Econ C103 Problem Set 2

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Exercise 1

(a) In this context, a mechanism consists of

- A set of messages M.
- A rule that maps from the set of messages to the (real-valued) tax the worker pays $t : M \to \mathbb{R}$.
- A rule that maps from the set of messages to the type of job the worker has $x : M \to \{l, h\}$.

(b) A direct mechanism is one in which the worker reports their ability θ. This implies $M = [\underline{\theta}, \bar{\theta}]$.

(c) An incentive compatible direct mechanism is one in which it is always optimal for the worker to tell the truth about their ability, that is, $x(m) = x(\theta)$. This implies

$$w_{x(\theta)} - t(\theta) - \frac{x(\theta)}{\theta} \geq \max_m \left\{ w_{x(m)} - t(m) - \frac{x(m)}{\theta} \right\}$$

(d) Incentive compatibility implies that an individual with true ability θ should prefer to report $\theta \neq \theta'$.

$$w_{x(\theta)} - t(\theta) - \frac{x(\theta)}{\theta} \geq w_{x(\theta')} - t(\theta') - \frac{x(\theta')}{\theta}$$

$$\frac{1}{\theta}(x(\theta') - x(\theta)) \geq w_{x(\theta')} - t(\theta') - (w_{x(\theta)} - t(\theta)) \tag{1}$$

Likewise, an individual with true ability θ' should prefer to report $\theta' \neq \theta$.

$$w_{x(\theta')} - t(\theta') - \frac{x(\theta')}{\theta'} \geq w_{x(\theta)} - t(\theta) - \frac{x(\theta)}{\theta'}$$

$$\frac{1}{\theta'}(x(\theta) - x(\theta')) \geq w_{x(\theta)} - t(\theta) - (w_{x(\theta')} - t(\theta')) \tag{2}$$

Adding (1) and (2)

$$(\frac{1}{\theta} - \frac{1}{\theta'})(x(\theta') - x(\theta)) \geq 0$$

Since $0 \leq \underline{\theta}$, both θ and θ' are positive, so we can rearrange to find

$$(\theta' - \theta)(x(\theta') - x(\theta)) \geq 0$$

Thus, for $\theta' > \theta$, $x(\theta') > x(\theta)$. This proves $x(\theta)$ is monotonically increasing.

(e) Since $x = \{l, h\}$, we define $\theta_0 = \sup\{\theta : x(\theta) = l\}$. Since a worker will always report the type that leads to the lowest taxes, the tax can only depend on the type of job the worker has, not on their ability.

$$t(\theta) = \begin{cases} t_1, & \theta > \theta_0 \\ t_0, & \theta < \theta_0 \end{cases}$$

Workers with ability $\theta < \theta_0$ should prefer the low-intensity job and associated tax

$$w_l - t_0 - \frac{l}{\theta_0} \geq w_h - t_1 - \frac{h}{\theta_0} \tag{3}$$

Likewise, workers with ability $\theta > \theta_0$ should prefer the high-intensity job and associated tax

$$w_l - t_0 - \frac{l}{\theta_0} \leq w_h - t_1 - \frac{h}{\theta_0} \tag{4}$$

Combining (3) and (4)

$$w_l - t_0 - \frac{l}{\theta_0} = w_h - t_1 - \frac{h}{\theta_0}$$
$$t_1 = t_0 + (w_h - w_l) - \frac{h - l}{\theta_0}$$

So

$$t(\theta) = \begin{cases} t_0 + (w_h - w_l) - \frac{h-l}{\theta_0}, & \theta > \theta_0 \\ t_0, & \theta < \theta_0 \end{cases}$$

(f) Since the outcome of any mechanism can be implemented in an incentive-compatible direct mechanism, the set of outcomes that can be implemented in any mechanism is characterized by the job allocation rule

$$x(\theta) = \begin{cases} h, & \theta > \theta_0 \\ l, & \theta < \theta_0 \end{cases}$$

And the taxation rule

$$t(\theta) = \begin{cases} t_0 + (w_h - w_l) - \frac{h-l}{\theta_0}, & \theta > \theta_0 \\ t_0, & \theta < \theta_0 \end{cases}$$

(g) Consider an indirect mechanism where the worker decides between high and low intensity jobs and reports only their wage $w_x \in \mathbb{R}$. Then $M = \mathbb{R}$. One outcome that can be implemented in this mechanism is a uniform tax τ on wages

$$t(w_x) = \tau w_x$$

The allocation rule is meaningless, since the worker decides their own job.

(h) The participation constraint implies the agent has positive utility in all cases

$$\max_m \left\{ w_{x(m)} - t(m) - \frac{x(m)}{\theta} \right\} \geq 0$$

In the context of taxation, this means that wages net of taxes and the cost of effort from working must be positive. If we consider a world in which unemployment is an option, i.e. the worker can choose to hold no job, receive no wage, and pay no tax ($x = w_x = t = 0$), then the participation constraint is binding in the sense that an agent can opt for unemployment and thereby refuse to participate in the mechanism. If we require that everyone hold a job and pay taxes, then the participation constraint does not apply.

(i) For $l = w_l = 0$, we have

$$E[t] = F(\theta_0)t_0 + (1 - F(\theta_0))(t_0 + w_h - \frac{h}{\theta_0})$$

If we restrict ourselves to mechanisms where $t \leq w_{x(\theta)}$, then $t_0 \leq 0$, since $w_l = 0$. Clearly, the expected tax revenue is increasing in t_0, so the maximum expected revenue occurs for $t_0 = 0$. This implies a tax structure

$$t(\theta) = \begin{cases} w_h - \frac{h}{\theta_0}, & \theta > \theta_0 \\ 0, & \theta < \theta_0 \end{cases}$$

(j) When $t = 0$, the worker will choose the high intensity job if

$$w_h - \frac{h}{\theta} > 0$$

$$\theta > \frac{h}{w_h}$$

With $w_h = 4$ and $h = 1$, workers with ability $\theta > \frac{1}{4}$ will choose the high intensity job. Given that θ is uniformly distributed, this implies the worker will choose the high intensity job with probability $\frac{3}{4}$.

(k) With $t_0 = 0$, $w_h = 4$, $h = 1$, $F(\theta) = \theta$

$$E[t] = (1 - \theta_0)(4 - \frac{1}{\theta_0}) = 5 - 4\theta_0 - \frac{1}{\theta_0}$$

The expected revenue is maximized for $\frac{dE[t]}{d\theta_0} = 0$

$$-4 + \frac{1}{\theta_0^2} = 0$$

$$\theta_0 = \frac{1}{2}$$

So, workers with ability $\theta > \frac{1}{2}$ will choose the high intensity job. Given that θ is uniformly distributed, this implies the worker will choose the high intensity job with probability $\frac{1}{2}$.