Book of Prob

Contents

Introduction

This book is the combined lecture notes of my course IE-231 Introduction to Probability at Istanbul Bilgi University, Turkey. You can find the original lecture notes and archives here. As the name suggests, these notes consist of an abbreviated version of the course as the objective is to give the essence of very significant parts of Probability. The course mainly follows (Walpole et al., 2012).

This book is periodically updated. Latest update is on 2018-05-09.

About the Author

The author of this book is Berk Orbay. I am the co-founder of Algopoly, a data science firm specialized on large scale forecasting problems, and a part-time academic (yes, still writing papers). My PhD was about pricing financial options with multiple models. I taught or am currently teaching Introduction to Probability, Computational Finance, Business Analytics and Essentials of Data Analysis courses.

Acknowledgements

Special thanks section.

Chapter 1

Initial Concepts of Probability

- **Probability** is the quantification of event uncertainty. For instance, probability of getting (H)eads in a coin toss is 1/2. Deterministic models will give the same results given the same inputs (e.g. 2 times 2 is 4), but probabilistic models might yield different outcomes.
- An **experiment** is a process that generates data. For instance, tossing a coin is an experiment. **Outcome** is the realization of an experiment. Possible outcomes for a coin toss is Heads and Tails.
- **Sample space** ($\mathbb{S}$) is the collection of all the possible outcomes of an experiment. Sample space of the coin toss is $\mathbb{S} = \{H, T\}$. Sample space of two coin tosses experiment is $\mathbb{S} = \{HH, HT, TH, TT\}$. Sample space can be discrete (i.e. coin tosses) as well as continuous (i.e. All real numbers between 1 and 3. $\mathbb{S} = \{x|1 \leq x \leq 3, x \in \mathbb{R}\}$) *(Side note: Sample space is not always well defined.)*
- An **event** is a subset of sample space. While outcome represents a realization, event is an information. Probability of an event $P(A)$, say getting two Heads in two coin tosses is $P(A) = 1/4$.
- A **random variable** represents an event is dependent on a probabilistic process. On the other hand, a **deterministic variable** is either a constant or a decision variable. For instance, value of the dollar tomorrow can be considered a random variable but the amount I will invest is a decision variable (subject to no probabilistic process) and spot (current) price of the dollar is a constant.

1.1 Set Operations

- **Complement** of an event (A') with respect to the sample space represents all elements of the sample space that are not included by the event (A). For instance, complement of event $A = \{HH\}$ is $A' = \{HT, TH, TT\}$
- **Union** of two events A and B ($A \cup B$) is a set of events which contains all elements of the respective events. For example, say A is the set that contains events which double Heads occur ($A = \{HH, HT, TH\}$) and B is the set which Tails occur at least once ($B = \{TT, HT, TH\}$). The union is $A \cup B = \{HH, TH, HT, TT\}$.
- **Intersection** of two events A and B ($A \cap B$) contains the common elements of the events. For example, say A is the set that contains events which Heads occur at least once ($A = \{HH, HT, TH\}$) and B is the set which Tails occur at least once ($B = \{TT, HT, TH\}$). The intersection is $A \cap B = \{TH, HT\}$.
- **Mutually exclusive** or disjoint events mean that two events have empty intersection ($A \cap B = \emptyset$) and their union ($A \cup B$) contains the same amount of elements as the sum of their respective number of elements. Also $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$. For example getting double Heads (HH) and double Tails (TT) are mutually exclusive events.

1.2 Axioms of Probability

1. Any event A belonging to the sample space $A \in \mathbb{S}$ should have nonnegative probability ($P(A) \geq 0$).
2. Probability of the sample space is one ($P(\mathbb{S}) = 1$).
3. Any disjoint events ($A_i \cap A_j = \emptyset \ \forall_{i,j \in 1 \dots n}$) satisfies $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$.

1.3 Other Set and Probability Rules

- $(A')' = A$
- $S' = \emptyset$
- $\emptyset' = S$
- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- $(A \cup B) \cup C = A \cup (C \cup B)$
- $(A \cap B) \cap C = A \cap (C \cap B)$
- $A \cup A' = \mathbb{S}$ and $A \cap A' = \emptyset$ so $P(A) = 1 - P(A')$. This is especially useful for many problems. For example the probability of getting at least one Heads in a three coin tosses in a row is $1 - P(\{TTT\}) = 7/8$, the complement of no Heads in a three coin tosses in a row. Otherwise, you should calculate the following expression.

$$P(\{HTT\}) + P(\{THT\}) + P(\{TTH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\}) + P(\{HHH\}) = 7/8$$

- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

1.4 Counting

Counting rules will help us enumerate the sample space. It will include multiplication rule, permutation and combination.

1.4.1 Multiplication Rule

If I have a series of independent events, say 1 to k, and number of possible outcomes are denoted with n_1 to n_k; total number of outcomes in the sample space would be $n_1 n_2 \dots n_k$.

Take a series of coin tosses in a row. If I toss a coin its sample space consists of 2 elements such as $\{H, T\}$. If I toss 2 coins the sample space would be 2*2 $\{HH, HT, TH, TT\}$. If I toss 3 coins, the sample space would be 2*2*2 $\{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$.

A poker card consists of a type and a rank. There are four types of playing cards (clubs, diamonds, hearts and spades) and 13 ranks (A - 2 to 10 - J - Q - K). Number of cards in a deck is 4*13 = 52.

1.4.2 Permutation Rule

Permutation is the arrangement of all or a subset of items.

- Given a set of items, say $A = a, b, c$ in how many different ways I can order the elements? Answer is n!. In our case it is, $3! = 3.2.1 = 6$.

$$A = \{a, b, c\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}, \{a, c, b\}$$

- Suppose there are 10 (n) participants in a competition and 3 (r) medals (gold, silver and bronze). How many possible outcomes are there? Answer is $n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!} = \frac{10!}{(10-3)!} = 720$.

- If there are more than one same type items in a sample, then the permutation becomes $\frac{n!}{n_1!n_2!\dots n_k!}$ where $\sum n_i = n$.

  For example enumerate the different outcomes of four coin tosses which result in 2 heads and 2 tails. Answer is $\frac{4!}{2!2!} = 6$

$$A = \{HHTT, HTTH, HTHT, THTH, THHT, TTHH\}$$

1.4.3 Combination Rule

Suppose we want to select r items from n items and the order does not matter. So the number of different outcomes can be found using $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Out of 10 students how many different groups of 2 students can we generate? Answer $\frac{10!}{8!2!} = 45$

1.5 Chapter Problems

1. Suppose I toss a coin, roll a die and draw a card from the deck. How many different number of outcomes are there for this experiment?

   Solution: Multiplication rule. $n_1 n_2 n_3 = 2.6.52 = 624$.

2. In how many ways can I order the Teletubbies? (Tinky-Winky, Dipsy, Laa Laa and Po) For instance, (TW - Dipsy - Po - Laa Laa) is an ordering and (Dipsy - Po - TW - Laa Laa) is another.

   Solution: Permutation rule. $n! = 4! = 24$

```
## [1] 24
```

3. I want to reorder the letters of the phrase “GOODGRADES”. In how many ways can I do it?.

   Solution: Remember the permutation rule with identical items. There are two “G”s, two “D”s and two “O”s. Remember the formula $\frac{n!}{n_1!n_2!\dots n_k!}$. So the result should be $\frac{10!}{2!2!2!1!1!1!1!} = 453600$.

```
##
## A D E G O R S
## 1 2 1 2 2 1 1
```

```
## [1] 453600
```

4. I want to make two letter words from "GRADES" such as "GA", "ED" or "DE" (it doesn't have to make sense). Find the number of permutations.

   Solution: Permutation of r items from n items is $\frac{n!}{(n-r)!}$. So the result is $\frac{6!}{4!} = 30$.

   ```
   ## [1] 30
   ```

5. Suppose I am drawing a hand of 5 cards from a playing deck of 52 cards. How many different hands there can be? (Each card is different. See the bottom of this document for details.)

   Solution: Since in a hand you do not care for the order, it is the combination $\binom{52}{5} = \frac{52!}{(52-5)!5!} =$ 2598960.

   ```
   ## [1] 2598960
   ```

1.6 Extra Problems

- Question 1 Suppose we draw three cards from a deck and roll two dice. Answer the following questions.

  a) What is the experiment?
     The experiment is "drawing three cards from a deck and rolling two dice".
  b) What is "getting two-sixes and three-kings or five-one (in any order) one queen one king one ace"? Pick one (Event / Outcome / Sample Space)
     Event.
  c) Give an example of two mutually exclusive events. (6 pts)
     Event A: Queen of Hearts / Queen of Spades / Queen of Diamonds / 6 / 5 Event B: Ace of Clubs / King of Clubs / Queen of Clubs / 4 / 4
  d) What is the probability of getting four-three (in any order) in dice roll and three queens in card draw?

     ```
     ## [1] 1.00553e-05
     ```

  e) How many different outcomes can there be? This time assume ordering is important (e.g. 6-1 and 1-6 are different outcomes).

     ```
     ## [1] 4773600
     ```

- Question 2 In how many ways can you arrange the letters of "HOUSEPARTY"?

  a) Any order.

     ```
     ## [1] 3628800
     ```

  b) Vowels together?

     ```
     ## [1] 120960
     ```

  c) Vowels in alphabetical order?

     ```
     ## [1] 151200
     ```

  d) There should be no consecutive vowels?

     ```
     ## [1] 604800
     ```

- Question 3 In how many ways can you arrange the letters of "CAMARADERIE"?

  a) Any order.

  ```
  ## [1] 1663200
  ```

  b) Vowels together?

  ```
  ## [1] 21600
  ```

  c) Vowels in alphabetical order?

  ```
  ## [1] 27720
  ```

  d) There should be no consecutive vowels?

  ```
  ## [1] 3600
  ```

- Question 4

  Suppose you are putting the top 12 basketball teams in 4 groups evenly (each group should consist of 3 teams). In how many different ways can you arrange the teams?

  ```
  ## [1] 369600
  ```

- Question 5 (20 pts - all equal)

  There are 18 people; 10 from Izmir, 8 from Mugla.

  a) Suppose you want to form a group of 5 people with at least 1 person from Izmir and Mugla. In how many ways can you form such a group?

  ```
  ## [1] 8260
  ```

  b) In how many ways can you form a group of 3 people from Izmir and 4 people from Mugla?

  ```
  ## [1] 8400
  ```

1.7 Coins, Dice and Cards

When questions mention about coins, dice and cards they are commonly referred items. Nevertheless, you can refer to .

- Coin tosses: Two possible outcomes. Heads or Tails.
- Dice rolling: Six possible outcomes. 1-2-3-4-5-6.
- Card drawing: 52 possible outcomes. There are 4 types (clubs, diamonds, spades and hearts) and 13 ranks for each type. (A)ce - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - (J)ack - (Q)ueen - (K)ing.

Chapter 2

Conditional Probability and Bayes' Rule

Let's recall some probability concepts.

- Probability is the quantification of uncertainty. For instance in a coin toss, $P(H) = 1/2$.
- Set of the union of all possible events is the sample space. For instance single coin toss sample space is $S = \{H, T\}$.
- Probability of an event is actually the relative "size" of the event to the sample space.
- In an experiment, that has equally likely outcomes, number of potential outcomes in an event relative to the total number of outcomes in the sample space corresponds to the probability of that event.
- Some events are mutually exclusive and some events are dependent.
- Sample space is not always easy to measure and is not always finite. Counting is one way of estimating a sample space (especially discrete ones).
- Estimation of probability of events is not always easy. One way to estimate is to repeat the experiment for a number of times. As we increase the number of repetitions, the probability will become more stable. For instance suppose we don't know the probability of getting (H)eads in a coin toss and we repeat coin tossing for a number of times.

```
## [1] 0.6
```

```
## [1] 0.57
```

```
## [1] 0.505
```

```
## [1] 0.4989
```

2.1 Some Examples

1. In a sports bar, there are 10 football fans, 15 basketball fans, 12 tennis fans and 6 curling fans. Suppose there is a single TV and there are matches of football, basketball, tennis and curling at the same time. If the TV remote is handed to one of them randomly, what is the probability that curling will be the show on TV?

