# Exercise 7

ECON / MATH C103 - Mathematical Economics
Philipp Strack

**due March 21, 4:59pm**

**Helpful Material:**
- Last week's lecture notes.

**Exercise 1:** Consider an all-pay auction auction where the highest bid wins and every agent pays her bid. Let $\theta_i$ be the value of agent $i$ for the object, and assume that the values are i.i.d. distributed according to $F : [\underline{\theta}, \overline{\theta}] \to [0,1]$. The utility of the agent equals $\theta_i - t_i$ if she gets the object and $-t_i$ otherwise.

(a) What is the expected revenue in this auction.

**Answer:** By the Revenue Equivalence Principle we know that we can reframe the All Pay Auction as a BNIC direct mechanism. Define $x_i(\theta)$ as 1 of agent $i$ gets the object and 0 otherwise, and $X_i(\theta_i) \triangleq \mathbb{E}[x_i(\theta_i, \theta_{-i})|\theta_i]$ the expected value of $x_i$.

As the auction is an all pay auction, the agent has to pay the transfer for sure. Then the envelope condition implies

$$t_i(\theta_i) = \theta_i X_i(\theta_i) - \int_{\underline{\theta}}^{\theta_i} X_i(s)ds$$

The expected revenue is

$$\begin{aligned}
\mathbb{E}[\Pi] &= \mathbb{E}\left[\sum_{i=1}^{n} t(\theta_i)\right] \\
&\vdots \\
&= \mathbb{E}\left[\sum_{i=1}^{n} X_i(\theta_i)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right]
\end{aligned}$$

Now suppose there is a symmetric equilibrium with strictly increasing strategies $b(\theta_i)$. Then agent $i$ gets the object if $b(\theta_i) > \max_{j \neq i} b(\theta_j)$, which is the same as $\theta_i > \max_{j \neq i} \theta_j$ since $b(\cdot)$ is strictly increasing[1]. Then we have that

$$X_i(\theta_i) = P\left(\theta_i > \max_{j \neq i} \theta_j \,\middle|\, \theta_i\right) = F^{n-1}(\theta_i)$$

[1]Note that the events in which $\theta_i = \max_{j \neq i} \theta_j$ have probability zero, so we will ignore them.

Replacing

$$\begin{aligned}\mathbb{E}[\Pi] &= \mathbb{E}\left[\sum_{i=1}^{n} F^{n-1}(\theta_i)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \\ &= \mathbb{E}\left[nF^{n-1}(\theta_i)\left(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right)\right]\end{aligned}$$

With some algebra, we can prove that

$$\mathbb{E}[\Pi] = \int_{\underline{\theta}}^{\overline{\theta}} \theta_i n(1-F(\theta_i))(n-1)F^{n-2}(\theta_i)f(\theta_i)d\theta_i$$

which is the expected value of the second highest valuation among $n$ agents.

(b) Find a symmetric equilibrium of this auction.

**Answer:** Suppose there is a symmetric equilibrium with strictly increasing strategies $b(\theta_i)$. Then agent $i$ gets the object if $b(\theta_i) > \max\limits_{j\neq i} b(\theta_j)$, which is the same as $\theta_i > \max\limits_{j\neq i} \theta_j$ since $b(\cdot)$ is strictly increasing[2]. Then we have that

$$X_i(\theta_i) = P\left(\theta_i > \max_{j\neq i}\theta_j \middle| \theta_i\right) = F^{n-1}(\theta_i)$$

The expected utility for agent $i$ of bidding $b^{AP}(\theta_i)$ is $\theta_i X_i(\theta_i) - b(\theta_i)$. It is straightforward that $b(\underline{\theta}) = 0$. Then by the envelope condition

$$\begin{aligned}\theta_i X_i(\theta_i) - b^{AP}(\theta_i) &= \int_{\underline{\theta}}^{\theta_i} X_i(s)ds \\ \theta_i F^{n-1}(\theta_i) - b^{AP}(\theta_i) &= \int_{\underline{\theta}}^{\theta_i} F^{n-1}(s)ds\end{aligned}$$

