Getting to Regression: The Workhorse of Quantitative Political Analysis

Department of Government
London School of Economics and Political Science

What do you think will be on the exam?

Correlation as Measure of Bivariate Relationship

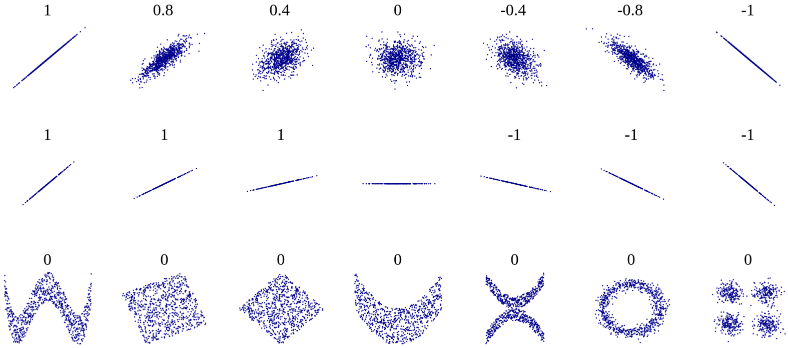
- Covariance:

$$Cov(X, Y) = \Sigma_{i=1}^{n} \frac{(X_i - \bar{X})(Y_i - \bar{Y})}{n - 1}$$

Correlation as Measure of Bivariate Relationship

- Covariance:
  $$Cov(X,Y) = \sum_{i=1}^{n} \frac{(X_i - \bar{X})(Y_i - \bar{Y})}{n-1}$$
- Correlation:
  $$Corr(X,Y) = r_{x,y} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})(Y_i - \bar{Y})}{(n-1)s_x s_y}$$
  where $s_x = \sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Correlation is linear!



Source: Wikimedia

Guess the Correlation!

1. Go to:
   `http://guessthecorrelation.com/`
2. Play a few rounds

Regression

- Definition: a statistical method for measuring the relationships between one variable and many other variables

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- Uses of Regression
  1. Description
  2. Prediction
  3. Causal Inference

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- *Ordinary least squares* (OLS) regression

Interpretations of OLS

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1. Line (or surface) of best fit
2. Ratio of $Cov(X, Y)$ and $Var(X)$
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4. Estimating Unit-level Causal Effect

Bivariate Regression I

- Y is continuous
- X is a randomized treatment indicator/dummy $(0, 1)$
- How do we know if the treatment X had an effect on Y?

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- X is a randomized treatment indicator/dummy $(0, 1)$
- How do we know if the treatment X had an effect on Y?
- Look at mean-difference: $E[Y_i|X_i = 1] - E[Y_i|X_i = 0]$

Bivariate Regression I

- Mean difference $(E[Y_i|X_i = 1] - E[Y_i|X_i = 0])$ is the regression line slope
- Slope (β) defined as $\frac{\Delta Y}{\Delta X}$

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- Mean difference $(E[Y_i|X_i = 1] - E[Y_i|X_i = 0])$ is the regression line slope
- Slope (β) defined as $\frac{\Delta Y}{\Delta X}$
  - $\Delta Y = E[Y_i|X = 1] - E[Y_i|X = 0]$
  - $\Delta X = 1 - 0 = 1$

Three Equations

1. Population:
   $Y = \beta_0 + \beta_1 X \;\; (+\epsilon)$

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Three Equations

1. Population:
   $Y = \beta_0 + \beta_1 X \ \ (+\epsilon)$

2. Sample estimate:
   $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x + e$

3. Unit:

$$\begin{aligned} y_i &= \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i \\ &= \bar{y}_{0i} + (y_{1i} - y_{0i}) x_i + (y_{0i} - \bar{y}_{0i}) \end{aligned}$$

Minimum Mathematical Requirements

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Minimum Mathematical Requirements