   Answer: There are a total of 43 fans in the sports bar and 6 curling fans. P(Curling) = 6/43.

2. Suppose there is a group of 20 people; 10 from Ankara, 10 from Istanbul. Suppose we randomly choose 5 people from that group.

   a. What is the probability that all of this subgroup is from Ankara?

      Answer: (# of combinations including people only from Ankara) / (# of total combinations)

      $$\frac{\binom{10}{5}}{\binom{20}{5}} = 0.01625387$$

      Alternative answer is: $(10/20) * (9/19) \cdots * (6/16) = 0.01625387$.

   b. What is the probability that at least two from Ankara and at least two from Istanbul?

      Answer: P(Ankara = 2, Istanbul = 3 OR Ankara = 3, Istanbul = 2) = (P(Ankara = 2, Istanbul = 3) + P(Ankara = 3, Istanbul = 2)) = ((# of comb. Ank 2, Ist 3) + (# of comb. Ank 3, Ist 2))/(# of total combinations)

      $$\frac{\binom{10}{2} * \binom{10}{3} + \binom{10}{3} * \binom{10}{2}}{\binom{20}{5}} = 0.6965944$$

3. ("Blackjack") Suppose you are drawing two cards from the deck. Assume all number rank cards (2-10) are points, Ace (A) is 11 points and (J)ack, (Q)ueen and (K)ing are 10 points. What is the probability that total points of these two cards exceed 20.

   Answer: Max score for two cards is 22 (two Aces). To exceed 20, one needs Ace-9 pairs or any pair of 10-Ace-J-Q-K cards. We know that there are 4 of each rank. So $P(Score \geq 20) = P(Aand9) + P(any\ two\ of\ 10 - Ace - J - Q - K)$. We know that total number of hands is $\binom{52}{2} = 1326$. So # of combinations with either A-9 or 9-A is $4 * 4 = 16$. # of combinations of getting two cards from 10-Ace-J-Q-K ranks is $\binom{20}{2} = 190$. So $P(Score \geq 20) = \dfrac{190 + 16}{1326} = 0.1553544$

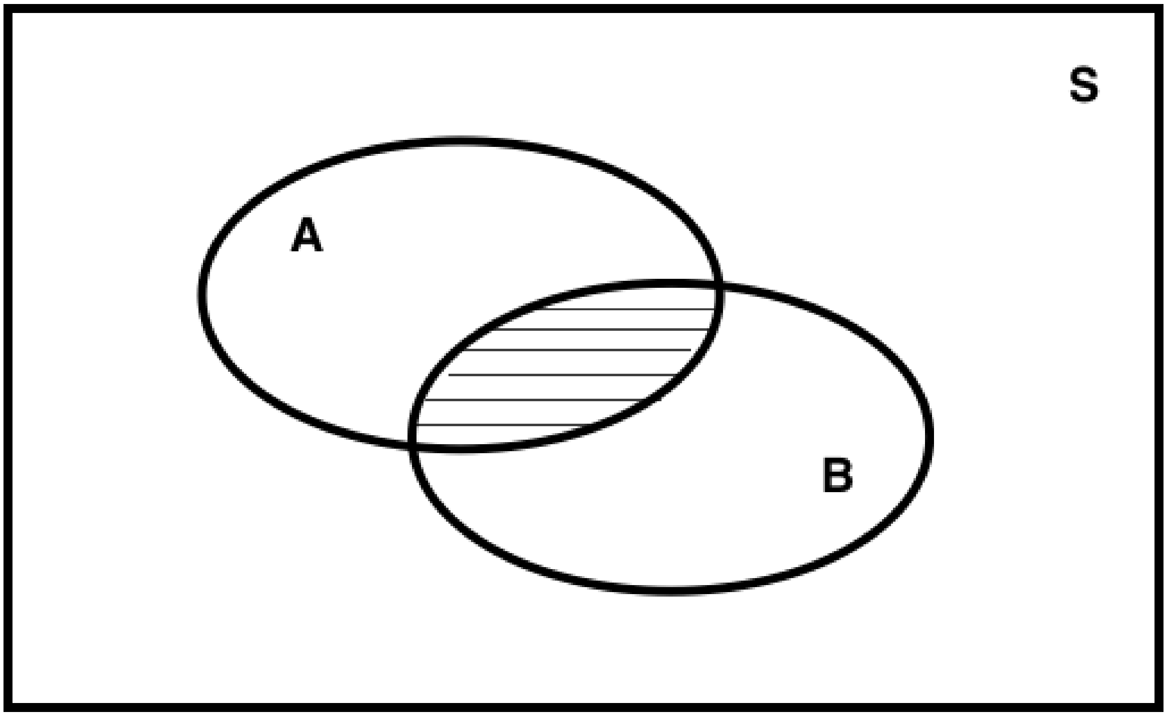
   Alternative answer: $P(A - 9or9 - A) = 4/52 * 4/51 + 4/52 * 4/51 = 0.01206637$ and $P(anytwoof10 - Ace - J - Q - K) = 20/52 * 19/51 = 0.1432881$. Total probability is $P(Score \geq 20) = 0.01206637 + 0.1432881 = 0.1553545$

4. Suppose you are the gatekeeper of a playing ground for 10 year old boys and try to assess their age by their height. Height of 90% of boys aged 10 falls between 130 and 150 cms. It is equally likely that a boy aged 10 has the height lower than 130 cm or higher than 150 cm. What is the probability of rejecting a child because he's too long?

   Answer: $P(130 \leq X \leq 150) = 0.9$ and $P(X \leq 130) = P(X \geq 150)$. So $P(X \geq 150) = 1 - P(130 \leq X \leq 150) - P(X \leq 130)$. $2.P(X \geq 150) = 0.1$ and $P(X \geq 150) = 0.05$.

2.2 Conditional Probability

Definition: $P(A|B)$ means that probability of event A given that event B already occured or probability of A **conditional on** B.



Example: What is the probability of getting two heads (HH) if we know that the first coin is H? (1/2)

Example: Suppose we are rolling two dice.

a. What is the probability of getting the sum equal to 8?
   $P(Sum = 8) = P(D_1 = 2, D_2 = 6) + P(3, 5) + P(4, 4) * 2 + P(5, 3) + P(6, 2) = 6/36 = 1/6$
b. What is the probability of getting the sum more than or equal to 8 given the first roll is 3?
   $P(Sum \geq 8|D_1 = 3) = P(3, 5) + P(3, 6) = 2/6 = 1/3$

Conditional probability of event A given B, can be defined as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) > 0$.

Example: We asked 1000 people, 400 men and 600 women: 'Where do you want to visit this summer? Amsterdam or London?'. The answers are as follows.

```
##       Amsterdam London
## Men         150    250
## Women       450    150
```

a. What is the probability of a randomly chosen person is a woman?
   Answer: $P(W) = (n(W|L) + n(W|P))/n(S) = (150 + 450)/1000 = 0.6$
b. What is the probability of a randomly chosen person chooses to to visit Amsterdam?
   Answer: $P(A) = (n(A|W) + n(A|M))/n(S) = (450 + 150)/1000 = 0.6$
c. What is the probability of a London visitor is a man?
   Answer: $P(M|L) = \dfrac{P(M \cap L)}{P(L)} = \dfrac{n(M \cap L)/n(S)}{n(L)/n(S)} = 250/400 = 0.625$

2.3 Independence

Two events can be independent. It means $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

If the events (A_i) are independent then $P(A_1 \cap A_2 \ldots A_k) = P(A_1)P(A_2)\ldots P(A_k)$.

Example: Suppose you rolled a die 5 times and they all ended up 4. What is the probability that, at the 6th time, the die will get 4 again?

Solution: Die rolls are independent events. So outcome of a roll does not affect the outcome of the other rolls. It is 1/6.

2.4 Product rule

If both $P(A) > 0$ and $P(B) > 0$, then $P(A \cap B) = P(A)P(B|A)$. We can also say $P(A \cap B) = P(B \cap A) = P(B)P(A|B)$

Example: You draw two cards from a deck. What is the probability that you get two (A)ces?

Solution: $P(A \cap B) = 4/52 * 3/52 = 0.00443787$

Example: You are at a tea shop. There are 10 (W)hite teas, 12 (B)lack teas and 8 (G)reen teas on the menu. According to the waiter all teas are equally tasty. You ask the waiter bring a tea but it should not be green tea. What is the probability that you will get a white tea?

Solution: $P(W|G') = P(W \cap G')/P(G') = P(W)/P(G') = \dfrac{10/30}{22/30} = 10/22$

2.5 Bayes' Rule

Theorem of total probability: Given the events B_i are collective parts of the sample space and $P(B_i) > 0$, then for any event A, $P(A) > 0$

$$P(A) = \sum_i^k P(B_i \cap A) = \sum_i^k P(B_i)P(A|B_i)$$

Bayes'rule: Given the events B_i are collective parts of the sample space and $P(B_i) > 0$, then for any event A, $P(A) > 0$

$$P(B_r|A) = \frac{P(B_r|A)}{\sum_i^k P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_i^k P(B_i)P(A|B_i)}$$

Example 2.41 from the book: In an assembly plant, there are three machines B_1, B_2 and B_3. These machines make the 30%, 45% and 25% of the products. 2%, 3% and 2% of their output is known to be defective. So, what is the probability of getting a defective product at a random time?

Solution:

Say, event of getting a defective item is A and we want to know $P(A)$. $P(B_i)$ is the probability of a product being manufactured by machine i. $P(B_1) = 0.3$, $P(B_2) = 0.45$, $P(B_3) = 0.25$. The defect probabilities given the machine are $P(A|B_1) = 0.02$, $P(A|B_1) = 0.03$, $P(A|B_1) = 0.02$.

So, $P(A) = \sum_{i=1}^3 P(B_i)P(A|B_i) = 0.3 * 0.02 + 0.45 * 0.03 + 0.25 * 0.02 = 0.0245$.

Example 2.42: If the product is defective, what is the probability that it came from machine 3?

$$P(B_3|A) = \frac{P(B_3 \cap A)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{P(B_3)P(A|B_3)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{0.25 * 0.02}{0.0245} = 10/49$$

Chapter 3

Random Variables and Distributions

3.1 Scales on Measurement

- Nominal scale: These are categorical values that has no relationship of order or rank among them. (e.g. colors, species)
- Ordinal scale: These are categorical values that has relationship of order or rank among them (e.g. military ranks, competiton results). Though the relative order has no defined magnitude (e.g. Champion can get 40 points, runner up 39 and third place 30).
- Interval scale: There is a numerical order but the difference can only be defined in intervals, *since there is no absolute minimum.* We cannot compare in relative values. For instance, we cannot say 10 degree celsius is twice as hot as 5 degree celsius; what about -5 vs +5?
- Ratio scale: Scale with an absolute minimum. (e.g. If I have 50TL and my friend has 100TL, I can say that she has twice the money that I have.) Height, weight, age are similar examples.

See more on https://en.wikipedia.org/wiki/Level_of_measurement. (p.s. Wikipedia wasn't banned when I prepared these notes)

3.2 Infinity

The concept of infinity is very broad. Currently, you just need to keep the distinction of countable and uncountable infinities in mind.