Solving for $b^{AP}(\theta_i)$

$$b^{AP}(\theta_i) = \theta_i F^{n-1}(\theta_i) - \int_{\underline{\theta}}^{\theta_i} F^{n-1}(s)ds$$

(c) Compare the bid agent $i$ makes in this auction to the bid she makes in a first or second price auction when she has the valuation $\theta_i$. Explain intuitively, why the bids are ordered between the different auction formats.

**Answer:** We know that the strategies for the first and second price auctions are

$$b^{FP}(\theta_i) = \theta_i - \frac{\int_{\underline{\theta}}^{\theta_i} F^{n-1}(s)ds}{F^{n-1}(\theta_i)}, \quad b^{SP}(\theta_i) = \theta_i$$

respectively. It is straightforward to see that

$$b^{AP}(\theta_i) \leq b^{FP}(\theta_i) \leq b^{SP}(\theta_i)$$

[2]Note that the events in which $\theta_i = \max\limits_{j\neq i}\theta_j$ have probability zero, so we will ignore them.

(d) Suppose now that the agent's utility is given by $\theta_i - c(t_i)$ if she gets the object and $-c(t_i)$ otherwise, where $c$ is a strictly increasing function. Characterize a symmetric equilibrium bidding strategy.

**Answer:** Let $b^d(\theta_i)$ be the optimal strategy. Assume the strategy is strictly increasing, so the definition of $X_i(\theta_i)$ is still valid. It is still straightforward that the expected utility of an agent with the lowest possible is zero. The envelope condition becomes

$$\theta_i X_i(\theta_i) - c\left(b^d(\theta_i)\right) = \int_{\underline{\theta}}^{\theta_i} X_i(s)ds$$

Replacing $X_i$ and solving for $b^d(\theta_i)$

$$b^d(\theta_i) = c^{-1}\left(\theta_i F^{n-1}(\theta_i) - \int_{\underline{\theta}}^{\theta_i} F^{n-1}(s)ds\right) = c^{-1}\left(b^{AP}(\theta_i)\right)$$

**Exercise 2:** Consider a situation with two agents $i \in \{1,2\}$ and a single object. Let $\theta_i$ be the value of agent $i$ for the object. Assume that agent 1's value is uniformly distributed on $[0, \overline{\theta}_1]$ and agent 2's value is uniformly distributed on $[0, \overline{\theta}_2]$, where $\overline{\theta}_2 > \overline{\theta}_1$.

(a) What is the expected valuation each agent assigns to the object according to the prior? Which agent values the object more on average? (4 pts)

**Answer:** Note that the CDF of agent $i$ is $F_i(\theta_i) = \frac{\theta_i}{\overline{\theta}_i}$ for $i = 1,2$. Then we have that

$$\mathbb{E}[\theta_1] = \frac{\overline{\theta}_1}{2}, \ \mathbb{E}[\theta_2] = \frac{\overline{\theta}_2}{2}$$

It is clear that agent 2 values the object more on average.

(b) What is the expected maximal valuation as a function of $(\overline{\theta}_1, \overline{\theta}_2)$? (4 pts)

**Answer:** Define the maximum valuation as $X \triangleq \max\{\theta_1, \theta_2\}$. Then its CDF is

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{\overline{\theta}_1 \overline{\theta}_2} & \text{if } 0 \leq x < \overline{\theta}_1 \\ \frac{x}{\overline{\theta}_2} & \text{if } \overline{\theta}_1 \leq x < \overline{\theta}_2 \\ 1 & \text{otherwise} \end{cases}$$

then its pdf is

$$f_X(x) = \begin{cases} 0 & \text{if } x \notin [0, \overline{\theta}_2] \\ \frac{2x}{\overline{\theta}_1 \overline{\theta}_2} & \text{if } 0 \leq x < \overline{\theta}_1 \\ \frac{1}{\overline{\theta}_2} & \text{if } \overline{\theta}_1 \leq x < \overline{\theta}_2 \end{cases}$$