1. Do we need variation in X?
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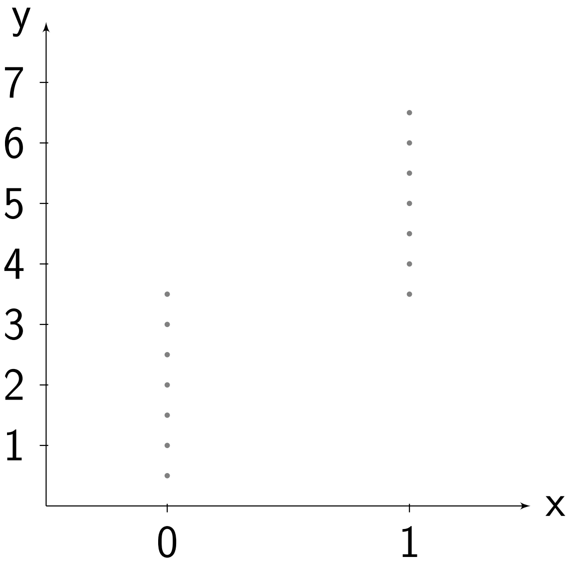
2. Do we need variation in Y?
   - No, $\hat{\beta}_1$ can equal zero
3. How many observations do we need?
   - $n \geq k$, where k is number of parameters to be estimated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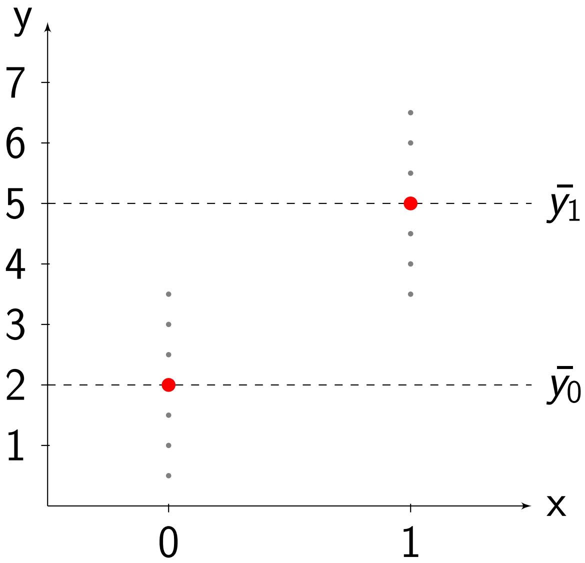
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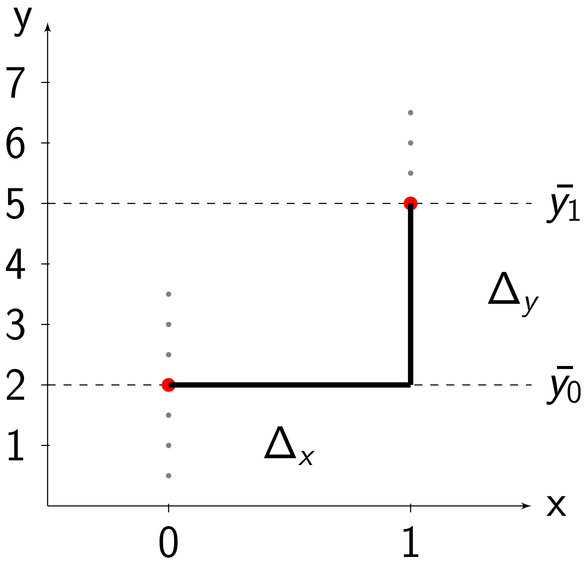
$\bar{y}_0$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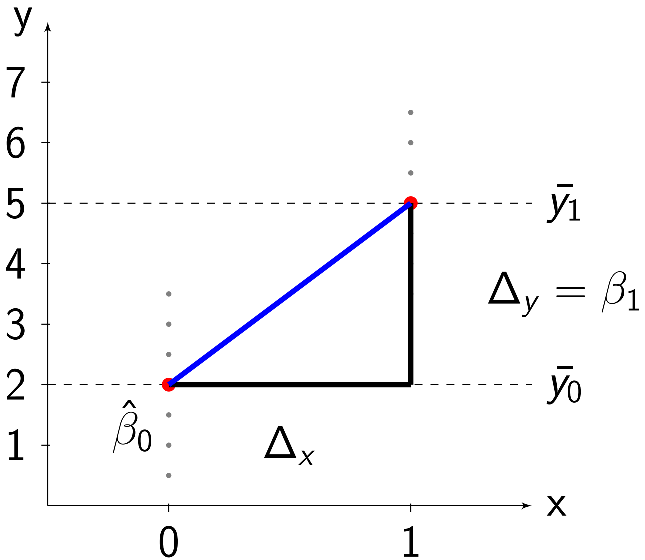