- Countably infinite: 1, 2, 3, 4, … (e.g. natural numbers, integers, **rational numbers**)
- Uncountably infinite: 1, 1.01, 1.001, 1.0001, 1.00001, … (e.g. real numbers)

How many real numbers are there between 0 and 1?

3.3 Descriptive Statistics

Here are brief descriptions of mean (expectation), median, mode, variance, standard deviation, quantile.

- Mean: $\bar{X} = \sum_i^N X_i$

- Median: Let's say X_k are ordered from smallest to largest and there are n values in the sample. Median(X)= $X_{(n+1)/2}$ if n is odd and (usually) Median(X)= $\dfrac{X_{(n/2)} + X_{(n/2+1)}}{2}$.
- Quantile: On an ordered list of values for quantile (α) provides the $(\alpha * n)^{th}$ smallest value of the list. For instance, if $\alpha = 70\% = 0.7$ quantile value is the 7th smallest value in a list of 10 values. $\alpha = 1$ means the maximum. Quantile is an important parameter in especially statistics.
- Mode: X_k with the highest frequency in the sample. In a sample of $(1, 2, 2, 3, 4, 5)$, 2 is the mode.
- Variance: $V(X) = \dfrac{\sum_i^N (X_i - \bar{X})^2}{n-1}$
- Standard Deviation: $\sigma(X) = \sqrt{\dfrac{\sum_i^N (X_i - \bar{X})^2}{n-1}}$

```
##  [1]  1  9 15 16 16 18 26 31 32 35
```

```
## [1] 19.9
```

```
## [1] 19.9
```

```
## [1] 17
```

```
## 77.77778%
##        31
```

```
## 0%
##  1
```

```
## 100%
##   35
```

```
## numbers
##  1  9 15 16 18 26 31 32 35
##  1  1  1  2  1  1  1  1  1
```

```
## [1] "16"
```

```
## [1] 118.7667
```

```
## [1] 118.7667
```

```
## [1] 10.89801
```

```
## [1] 10.89801
```

3.4 Random Variables

A random variable (usually defined with a capital letter or symbol i.e. X) is a quantity determined by the outcome of the experiment. Its realizations are usually symbolized with lowercase letter (x).

Example: Suppose there are 10 balls in an urn, 5 black and 5 red. Two balls are randomly drawn from the urn without replacement. Define the random variable X as the number of black balls. Then, $X = x$ can get the values of 0, 1 and 2. Let's enumerate $P(X = x)$.

$$P(X = 0) = P(RR) = 5/10 * 4/9 = 2/9$$

$$P(X = 1) = P(BR) + P(RB) = 5/10 + 5/9 + 5/10 * 5/9 = 5/9$$

$$P(X = 2) = P(BB) = 5/10 * 4/9 = 2/9$$

3.5 Discrete Random Variables and Distributions

If a sample space has finite number of possibilities or countably infinite number of elements, it is called a discrete sample space. Discrete random variable probabilities are shown as point probabilities $P(X = x)$. The probability distribution of discrete random variables is also called probability mass function (pmf).

$$P(X = x) = f(x)$$

$$\sum_x f(x) = 1$$

$$f(x) \geq 0$$

Example: (Same as above) Enumerate the probability distribution.

Solution: Random variable X can take values (x) 0, 1 and 2. So $f(1) = 2/9$, $f(2) = 5/9$ and $f(3) = 2/9$.

3.5.1 Cumulative Distribution Function (CDF)

Cumulative distribution function is a special defined function yielding the cumulative probability of random variables up to a value. It is usually symbolised as $F(x)$

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

$$F(x) = P(X \leq x) = \sum_{\infty} f(t) = 1$$

Example: (Same as above) Enumerate the cdf.

$$F(0) = P(X \leq 0) = P(X = 0) = 2/9$$

$$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 7/9$$

$$F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 1$$

3.6 Continuous Random Variables and Distributions

If a sample space has uncountably infinite number of possibilities, it is called a continuous sample space. Continuous random variables' probabilities are defined in intervals $P(a < X < b)$. The probability distribution of continuous random variables is also called probability density function (pdf).

$$f(x) \geq 0$$
$$P(a < X < b) = f(x) = \int_a^b f(x)dx$$
$$\int_{-\infty}^{\infty} f(x) = 1$$

Example (from the book): Suppose the probability function of a continuous distribution $f(x) = x^2/3$ defined between $-1 < x < 2$ and 0 everywhere else. Verify that it is a density function (i.e. the integral in the defined interval is 1) and calculate $P(0 < x < 1)$.

a. $\int_{-1}^{2} x^2/3dx = x^3/9|_{-1}^{2} = 8/9 - (-1/9) = 1$. Verified.
b. $\int_0^1 x^2/3dx = x^3/9|_0^1 = 1/9 - (0) = 1/9$. Verified.

Cumulative distribution function (CDF) for continuous random variables is defined with the integral.

$$F(x) = P(X < x) = \int_{-\infty}^{a} f(x)dx$$

Example: (same as above) Calculate the cdf $F(3/2)$

Solution: $F(1.5) = \int_{-\infty}^{3/2} f(x) = x^3/9|_{-1}^{3/2} = 3/8 - (-1/9) = 35/72$

Example: Calculate $P(X > 1)$.

Solution: $P(X > 1) = 1 - P(X < 1) = 1 - F(1) = 1 - \int_{-\infty}^{1} = 1 - ((1/9) - (-1/9)) = 7/9$.

3.7 Joint Distribution

So far we had distributions with only one random variable. What if we had more than one random variable in a distribution? It is not that different from univariate distributions.

$$f(x,y) \geq 0$$
$$\sum_x \sum_y f(x,y) = 1$$
$$P(X = x, Y = y) = f(x,y)$$

Example (from the book): Two pens are selected at random from a box of 3 blue, 2 red and 3 green pens. Define X as the number of blue pens and Y as the red pens selected. Find

a. Joint probability function $f(x,y)$
b. $P[(X,Y)\ inA]$ where A is the region $\{(x,y)|x + y \leq 1\}$.

Solution:

a. The possible cases for (x, y) are $(0,0),(0,1),(0,2),(1,0),(2,0),(1,1)$. For instance $(0,1)$ is one green and one red pen selected. There are a total of 8 pens. Then sample space size for two pens selected is $\binom{8}{2} = 28$. There are $\binom{2}{1}\binom{3}{1} = 6$ ways of selecting 2 pens from green and red pens. So the probability is $f(0,1) = P(X = 0, Y = 1) = 6/28 = 3/14$. It is possible to calculate other possible outcomes in a similar way. A generalized formula would be as follows.

$$\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}$$

b. Possible outcomes satisfying $A = (x, y) \leq 1$ are $(0,0),(0,1),(1,0)$. So $P(X + Y \leq 1) = P(0,0) + P(0,1) + P(1,0) = 9/14$.

In the continuous case it is similar. It is now called joint probability density function.

$$f(x, y) \geq 0$$

$$\int_x \int_y f(x, y)dxdy = 1$$

$$P(X = x, Y = y) \in A = \int \int_A f(x, y)dxdy$$

Example: (from the book)

A privately owned business operates both a drive in and a walk in facility. Define X and Y as the proportions of using the drive in and walk in facilities. Suppose the joint density function is $f(x, y) = 2/5 * (2x + 3y)$ where $0 \leq x \leq 1$ and $0 \leq y \leq 1$ (0, otherwise).

a. Verify it is a distribution function.
b. Find $P[(X, Y)] \in A$, where $A = \{(x, y)|0 < x < 1/2, 1/4 < y < 1/2\}$.

Solution:

a. $\int \int f(x, y)dxdy = \int 2/5 * (x^2 + 3xy)dy|_0^1 = \int (2/5 + 6/5 * y)dy = 2y/5 + 3/5 * y^2|_0^1 = 2/5 + 3/5 = 1$.

b. $\int_{1/4}^{1/2} \int_0^{1/2} f(x, y)dxdy = \int 2/5 * (x^2 + 3xy)dy|_0^{1/2} = \int_{1/4}^{1/2}(1/10 + 3/5 * y)dy$, $y/10 + 3y^2/10|_{1/4}^{1/2} = 13/160$.

3.8 Marginal Distribution

In a joint distribution, marginal distribution is the probability distributions of individual random variables. Define $g(x)$ and $h(y)$ as the marginal distributions of X and Y.

$$g(x) = \sum_y f(x, y), h(y) = \sum_y f(x, y)$$

$$g(x) = \int_y f(x, y)dy, h(y) = \int_x f(x, y)dx$$

Example: Go back to pen and walk in examples and calculate marginal probabilities.

3.9 Conditional Distribution

Remember the conditional probability rule $P(A|B) = P(A \cap B)/P(B)$ given $P(B) > 0$. We can define conditional distribution as $f(y|x) = f(x,y)/g(x)$, provided $g(x) > 0$ whether they are discrete or continuous.

Example: (pen example) Calculate $P(X = 0|Y = 1)$

Solution: We know that $h(1) = 3/7$. $f(x = 0, y = 1) = 3/14$. $f(x = 0|y = 1) = f(x = 0, y = 1)/h(1) = 1/2$

3.10 Statistical Independence

Two random variables distributions are statistically independent if and only if $f(x,y) = g(x)h(y)$.

Proof:

$$f(x,y) = f(x|y)h(y)$$
$$g(x) = \int f(x,y)dy = \int f(x|y)h(y)dy$$

If $f(x|y)$ does not depend on y we can write $f(x|y)\int h(y)dy$. $f(x|y) * 1$. Therefore, $g(x) = f(x|y)$ and $f(x,y) = g(x)h(y)$.

Any number of random variables $(X_1 \dots X_n)$ are statistically independent if and only if $f(x_1, \dots, x_n) = f(x_1) \dots f(x_n)$.

Chapter 4

Expectation and Variance

This chapter will be improved

4.1 Mathematical Expectation

$$E[X] = \sum xf(x)$$
$$E[X] = \int xf(x)dx$$

4.2 Variance

$$V(X) = E[X^2] - E[X]^2$$

Chapter 5

Some Discrete Distributions

5.1 Bernoulli Distribution

It can also be called “single coin toss distribution”. For a single event with probability of success p and failure $q = 1 - p$, the distribution is called Bernoulli.

- pmf: $f(x = 0; p) = q,\ f(x = 1) = p$
- $E[X] = 0 * (1 - p) + 1 * p = p$
- $V[X] = pq$

Example: Coin Toss

- $p = 0.5,\ q = 1 - p = 0.5$
- pmf: $f(x = 0) = 0.5,\ f(x = 1) = 0.5$
- $E[X] = 0 * (1 - 0.5) + 1 * 0.5 = 0.5$
- $V(X) = 0.5 * 0.5 = 0.25$

5.2 Binomial Distribution

Think of multiple Bernoulli trials (e.g. several coin tosses).

- pmf: $f(x; p, n) = \binom{n}{x} p^x q^{(n-x)}$
- $E[X] = np$
- $V(X) = npq$
- cdf: $F(X \leq x) = \sum_{i=0}^{n} f(i)$

Example: Multiple Coin Tosses (x5 coins, p = 0.5)

- pmf: $f(x = 3; n = 5) = \binom{5}{3}(0.5)^3(1 - 0.5)^{(5-3)} = 0.3125$

```
## [1] 0.3125
```

- $E[X] = 5 * 0.5 = 2.5$
- $V(X) = 5 * 0.5 * 0.5 = 1.25$
- cdf: $F(X \leq 3; n = 5) = \sum_{i=0}^{5} f(i) = 0.8125$

```
## [1] 0.8125
```

5.3 Multinomial Distribution

Now suppose there is not one probability (p) but there are many probabilities ($p_1, p_2, \ldots, p_k$).

