Exercise 6 - Suggested Solutions

due March 7, 4:59pm

Helpful Material:

- Last week's lecture notes.

Exercise 1: Consider a set up with n agents and $k \geq 1$ identical objects. The designer can decide whether agent i receives an object $x_i = 1$ or does not receive an object $x_i = 0$. Each agent can receive at most one object. Agents have quasi-linear preferences of the form

$$u_i((x, t_i), \theta_i) = x_i \theta_i - t_i\,,$$

where $\theta_i \geq 0$ is agent i's value for an object. Each agent's value for an object θ_i is drawn independently from the uniform distribution on $[1, 10]$.

Economic Example: As an interpretation of this exercise consider for example k cell-phone carriers who want to by spectrum from the government. The government is interested in allocating the spectrum efficiently, and conditional on efficient allocation wants to raise as much revenue as possible.

(a) Derive the efficient allocation rule. (4pts)

NOTE: For all the questions in this exercise I omit the cases when two or more agents have the same valuation since this cases have probability zero.

Answer: Define $\Theta \triangleq [1,10]^n$ and $X = \{(x_1, \ldots, x_n) : x_i \in \{0,1\}, \sum_{i=1}^n x_i \leq k\}$. The allocation rule $x^* : \Theta \to X$ is efficient if for all $\theta \in \Theta$

$$x^*(\theta) \in \arg\max \sum_{i=1}^{n} x_i^*(\theta)\theta_i$$

Since $\theta_i > 0$ and $x_i \geq 0$ for all i, it is clear that the rule that assigns the objects to the k agents with the highest valuations, i.e. $x^*(\theta) = (x_1^*(\theta), x_2^*(\theta), \ldots, x_n^*(\theta))$ is defined by

$$x_i^*(\theta) = \begin{cases} 1 & \text{if } \left|\{j \neq i : \theta_j > \theta_i\}\right| < k \\ 0 & \text{otherwise} \end{cases}$$

To prove that this is the only efficient allocation, by way of contradiction suppose there is another different efficient allocation $x'(\theta) \neq x^*(\theta)$. It is straightforward to show that $\sum_{i=1}^n x_i^*(\theta)\theta_i > \sum_{i=1}^n x_i'(\theta)\theta_i$, which contradicts the fact that $x'(\theta)$ is efficient.

(b) Characterize all efficient, dominant strategy incentive compatible (DIC), direct mechanisms. (4pts)

Answer: Green and Laffont (1979)[1] show that Vickrey-Clarke-Groves (VCG) mechanisms are the only mechanisms that make truthful revelation a dominant strategy for the efficient transfer rule $x^*(\theta)$. We know that VCG mechanisms can be characterized by

- The allocation rule is efficient, i.e. $x(\theta) = x^*(\theta)$
- The transfer rule has the form

$$t_i(\theta) = -\sum_{j \neq i} x_j(\theta)\theta_j + \tau_i(\theta_{-i})$$

for an arbitrary function $\tau_i : \Theta_{-i} \to \mathbb{R}$, where $\Theta_{-i} = [1, 10]^{n-1}$

(c) Derive the Pivot mechanism. (4pts)

Answer: The pivot mechanism is the VCG mechanism in which

$$\tau_i = \max_{x \in X} \sum_{j \neq i} x_j \theta_j$$

Then the mechanism is given by

$$\begin{aligned} M &= \Theta \\ x(\theta) &= x^*(\theta) \\ t_i(\theta) &= -\sum_{j \neq i} x_j(\theta)\theta_j + \max_{x \in X} \sum_{j \neq i} x_j \theta_j \end{aligned}$$

Let $\theta_{(n-k)}$ be the $k+1^{\text{th}}$ highest valuation. Then the mechanism is given by

$$\begin{aligned} M &= \Theta \\ x(\theta) &= x^*(\theta) \\ t_i(\theta) &= \begin{cases} \theta_{(n-k)} & \text{if } \left|\{j \neq i : \theta_j > \theta_i\}\right| < k \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(d) Derive the DIC direct mechanisms that maximizes revenue in the over the set of all efficient mechanisms. (4pts)

Answer: We know that the *revenue pivotal mechanism* is the one that maximizes revenue. It is a VCG mechanism in which the transfer rule is given by

$$t_i^P(\theta_i) = -\sum_{j \neq i} x_j(\theta)\,\theta_j + \sum_{j \neq i} x_j\left(\underline{\theta_i}, \theta_{-i}\right)\theta_j + x_i\left(\underline{\theta_i}, \theta_{-i}\right)\underline{\theta_i}$$

[1] Green, J. R., & Laffont, J. J. (1979). *Incentives in public decision making*. Amsterdam: North-Holland.

If $k < n$, then as the values are i.i.d. we have that $x_i\left(\underline{\theta_i}, \theta_{-i}\right) = 0$ and the DIC direct mechanisms that maximizes revenue in the over the set of all efficient mechanisms is the pivotal mechanism described above.

If $k \geq n$ we have that $x^*(\theta) = (1, 1, \ldots, 1)$, and $t_i^P(\theta_i) = 1$ (According to Juuso Välimäki's notes: *It is immediately clear that with these transfers, the lowest type of each agent has a zero expected payoff* (p. 21). However, this is not true in this case!).

Exercise 2: Suppose there are two units of a good that the social planner can distribute among three agents. Each agent has quasilinear utility over the outcome and her transfer to the planner, where her value over units is described below.

	zero units	one unit	two units
Agent 1	0	20	20
Agent 2	0	0	25
Agent 3	0	10	20

For example, Agent 3's utility from receiving one unit and paying 8 dollars is $10-8=2$, and Agent 2's utility from receiving both units and paying 18 dollars is $25-18=7$.

(a) Which allocations are Pareto efficient if **no** transfers between the agents are allowed.

Answer: Let (x_1, x_2, x_3) represent the allocation in which player i receives x_i unites, $i = 1,2,3$. Note that $x_i \in 0,1,2$ and $\sum x_i = 2$. If there are no transfers, the Pareto Efficient allocations are

- (0,2,0)
- (0,0,2)
- (1,0,1)

Note that the only way to find these allocation is by inspection of every possible case.

(b) Derive the efficient allocation when transfers are allowed.

Answer: When transfers are allowed we know that the efficient allocation refers to both the Pareto efficiency and Utilitarian efficiency. Since the allocation (1,0,1) yields a higher sum of utilities than any other of the Pareto efficient allocations found in (a), this is the efficient allocation when transfers are allowed.

(c) Find the transfer for each agent in the pivot mechanism.

Answer: We can interpret the transfers in the pivot mechanism as the total utility the agent is taking from other agents by her presence. Then the transfers are:

- Player 1: If player 1 were not present the efficient allocation would be to give both units to player 2, i.e. (0,2,0). Going from (1,0,1) to (0,2,0) increases player 2's utility in 25 ($\Delta_2 = 25$) and reduces player 3's utility in 10 ($\Delta_3 = -10$). Then player 1's transfer is $t_1 = \Delta_2 + \Delta_3 = 15$.
- Player 2: If player 2 were not present the efficient allocation would still be (1,0,1). Then player 2's transfer is $t_2 = 0$.
- Player 3: If player 3 were not present the efficient allocation would be to give both units to player 2, i.e. (0,2,0). Going from (1,0,1) to (1,0,1) increases player 2's utility in 25 ($\Delta_2 = 25$) and reduces player 1's utility in 20 ($\Delta_1 = -20$). Then player 3's transfer is $t_3 = \Delta_1 + \Delta_2 = 5$.