Exercise 3 - Suggested Solutions

ECON / MATH C103 - Mathematical Economics
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due Tue Feb 7, 4:59pm

Each sub-exercise is weighted equally.

Helpful Material:
- Last week's lecture notes.

Exercise 1: (12 points) Consider a general mechanism a design problem with a single agent. The agent has a single dimensional type $\theta \in [\underline{\theta}, \overline{\theta}] \triangleq \Theta \subseteq \mathbb{R}$. Types are distributed according to $F : \Theta \to [0,1]$, where the support of F equals Θ. The principal chooses an allocation $a \in A$, where we do not make any assumption on A. For example the allocation might or might not involve monetary transfers. The agent's utility is given by $u : A \times \Theta \to \mathbb{R}$ and we assume that u is uniformly Lipschitz continuous in $\theta \in \Theta$. Prove that the outcome

$$a : \theta \to A$$

of *any* mechanism satisfies

$$u(a(\theta), \theta) = u(a(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(a(s), s) \mathrm{d}s \,.$$

Answer: Let M be the space of messages the agent can send, $x : M \to A$ the allocation rule, and $m^*(\theta)$ the optimal strategy, i.e. $\forall \theta \in \Theta$ we have

$$m^*(\theta) \in \arg\max_{m \in M} u(x(m), \theta)$$

By definition the outcome $a(\theta)$ is given by

$$a(\theta) = x(m^*(\theta))$$

Now (following the proof of the Revelation Principle) build a new mechanism in which the set of messages is Θ (is a direct mechanism) and the allocation rule is $a(\theta)$. Since $a(\theta)$ was build with the optimal message, this new mechanism is incentive compatible.

As F has full support, the agent in this new mechanism presents a family of optimization problems $\{u(a, \theta)\}$ parametrized by a continuous real valued parameter $\theta \in \Theta$. Let $v(\theta) \triangleq \sup_{a \in A} u(a, \theta)$,

$S^*(\theta) \triangleq \arg\max_{a\in A} u(a,\theta)$ the set of optimal allocations in the new mechanism, and $s^*(\theta)$ a generic element of $S^*(\theta)$, i.e. $s^*(\theta) \in S^*(\theta)$. Since the mechanism is IC we know that $a(\theta) \in S^*(\theta)$ and $v(\theta) = u(a(\theta),\theta)$.

Since u is uniformly Lipschitz continuous in $\theta \in \Theta$ it is differentiable and also its partial derivative is bounded, i.e. $\left|\frac{\partial u(a,\theta)}{\theta}\right| \leq B$. Then, the Envelope Theorem can be applied.

The Envelope Theorem implies that $v(\theta)$ is absolutely continuous and for every maximizer $s^*(\theta)$ for almost all $\theta \in \Theta$

$$v'(\theta) = \left.\frac{\partial u(a,\theta)}{\partial \theta}\right|_{a=s^*(\theta)} = u_\theta(s^*(\theta),\theta)$$

In particular

$$v'(\theta) = \left.\frac{\partial u(a,\theta)}{\partial \theta}\right|_{a=a(\theta)} = u_\theta(a(\theta),\theta)$$

Therefore

$$u(a(\theta),\theta) = v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} v'(s)ds = u(a(\underline{\theta}),\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_\theta(a(s),s)ds$$

Exercise 2: (16 points) Let the set of physical allocations X be single dimensional $X = [0,\bar{x}] \subset \mathbb{R}$. Assume quasilinear preferences described by the utility function

$$u((x,t),\theta) = \sqrt{x}\,\theta - t$$

and a single dimensional type $\theta \in [\underline{\theta}, \overline{\theta}] \triangleq \Theta \subseteq \mathbb{R}$, distributed according to $F : \Theta \to [0,1]$, with full support, and density $f = F'$.

(a) Characterize the set if incentive compatible direct mechanisms.

Answer: Since preferences take the form $u((x,t),\theta) = w(x,\theta) - t$ they are quasilinear. Hence we know that every incentive compatible direct mechanism can be characterized by having the message space defined by $M = \Theta$ (which makes it direct) and by the two following conditions that assure incentive compatibility:

- The allocation function x is nondecreasing
- The transfer function t satisfies

$$t(\theta) = w(x(\theta),\theta) - \int_{\underline{\theta}}^{\theta} w_\theta(x(s),s)ds - (w(x(\underline{\theta}),\underline{\theta}) - t(\underline{\theta}))$$

Replacing

$$t(\theta) = \sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds - \sqrt{x(\underline{\theta})}\underline{\theta} + t(\underline{\theta})$$

Remark: In mathematics a characterization is usually defined as "A description of an object by properties that are different from those mentioned in its definition, but are equivalent to them" (see for example `http://mathworld.wolfram.com/Characterization.html`). Therefore the definition of incentive compatibility is not a characterization by itself. Given that some of you might not have been familiar with this language we made an exception this time and assigned full-points to everyone who provided a correct definition and bonus points to everyone who provided a characterization.