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$\Delta_y = \beta_1$
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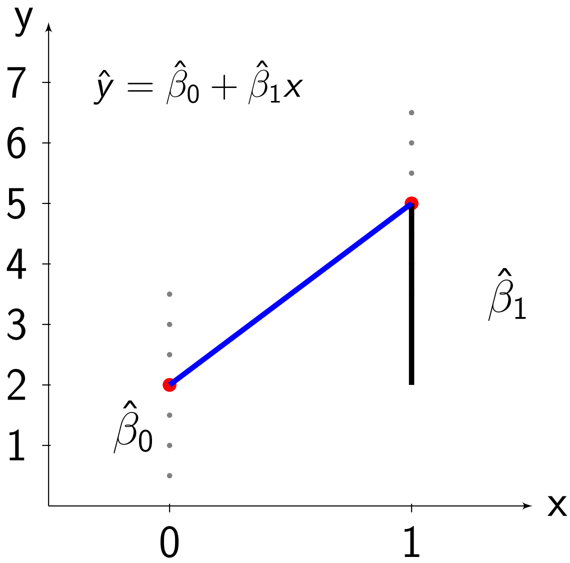
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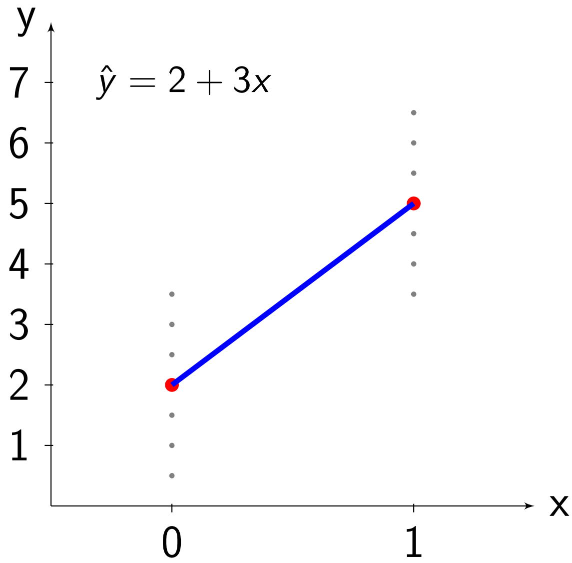
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
$\hat{\beta}_1$
$\hat{\beta}_0$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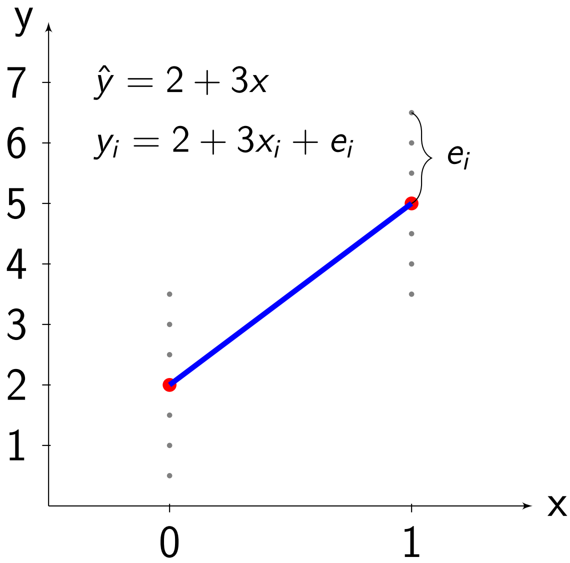
$\hat{y} = 2 + 3x$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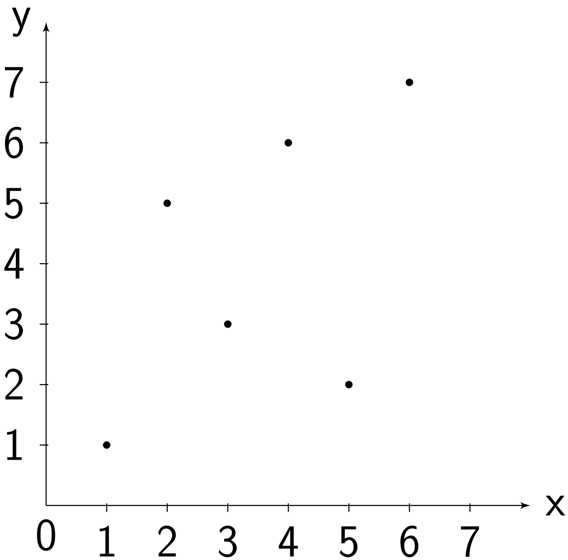
$\hat{y} = 2 + 3x$
$y_i = 2 + 3x_i + e_i$
e_i

