation, except possibly to check your answer.

1.7 Variance

Is the variance of person's score for an entire game of bowling less than, equal to, or greater than ten times the variance of a person's score on a single frame of bowling? You do not need to compute any variance exactly but do need to explain why you reached that conclusion.

2 Probability in Poker

This question refers to the following hand of poker (you can skip over any commercials that may pop up)

https://youtu.be/dZyGfk2HDaY

If you do not know the rules of this game (no limit Texas Hold 'Em), you can read about them here

https://en.wikipedia.org/wiki/Texas_hold_%27em

Basically the two (face down) cards in your hand can be combined with any three (face up) cards that are in the middle to form a five card poker hand and there are several opportunities to fold or to bet at least as much as the other player(s).

Since this is a poker tournament, you cannot take your poker chips to the cashier, exchange them for cash, and leave. Players are eliminated from the poker tournament when they have zero chips left and win prize money for doing well.

Before the hand is dealt, each of the five players puts $120,000$ worth of chips into the pot as an “ante”, one player (Rick Salomon) puts an additional $600,000$ worth of chips into the pot as the “big blind”, and one player (Dan Smith) puts an additional $300,000$ worth of chips into the pot as the “small blind”. At this point, there are $1,500,000$ worth of chips in the pot and as shown at the top of the screen, the players have the following amounts of chips left

```r
(stacks <- c(Salomon = 27450000, Kaverman = 8025000, Holz = 28675000,
             Bonomo = 51150000, Smith = 19700000) / 600000)
```

```
##   Salomon Kaverman     Holz   Bonomo    Smith
## 45.75000 13.37500 47.79167 85.25000 32.83333
```

The units of poker chips are essentially arbitrary so it is customary to divide by the amount of the “big blind” (here $600,000$). In other words, Byron Kaverman — who has to go first on this hand — only has enough chips to pay the ante and blinds for about 45 more hands, even if he folded every single time, which puts him much closer to being eliminaed than the other four players.

2.1 Going “All In”

At 0:13 of the video, Kaverman (although it is hard to hear over the announcer) says that he is “all in” — meaning he is betting all of his remaining chips — when his two face down cards are the Ace of clubs and the five of clubs.

People have done simulations that show the expected change in *chips* for someone in Kaverman’s situation (having to act first, starting the hand with about 14 big blinds worth of chips) going “all in” is positive if and only if you have one of the following 138 out of $\binom{52}{2} = 1326$ possible hands:

- Pairs that are better than fives (9 pairs × 6 suit combinations = 54 hands)
  - 🂡 🂱, 🂡 🃁, 🂡 🃑, 🂱 🃁, 🂱 🃑, 🃁 🃑, 🂮 🂾, 🂮 🃎, 🂮 🃞, 🂾 🃎, 🂾 🃞, 🃎 🃞,
    🂭 🂽, 🂭 🃍, 🂭 🃝, 🂽 🃍, 🂽 🃝, 🃍 🃝, 🂫 🂻, 🂫 🃋, 🂫 🃛, 🂻 🃋, 🂻 🃛, 🃋 🃛,
    🂪 🂺, 🂪 🃊, 🂪 🃚, 🂺 🃊, 🂺 🃚, 🃊 🃚, 🂩 🂹, 🂩 🃉, 🂩 🃙, 🂹 🃉, 🂹 🃙, 🃉 🃙,
    🂨 🂸, 🂨 🃈, 🂨 🃘, 🂸 🃈, 🂸 🃘, 🃈 🃘, 🂧 🂷, 🂧 🃇, 🂧 🃗, 🂷 🃇, 🂷 🃗, 🃇 🃗,
    🂦 🂶, 🂦 🃆, 🂦 🃖, 🂶 🃆, 🂶 🃖, 🃆 🃖
- Failing that, an Ace and at least a jack of different suit (6 combinations × 6 suit pairs = 36 hands)
  - 🂡 🂾, 🂡 🃎, 🂡 🃞, 🂡 🂽, 🂡 🃍, 🂡 🃝, 🂡 🂻, 🂡 🃋, 🂡 🃛, 🂱 🂮, 🂱 🃎, 🂱 🃞,
    🂱 🂭, 🂱 🃍, 🂱 🃝, 🂱 🂫, 🂱 🃋, 🂱 🃛, 🃁 🂮, 🃁 🂾, 🃁 🃞, 🃁 🂭, 🃁 🂽, 🃁 🃝,
    🃁 🂫, 🃁 🂻, 🃁 🃛, 🃑 🂮, 🃑 🂾, 🃑 🃎, 🃑 🂭, 🃑 🂽, 🃑 🃍, 🃑 🂫, 🃑 🂻, 🃑 🃋,
- Or an Ace and at least a nine of the same suit (5 combinations × 4 suits = 20 hands)
  - 🂡 🂮, 🂡 🂭, 🂡 🂫, 🂡 🂪, 🂡 🂩, 🂱 🂾, 🂱 🂽, 🂱 🂻, 🂱 🂺, 🂱 🂹, 🃁 🃎, 🃁 🃍,
    🃁 🃋, 🃁 🃊, 🃁 🃉, 🃑 🃞, 🃑 🃝, 🃑 🃛, 🃑 🃚, 🃑 🃙
- Failing that, both cards different but at least 10 of the same suit (6 combinations × 4 suits = 24 hands)
  - 🂮 🂭, 🂮 🂫, 🂮 🂪, 🂭 🂫, 🂭 🂪, 🂫 🂪, 🂾 🂽, 🂾 🂻, 🂾 🂺, 🂽 🂻, 🂽 🂺, 🂻 🂺,
    🃎 🃍, 🃎 🃋, 🃎 🃊, 🃍 🃋, 🃍 🃊, 🃋 🃊, 🃞 🃝, 🃞 🃛, 🃞 🃚, 🃝 🃛, 🃝 🃚, 🃛 🃚
- Finally, an Ace and a five of the same suit (4 suits = 4 hands): 🂡 🂥, 🂱 🂵, 🃁 🃅, 🃑 🃕