- pmf: $f(x_1, \ldots, x_k; p_1, \ldots, p_k; n) = \binom{n}{x_1, \ldots, x_k} p_1^{x_1} * \cdots * p_k^{x_k}$
  where $\binom{n}{x_1, \ldots, x_k} = \dfrac{n!}{x_1! \ldots x_k!}$, $\sum_i^k x_i = n$ and $\sum_i^k p_i = 1$.

Example: Customers of a coffee shop prefer Turkish coffee with probability 0.4, espresso 0.25 and filter coffee 0.35. What is the probability that out of the first 10 customers, 3 will prefer Turkish coffee, 5 will prefer espresso and 2 will prefer filter coffee?

$f(3, 5, 2; 0.4, 0.25, 0.35; 10) = \binom{10}{3,5,2} * 0.4^3 * 0.25^5 * 0.35^1 0 = 4.3 * 10^{-6} = 0.0193$

```
## [1] 0.01929375
```

```
## [1] 0.01929375
```

Binomial distribution is a special case of multinomial distribution.

5.4 Hypergeometric Distribution

Hypergeometric distribution can be used in case the sample is divided in two such as defective/nondefective, white/black, Ankara/Istanbul. Suppose there are a total of N items, k of them are from group 1 and $N - k$ of them are from group 2. We want to know the probability of getting x items from group 1 and $n - k$ items from group 2.

- pmf: $f(x, n; k, N) = \dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$
- $E[X] = \dfrac{nk}{N}$
- $V[X] = \dfrac{N-n}{N-1} * n * \dfrac{k}{N} * (1 - \dfrac{k}{N})$

Example: Suppose we have a group of 20 people, 12 from Istanbul and 8 from Ankara. If we randomly select 5 people from it what is the probability that 1 of them is from Ankara and 4 of them from Istanbul.

$f(1, 5; 8, 20) = \dfrac{\binom{8}{1}\binom{20-8}{5-1}}{\binom{20}{5}} = 0.256$

```
## [1] 0.255418
```

```
## [1] 0.255418
```

5.5 Negative Binomial Distribution

Negative Binomial distribution answers the question "What is the probability that k-th success occurs in n trials?". Differently from the binomial case, we fix the last attempt as success.

- pmf: $f(x;p,n) = \binom{n-1}{x-1}p^x q^{(n-x)}$

Example: Suppose I'm repeatedly tossing coins. What is the probability that 3rd Heads come in the 5th toss?
$f(3;0.5,5) = \binom{5-1}{3-1}0.5^3 0.5^{(5-3)} = 0.1875$

```
## [1] 0.1875
```

```
## [1] 0.1875
```

```
## [1] 0.1875
```

5.6 Geometric Distribution

Geometric distribution answers "What is the probability that first success comes in the n-th trial?"

- pmf: $f(x;p,n) = q^{(n-1)}p$
- $E[X] = 1/p$
- $V[X] = \dfrac{1-p}{p^2}$

5.7 Poisson Distribution

Poisson distribution is widely used to represent occurences in an interval, mostly time but sometimes area. Examples include arrivals to queues in a day, number of breakdowns in a machine in a year, typos in a letter, oil reserve in a region.

5.7.1 Binomial Approximation to Poisson Distribution

We know from binomial distribution that k occurences in n trials with probability p has the following function.

$$P\{X=k\} = \binom{n}{k}p^k(1-p)^{n-k} = \frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}$$

and expected value is $E[X] = np$. Now define $\lambda = np$.

$$\begin{aligned}
P\{X=k\} =& \frac{n!}{(n-k)!k!}\left(\frac{\lambda}{n}\right)^k\left(1-\left(\frac{\lambda}{n}\right)\right)^{n-k} \\
=& \frac{n(n-1)\dots(n-k+1)}{n^k}\left(\frac{\lambda^k}{k!}\right)\frac{(1-\dfrac{\lambda}{n})^n}{(1-\dfrac{\lambda}{n})^k}
\end{aligned}$$

For very large n and very small p the resulting pdf becomes $\dfrac{\lambda^k e^{-\lambda}}{k!}$.

5.7.2 Properties of Poisson Distribution

- PMF: $P\{X = k\} = \dfrac{\lambda^k e^{-\lambda}}{k!}$
- CDF: $P\{X \leq k\} = \sum_{i=0}^{k} \dfrac{\lambda^i e^{-\lambda}}{i!}$
- $E[X] = \lambda (due to \sum_{i=0}^{\infty} \dfrac{x^i}{i!} = e^x)$
- $V(X) = \lambda$

Rate parameter λ can also be defined as λt, t being the scale parameter. For instance, let arrivals in 30 minutes interval be $\lambda t_{30} = 4$. If we would want to work on hourly intervals, we should simply rescale, $\lambda t_{60} = 8$.

5.8 Chapter Questions

1. A boutique shop offers three types of breads; olive, rye and white. 15% of its customers buy olive bread, 55% rye bread and the rest white bread. What is the probability that the first 10 customers of the shop buys 2 olive, 5 rye and 3 white bread?

   Solution: Multinomial distribution.

$$\binom{10}{2,5,3} 0.15^2 * 0.55^5 * 0.3^3 = 0.0770478$$

```
## [1] 0.0770478
```

2. An urn contains 32 balls; 18 red, 14 white. If I take 8 balls randomly from the urn, what is the probability of getting 4 red balls?

   Solution: Hypergeometric distribution.

$$\frac{\binom{18}{4}\binom{14}{4}}{\binom{32}{8}} = 0.2912125$$

```
## [1] 0.2912125
```

3. A UNICEF activist asks bypassers to contribute to their efforts. Each of her attempts has a 10% chance of success. What is the probability that she got the first contribution at the 10th attempt?

   Solution: Geometric distribution.

$$(0.9)^9 * (0.1) = 0.03874205$$

```
## [1] 0.03874205
```

4. Suppose there are 10 cards in a deck numbered from 1 to 10. If I draw four cards out of the deck without putting them back, what is the probability that they are in increasing order (e.g. 3-4-5-8)?

   Solution: Increasing order is a special order. There is only one decreasing order in every permutation set. For instance

$$8-5-4-3$$
$$8-5-3-4$$
$$8-4-5-3$$
$$4-5-3-8$$
$$\dots$$
$$**3-4-5-8**$$

   So, the answer is $1/(4!) = 1/24$.

5. A tennis player has the probability of 0.15 to ace a serve (i.e. he serves and gets the point without the opponent touching the ball) at each shot. His trainer challenged him that he will pay the square of the aces he makes out of 10 serves (e.g. if he serves 8 aces out of 10, the trainer pays him $8^2 = 64TL$). But, if he serves less than or equal to 6 aces, the player should pay the trainer 5 TL for each ace he missed (e.g. 4 aces 6 misses, player pays up 6*5 = 30TL).

   a) What is the maximum amount of money he can get?
   b) What is the expected earnings of the player?
   c) What is the variance of his earnings?

   Solution:

   a) 10 out of 10 means 10^2 = 100.
   b) $g(X=x) = x^2$ if $x \geq 7$, $-x$ else, $f(g(X)=x^2) = f(X=x) = \binom{10}{x} 0.15^x * 0.85^(10-k)$, so $E[g(x)] = \sum_{i=0}^{1} 0 g(x) f(x)$.
   So $49 * \binom{10}{7} 0.15^7 * 0.85^3 + 64 * \binom{10}{8} 0.15^8 * 0.85^2 + 81 * \binom{10}{9} 0.15^9 * 0.85 + 100 * (0.15)^10 - 20 * \binom{10}{6} 0.15^6 * 0.85^4 - 25 * \binom{10}{5} 0.15^5 * 0.85^5 - 30 * \binom{10}{4} 0.15^4 * 0.85^6 - 35 * \binom{10}{3} 0.15^3 * 0.85^7 - 40 * \binom{10}{2} 0.15^2 * 0.85^8 - 45 * \binom{10}{1} 0.15^1 * 0.85^9 - 50 * 0.85^10 = -42.5TL$.

```
## [1] -42.4913
```

   c) $V(X) = E[X^2] - (E[X])^2$. We know that $E[X] = -42.5$, then $(E[X])^2 = 1805.51$. Also $E[X^2] = 1838.434$ and $V(X) = 32.92$.

```
## [1] 32.92423
```

6. Suppose a machine has a probability of failure 0.001 per hour. What is the probability that the machine had failed at least three times within 2000 hours.

   Binomial solution

$$\begin{aligned} P\{X \geq 3\} &= 1 - \binom{2000}{0} 0.001^0 0.999^{750} - \binom{2000}{1} 0.001^1 0.999^{749} - \binom{2000}{2} 0.001^2 0.999^{748} \\ &= 0.3233236 \end{aligned}$$

```
## [1] 0.3233236
```

Poisson solution

$$\begin{aligned}\lambda &= np = 2000 * 0.001 = 2\\ P\{X \geq 3\} &= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} - \frac{e^{-2}2^2}{2!}\\ &= 0.3233236\end{aligned}$$

```
## [1] 0.3233236
```

7. People arrive at a bank with rate $\lambda = 5$ every 10 minutes. What is the probability that 10 people arrive in 30 minutes?

$$\lambda t_{10} = 5$$

$$\lambda' = \lambda t_{30} = 15$$

$$P\{X = 10, t = 30\} = \frac{e^{-15}15^{10}}{10!} = 0.049$$

```
## [1] 0.04861075
```

8. A machine breaks down with a poisson rate of $\lambda = 10$ per year. A new method is tried to reduce the failure rate to $\lambda = 3$, but there is a 50% chance that it won't work. If the method is tried and the machine fails only 3 times that year, what is the probability that the method worked on the machine?

$$P\{Works|X = 3\} = \frac{P\{WorksandX = 3\}}{P\{X = 3\}}$$

$$P\{Works\} = 0.5$$

$$P\{WorksandX = 3\} = 0.5 * \frac{e^{-3}3^3}{3!} = 0.1120209$$

$$\begin{aligned}P\{X = 3\} &= P\{WorksandX = 3\} + P\{Doesn't\ WorkandX = 3\} = 0.5 * \frac{e^{-3}3^3}{3!} + 0.5 * \frac{e^{-10}10^3}{3!}\\ &= 0.96733\end{aligned}$$

```
## [1] 0.96733
```

Chapter 6

Exercise Questions

Question 1

How many ways are there to arrange "ECONOMETRICS".

Solution: There 12 characters, 5 vowels 7 consonants. 2 Cs, 2 Es and 2 Os.

a) In any order.

Solution: $\frac{12!}{2!2!2!}$.

b) Vowels together.

Solution: Add one representative letter to consonants to denote vowels as a single letter. $\frac{8!}{2!}\frac{5!}{2!2!}$.

c) No consecutive vowels.

Solution: $\frac{7!}{2!}\frac{8!}{(8-5)!2!2!}$

Question 2

7 people from Istanbul (suppose names A-B-C-D-E-F-G) and 7 people from Ankara (M-N-O-P-Q-R-S) will sit around a round table.

a) In how many different ways can they sit around the table?

Solution: $(14-1)! = 13!$

b) Same as (a) but no two people from Ankara should sit together.

Solution: Fix one from Istanbul to a position. $(7-1)!(7)!$

c) Same as (a) but all the people from Istanbul should sit together.

Solution: Suppose there are 8 people from Ankara, 1 reserved for Istanbul group and Istanbul people have their own permutation within. $(8-1)!7!$

Question 3

You roll a die once, and assume the number your rolled is X. Then continue rolling the die until you either match or exceed X. What is the expected number of additional rolls?

Solution:

$$E[n|X_1 = i] = 1/P(X_2 \geq i|X_1 = i) = 6/(6-i+1)$$

$$\sum_{i=1}^{6} = 6/(6-i+1) * P(X = i) = 2.45$$

Question 4

In a three dice roll, if at least two dice have the same number (e.g. 5-5-4 or 3-3-3) you win.

a) What is the probability that you win at least three times in a 30 rolls game?