Finally, its expected value is

$$\mathbb{E}[X] = \int_0^{\overline{\theta}_2} x f_X(x) dx = \frac{2}{\overline{\theta}_1 \overline{\theta}_2} \int_0^{\overline{\theta}_1} x^2 dx + \frac{1}{\overline{\theta}_2} \int_{\overline{\theta}_1}^{\overline{\theta}_2} x dx = \frac{1}{\overline{\theta}_2} \left( \frac{\overline{\theta}_1^2}{6} + \frac{\overline{\theta}_2^2}{2} \right)$$

(c) Calculate the virtual valuations for each agent. (4 pts)

**Answer:** The virtual valuations are

$$J_1(\theta_1) = \theta_1 - \frac{1 - \frac{\theta_1}{\overline{\theta}_1}}{\frac{1}{\overline{\theta}_1}} = 2\theta_1 - \overline{\theta}_1$$

$$J_2(\theta_1) = \theta_2 - \frac{1 - \frac{\theta_2}{\overline{\theta}_2}}{\frac{1}{\overline{\theta}_2}} = 2\theta_2 - \overline{\theta}_2$$

(d) Construct a revenue maximizing mechanism, where both agents participate voluntarily. Provide an economic intuition for the asymmetry in the mechanism. (8 pts)

**Answer:** Since the virtual valuations are increasing, we know that the revenue maximizing mechanism gives the object to the agent with the highest virtual value as long as it is positive. Then the optimal allocation rule is

$$x^*(\theta) = \begin{cases} (1,0) & \text{if } \theta_1 > \theta_2 - \frac{\overline{\theta}_2 - \overline{\theta}_1}{2} \text{ and } \theta_1 \geq \frac{\overline{\theta}_1}{2} \\ (0,1) & \text{if } \theta_1 \leq \theta_2 - \frac{\overline{\theta}_2 - \overline{\theta}_1}{2} \text{ and } \theta_2 \geq \frac{\overline{\theta}_2}{2} \\ (0,0) & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} X_1(\theta_1) &= P\left(\theta_1 > \theta_2 - \frac{\overline{\theta}_2 - \overline{\theta}_1}{2} \text{ and } \theta_1 \geq \frac{\overline{\theta}_1}{2} \middle| \theta_1\right) = \begin{cases} 0 & \text{if } \theta_1 < \frac{\overline{\theta}_1}{2} \\ \frac{1}{2} - \frac{\overline{\theta}_1 - 2\theta_1}{2\overline{\theta}_2} & \text{if } \frac{\overline{\theta}_1}{2} \leq \theta_1 < \overline{\theta}_1 \end{cases} \\ X_2(\theta_1) &= P\left(\theta_2 > \theta_1 + \frac{\overline{\theta}_2 - \overline{\theta}_1}{2} \text{ and } \theta_2 \geq \frac{\overline{\theta}_2}{2} \middle| \theta_1\right) = \begin{cases} 0 & \text{if } \theta_2 < \frac{\overline{\theta}_2}{2} \\ \frac{1}{2} - \frac{\overline{\theta}_2 - 2\theta_2}{2\overline{\theta}_1} & \text{if } \frac{\overline{\theta}_2}{2} \leq \theta_2 < \frac{\overline{\theta}_1 + \overline{\theta}_2}{2} \\ 1 & \text{if } \theta_2 \geq \frac{\overline{\theta}_1 + \overline{\theta}_2}{2} \end{cases} \end{aligned}$$