Calculations

If x is continuous, calculation is more complicated:
$\widehat{\beta_1} = Cov(x, y)/Var(x)$

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	1	?	?	?	?
2	5	?	?	?	?
3	3	?	?	?	?
4	6	?	?	?	?
5	2	?	?	?	?
6	7	?	?	?	?
$\bar{x}$	$\bar{y}$			$Cov(x, y)$	$Var(x)$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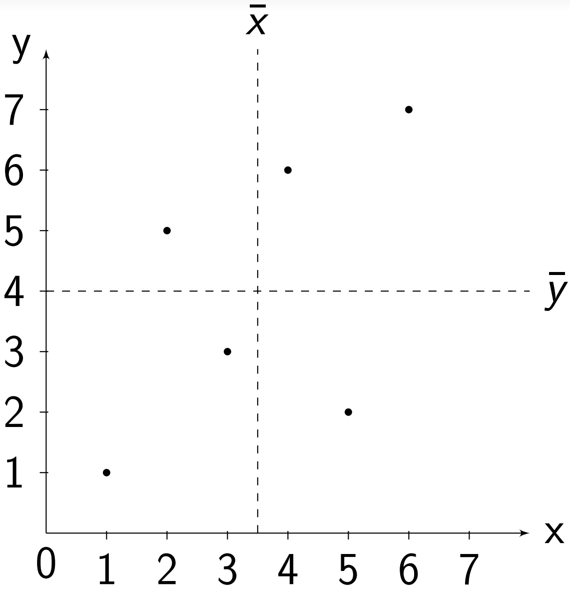
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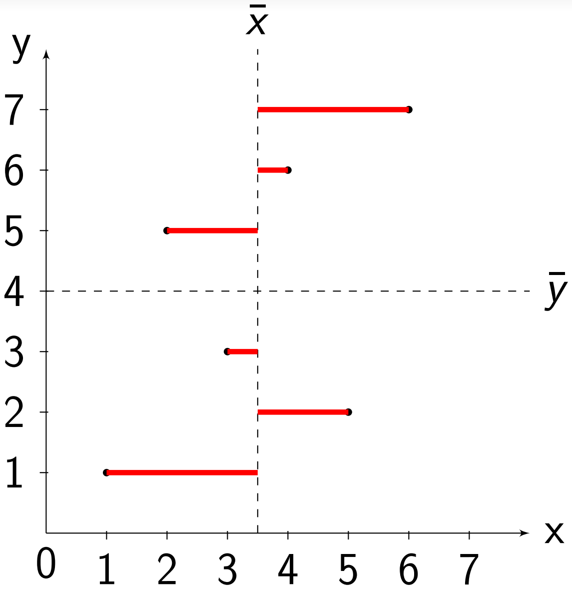
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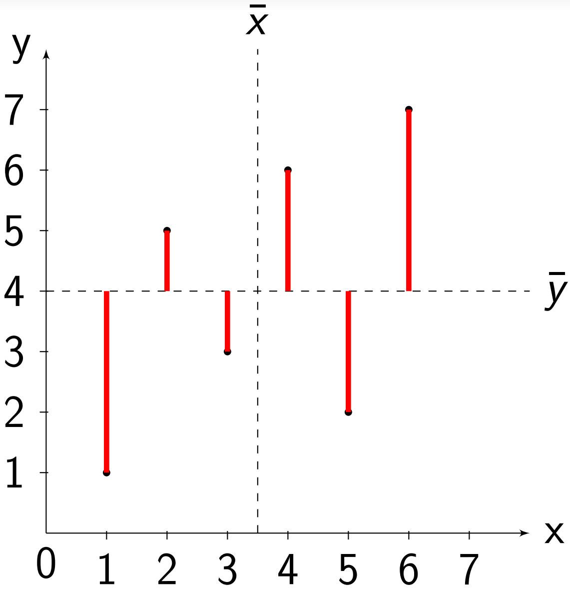
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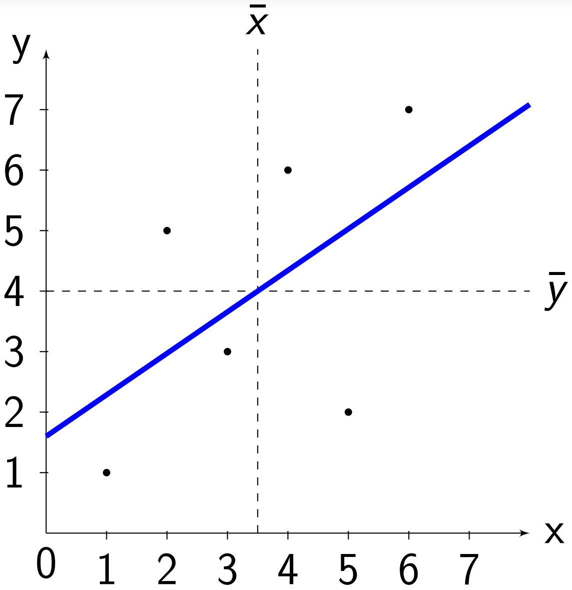
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2	5	?	?	?	?
3	3	?	?	?	?
4	6	?	?	?	?
5	2	?	?	?	?
6	7	?	?	?	?
$\bar{x}$	$\bar{y}$			$Cov(x, y)$	$Var(x)$

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1	1	$-2.\bar{6}$	-3	$-6.6\bar{6}$	6.25
2	5	$-1.\bar{3}$	$+1$	-2.00	2.25
3	3	$-0.\bar{6}$	-1	$-0.3\bar{3}$	0.25
4	6	$+0.\bar{3}$	$+2$	$-0.1\bar{6}$	0.25
5	2	$+1.\bar{6}$	-2	-2.50	2.25
6	7	$+2.\bar{3}$	$+3$	$-8.3\bar{3}$	6.25
3.5	$3.\bar{6}$			11	17.5

Calculations

If x is continuous, calculation is more complicated:
$\widehat{\beta_1} = Cov(x,y)/Var(x) = 11/17.5 = \mathbf{0.627}$

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- Ex.: $\hat{\beta}_0 = 3.\bar{6} - 0.627 * 3.5 = 1.4\bar{6}$