Solution: All combinations $N = 6^3 = 216$. Non repeating permutations $n_{lose} = \frac{6!}{3!} = 120$. Probability of win is $1 - \frac{n_lose}{N} = 0.444$. Probability of winning at least 3 times is

$$P(X \geq 3|T = 30) = 1 - P(X = 0|T = 30) - P(X = 1|T = 30) - P(X = 2|T = 30)$$

$$P(X \geq 3|T = 30) = 1 - 0.9999933$$

b) What is the expected number and variance of the number of wins?

$$E[X] = np = 30 * (0.444) = 13.33$$
$$V[X] = np(1-p) = 30 * (0.444) * (1-0.444) = 7.407407$$

Question 5

In a Go game, a player that wins three games out of five is the winner. Suppose the artificial intelligence Alpha Go has probability 0.65 of winning against the World's (Human) Go Champion. What is the probability that Alpha Go wins the game at the 4th game?

Solution: AI should win 2 games in the first 3 and win the 4th game. $\binom{3}{2}(0.65)^2(0.35)(0.65) = 0.289$

Question 6

An egg basket contains 8 eggs, 4 of which are broken. 3 eggs are selected to make an omelette.

a) What is the probability that all 3 eggs are intact?

Solution: $\frac{\binom{4}{3}}{\binom{8}{3}} = 0.0714$

b) What is the probability that the 2nd broken egg is the 3rd egg?

Solution: Probability of getting 1 intact and 1 broken egg $\frac{\binom{4}{1}\binom{4}{1}}{\binom{8}{2}} = 0.571$. At 3rd egg there should be 3 intact and 3 broken eggs remaining, so getting the 3rd broken egg is 1/2. Final probability is 0.286.

Question 7

Nejat is a film critic and he will attend to IKSV Film Festival between April 5-15. Nejat likes a movie with probability 0.6 if the genre of the movie is mystery. For other genres, he likes the movie with probability 0.4. There will be 40 movies during the festival, 10 of which are mystery.

a) What is the probability that Nejat will like a randomly selected movie?

Solution: (L)ike, (D)islike, (M)ystery, (O)ther. $P(M) = 10/40 = 0.25$.

$$P(L) = P(L|M)P(M) + P(L|O)P(O) = 0.6 * 0.25 + 0.4 * 0.75 = 0.45$$

b) Suppose Nejat did not like the film. What is the probability that the selected movie is a non-mystery film?

Solution: $P(O|D) = \frac{P(D|O)P(O)}{P(D)} = \frac{P(D|O)P(O)}{1 - P(L)} = 0.6 * 0.75/0.55 = 0.818$

Question 8

A student applies for internships to 11 companies. She has a 0.6 probability to get an offer for an internship.

a) What is the probability that she will get offers from at least 4 companies?

Solution: $P(X \geq 4) = \sum_{i=4}^{11} \binom{8}{i}(0.6)^i(0.4)^{n-i} = 0.97$

b) What is the probability that she gets her fourth offer at the seventh application?

Solution: $\binom{6}{3}(0.6)^3(0.4)^3(0.6) = 0.166$

c) What is the expected value and variance of the applications?

Solution: $E[X] = np = 11 * 0.6 = 6.6$, $V(X) = np(1 - p) = 11 * 0.6 * 0.4 = 2.64$.

Question 9

A steakhouse serves (C)hateaubriand, (K)obe Beef Tenderloin and (T)-Bone. Customers order C with probability 0.35, K w.p. 0.2 and T w.p. 0.45.

a) 12 customers arrive. What is the probability that 4 of them order C, 3 order K and 5 order T?

Solution: $\binom{12}{4,3,5}(0.35)^4(0.2)^3(0.45)^5 = 0.0614$.

b) Suppose C sells for 90TL, K for 150TL and T for 120TL. If 100 customers are served that day, what is the expected revenue?

Solution: $100 * (90 * 0.35 + 150 * 0.2 + 120 * 0.45) = 11550TL$.

Question 10

An international student group will select a committee of 4 people at random. There are 7 Turkish, 6 Greek, 4 Italian and 5 Irish students in the group.

a) What is the probability that all countries are represented?

Solution: Total number students is 22. $\dfrac{7*6*4*5}{\binom{22}{4}} = 0.1148325$

b) If the committee consisted of 5 people what would be the probability of (a)?

Solution: $\dfrac{7*6*4*\binom{5}{2} + 7*6*5*\binom{4}{2} + 7*5*4*\binom{6}{2} + 6*5*4*\binom{7}{2}}{\binom{22}{4}} = 0.287$

Question 11

There are 3 balls in an urn, each of them is either (B)lack or (W)hite. At each step i, a ball is drawn from the urn randomly and another ball is thrown the urn with equal probability for both colors. So there are three balls at the end of each step. Let X_i be the number of White balls at time i. We know that at step 0, there are two white balls and one black ball in the urn (i.e. $X_0 = 2$). Calculate the following.

a) $P(X_1 = 2|X_0 = 2)$

b) $P(X_2 = 1|X_0 = 2)$

Now consider that the process is switched. First, a ball is thrown at random inside the urn then a ball is drawn from the urn. Again at step 0, there are two white balls and one black ball (i.e. $X_0 = 2$). Calculate the following.

c) $P(X_1 = 1|X_0 = 2)$

d) $P(X_2 = 2|X_0 = 2)$

Solution: Define Y_i as the ball drawn the urn and Z_i as the ball added at stage i. Probabilities of Z will always be the same. $P(Z_i = W) = P(Z_i = B) = 0.5$. For a and b, probabilities of Y_i is dependent only on X_{i-1}. But, for c and d, Y depends on both X_{i-1} and Z_i.

a) $P(X_1 = 2|X_0 = 2) = P(Y_1 = W|X_0 = 2)P(Z_1 = W) + P(Y_1 = B|X_0 = 2)P(Z_1 = B) = 2/3 * 1/2 + 1/3 * 1/2 = 1/2$. b) $P(X_2 = 1|X_0 = 2) = P(X_2 = 1|X_1 = 2, X_0 = 2) + P(X_2 = 1|X_1 = 1, X_0 = 2) = P(X_2 = 1|X_1 = 2)P(X_1 = 2|X_0 = 2) + P(X_2 = 1|X_1 = 1)P(X_1 = 1|X_0 = 2)$. We can say that P(X_i = x_i|X_{i-1} = x_{i-1}) is equal for all i. So $P(X_2 = 1|X_1 = 2)P(X_1 = 2|X_0 = 2) + P(X_2 = 1|X_1 = 1)P(X_1 = 1|X_0 = 2) = P(X_1 = 1|X_0 = 2)P(X_1 = 2|X_0 = 2) + P(X_1 = 1|X_0 = 1)P(X_1 = 1|X_0 = 2) = P(X_1 = 1|X_0 = 2)(P(X_1 = 2|X_0 = 2) + P(X_1 = 1|X_0 = 1))$. Then $P(X_1 = 1|X_0 = 2) = P(Y_1 = W|X_0 = 2)P(Z_1 = B) = 2/3 * 1/2 = 1/3$. $P(X_1 = 1|X_0 = 1) = P(Y_1 = W|X_0 = 1)P(Z_1 = W) + P(Y_1 = B|X_0 = 1)P(Z_1 = B) = 1/3 * 1/2 + 2/3 * 1/2 = 1/2$.

So, $P(X_1 = 2|X_0 = 2) = 1/3 * (1/2 + 1/2) = 1/3$.

c) Similar but this time Z_i is more important. $P(X_1 = 1|X_0 = 2) = P(Z_1 = B)P(Y_1 = B|X_0 = 2, Z_1 = B) = 1/2 * 1/2 = 1/4$.

d) $P(X_2 = 2|X_0 = 2) = P(X_2 = 2|X_1 = 2)P(X_1 = 2|X_0 = 2) + P(X_2 = 2|X_1 = 1)P(X_1 = 1|X_0 = 2) + P(X_2 = 2|X_1 = 3)P(X_1 = 3|X_0 = 2)$. Let's rephrase $P(X_1 = 2|X_0 = 2)^2 + P(X_1 = 2|X_0 = 1)P(X_1 = 1|X_0 = 2) + P(X_1 = 2|X_0 = 3)P(X_1 = 3|X_0 = 2)$. $P(X_1 = 2|X_0 = 2) = 1/2*3/4 + 1/2*1/4 = 1/2$. $P(X_1 = 2|X_0 = 1) = 1/2 * 1/2 = 1/4$. $P(X_1 = 3|X_0 = 2) = 1/2 * 1/4 = 1/8$. $P(X_1 = 2|X_0 = 3) = 1/2 * 3/4 = 3/8$.

$$(1/2)^2 + 1/4 * 1/4 + 1/8 * 3/8 = 23/64$$

Question 12

Calculate the number of permutations of the word "SUCCESSFUL" with vowels together.

Solution: 3S, 2C, 1L, 1F, 2U and 1F. 7 consonants, 3 vowels. $\dfrac{(7+1)!}{3!2!}\dfrac{3!}{2!} = 10800$.

Question 13

10 students enter the cafeteria. There are three different meal choices; red meat, vegeterian and fish. A student prefers red meat with 50% probability, vegeterian 20% and fish 30%. What is the probability that, of this group of 10, 4 students prefer eat red meat, 3 students prefer vegeterian and 3 students prefer fish?

Solution: Multinomial Distribution $\binom{10}{4,3,3}(0.5)^4(0.2)^3(0.3)^3 = 0.0567$

Question 14

Three students are going to start an internship on January. There are five internship posts on the department website. Each has to apply to only one internship. What is the probability that none of the students apply to the same internship?

Solution: Similar to birthday problem. $\dfrac{5!}{2!}/5^3 = 0.48$.

Question 15

Baran is an activist who raises funds for UNICEF on the street. 80% of people walking by stops when Baran asks to tell about UNICEF. 50% of those who stop donates 10TL to UNICEF. What is the expected amount of donations if Baran tried to reach 15 people that day.

Solution: P(Donation) = P(Stop)P(Donation|Stop) = 0.8*0.5 = 0.4. Expectation of binomial distribution np = 15*0.4 = 60.

Question 16

Merve is an aspiring singer. She has 25% chance to be offered to sing at a concert, if she applies. She applied to 8 concerts for the next season. What is the probability that she got at least 2 offers?

Solution: P(Reject) = 0.75. $P(X \geq 2) = 1 - P(0) - P(1) = 1 - (0.75^8 + 8 * 0.75^7 * 0.25) = 0.64$

Question 17

Bureau of Statistics provide you with the following information. 40% of Architects, 60% of Engineers and 30% of Lawyers make more than 5000TL per month in the first two years of their career. Of those interested in these occupations 20% choose Architecture, 50% choose Engineering and, 30% choose Law.

a) What is the probability that a randomly selected person from these careers make less than 5000TL per month in the first two years of his/her career?

b) If a person makes more than 5000TL, what is the probability that he/she is an Architect?

Solution: Define W as making more than 5000TL, W^c as less and (A) as choosing Architecture career, (E) Engineering and (L) Law.

a) $P(W^c|A) = 1 - P(W|A) = 0.6$, $P(W^c|E) = 1 - P(W|E) = 0.4$, $P(W^c|L) = 1 - P(W|L) = 0.7$. $P(W^c) = P(A)P(W^c|A) + P(E)P(W^c|E) + P(L)P(W^c|L) = 0.2 * 0.6 + 0.5 * 0.4 + 0.3 * 0.7 = 0.53$.

b) $P(A|W) = P(A \cap W)/P(W)$. $P(A \cap W) = P(A)P(W|A) = 0.2 * 0.4 = 0.08$. $P(W) = 1 - P(W^c) = 0.47$. $P(A|W) = 0.08/0.47 = 0.17$.