Finally we can set the expected value of the transfer to be equal to the actual value of the transfer (set the mechanism as something similar to an all pay auction), then

$$\begin{aligned} t_1^*(\theta_1) &= X_1(\theta_1)\theta_1 - \int_0^{\theta_1} X_1(s)ds = \cdots = \begin{cases} 0 & \text{if } \theta_1 < \frac{\overline{\theta}_1}{2} \\ \frac{\theta_1^2}{2\overline{\theta}_2} + \frac{\overline{\theta}_1}{4}\left(1 + \frac{1-\overline{\theta}_1}{\overline{\theta}_2}\right) & \text{otherwise} \end{cases} \\ t_2^*(\theta_2) &= X_2(\theta_2)\theta_2 - \int_0^{\theta_2} X_2(s)ds = \cdots = \begin{cases} 0 & \text{if } \theta_2 < \frac{\overline{\theta}_2}{2} \\ \frac{\theta_2^2}{2\overline{\theta}_1} + \frac{\overline{\theta}_2}{4}\left(1 + \frac{1-\overline{\theta}_2}{\overline{\theta}_1}\right) & \text{if } \frac{\overline{\theta}_2}{2} \leq \theta_2 < \frac{\overline{\theta}_1 + \overline{\theta}_2}{2} \\ \frac{\overline{\theta}_2}{2} & \text{otherwise} \end{cases} \end{aligned}$$

The economic intuition for the asymmetry in the mechanism is that agent 1's low valuations could be an incentive for agent 2 to submit lower bids. Therefore to give advantage to agent 1 gives incentives to agent 2 to raise her bid.

(e) What is the expected revenue in that mechanism as a function of $(\overline{\theta}_1, \overline{\theta}_2)$. (4 pts)

