Exercise 2 - Suggested Solutions

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1. (a) In this context, a mechanism is a tuple $(M, (t, x))$ where:
   - M is the set of messages that the worker can send to the principal.
   - t is a transfer rule $t : M \to \mathbb{R}$.
   - x is an allocation rule $x : M \to \{l, h\}$.

   (b) A direct mechanism (by definition) is a mechanism in which the message space is the same as the type space, i.e., $M = \Theta$

   (c) Incentive Compatibility is the idea that it's in the worker's interest to report his true type/ability. Formally, $\forall \theta, \theta' \in \Theta$,

$$u((x(\theta), t(\theta)), \theta) \quad \geq \quad u((x(\theta'), t(\theta')), \theta) \tag{1}$$

   Or equivalently,

$$u((x(\theta), t(\theta)), \theta) \quad = \quad \max_{m \in \Theta} u((x(m), t(m)), \theta) \tag{2}$$

   (d) Suppose there is an IC direct mechanism in which x is not increasing. Then $\exists \theta, \theta' \in \Theta$ such that $\theta > \theta'$ and $x(\theta) < x(\theta')$. As the mechanism is IC, for a worker of ability θ is true that

$$\begin{aligned}
u((x(\theta), t(\theta)), \theta) \quad &\geq \quad u((x(\theta'), t(\theta')), \theta) \\
\big(w_{x(\theta)} - t(\theta)\big) - \frac{x(\theta)}{\theta} \quad &\geq \quad \big(w_{x(\theta')} - t(\theta')\big) - \frac{x(\theta')}{\theta} \\
\frac{x(\theta') - x(\theta)}{\theta} \quad &\geq \quad w_{x(\theta')} - w_{x(\theta)} - t(\theta') + t(\theta) \qquad (3)
\end{aligned}$$

   Similarly for a worker with ability θ'

$$\begin{aligned}
u((x(\theta'), t(\theta')), \theta') \quad &\geq \quad u((x(\theta), t(\theta)), \theta') \\
\big(w_{x(\theta')} - t(\theta')\big) - \frac{x(\theta')}{\theta'} \quad &\geq \quad \big(w_{x(\theta)} - t(\theta)\big) - \frac{x(\theta)}{\theta'} \\
w_{x(\theta')} - w_{x(\theta)} - t(\theta') + t(\theta) \quad &\geq \quad \frac{x(\theta') - x(\theta)}{\theta'} \qquad (4)
\end{aligned}$$

   Equations (3) and (4) imply that

$$\frac{x(\theta') - x(\theta)}{\theta} \quad \geq \quad \frac{x(\theta') - x(\theta)}{\theta'}$$

   Given that $x(\theta) < x(\theta')$ equation (7) implies that $\theta' \geq \theta$ which contradicts the original assumption $\theta > \theta'$. Therefore in every IC direct mechanism the allocation rule x has to be increasing.

(e) As x is nondecreasing, we can define $\theta^* \in \Theta$ as

$$\theta^* \triangleq \sup\{\theta \in \mathbb{R} : x(\theta) = l\}$$

Since utility is decreasing in the size of the transfer $t(\theta)$ the transfer can only depend on the allocation for the mechanism to be IC.Therefore it can be written as

$$t(\theta) = \begin{cases} t_l & \text{if } \theta \leq \theta^* \\ t_h & \text{otherwise} \end{cases}$$

As the worker with ability θ^* gets the allocation $m(\theta^*) = l$ and the mechanism is IC we have that

$$w_l - t_l - \frac{l}{\theta^*} \geq w_h - t_h - \frac{h}{\theta^*} \tag{5}$$

Also, as all the workers with ability $\theta > \theta^*$ get the allocation $x(\theta) = h$ and the mechanism is IC we have that $\forall \theta > \theta^*$

$$w_h - t_h - \frac{h}{\theta} \geq w_l - t_l - \frac{l}{\theta} \tag{6}$$

As the utility is continuous in θ (since $\theta > 0$) we have that equations (5) and (6) imply that

$$w_l - t_l - \frac{l}{\theta^*} = w_h - t_h - \frac{h}{\theta^*} \Leftrightarrow t_h = w_h - w_l + t_l - \frac{h-l}{\theta^*}$$