(b) Characterize the set of incentive compatible mechanism which satisfy the participation constrained.

Answer: If the agent does not participate gets an allocation of $x = 0$ and pays a transfer of $t = 0$. Therefore her utility of not participating is $u((0,0),\theta) = 0$.

Using the previous characterization of incentive compatible mechanism, we only need to add the condition that the participation constrained is satisfied:

- The allocation function x is nondecreasing
- The transfer function t satisfies

$$t(\theta) = \sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds - \sqrt{x(\underline{\theta})}\underline{\theta} + t(\underline{\theta})$$

- $u((x(\theta),t(\theta)),\theta) \geq 0 \; \forall \theta \in \Theta$.

From the form of $t(\theta)$ we can see that the utility an agent of type θ receives is

$$\sqrt{x(\theta)}\theta - t(\theta) = \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds + \sqrt{x(\underline{\theta})}\underline{\theta} - t(\underline{\theta})$$

which is increasing in θ. Thus if the participation constraint holds for the agent of type $\underline{\theta}$ then it will hold for all the types. Then the characteristics needed to a direct mechanism to be incentive compatible and satisfy the participation constraint are

- The allocation function x is nondecreasing
- The transfer function t satisfies

$$t(\theta) = \sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds - \sqrt{x(\underline{\theta})}\underline{\theta} + t(\underline{\theta})$$

- $t(\underline{\theta}) \leq \sqrt{x(\underline{\theta})}\underline{\theta}$

(c) What is the information rent an agent of type θ receives? Explain in your own words the economic meaning of this information rent.

Answer: Suppose the principal knows the type of the agent. If this is the case the principal's optimal strategy will be to extract all the surplus from the agent, i.e. the transfer will be equal to $w(x,\theta)$. Then the rents the agent obtains because her type is private information are equal to her utility.

Let R be the information rent an agent of type θ receives. Then

$$R(\theta) = \max_{m\in\Theta} u(x(m),t(m),\theta) = \max_{m\in\Theta} \sqrt{x(m)}\theta - t(m)$$

Since the mechanism is IC the rent is

$$R(\theta) = \sqrt{x(\theta)}\theta - t(\theta)$$

Replacing the formula for transfers described in question (a) we have that

$$R(\theta) = \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds + \sqrt{x(\underline{\theta})}\underline{\theta} - t(\underline{\theta})$$

Which is the cost the principal needs to incur in order to assure that the agent reveals her true type.

Note that the information rent $R(\theta)$ in increasing in θ. When the agent' type θ is higher the principal needs to transfer a higher rent to give the agent enough incentives to reveal her true type.

(d) Derive the maximal expected revenue which can be generated in a mechanism which implements the physical allocation $x : \Theta \to [0,\bar{x}]$ and satisfies the participation constrained.

Answer: We know that

$$t(\theta) = \sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds - \sqrt{x(\underline{\theta})}\underline{\theta} + t(\underline{\theta})$$

Then, the expected revenue is

$$\begin{aligned}
\mathbb{E}[t(\theta)] &= \int_{\Theta} t(\theta)dF(\theta) \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \left(\sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds - \sqrt{x(\underline{\theta})}\underline{\theta} + t(\underline{\theta})\right) f(\theta)d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \left(\sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds\right) f(\theta)d\theta + \left(t(\underline{\theta}) - \sqrt{x(\underline{\theta})}\underline{\theta}\right) \int_{\underline{\theta}}^{\overline{\theta}} f(\theta)d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \left(\sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds\right) f(\theta)d\theta + t(\underline{\theta}) - \sqrt{x(\underline{\theta})}\underline{\theta}
\end{aligned}$$

We know that the condition for the participation constraint to be satisfied for all types is $t(\underline{\theta}) \leq \sqrt{x(\underline{\theta})}\underline{\theta}$. Since the revenue is increasing in $t(\underline{\theta})$, the revenue maximizing mechanism must have $t(\underline{\theta}) = \sqrt{x(\underline{\theta})}\underline{\theta}$. Therefore it takes the form

$$\mathbb{E}\left[t^*(\theta)\right] = \int_{\underline{\theta}}^{\overline{\theta}} \left(\sqrt{x(\theta)}\theta - \int_{\underline{\theta}}^{\theta} \sqrt{x(s)}ds\right) f(\theta)d\theta$$