The announcer subsequently says “[Kaverman] didn’t want to [go all in] with two eights two hands ago” when there were still six players remaining (one of whom lost his remaining $8,000,000$ chips and was eliminated). The prize structure for this tournament is as follows:

- 6th place and worse: $0
- 5th place: $2 million
- 4th place: $2.84 million
- 3rd place: $4 million
- 2nd place: $6 million
- 1st place: $10 million

Assuming each poker player’s utility function is equal to the amount of prize money received, explain qualitatively why it could be rational for Kaverman to go all in with the Ace of clubs and the five of clubs now that there are only five players but not go all in with a pair of eights two hands ago when there were six players, even though the expected change in chips is about the same and is positive.

2.2 Holz's First Decision

The next player, Fedor Holz, — who now knows Kaverman has one of those 138 hands — has a pair of tens.

1. If Kaverman has a two cards with the same value that are higher than a pair of tens, what is the probability that Holz nevertheless wins the hand by getting three or four of a kind? For simplicity, you can assume that everyone else folds, and ignore the slim possibility that Kaverman or Holz gets a straight, flush, or full house.
2. If Kaverman has a two cards with different values that are both higher than a ten, what is the probability that Holz wins the hand? Again, for simplicity you can assume that everyone else folds, and ignore the slim possibility that Kaverman or Holz gets a straight, flush, or full house.
3. Based on your answers to the previous two subquestions and additional considerations, explain why the announcer says at 0:54 that "Holz is obviously calling".

2.3 Holz's Second Decision

After Holz calls Kaverman's bet of just over 13 big blinds, the next two players (Justin Bonomo and Dan Smith) fold and then Rick Salomon goes all in for almost 46 big blinds, leaving Holz to decide whether to call for almost (but not quite) all of his remaining chips. As the announcer implies, given that both Kaverman and Holz have strong hands (or else they would have folded), the expected value of Salomon going all in is positive if and only if Salomon either has an Ace and a King or has a pair of Jacks, Queens, Kings, or Aces.

However, between 1:03 and 1:04 of the video, Salomon picks up the Ace of hearts with his right hand in such a way that everyone could see it. By rule, if someone's face down card becomes visible to anyone else, then it has to be turned face up. Given that Salomon has the Ace of hearts, Holz knows that Salomon's other card is either one of the three remaining Aces or one of the four Kings (In fact, Smith folded a hand with the Ace of spades and Bonomo folded a hand with the King of diamonds, but Holz does not know that so he cannot condition on it.)

Holz thinks for a while before eventually calling, at which point the video says that Holz has a 50% chance to win the hand, Salomon has a 35% chance, and Kaverman has a 15% chance.

1. What would their three chances of winning the hand be if Salomon instead had a pair of Aces?
2. From Holz's perspective, what is the probability that Salomon has a pair of Aces?
3. Based on your answer's to the previous two subquestions and additional considerations, was Holz's decision to call the second time a good decision or a bad decision. Why?

2.4 Independent Chip Model

The popular Independent Chip Model (ICM) assumes — late in a poker tournament — that the probability of player j finishing in first place (before the next hand is dealt) is equal to the proportion of chips that player j has. Let's assume that if Salomon had *not* exposed the Ace of hearts, then Holz would have folded and Salomon's two pair would have eliminated Kaverman, leaving the chip stacks at

```
stacks <- c(Salomon = 45000000, Holz = 20650000, Bonomo = 51150000, Smith = 19700000)
(prob <- stacks / sum(stacks))
```

```
##   Salomon      Holz    Bonomo     Smith
## 0.3296703 0.1512821 0.3747253 0.1443223
```

The ICM can then be used to compute the probability that player j finishes in second place, given that player $i \neq j$ finishes in first place by computing the proportion of chips among all players except the i-th, although there would be three different sets of proportions because any of the three players besides the j-th could finish in first place. And so on for the probability of finishing in third place or fourth place.

Use the ICM compute the expected amount of prize money Salomon would have won had he not exposed the Ace of hearts.

2.5 Perspectives on Probability

Explain how poker involves the classical, the frequentist, and the Bayesian perspectives on probability.