Econ C103 Problem Set 1

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January 23, 2017

Exercise 1

(a) $E[x] = \int\limits_{\mathbb{R}} xf(x)dx$

$V[x] = \int\limits_{\mathbb{R}} x^2 f(x)dx - (E[x])^2$

(b) $\int\limits_{\mathbb{R}} v(x)f(x)dx = -\int\limits_{\mathbb{R}} v(x)\frac{d}{dx}(1-F(x))dx = \int\limits_{\mathbb{R}} v'(x)(1-F(x))dx - v(x)(1-F(x))\Big|_{-\infty}^{+\infty}$, using integration by parts. This equals $\int\limits_{\mathbb{R}} v'(x)(1-F(x))dx + \lim\limits_{x\to-\infty} v(x)$, since $\lim\limits_{x\to+\infty}(1-F(x)) = 0$ and $\lim\limits_{x\to-\infty}(1-F(x)) = 1$.

(c) $P(y_1 < x) = P(x_1 < x) \cap P(x_2 < x)$, which implies $F_{y_1}(x) = F(x)^2$. Thus $f_{y_1}(x) = 2F(x)f(x)$.

$P(y_2 < x) = 1 - P(y_2 \geq x) = 1 - (P(x_1 \geq x) \cap P(x_2 \geq x))$, which implies $F_{y_2}(x) = 1 - (1 - F(x))^2$. Thus $f_{y_2}(x) = 2(1 - F(x))f(x)$.

(d) $E[y_1] = 2\int\limits_{\mathbb{R}} xF(x)f(x)dx$

$E[y_2] = 2\int\limits_{\mathbb{R}} x(1 - F(x))f(x)dx$

(e) For $x \geq z$, $f_{y_1}(x|x \geq z) = \frac{P(y_1=x)}{P(y_1 \geq z)} = \frac{2F(x)f(x)}{1-F(z)^2}$. For $x < z$, $f_{y_1}(x|x \geq z) = 0$. So $E[y_1|y_1 > z] = \frac{2}{1-F(z)^2}\int\limits_z^\infty xF(x)f(x)dx$.

(f) For $x \geq z$, $f_{y_1}(x|y_2 = z) = \frac{P(y_1=x)\cap P(y_2=z)}{P(y_2=z)} = \frac{2f(x)f(z)}{2(1-F(z))f(z)} = \frac{f(x)}{1-F(z)}$. For $x < z$, $f_{y_1}(x|y_2 = z) = 0$, since the maximum cannot be less than the minimum. So $E[y_1|y_2 = z] = \frac{1}{1-F(z)}\int\limits_z^\infty xf(x)dx$.

Exercise 2

(a) Since $m(\theta) = x^*$ is a maximum, it is defined implicitly by $f(x^*, \theta) = \frac{\partial w}{\partial x} = 0$. By the implicit function theorem, $m'(\theta) = \frac{\partial x^*}{\partial \theta} = -\frac{\partial f/\partial\theta}{\partial f/\partial x^*} = -\frac{\partial^2 w/\partial x\partial\theta}{\partial^2 w/\partial x^2}$.

(b) We are given that w is concave in x, so $\frac{\partial^2 w}{\partial x^2} < 0$. Thus $m(\theta)$ is increasing for all $\theta \in \Theta$ when $\frac{\partial^2 w}{\partial x\partial\theta} > 0$ and decreasing when $\frac{\partial^2 w}{\partial x\partial\theta} < 0$.

(c) $v(\theta) = w(m(\theta), \theta)$, so by the chain rule $v'(\theta) = \frac{\partial w}{\partial x^*}m'(\theta) + \frac{\partial w}{\partial\theta}$. But $\frac{\partial w}{\partial x^*} = 0$, since the derivative of the objective function is 0 at the maximum. So $v'(\theta) = \frac{\partial w}{\partial\theta} = w_\theta(m(\theta), \theta)$. This is an illustration of the envelope theorem.