(f) The answer depends on your assumptions.

case 1: You could assume this is a society where people have no choice but to work - they can only choose between jobs l and h, and be taxed accordingly. Since they can't "opt out" of work (and taxation), PC does not apply.

case 2: people are allowed to opt out – and *do* opt out – if their utility turns negative. In this case, if the participation constraint (PC) is satisfied for the worker with lowest ability $\theta = \underline{\theta}$ then it is satisfied for all the workers. The proof is straightforward from the fact that given (x, t) the utility is increasing in θ, then all the workers with $x = l$ ($\theta \leq \theta^*$) have non-negative utility.

Since the worker with $\theta = \theta^*$ is indifferent between $x = h$ and $x = l$, then her utility with $x = h$ has to be non-negative, and therefore the utility for all the workers with $\theta > \theta^*$ also has to be.

Assuming that the PC is satisfied the set of outcomes (x, t) that can be implemented in any mechanism are of the form

$$x(\theta) = \begin{cases} l & \text{if } \theta \leq \theta^* \\ h & \text{otherwise} \end{cases}, \quad t(\theta) = \begin{cases} t_l & \text{if } \theta \leq \theta^* \\ w_h - w_l + t_l - \frac{h-l}{\theta^*} & \text{otherwise} \end{cases}, \quad t_l \leq w_l - \frac{l}{\underline{\theta}}$$

for any $\theta^* \in \Theta$.

(g) An implementation could be a mechanism in which the worker has to decide between taking the high or the low intensity job and the taxes are $t_l = \frac{l}{\underline{\theta}}$ and $t_h = w_h - w_l + t_l - \frac{h-l}{\theta^*}$.

(h) Since $t_l \leq w_l = 0$ and given that t_l does not affect the IC constraint it is clear that the revenue maximizing mechanism has $t_l = 0$. Given the characteristics of the transfer function given in answer (e) we have that the structure of the direct mechanism that maximizes $\mathbb{E}[t]$ is

$$x(\theta) = \begin{cases} l & \text{if } \theta \leq \theta^* \\ h & \text{otherwise} \end{cases}, \quad t(\theta) = \begin{cases} 0 & \text{if } \theta \leq \theta^* \\ w_h - \frac{h}{\theta^*} & \text{otherwise} \end{cases}$$

(i) If $w_l = l = 0$, $w_h = 4$, $h = 1$, and $t = 0$, the agent chooses the high intensity job if

$$\begin{aligned} w_h - t - \frac{h}{\theta} &\geq w_l - t - \frac{l}{\theta} \\ \theta &\geq \frac{1}{4} \end{aligned}$$

The probability that he takes the job is

$$P\left(\theta \geq 1/4\right) = 1 - P\left(\theta < 1/4\right) = 1 - F\left(1/4\right) = 1 - 1/4 = 3/4$$

(j) Given the answer in (h) we can write θ^* as a function of t_h as

$$\theta^* = \frac{h}{w_h - t_h}$$

Then the expected revenue collected from taxes can be written as

$$\begin{aligned} \mathbb{E}[t] &= \int_{\theta^*}^{\overline{\theta}} t_h dF(\theta) \\ &= t_h\left(1 - F(\theta^*)\right) \\ &= t_h\left(1 - \theta^*\right) \\ &= t_h\left(1 - \frac{h}{w_h - t_h}\right) \\ &= t_h - \frac{t_h}{4 - t_h} \end{aligned}$$

Let t^* be tax for workers with high income that maximizes $\mathbb{E}[t]$. The first order condition is

$$\left.\frac{d\,\mathbb{E}[t]}{dt_h}\right|_{t_h = t^*} = 1 - \frac{4}{(4 - t^*)^2} = 0$$

Whose solutions are $t_1^* = 2$ and $t_2^* = 6$. Given the participation constraints it is clear that the maximizer is $t^* = 2$.