Exercise 5

ECON / MATH C103 - Mathematical Economics
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due Tue Feb 28, 4:59pm

Helpful Material:

- Last week's lecture notes.

Exercise 1: Each of two agents $i \in \{1,2\}$ owns an object. The value of an agent's object is his private information, i.e. known to him but not the other agent or the designer. We assume that the values of both objects are independently uniformly distributed on $[0,1]$. Both agents assign the same value to the objects.

For example if the object of agent 1 has a value of 0.5 and the object of agent 2 has a value of 0.3 and agent 1 gets agent $2's$ object $1's$ utility is given by 0.3 and $2's$ utility by 0.5.

Each agent is asked independently and simultaneously whether he wants to exchange his object for the other agent's object. If both agents agree to trade then the objects are exchanged; otherwise each agent keeps his own object. Each agents objective is to maximize his expected payoff, there are no transfers.

(a) Describe the above situation formally as a mechanism design problem, by specifying utilities, types, and the mechanism. (3pts)

(Hint: An agent's utility here needs to depend on the other agent's type.)
Answer:

- $M = \{y(es), n(o)\}$
- $\Theta \sim U[0,1]$
- $x(m_i) = \begin{cases} \theta_j & \text{if } m_i = m_j = yes \\ \theta_i & \text{otherwise} \end{cases}$
- $t(m_i) = 0$
- $u(x,t) = x$

(b) Describe the set of Pareto efficient allocations, as well as the set of allocations that maximize the sum of the agents' utilities. (2pt)
Answer: In this case, any allocation is Pareto efficient; and any allocation maximizes the sum of utilities.

(c) Derive formally the dominant strategy equilibria of the mechanism, or show none exists. (4pts)

Answer: No dominant strategy exists. Suppose there were one, called $\sigma(\theta)$. Partition $[0,1]$ into two sets Y, N such that:

$$\theta \in \begin{cases} Y & \iff \sigma(\theta) = yes \\ N & \iff \sigma(\theta) = no \end{cases}$$

Suppose $\theta \in Y$. It's possible the other agent has an object of lower value and agrees to the exchange. In this case, saying *no* would've been optimal; and therefore σ is not an optimal strategy. Similarly, suppose $\theta \in N$. It's possible the other agent has an object of higher value and has agreed to the exchange. In this case, saying *yes* would've been optimal. Thus, it's not possible to construct an optimal strategy that works irrespective of the other agent's type & message.

(d) Derive formally the Bayes Nash equilibria of the mechanism. (if multiple equilibria exist, derive any one.)(8pts)

Answer: Let's consider the case for player 1. Her utility from saying no is:

$$u(n, (\theta_1, \theta_2)) = \theta_1$$

Her *expected* utility from saying yes is:

$$u(y, (\theta_1, \theta_2)) \quad = \quad E[\,(1_{\{m_2=y\}}\theta_2 + 1_{\{m_2=n\}}\theta_1)\ \mid \theta_1]$$

(We condition on θ_1 since player 1 only knows her own type and takes it as given)

Player 1 says yes only if her expected utility from saying yes is at least as much as from saying no. That is, $m_1 = y$ *iff*:

$$\begin{array}{rrcl} & u(n,(\theta_1,\theta_2)) & \leq & u(y,(\theta_1,\theta_2)) \\ \Longrightarrow & \theta_1 & \leq & E[\,(1_{\{m_2=y\}}\theta_2 + 1_{\{m_2=n\}}\theta_1)\ \mid \theta_1] \\ \Longrightarrow & \theta_1 & \leq & E[1_{\{m_2=y\}}\theta_2] + P(m_2=n)\theta_1 \\ \Longrightarrow & \theta_1 & \leq & \frac{E[1_{\{m_2=y\}}\theta_2]}{1-P(m_2=n)} \\ \Longrightarrow & \theta_1 & \leq & \frac{E[1_{\{m_2=y\}}\theta_2]}{P(m_2=y)} = E[\theta_2|m_2=y] \\ \Longrightarrow & \theta_1 & \leq & k_2 \end{array}$$

where $k_2 := E[\theta_2|m_2=y]$ is some real number that depends on the set of θ_2 for which player 2 trades. In other words, Player 1's strategy is to say yes if and only if her type is below k_2, and no otherwise. By the same argument player 2 trades if and only if her type is below $k_1 := E[\theta_1|m_1=y]$. Consequently, $m_2 = y \Leftrightarrow \theta_2 \leq k_1$ and thus

$$k_2 = E[\theta_2|m_2=y] = E[\theta_2|\theta_2 \leq k_1] = \frac{k_1}{2}.$$

Intuitively, player 1's best response is to chose a cut-off half as high as the cut-off used by player 2. By symmetry the same holds true for player 2 and we have that

$$k_2 = \frac{1}{2}k_1$$
$$k_1 = \frac{1}{2}k_2\,.$$

The only solution to this system of equations is $k_1 = k_2 = 0$ which means they never trade. Thus, each player saying 'No' for all types greater zero is the unique BNE.