Question 18

Let X and Y be the random variables and $f(x, y)$ is the probability density function of the joint distribution. Suppose $f(x, y)$ is defined as $k(2x^2 + y)$ if $0 < x < 2$ and $0 < y < 1$ (0 otherwise).

a) Find k.

b) Find the marginal distribution of y ($h(y)$) and $h(y < 0.5)$.

c) Find the conditional distribution of $f(y|x)$.

Solution:

a) $\int_0^1 \int_0^2 k(2x^2 + y)dxdy = \int_0^1 k((2/3) * x^3 + xy)dy|_0^2 = \int_0^1 k(16/3 + 2y)dy$. (This is also $h(y)$ if k is known.) $k(16y/3 + y^2)|_0^1 = k(19/3)$. In order to be a distribution it should be equal to 1. So $k = 3/19$.

b) As given in (a) $h(y) = 3/19(16/3 + 2y)$. So $h(y < 0.5) = \int_0^{0.5} 3/19(16/3 + 2y)dy = 3/19(16y/3 + y^2)|_0^{0.5} = 3/19(8/3 + 1/4) = 3/19 * 33/12 = 0.43$

c) We need to find $g(x) = \int_0^1 3/19(2x^2 + y)dy = 3/19(2x^2 + 1/2)$. $f(y|x) = f(x, y)/g(x) = \dfrac{3/19(2x^2 + y)}{3/19(2x^2 + 1/2)} = \dfrac{4x^2 + 2y}{4x^2 + 1}$.

Chapter 7

Some Continuous Distributions

So far we had only seen discrete distributions which only specific values have positive probability values. Now we are going to see continuous variables where each real value defined in the domain of the distribution has a positive probability.

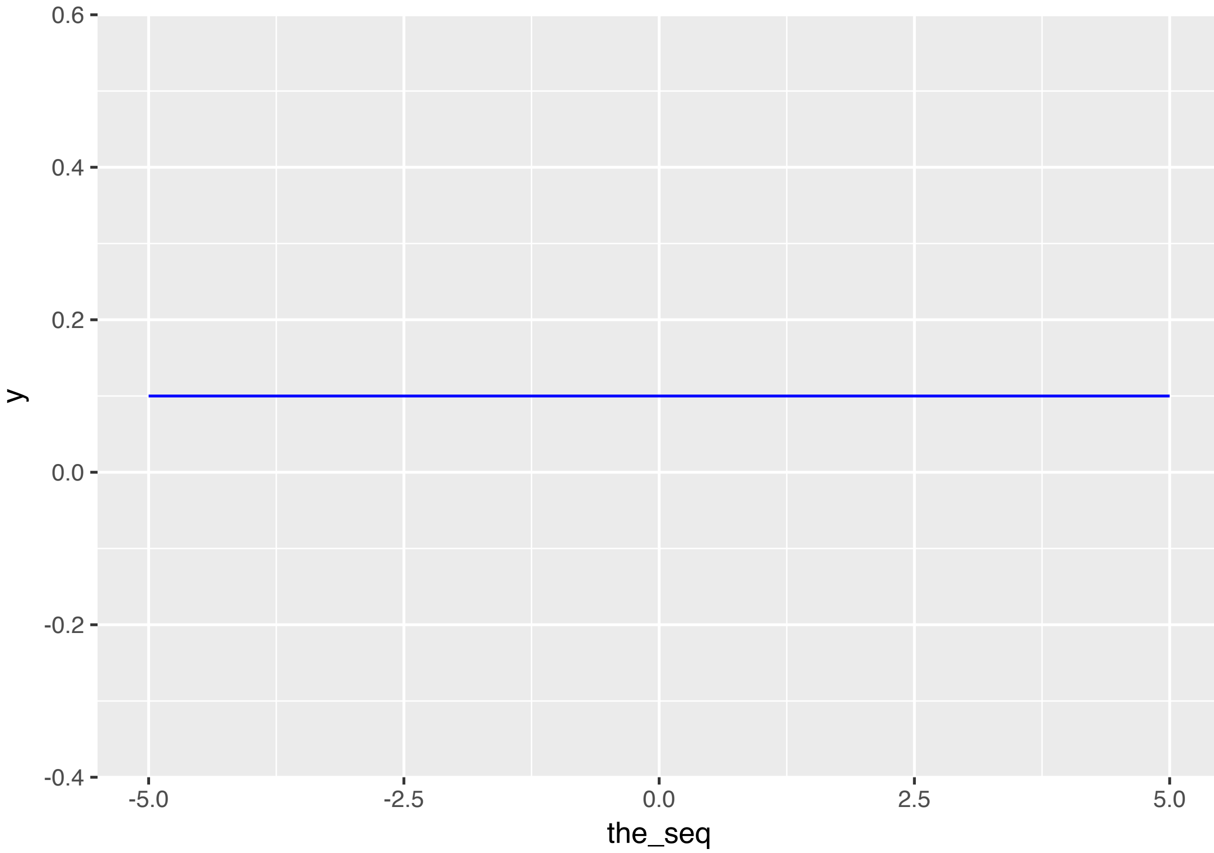
In this class we are going to see uniform, exponential, normal, gamma and weibull distributions. Of those, we will see uniform, exponential and normal distributions in detail.

7.1 Uniform Distribution

Given an interval $[a, b]$, each value within the interval has equal probability in uniform distribution.

$$X \sim U[a, b]$$

- Density: $f(X) = \dfrac{1}{b-a}$
- CDF: $F(X \leq x) = \dfrac{x-a}{b-a}$
- $E[X] = (b+a)/2$
- $V(X) = 1/12(b-a)^2$



Example: Suppose there is a lecture that can end anytime between 11:00 and 13:00. What is the probability that it ends before 12:30?

Solution: There are 120 minutes betweeen 11:00 (a) and 13:00 (b) and 90 minutes between 11:00 (a) and 12:30 (x).

$$P(X \leq 90) = 90/120 = 3/4$$

7.2 Exponential Distribution

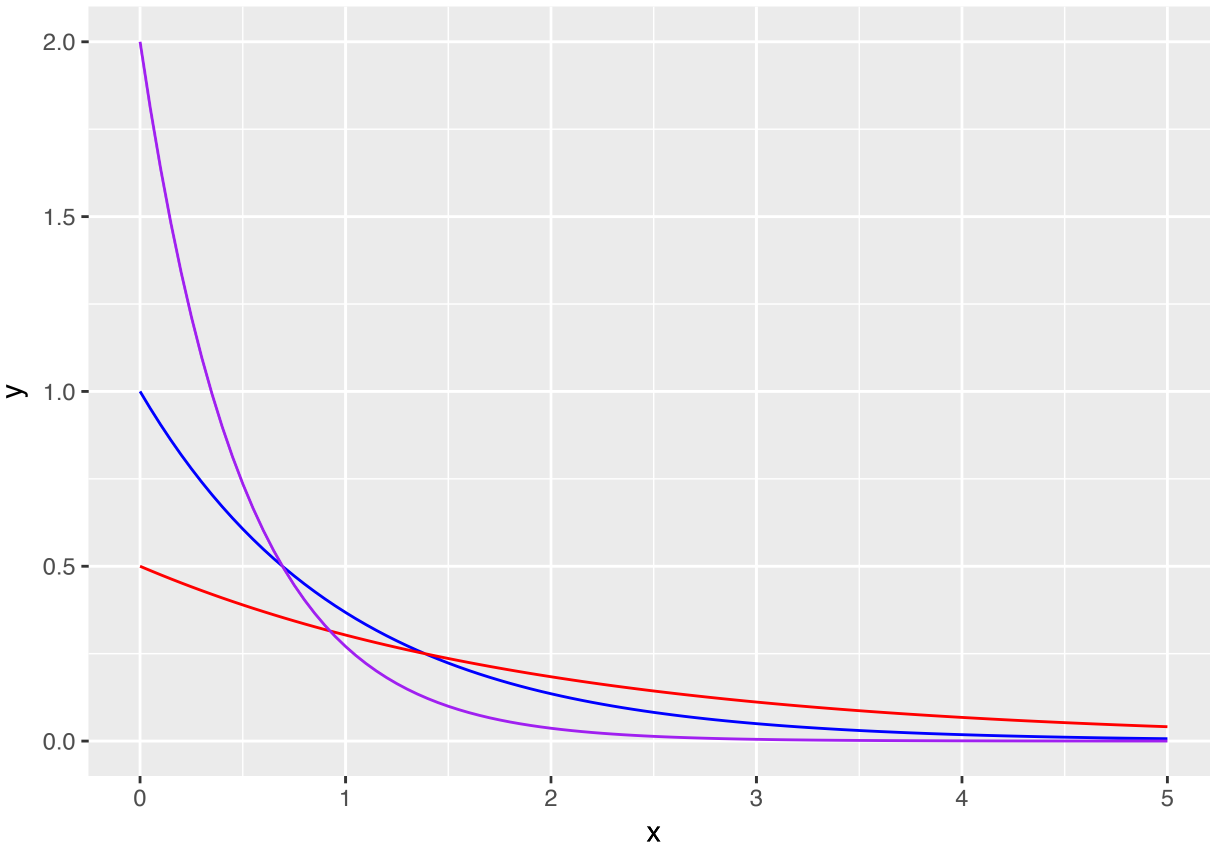
Exponential distribution is generally used to measure time before an event happens. Common examples are component (i.e. light bulb) lifetime and job processing (i.e. queue serving). It is closely related to Poisson distribution. While Poisson is used to estimate number of events in a given time period, Exponential distribution estimates the time of an event.

- Density: $f(X) = \lambda e^{-\lambda x}$
- CDF: $F(X \leq x) = 1 - e^{-\lambda x}$
- $E[X] = 1/\lambda$
- $V(X) = 1/\lambda^2$

Exponential distribution has memoryless property.

$$P(X > t + s|X > t) = \frac{P(X > t + s)}{P(X > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

Note: Exponential distribution is frequently used in maintenance themed questions (e.g. “A machine breaks down…”). Though, memoryless property in exponential distribution might seem a bit off as it kind of assumes replenished starting point and no “wear and tear effect”. Don’t forget these are mathematical questions and applications might differ.



Example: Lifetime of a bulb is expected to be 10,000 hours, estimated with exponential distribution. What is the probability that the bulb will fail in the first 3,000 hours?

$$\lambda = 1/10^5$$

$$P(X < 3,000) = 1 - e^{-\lambda x} = 1 - e^{-10^{-5}*3*10^3} = 0.0296$$

Example: Using the same properties of the question above, what is the probability that the bulb will last more than 7,000 hours if it didn’t fail in the first 5,000 hours?

