Econ C103 Problem Set 4

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Exercise 1

(a) Let $\hat{m}_i = \max_{j \neq i}\{m_j\}$ denote the maximum bid submitted by agents $j \neq i$.

- The set of possible messages for each agent is $M_i = \mathbb{R}_+$.
- The allocation is

$$x_i = \begin{cases} 1 & \text{if } m_i > \hat{m}_i \\ 0 & \text{otherwise} \end{cases}.$$

- The transfer is

$$t_i = \begin{cases} \hat{m}_i & \text{if } m_i > \hat{m}_i \\ 0 & \text{otherwise} \end{cases}.$$

(b) We proceed by cases.

1. If $\theta_i > \hat{m}_i$, then
   - Bids $m_i > \theta_i$ result in winning the auction with positive payoff $\theta_i - \hat{m}_i > 0$.
   - Bids $\hat{m}_i < m_i < \theta_i$ result in winning the auction with positive payoff $\theta_i - \hat{m}_i > 0$.
   - Bids $m_i < \hat{m}_i$ result in losing the auction with payoff 0.

   So the utility-maximizing bid is any $m_i > \hat{m}_i$.
2. If $\theta_i < \hat{m}_i$, then
   - Bids $m_i > \hat{m}_i$ result in winning the auction with negative payoff $\theta_i - \hat{m}_i < 0$.
   - Bids $\theta_i < m_i < \hat{m}_i$ result in losing the auction with payoff 0.
   - Bids $m_i < \theta_i$ result in losing the auction with payoff 0.

   So the utility-maximizing bid is any $m_i < \hat{m}_i$.

(c) Bidding truthfully, i.e. $m_i = \theta_i$, maximizes the agent's utility. We can see this from the previous section: If $\theta_i > \hat{m}_i$, then any bid $m_i > \hat{m}_i$ is optimal, so $m_i = \theta_i$ is optimal. Likewise, if $\theta_i < \hat{m}_i$, then any bid $m_i < \hat{m}_i$ is optimal, so $m_i = \theta_i$ is optimal.

(d) We have just shown that the strategy of bidding truthfully, $s_i(\theta_i) = \theta_i$, is optimal for each agent independent of what the other agents do. Bidding truthfully is thus by definition a dominant strategy equilibrium. It is a *unique* dominant strategy equilibrium because any deviation from this bid $s_i(\theta_i) = \theta_i \pm \epsilon$ would not be optimal in some cases ($\theta_i + \epsilon$ is not optimal if $\theta_i < \hat{m}$ and $\theta_i - \epsilon$ is not optimal if $\theta_i > \hat{m}$).

The expected revenue is thus the expected value of the second of n draws from the distribution of types F. For θ to be the second-highest bid, we need exactly $n-2$ of the other $n-1$ agents to bid less than θ, which occurs with probability $F(\theta)^{(n-2)}$, and we need exactly one of the other $n-1$ agents to bid more than θ, which occurs with probability $(n-1)(1-F(\theta))$. So

$$\mathbb{E}[t] = \int_0^{\bar{\theta}} \theta f(\theta) F(\theta)^{(n-2)}(n-1)(1-F(\theta))\, d\theta$$

(e) No. Consider the best response of player 1. If all other players bid 0, then player 1 could bid $\bar{\theta}$ and win the auction with transfer 0. But if one other player bids m, and $\theta_1 < m$, then the best response is to bid $m_1 < m$. So, the best response of player 1 depends on the behavior of other players; bidding $\bar{\theta}$ is not always optimal and thus this not a dominant strategy equilibrium.

(f) Yes. Player 1 expects $\hat{m}_1 = 0$, so any $m_1 > 0$ results in winning the auction, with positive payoff. The other players expect $\hat{m}_i = \bar{\theta}$, so they expect to lose the auction no matter what message they send; any $m_i \in [0, \bar{\theta}]$ results in payoff 0. Each agent's strategy is optimal given their expectation of the other agents' strategies, so this is a Bayes-Nash equilibrium.

(g) Given this strategy profile, the good will be allocated to player 1 with transfer $\hat{m}_1 = 0$, so the expected revenue is 0.

Exercise 2

(a) The intuition is that if s_i is the best response to *every* possible set of messages m_{-i}, it is also the best response to the expectation of the Bayes-Nash equilibrium set of messages $\mathbb{E}[s_{-i}(\theta) \mid \theta]$.

Formally, if s_i is a dominant strategy equilibrium, then

$$\forall m_{-i} : s_i(\theta_i) \in \arg\max_{m_i \in M_i} u(a(m_i, m_{-i}), \theta_i)$$

so

$$\forall m_i \; \forall m_{-i} : u(a(s_i(\theta_i), m_{-i}), \theta_i) > u(a(m_i, m_{-i}), \theta_i).$$

Then

$$\forall \theta_i \; \forall m_i : u(a(s_i(\theta_i), s_{-i}(\theta_i)), \theta_i) > u(a(m_i, s_{-i}(\theta_i)), \theta_i).$$

and if θ_i is distributed with density $f(\theta_i)$ and support $[\underline{\theta_i}, \bar{\theta}_i]$

$$\forall m_i : \int_{\underline{\theta_i}}^{\bar{\theta}_i} f(\theta_i)\; u(a(s_i(\theta_i), s_{-i}(\theta_i)), \theta_i)\, d\theta_i > \int_{\underline{\theta_i}}^{\bar{\theta}_i} f(\theta_i)\; u(a(m_i, s_{-i}(\theta_i)), \theta_i)\, d\theta_i.$$

Thus

$$\forall m_i : \mathbb{E}[u(a(s_i(\theta), s_{-i}(\theta_i)), \theta_i) \mid \theta_i] > \mathbb{E}[u(a(m_i, s_{-i}(\theta_i)), \theta_i) \mid \theta_i]$$

and

$$s_i(\theta_i) \in \arg\max_{m_i \in M_i} \mathbb{E}[u(a(m_i, s_{-i}(\theta_i)), \theta_i) \mid \theta_i]$$

so s_i is a Bayes-Nash equilibrium.

(b) Consider a mechanism in which two agents A and B are each allocated a good if and only if they send the same message from the set $\{0, 1\}$. Formally, $M_i = \{0, 1\}$, $x_i = 1$ if $m_A = m_B = 0$ or $m_A = m_B = 1$, otherwise $x_i = 0$. Assume the agents both have positive valuation for the good and that there are no transfers, i.e. $t_i = 0$.

Then the best response for A is $m_A = 0$ if $m_B = 0$, or $m_A = 1$ if $m_B = 1$. The situation is symmetric for B. There is no response that is optimal *independent* of what the other agent does, thus there is no dominant strategy equilibrium.

There are, however, two Bayes-Nash equilibria: $m_A = m_B = 1$ and $m_A = m_B = 0$.