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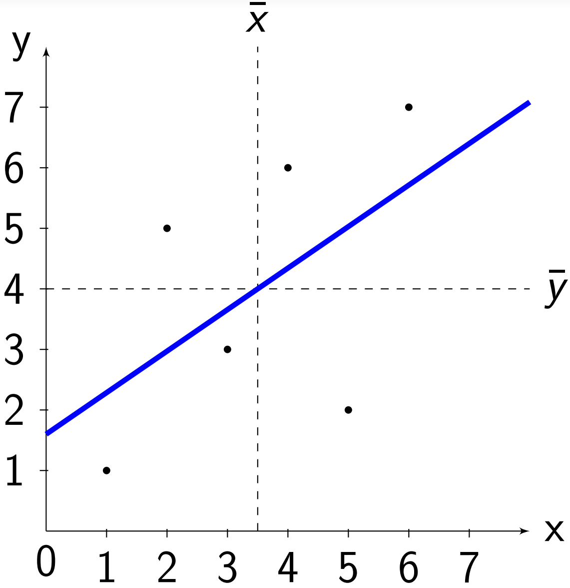
- Ex.: $\hat{\beta}_0 = 3.\bar{6} - 0.627 * 3.5 = 1.4\bar{6}$
- $\hat{y} = 1.4\bar{6} + 0.6857\hat{x}$



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Systematic versus unsystematic components

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  - Linear regression estimates the conditional means of the population data (i.e., $E[Y|X]$)

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  - Linear regression estimates the conditional means of the population data (i.e., $E[Y|X]$)

- Unsystematic: Error term is the deviation of observations from the line
  - The difference between each value y_i and $\hat{y}_i$ is the *residual*: e_i
  - OLS produces an estimate of β that minimizes the *residual sum of squares*

Why are there residuals?

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- Fundamental randomness

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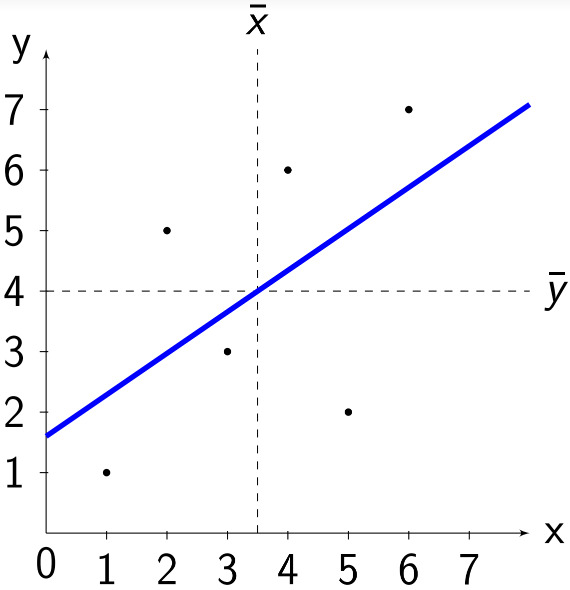
- Measurement error

Why are there residuals?

- Fundamental randomness
- Measurement error
- Omitted variables

OLS Minimizes SSR

- Total Sum of Squares (SST): $\sum_{i=1}^{n}(y_i - \bar{y})^2$
- We can partition SST into two parts (ANOVA):
  - Explained Sum of Squares (SSE)
  - Residual Sum of Squares (SSR)
- $SST = SSE + SSR$
- OLS is the line with the lowest SSR



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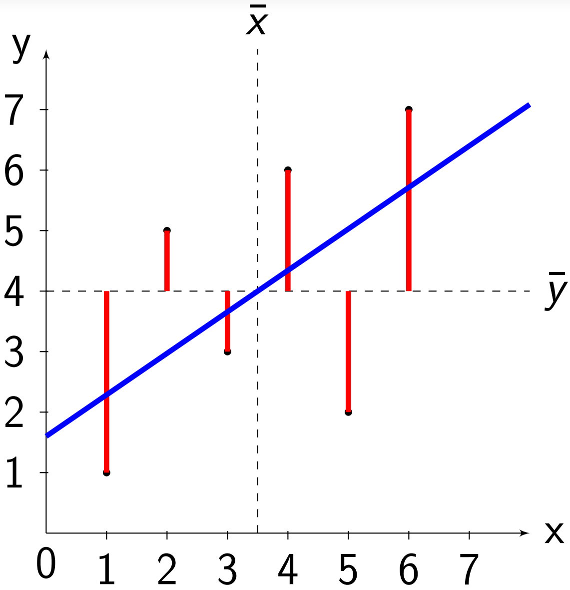