From memoryless property.

$$P(X > 7000|X > 5000) = P(X > 5000 + 2000)/P(X > 5000) = P(X > 2000) = e^{-\lambda x} = 0.9801987$$

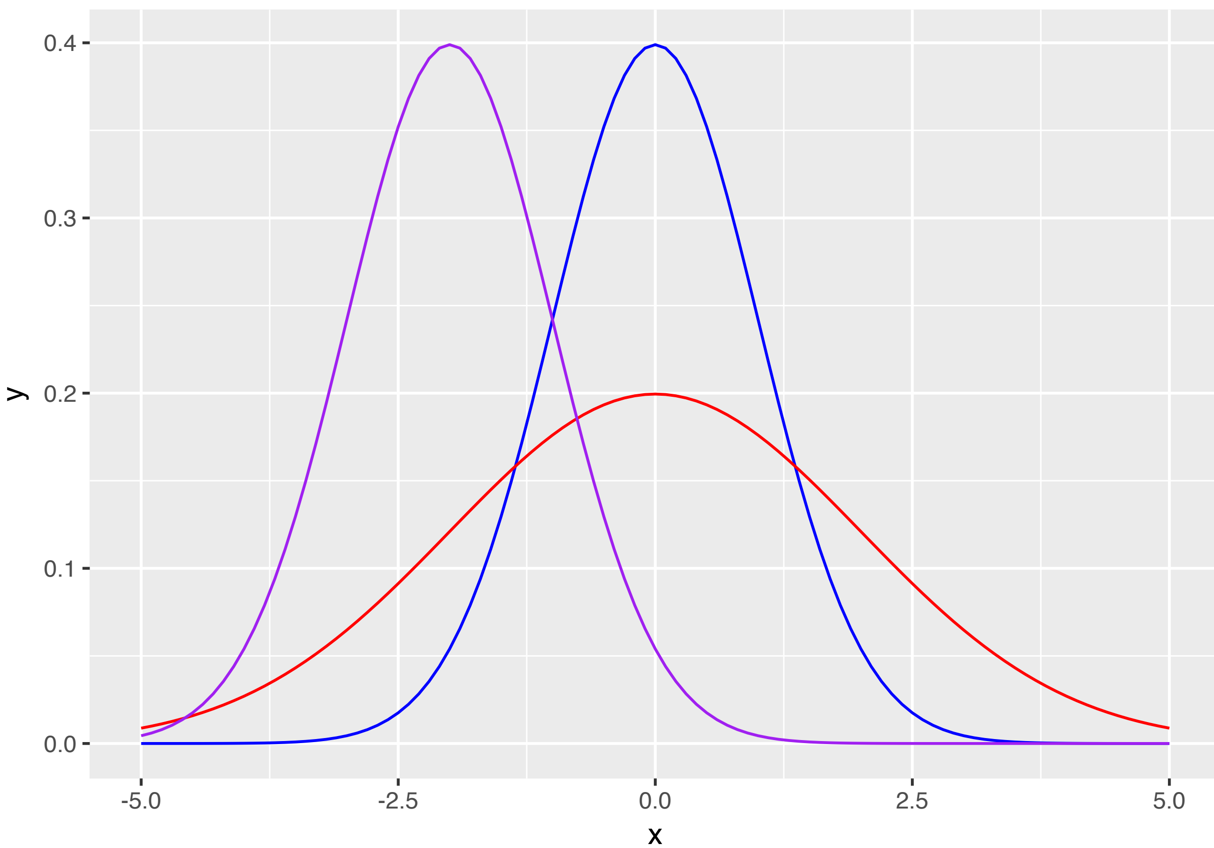
7.3 Normal Distribution

It is the most popular continuous distribution with uniform distribution and it has many applications. Also several discrete and continuous distributions (i.e. Binomial, t and chi-squared) converge to normal distribution when data size increases. It is also called Gaussian distribution. Many miscalculations or failed prediction happen because people approximate empirical distributions to normal distribution. It has two main parameters mean (location) μ and standard deviation (scale) σ.

$$X \sim N(\mu, \sigma)$$

- Density: $f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$
- $E[X] = \mu$
- $V(X) = \sigma^2$

If $X \sim N(0, 1)$, it is called standard normal distribution.



7.3.1 Standard Normal Distribution

Standard Normal Distribution is a special case of normal distribution with mean (μ) 0 and standard deviation (σ) 1 X $N(0, 1)$. Probability calculations in normal distribution is usually done with converting the parameters to standard normal parameters and finding the probabilities from the standard normal table (or z-table). Conversion to standard normal is done as follows. (*You can find the z-table on course website.*)

$$\frac{X-\mu}{\sigma}$$

Example: In a population height of the individuals are normally distributed with mean 170cm and standard deviation 5cm. What is the probability of a randomly selected person's height is 160cm or lower?

Solution: $P(X < 160; \mu = 170, \sigma = 5) = P(X < \dfrac{160-170}{5}) = \phi(-2) = 0.0228.$

7.4 Gamma Distribution

First, let's define the gamma function.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

By extensions there are some interesting properties.

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

If n is a positive integer then

$$\Gamma(n) = (n-1)!$$

.

You can use Gamma Distribution in reliability calculations with multiple components.

- Density: $f(X) = \dfrac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}, x > 0.$
- $E[X] = r/\lambda$
- $V(X) = r/\lambda^2$

Exponential distribution is a special case of Gamma distribution with $r = 1$.

7.5 Weibull Distribution

Some application of Weibull distribution are to estimate the time for failure in multi component electrical or mechanical systems, and modelling wind speed.

- Density: $f(X) = \dfrac{\beta}{\delta}\left(\dfrac{x-\gamma}{\delta}\right)^{\beta-1} e^{-\left(\dfrac{x-\gamma}{\delta}\right)^{\beta}}.$
- CDF: $F(X) = 1 - e^{-\left(\dfrac{x-\gamma}{\delta}\right)^{\beta}}$
- $E[X] = \gamma + \delta\Gamma(1 + 1/\beta)$
- $V(X) = \delta^2\left[\Gamma\left(1+\dfrac{2}{\beta}\right) - \left[\Gamma\left(1+\dfrac{1}{\beta}\right)\right]^2\right]$

7.6 R Functions

R has predefined functions for virtually all distributions. Just write `?Distributions` to the console.

7.7 Mini In-Class Exercises

These exercises are at will attempt questions. They will not be graded, you will not turn in any submission.

1. Waiters in Çengelköy Çınaraltı Cafe, goes on a tea serving tour every 5 minutes. What is the probability that a newly arrived customer waits between 2 and 3 minutes?
2. The waiting time at Çengelköy Börekçisi queue is exponentially distributed with a mean of 10 minutes. What is the probability that a customer would wait less than 8 minutes?
3. Suppose Z is a standard normal distributed random variable ($Z\ N(0,1)$).

   a. $P(-0.2 \leq Z \leq 0.4) = ?$
   b. If $P(Z < k) = 0.0987$ what is k?
   c. If $P(|Z| < k) = 0.95$ what is k?

7.8 Chapter Exercises

1. Two friends (A and B) agree to meet on 4:00 PM. A usually arrives between 5 minutes early and 5 minutes late. B usually arrives between 5 minutes early and 15 minutes late. Their times of arrival are independent from each other.

   a) What is the probability that B arrives definitely later than A?
   b) In terms of expected values, for how many minutes does A come earlier than B?
   c) What is the probability that both meet early?

   a) $P(B > 5) = \dfrac{5 - (-5)}{15 - (-5)} = 1/2$

   b) $E[B] - E[A] = 5 - 0 = 5$

   c) $P(B < 0, A < 0) = P(B < 0)P(A < 0) = 5/20 * 5/10 = 1/8$

2. There are three computers, which provide answers to questions with speed according to exponential distribution with means $(1/\lambda)$ 6, 4 and 3 per hour, respectively. What is the probability that at least one machine provides an answer within the first hour?

$$P(X < x) = 1 - e^{-\lambda x}$$

$$P(X > x) = e^{-\lambda x}$$

The solution is 1 - no machine provides an answer within the hour $1 - P(X_1 > 1, X_2 > 1, X_3 > 1)$.

$$1 - P(X_1 > 1, X_2 > 1, X_3 > 1) = 1 - e^{-\lambda_1 - \lambda_2 - \lambda_3}$$

```
## [1] 2.260329e-06
```

3. Time between customer arrivals in a cafe is exponential with the mean value of 6 minutes.

a) What is the probability that no customers arrive in 15 minutes?

$P(X > 15) = 1 - P(X < 15) = 1 - (1 - e^{-\lambda x}) = e^{-15/6} = 0.08$

b) What is the inter-arrival time if the probability of a customer to arrive is 0.9?

$P(X > t) = e^{-t/6} = 0.1$, then $t = 13.81$.

c) What is the probability that 10 customers arrive in the first hour?

$\lambda * t = 1/6 * 60 = 10$

$P(X = 10) = \dfrac{e^{-\lambda * t}(\lambda t)^1 0}{10!} = 0.125.$

d) What is the probability of getting the first customer in 15 minutes if no customer arrived in the first 10 minutes?

$P(X < 15|X > 10) = 1 - P(X > 15|X > 10) = 1 - P(X > 5) = 1 - e^{-5/6} = 0.565$ (memoryless property)

Hint: Check the relationship between Poisson and Exponential distributions.

4. A pack of flour contains 1 kg of flour. Though a flour pouring machine has a standard deviation of 50 gr.

   a) What is the probability that a randomly selected package contains between 925-1075 grams of flour?
   b) If a proper flour package should contain between 1000-x and 1000+x grams of flour, what should x be that 80% of the packages are deemed proper?
   c) Your customer strictly declared that 95% of the packages should contain at least 1000 grams of flour, so you should adjust the mean value. What should be the new mean value?

We have $\mu = 1000,\ \sigma = 50$. Define $\Phi(.)$ as the cdf of the standard normal distribution.

a) $\Phi(\dfrac{1075 - 1000}{50}) - \Phi(\dfrac{925 - 1000}{50})$
b) Find an x that $\Phi(\dfrac{1075 - 1000}{50}) - \Phi(\dfrac{925 - 1000}{50}) = 0.8$ approximately. The answer is more like how many standard deviations. We can find it by searching for a quantile of $\alpha = (1 - 0.8)/2 = 0.1$. Then find the y $\Phi(y) = 1 - \alpha$. $y = 1.282$ so reverse the procedure from standard normal to N($\mu = 1000,\ \sigma = 50$) by $y * \sigma$ to find x.
c) Your $\Phi(\dfrac{1000 - \mu}{50}) = 0.05$. The quantile value for 0.05 is -1.645. So, 1000 - 50*(-1.645) = 1082.25.

```
## [1] 0.8663856
```

```
## [1] 64.07758
```

```
## [1] 1082.243
```

5. There are two different roads to get to Sarıyer. Road A takes 35 minutes on average with standard deviation 5 minutes. Road B takes 32 minutes on average with standard deviation 8 minutes.

   a) Which road has the higher advantage if one wants to reach Sarıyer in 42 minutes?
   b) What is the maximum time of arrival with 90% probability? Calculate for each road.

a) For Road A it will take 42 minutes or less with probability $\Phi(\frac{42-35}{5}) = \Phi(1.4) = 0.92$. For Road B, $\Phi(\frac{42-32}{8}) = \Phi(1.4) = 0.89$. Take Road A. b) It is the quantile value of $\alpha = 0.9$. Then we need to find $\Phi^{-1}(\alpha) = 1.28 = \frac{x-\mu}{\sigma}$. For Road A $1.28 * 5 + 35 = 41.4$, B $1.28 * 8 + 32 = 42.2$.

Note: b is phrased weakly. It is also possible to understand it as a probability interval as the probability of an event occurring in normal distribution happens in intervals (i.e. originates from mu). In that case the quantile max value is $\Phi^{-1}(0.95) = 1.64$ as there should be 5% slack at each side of the distribution.

Chapter 8

Joint Distributions

So far, we learned about joint probabilities in Bayesian context such as $P(A|B) = P(A,B)/P(B)$. Now, we are going to expand this concept into discrete and continuous distributions. Define $P(X = x, Y = y) = f(x,y)$ as the probability mass function (discrete) or probability density function (continuous).

Same probability laws apply to joint distributions as well.

- $f(x,y) \geq 0$ for all (x,y).
- $\sum_x \sum_y f(x,y) = 1$ or $\int_x \int_y f(x,y)dxdy = 1$

Example (Discrete): Suppose there are 10 balls in a box; 3 white, 4 black and 3 red. Two balls are randomly selected. Let's say random variable X is the number of white balls picked and r.v. Y is the number of black balls picked. (a) Find the joint probability function and (b) find the probabilities.

