Exercise 9
Suggested Solutions

ECON / MATH C103 - Mathematical Economics
Philipp Strack

due April 18, 4:59pm

Helpful Material:
- Last week's lecture notes.

Exercise 1: Consider a context with three possible allocations $A = \{\alpha, \beta, \gamma\} \subset \mathbb{R}$ with $\alpha < \beta < \gamma$. An ordinal preference is a ranking over alternatives. For example if agent i prefers α over β over γ the corresponding ordinal preference is

$$\alpha \prec_i \beta \prec_i \gamma.$$

A cardinal preference is described by a utility function $u_i : A \to \mathbb{R}$ and induces an ordinal preference by

$$u_i(a) < u_i(a') \Leftrightarrow a \prec_i a'.$$

In this exercise we consider only mechanisms where the allocation is non-randomized.

(a) Show that any dominant strategy incentive compatible mechanism conditions only on the ordinal preferences of the agents, but not the cardinal preferences. (8pts)

Answer: Let

- n be the number of agents,
- $N \triangleq \{1, 2, \ldots, n\}$ the set of agents
- P the set of all possible strict preference relations over A,
- $\vec{\prec} = (\prec_1, \prec_2, \ldots, \prec_n) \in P^n$ the preference profile, and
- $a : P^n \to A$ the mechanism.

For all $x, y \in A$ define the relation $x \preceq_i y$ by

$$x \preceq_i y \Leftrightarrow (x \prec_i y \text{ or } x = y)$$

If the mechanism is dominant strategy incentive compatible, we have that

$$a(\prec_i', \vec{\prec}_{-i}) \preceq_i a(\prec_i, \vec{\prec}_{-i}) \;\; \forall \prec_i, \prec_i' \in P, \; \vec{\prec}_{-i} \in P^{n-1}, \; i \in N$$

This condition clearly does not depend on the cardinal preferences. □

Intuitively, since the mechanism is DSIC, then the outcome obtained by reporting truthfully has to be preferred in all possible situations. Hence, the "strength" of the preference is not important, as long as it is the most preferred alternative that the agent can achieve. Note that this argument also depends on the assumption on the allocation to be non-randomized.

(b) List all strict ordinal preferences over A. (4pts)

Answer: The list of all possible strict ordinal preferences over A is:

$$
\begin{aligned}
&\alpha \prec \beta \prec \gamma \\
&\alpha \prec \gamma \prec \beta \\
&\beta \prec \alpha \prec \gamma \\
&\beta \prec \gamma \prec \alpha \\
&\gamma \prec \alpha \prec \beta \\
&\gamma \prec \beta \prec \alpha
\end{aligned}
$$

(c) Which of those preferences are single peaked? (4pts)

Answer: The preferences that are single peaked are:

$$
\begin{aligned}
&\alpha \prec \beta \prec \gamma \\
&\beta \prec \alpha \prec \gamma \\
&\beta \prec \gamma \prec \alpha \\
&\gamma \prec \beta \prec \alpha
\end{aligned}
$$

(d) Consider the following preference with $n = 3$ agents

$$
\begin{aligned}
&\alpha \prec_1 \gamma \prec_1 \beta \\
&\gamma \prec_2 \beta \prec_2 \alpha \\
&\beta \prec_3 \alpha \prec_3 \gamma.
\end{aligned}
$$

Do the agents in this case have single peaked preferences? Which allocation would be implemented with this preference profile in the median voting mechanism where the media of the agents' most preferred allocations is implemented. (2pts)

Answer: The agents do not have single peaked preferences. In particular agent 1 does not have single peaked preferences, while agents 2 and 3 do.

The median voter mechanism would take the median from the points α, β, γ. It would implement the allocation β.

(e) Could some agent improve by misreporting her/his most preferred allocation under the preference profile given in (d)? (4pts)

Answer: Suppose agent 3 reports the preference relation $\prec_3'$ defined by

$$\gamma \prec_3' \beta \prec_3' \alpha$$

Then the mechanism would choose the median from the points β, α, α, which is α. Since $\beta \prec_3 \alpha$, it is not optimal for agent 3 to report truthfully.