**Answer:** Since the payment of the transfer is independent of who gets the object, the expected revenue is

$$\begin{aligned} \mathbb{E}[\Pi] &= \mathbb{E}\left[t_1^*(\theta_1) + t_2^*(\theta_2)\right] \\ &= \mathbb{E}\left[\frac{\theta_1^2}{2\overline{\theta}_2} + \frac{\overline{\theta}_1}{4}\left(1 + \frac{1-\overline{\theta}_1}{\overline{\theta}_2}\right) \middle| \frac{\overline{\theta}_1}{2} \leq \theta_1 < \overline{\theta}_1\right] \cdot P\left(\frac{\overline{\theta}_1}{2} \leq \theta_1 < \overline{\theta}_1\right) + \\ &\quad + \mathbb{E}\left[\frac{\theta_2^2}{2\overline{\theta}_1} + \frac{\overline{\theta}_2}{4}\left(1 + \frac{1-\overline{\theta}_2}{\overline{\theta}_1}\right) \middle| \frac{\overline{\theta}_2}{2} \leq \theta_2 < \frac{\overline{\theta}_1 + \overline{\theta}_2}{2}\right] \cdot P\left(\frac{\overline{\theta}_2}{2} \leq \theta_2 < \frac{\overline{\theta}_1 + \overline{\theta}_2}{2}\right) + \\ &\quad + \mathbb{E}\left[\frac{\overline{\theta}_2}{2} \middle| \frac{\overline{\theta}_1 + \overline{\theta}_2}{2} \leq \theta_2 < \overline{\theta}_2\right] \cdot P\left(\frac{\overline{\theta}_1 + \overline{\theta}_2}{2} \leq \theta_2 < \overline{\theta}_2\right) \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[\Pi] &= \mathbb{E}\left[\frac{\theta_1^2}{2\overline{\theta}_2}\middle|\frac{\overline{\theta}_1}{2}\leq\theta_1<\overline{\theta}_1\right]\cdot P\left(\frac{\overline{\theta}_1}{2}\leq\theta_1<\overline{\theta}_1\right)+\\
&+\left(\frac{\overline{\theta}_1}{4}+\frac{\overline{\theta}_1}{4\overline{\theta}_2}-\frac{\overline{\theta}_1^2}{4\overline{\theta}_2}\right)\cdot P\left(\frac{\overline{\theta}_1}{2}\leq\theta_1<\overline{\theta}_1\right)+\\
&+\mathbb{E}\left[\frac{\theta_2^2}{2\overline{\theta}_1}\middle|\frac{\overline{\theta}_2}{2}\leq\theta_2<\frac{\overline{\theta}_1+\overline{\theta}_2}{2}\right]\cdot P\left(\frac{\overline{\theta}_2}{2}\leq\theta_2<\frac{\overline{\theta}_1+\overline{\theta}_2}{2}\right)+\\
&+\left(\frac{\overline{\theta}_2}{4}+\frac{\overline{\theta}_2}{4\overline{\theta}_1}-\frac{\overline{\theta}_2^2}{4\overline{\theta}_1}\right)\cdot P\left(\frac{\overline{\theta}_2}{2}\leq\theta_2<\frac{\overline{\theta}_1+\overline{\theta}_2}{2}\right)+\\
&+\frac{\overline{\theta}_2}{2}\cdot P\left(\frac{\overline{\theta}_1+\overline{\theta}_2}{2}\leq\theta_2<\overline{\theta}_2\right)\\
&= \left(\frac{7\overline{\theta}_1^2}{48\overline{\theta}_2}\right)+\left(\frac{\overline{\theta}_1}{4}+\frac{\overline{\theta}_1}{4\overline{\theta}_2}-\frac{\overline{\theta}_1^2}{4\overline{\theta}_2}\right)\frac{1}{2}+\\
&+\left(\frac{\overline{\theta}_1^2}{48\overline{\theta}_2}+\frac{3\overline{\theta}_1}{48}+\frac{3\overline{\theta}_2}{48}\right)+\left(\frac{\overline{\theta}_2}{4}+\frac{\overline{\theta}_2}{4\overline{\theta}_1}-\frac{\overline{\theta}_2^2}{4\overline{\theta}_1}\right)\frac{\overline{\theta}_1}{2\overline{\theta}_2}+\frac{\overline{\theta}_2}{2}\cdot\left(\frac{1}{2}-\frac{\overline{\theta}_1}{2\overline{\theta}_2}\right)\\
&= \frac{1}{8\overline{\theta}_2}\left(\frac{\overline{\theta}_1^2}{3}+\frac{3\overline{\theta}_2^2}{2}+\frac{\overline{\theta}_1\overline{\theta}_2}{2}+\overline{\theta}_1+\overline{\theta}_2\right)
\end{aligned}
$$

(f) Can you find two values of $(\overline{\theta}_1,\overline{\theta}_2)$ such that the expected maximal valuation is the same, but the revenue raised in the mechanism is different. (4 pts)

**Answer:** Given the previous results, to find these two pairs of values is trivial. For example, set $\overline{\theta}_1=1$ and $\overline{\theta}_2=2$. Then

$$
\begin{aligned}
\mathbb{E}[X|\overline{\theta}_1=1,\overline{\theta}_2=10] &= \frac{13}{12}\\
\mathbb{E}[\Pi|\overline{\theta}_1=2,\overline{\theta}_2=10] &= \frac{31}{48}\approx 0.65
\end{aligned}
$$

Note that for $\overline{\theta}_1'=\frac{1}{2}$ and $\overline{\theta}_2'=\frac{13+\sqrt{157}}{12}\approx 2.13$ we have

$$
\begin{aligned}
\mathbb{E}[X|\overline{\theta}_1'=\frac{1}{2},\overline{\theta}_2=\frac{13+\sqrt{157}}{12}] &= \frac{13}{12}\\
\mathbb{E}[\Pi|\overline{\theta}_1'=\frac{1}{2},\overline{\theta}_2=\frac{13+\sqrt{157}}{12}] &= \frac{251-11\sqrt{157}}{192}\approx 0.59
\end{aligned}
$$