(a) Let's first enumerate the alternatives. (x,y) pair can be either of $(0,0),(0,1),(0,2),(1,1),(2,0),(1,0)$. Total number of alternatives are $\binom{10}{2}$. To calculate, number of ways of getting 1 white and 1 black ball is $\binom{3}{1}\binom{4}{1}\binom{3}{0}$. So, the probability will be $\frac{\binom{3}{1}\binom{4}{1}\binom{3}{0}}{\binom{10}{2}}$. We can generalize it to a function.

$$f(x,y) = \frac{\binom{3}{x}\binom{4}{y}\binom{3}{2-x-y}}{\binom{10}{2}}$$

Let's also make it into an R function

```
## [1] 0.2666667
```

(b) Using the above formula we can calculate all the probabilities within the specified region $x + y \leq 2$.

```
##      x_0  x_1  x_2
## y_0 0.07 0.20 0.07
## y_1 0.27 0.27 0.00
## y_2 0.13 0.00 0.00
```

Example (continuous): *(This is from the textbook, Example 3.15)* A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of time that the drive-in and the walk-in facilities are in use and suppose that the joint density function of these random variables is

$$f(x,y) = \frac{2}{5}(2x+3y), 0 \leq x \leq 1, 0 \leq y \leq 1$$

and 0 for other values of x and y.

(a) Verify $\int_x \int_y f(x,y)dxdy = 1$

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y)|0 < x < 1/2, 1/4 < y < 1/2\}$

(c) (see the book for the full calculations)

$$\int_x \int_y f(x,y)dxdy = \int_0^1 \int_0^1 \frac{2}{5}(2x+3y)dxdy = 1$$

(a) (see the book for the full calculations)

$$\int_x \int_y f(x,y)dxdy = \int_{1/4}^{1/2} \int_0^{1/2} \frac{2}{5}(2x+3y)dxdy = 13/160$$

Example (with special distributions)

1. Patients arrive at the doctor's office according to Poisson distribution with $\lambda = 2$/hour.

   a) What is the probability of getting less than or equal to 2 patients within 2 hours?
   b) Suppose each arriving patient has 50% chance to bring a person to accompany. There are 10 seats in the waiting room. At least many hours should pass that there is at least 50% probability that the waiting room is filled with patients and their relatives?

Solution

a) $P(X \leq 2|\lambda t = 2) = \sum_{i=0}^{2} \dfrac{e^{-\lambda t}(\lambda t)^i}{i!}$

```
## [1] 0.2381033
```

b) First let's define the problem. Define n_p as the number of patients and n_c is the number of company. We want $n_p + n_c \geq 10$ with probability 50% or higher for a given t^*. Or to paraphrase, we want $n_p + n_c \leq 9$ w.p. 50% or lower.

   What is n_c affected by? n_p. It is actually a binomial distribution problem. $P(n_c = i|n_p) = \binom{n_p}{i}(0.5)^i * (0.5)^{n_p - i}$. It is even better if we use cdf $P(n_c \leq k|n_p) = \sum_{i=0}^{k} \binom{n_p}{i}(0.5)^i * (0.5)^{n_p-i}$.

   We know the arrival of the patients is distributed with poisson. So, $P(n_p = j|\lambda t^*) = \dfrac{e^{-\lambda t}(\lambda t)^j}{j!}$. So $P(j + k \leq N) = \sum_{a=0}^{j} P(n_p = a|\lambda t^*) * P(n_c \leq N - a|n_p = a)$. Remember it is always $n_c \leq n_p$.

```
## [1] 0.8631867
```

```
## [1] 0.5810261
```

```
## [1] 0.4905249
```

8.0.1 Marginal Distributions

You can get the marginal distributions by just summing up or integrating the other random variable such as $P(Y = y) = \sum_x f(x, y)$ or $f(y) = \int_x f(x, y)dx$. Let's calculate the marginal distribution of black balls (rv Y) in the above example.

```
##      x_0  x_1  x_2
## y_0 0.07 0.20 0.07
## y_1 0.27 0.27 0.00
## y_2 0.13 0.00 0.00
```

```
##       y_0       y_1       y_2
## 0.3333333 0.5333333 0.1333333
```

Marginal distribution of y in the second example is calculated as follows.

$$\int_x \frac{2}{5}(2x + 3y)dx = \frac{2(1 + 3y)}{5}$$

8.0.2 Conditional Distribution

Similar to Bayes' Rule, it is possible to calculate conditional probabilities of joint distributions. Let's denote g(x) as the marginal distribution of x and h(y) as the marginal distribution of y. The formula of conditional distribution of x given y is as follows.

$$f(x|y) = f(x, y)/h(y)$$

Note that conditional distribution function is useless if x and y are independent. ($f(x|y) = f(x)$)

Chapter 9

Conditional Expectation

We learned about conditional distributions, but what about expectations? ($E[X|Y = y]$)

$$E[X|Y = y] = \sum_x xP(X = x|Y = y) E[X|Y = y] = \int_x xf(x|y)dx E[E[X|Y]] = \sum_y E[X|Y = y]P(Y = y) = E[X]$$

Example: A mouse is put into a labyrinth with 3 passages, at the end of the labyrinth there is cheese. First passage leads to the cheese in 3 mins. Second passage delays the mouse for 5 minutes and returns the mouse to the starting point. Third is the same as the second but the travel time is 10 minutes. It is equally likely that the mouse chooses any of those passages. What is the expected amount of time that the mouse will get to cheese?

Say T is time and Y is the passage chosen.

$$E[T] = E[E[T|Y]] = 1/3E[T|Y = 1] + 1/3E[T|Y = 3] + 1/3E[T|Y = 3]$$

$$E[T|Y = 1] = 3$$
$$E[T|Y = 2] = 5 + E[T]$$
$$E[T|Y = 3] = 10 + E[T]$$

$$E[T] = 1/3(3 + 5 + E[T] + 10 + E[T]) = 18$$

Chapter 10

Further Topics

(not included)

10.1 Moment Generating Function (MGF)

If we define a function $g(X) = X^r$ of r.v. X, the expected value $E[g(X)]$ is called the rth moment about the origin.

$$E[X^r] = \sum_x x^r f(x)$$

$$E[X^r] = \int_x x^r f(x) dx$$

The first moment gives us the expectation $E[X^1]$. With the second moment $E[X^2]$ we can calculate the variance $V(X) = E[X^2] - E[X]^2$.

The moment generating functon $M_X(t)$ is defined as follows.

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} f(x)$$

$$M_X(t) = E[e^{tX}] = \int_x e^{tx} f(x) dx$$

If the sum or interval above converges, then MGF exists. If MGF exists then all moments can be calculated using the following derivative.

$$\frac{d^r M_X(t)}{dt^r} = E[X^r], at\ t = 0$$

For instance, the MGF of binomial distribution is $M_X(t) = \sum_0^n e^{tx} \binom{n}{x} p^x q^{n-x}$.

10.2 Covariance

We know about the variance ($V(X) = \sigma_x^2 =$ E[(X-E[X])^2]). But what about the variance of two random variables? Then we talk about the **covariance** of the joint distribution ($V(X,Y) = E[(X-E[X])(Y-E[Y])]$) or ($E[XY] - E[X]E[Y]$).

10.3 Correlation

Simply put, it is the magnitude of (linear) relationship between random processes X and Y. Correlation coefficient can be found by using covariance and variances of the marginal distributions. ($\frac{\sigma_{XY}}{\sigma_X \sigma_Y}$).

Correlation is frequently used to indicate the similarity between two processes. Though, there is a popular saying that 'correlation does not imply causation', meaning seemingly correlated processes might actually be independent. Ask your instructor (or Google) about 'spurious correlations'.

Appendix A

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Solving normal distribution problems is quite easy if you remember some simple items.

A.1 Item 1: Standard Normal Distribution and z-table

Standard normal distribution is special case of normal distribution with mean (μ) of 0 and standard deviation (σ) of 1. The cumulative distribution function (CDF) values ($P(X \leq x)$) of std normal distribution are given in the z-table.

Note: CDF of the standard normal is also denoted with $\Phi(x) = P(X \leq x)$. Note 2: For those of you who did not grasp the concept of CDF; it is basically the probability that the outcome of the event is less than the value specified. For instance $\Phi(0)$

A.2 Item 2: Convert and normal process to standard normal.

Suppose you have a process (say, you pour cream on cakes) with mean 50 and standard deviation 5 and you want to know the probability of getting less than 52. Follow this formula.

$$\Phi\left(\frac{X-\mu}{\sigma}\right)$$

So in our case, $\Phi\left(\frac{52-50}{5}\right) = \Phi(0.4)$.

A.3 Item 3: How to read z-table.

In the z-table sum of row and column names gives us the value (x) and their corresponding value gives us the probability $\Phi(x)$. In our example $\Phi(0.4) = \Phi(row + column) = \Phi(0.4 + 0.00) = 0.6554$

The notation for getting the (quantile) value is $\Phi^{-1}(0.6554) = 0.4$.

A.4 Item 4: Symmetry

The standard normal distribution is symmetric around 0. It gives us the following useful property $\Phi(-x) = 1 - \Phi(x)$. See $\Phi(-0.4) = 1 - 0.6554 = 0.3446$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

A.5 Item 5: Understand intervals.

Due to the nature of normal distribution events mainly occur around the mean. That's why central intervals are given with some probability instead of from the bottom to the end. For instance suppose we would like to know the probability around +-9.8 liters around the mean.

$$P(50-9.8 < X < 50+9.8) = P(40.2 < X < 59.8) = P(X < 59.8, X > 40,2) = P(X < 59.8) - P(X < 40.2)$$

Convert to standard normal.

$$P(X < (59.8-50)/5) - P(X < (40.2-50)/5) = \Phi(1.96) - \Phi(-1.96)$$

We know that $\Phi(-x) = 1 - \Phi(x)$, so $\Phi(-1.96) = 1 - \Phi(1.96)$.

$$\Phi(1.96) - \Phi(-1.96) = \Phi(1.96) - (1 - \Phi(1.96)) = 2 * \Phi(1.96) - 1$$

Check the z-table

$\Phi(1.96)$ is 0.975. So $2 * 0.975 - 1 = 0.95$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379

A.6 Item 6: Moving the mean

It is possible to desire an increased level of quality. For instance, if we would like to have 90% of the cakes have at least 50 grams of cream instead of 50 grams just being the mean. So $P(X > 50) = 1 - P(X < 50) = 0.9$, then $P(X < 50) = 0.1$. You have the same standard deviation of 5 but you need a new mean.

$$\Phi\left(\frac{50 - \mu^*}{5}\right) = 0.1$$

We should invert the standard normal variate $\Phi^{-1}(0.1)$. Let's check the z-table for the values corresponding to 0.1.

Well, closest value is 0.1003 so we go with it (usually linear interpolation is done, but we'll skip that). The corresponding value is -1.28.

$$\begin{aligned}\frac{50 - \mu^*}{5} &= -1.28 \\ 50 - \mu^* &= -1.28 * 5 = -6.4 \\ \mu^* &= 50 + 6.4 = 56.4\end{aligned}$$

So we should align ourself to pour 56.4 grams of cream per cake on average in order to have 90% of the cakes with 50 grams or more cream.

A.7 Conclusion

By the end of this tutorial you should know your way around the normal distribution problems. It is not hard but you might need some practice till you get adept on the topic.

Bibliography

Walpole, R. E., Myers, R. H., Myers, S. L., and Ye, K. (2012). *Probability and Statistics for Engineers and Scientists*. Pearson, Boston, MA, 9th edition. ISBN 978-0-321-74823-2.