BI476: Biostatistics - Case Studies

Assignment 1: Math Fundamentals

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1 Calculus

1.1 For $\mathbf{y} = \mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{x}; \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{n\times n}$

(1) Prove that $\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{A}\mathbf{x}$

(2) Compute $\frac{\partial \mathbf{y}}{\partial \mathbf{x}^T}$

1.2 For $\mathbf{z} = \mathbf{x}^\mathbf{T}\mathbf{A}\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m\times n}$

(1) Compute $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$

(2) Compute $\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$

1.3 Linear regression

A linear regression problem $\mathbf{y} = \mathbf{X}\beta$ can be solved either using least squares or maximum likelihood. Can you write down the two objective functions and reach the normal equations.

2 Linear algebra

2.1 Trace

Here is definition of the trace for a square matrix $\mathbf{X}$:

$$\mathrm{tr}(X) = |X| = \sum_{i=1}^{n} x_{ii}$$

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ are two square matrices, and $c \in \mathbb{R}$. Prove that:

(1) $tr(A \pm B) = tr(A) + tr(B)$

(2) $tr(A^T) = tr(A)$

(3) $tr(cA) = c\text{tr}(A)$

(4) $tr(AB) = tr(BA)$

(5) $tr(AA^T) = tr(A^T A)$

2.2 Determinant

We know that the determinant of a square matrix $A \in \mathbb{R}^{n \times n}$ is defined by

$$\|A\| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \|M_{ij}\|, M_{ij} = A_{-i,-j}$$

Prove that

(1) If A is diagonal or triangular, then $\|A\| = \prod_{i=1}^{n} a_{ii}$

2.3 Spectral Decomposition

The spectral decomposition of a square matrix $A \in \mathbb{R}^{n \times n}$ can be defined as

$$\mathbf{A} = \mathbf{\Gamma \Lambda \Gamma^T}$$

where Γ are orthonormal matrix such that $\Gamma\Gamma^T = I$ and $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues of $\mathbf{A}$. Write down the form of $\mathbf{A}^m$.

2.4 Eigen-decomposition

(1) What are positive-definite (PD) matrix and positive semi-definite (PS) matrices?

(2) Can you use eigen-decomposition to determine if a square matrix is positive definite or not?

2.5 Other Matrix Decomposition Techniques

(1) What is QR-decomposition and Cholesky-decomposition?

(2) Write some comments on the applications of the above matrix decomposition techniques.

(3) `base::qr` and `base::chol` can be used to conduct the two decompositions. Give an example to illustrate the usage of the two decompositions?

3 Mathematical Optimization

4 Probability

4.1 Sufficient statistic

(1) What is sufficient statistic?

(2) How to prove that a statistic is a sufficient statistic for a parameter?

4.2 Negative binomial distribution

(1) What is negative-binomial distribution?

(2) Can you write down the expectation and variance of the distribution?

(3) Both Poisson and negative binomial distributions can be used to fit the counts. What are the differences between these two distributions?

4.3 Beta-Binomial model

```
y <- 13
N <- 35

a <- 1
b <- 1

theta <- seq(0, 1, len = 100)
likelihood <- theta ^ y * (1-theta)^(N-y) #dbinom(y, N
    , theta)
prior <- theta ^ (a-1) * (1-theta)^(b-1) #dbeta(theta,
     a, b)
posterior1 <- prior * likelihood
posterior2 <- dbeta(theta, y + a, N - y + b)

plot(theta, likelihood, ty='l')
abline(v = y/N, col="red")

par(mfrow=c(1,3))
plot(theta, prior, ty='l')
plot(theta, posterior1, ty='l')
plot(theta, posterior2, ty='l', col="blue")
```