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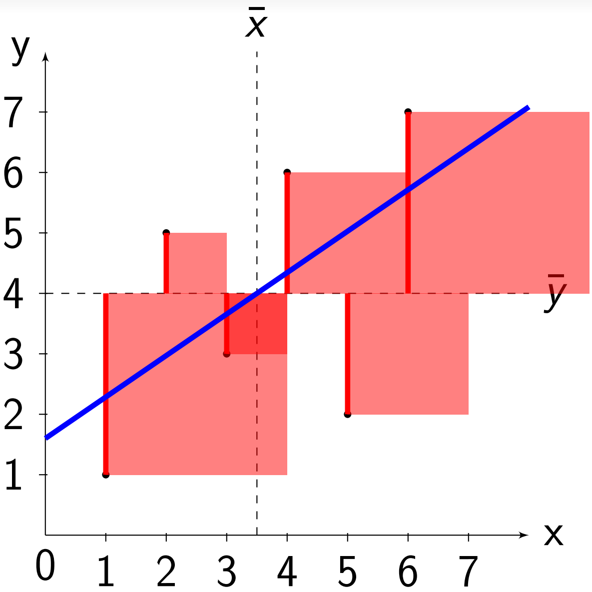

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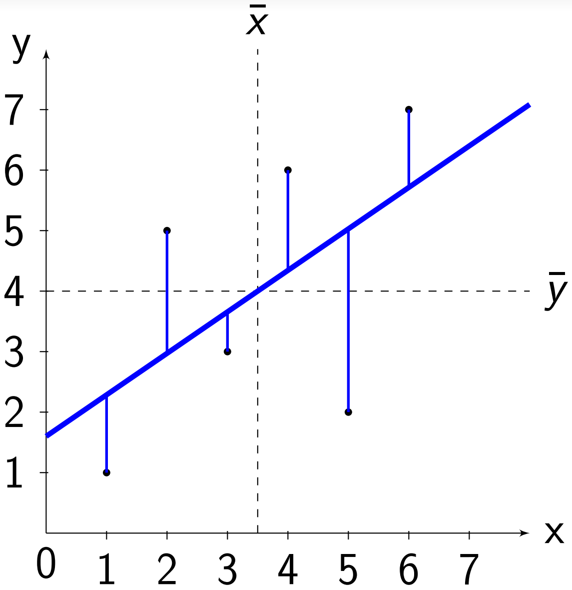
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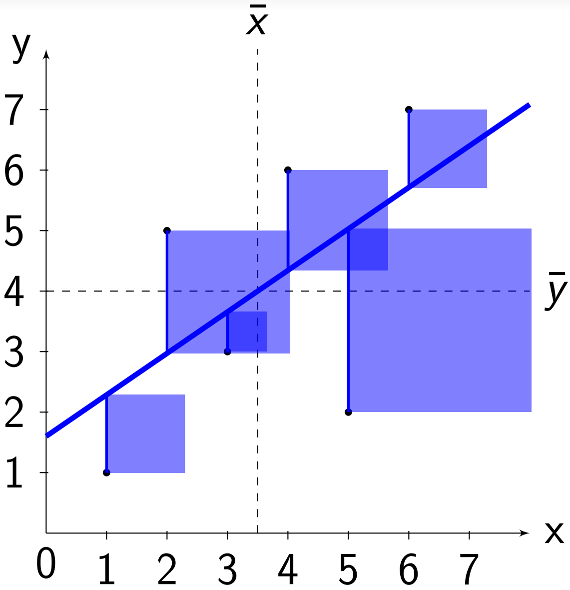
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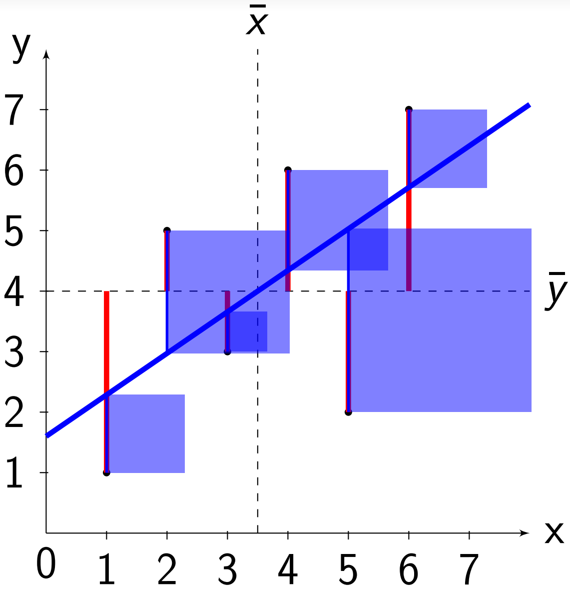
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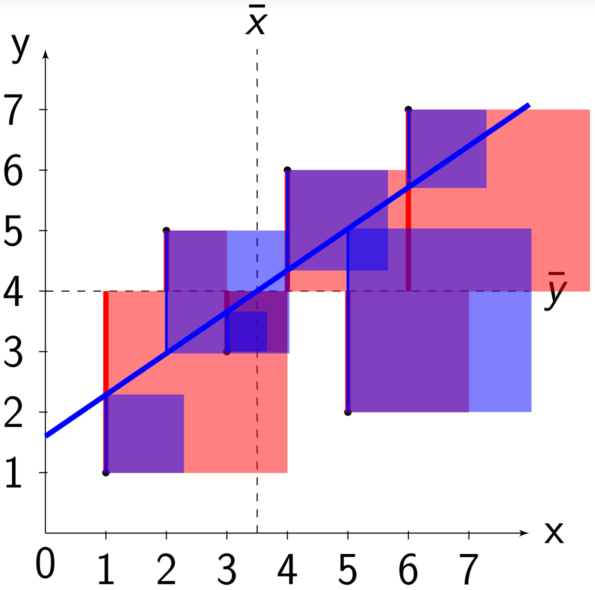
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$\bar{y}$

RMSE (σ)

- Definition: $\hat{\sigma} = \sqrt{\frac{SSR}{n-p}}$, where p is number of parameters estimated
- Interpretation:
  - How far, on average, are the observed y values from their corresponding fitted values $\hat{y}$
  - $sd(y)$ is how far, on average, a given y_i is from $\bar{y}$
  - σ is how far, on average, a given y_i is from $\hat{y}_i$
- Units: same as y (range 0 to $sd(y)$)

Correlation/Regression Equivalence

- Definition: $Corr(x, y) = \hat{r}_{x,y} = \frac{Cov(x,y)}{(n-1)s_x s_y}$
- Slope $\hat{\beta}_1$ and correlation $\hat{r}_{x,y}$ are simply different scalings of $Cov(x, y)$

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- Slope $\hat{\beta}_1$ and correlation $\hat{r}_{x,y}$ are simply different scalings of $Cov(x, y)$
- $R^2 = \hat{r}^2_{x,y} = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$

Questions about OLS calculations?

Are Our Estimates Any Good?

Are Our Estimates Any Good?

1. Works mathematically
2. Linear relationship between X and Y
3. X is measured without error
4. No missing data (or MCAR)
5. No confounding (next week)

Linear Relationship

- If linear, no problems
- If non-linear, we need to transform
  - Power terms (e.g., x^2, x^3)
  - log (e.g., $log(x)$)
  - Other transformations
  - If categorical: convert to set of indicators
  - Multivariate interactions (next week)

Coefficient Interpretation Activity

- Four types of variables:
  1. Indicator (0,1)
  2. Categorical
  3. Ordinal
  4. Interval

- How do we interpret a coefficient on each of these types of variables?

Questions?