Exercise 8
Suggested Solutions

ECON / MATH C103 - Mathematical Economics
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due April 11, 4:59pm

Helpful Material:
- Last week's lecture notes.

Exercise 1: Consider an two agent single object allocation problem. Each agent's value θ_i for the object takes only two values $\theta_i \in \{1,2\}$. Suppose, that the distribution of types is given by

$\theta_1 \backslash \theta_2$	2	1
2	p_{11}	p_{10}
1	p_{10}	p_{00}

Furthermore, assume linear preferences $u_i = \theta_i x_i - t_i$.

(a) What is the expected revenue of a second price auction with reserve price of zero and with reserve price of one. (4pts)

Answer: Assume in case of a tie the object is randomly allocated and the agent who gets the object pays the other agent's bid (which is equal to her bid). We know that in a second price auction to report truthfully is a dominant strategy. The expected revenue for different reserve prices r are:

- $r = 1$: $2p_{11} + 1(2p_{10} + p_{00}) = p_{11} + (p_{11} + 2p_{10} + p_{00}) = 1 + p_{11}$. Note that this is the same revenue as if there is no reservation price since the reservation price is never binding.
- $r = 2$: $2(p_{11} + 2p_{10}) + 0p_{00} = 2p_{11} + 4p_{10} = 1 + p_{11} + 2p_{10} - p_{00}$

Then the second price auction with $r = 1$ yields a higher revenue than the second price auction with $r = 2$ if

$$1 + p_{11} \geq 1 + p_{11} + 2p_{10} - p_{00} \Leftrightarrow p_{00} \geq 2p_{10}$$

(b) Derive the revenue maximizing dominant strategy incentive compatible mechanism, where the utility for each type from participating in the mechanism is greater zero for any realization of types. (4pts)

Answer: First, note that any IC mechanism needs to have an increasing allocation rule (see first lecture's notes). Since the game is symmetric we focus only on symmetric mechanisms[1]. Let $x_1(m_1, m_2)$ and $t_1(m_1, m_2)$ be the probability agent 1 gets the object and the payment she has to make if she gets the object when she reports a type m_2 and agent 2 reports a type m_2. To simplify notation, let

$$\begin{array}{rcl} x_{11} \triangleq x_1(1,1) = x_2(1,1) & , & x_{12} \triangleq x_1(1,2) = x_2(2,1), \\ x_{21} \triangleq x_1(2,1) = x_2(2,1) & , & x_{22} \triangleq x_1(2,2) = x_2(2,2), \\ t_{11} \triangleq t_1(1,1) = t_2(1,1) & , & t_{12} \triangleq t_1(1,2) = t_2(2,1), \\ t_{21} \triangleq t_1(2,1) = t_2(2,1) & , & t_{22} \triangleq t_1(2,2) = t_2(2,2) \end{array}$$

Note that symmetry implies $x_{11}, x_{22} \in \left[0, \frac{1}{2}\right]$ and $x_{12} + x_{21} \in [0,1]$

To find the optimal transfer, let's focus on agent i's IC constraint corresponding to reporting a type of 1 when her real type is 2.[2]

$$\begin{array}{rcl} x_{22}(2 - t_{22}) & \geq & x_{12}(2 - t_{12}) \\ x_{21}(2 - t_{21}) & \geq & x_{11}(2 - t_{11}) \end{array}$$

To maximize transfers it is optimal to make those constraints binding, i.e. hold with equality. As increasing t_{12} or t_{11} relaxes the constraint and increases the revenue it is optimal to chose them equal to their maximal value such that the participation constraint is satisfied $t_{11} = t_{12} = 1$. As a consequence the incentive constraints become

$$x_{22}(2 - t_{22}) = x_{12} \quad \Rightarrow \quad t_{22}x_{22} = 2x_{22} - x_{12} \tag{1}$$

$$x_{21}(2 - t_{21}) = x_{11} \quad \Rightarrow \quad t_{21}x_{21} = 2x_{21} - x_{11} \tag{2}$$

Note, that x_{12} and x_{11} are thus the information rents the principal has to pay. The expected revenue raised from agent i (and by symmetry half of the total expected revenue) equals

$$\begin{array}{rcl} R & = & p_{00}x_{11}t_{11} + p_{10}x_{12}t_{12} + p_{10}x_{21}t_{21} + p_{11}x_{22}t_{22} \\ & = & p_{00}x_{11} + p_{10}x_{12} + p_{10}(2x_{21} - x_{11}) + p_{11}(2x_{22} - x_{12}) \\ & = & (p_{00} - p_{10})x_{11} + (p_{10} - p_{11})x_{12} + 2x_{21}p_{10} + 2x_{22}p_{11} \end{array}$$

Since we have the restriction $0 \leq x_{12} + x_{21} \leq 1$ it is optimal to set $x_{12} = 0$ and $x_{21} = 1$. Finally, there are two cases:

[1] To see that an asymmetric mechanism could not lead to a strictly higher revenue, take a convex combination of that mechanism and the same mechanism interchanging the agents. If the weights in the combination are 0.5 for each mechanism you get a symmetric mechanism with the same expected revenue.

[2] To see why these are the relevan constraints, remember that the information rents can be interpreted as coming only from the possibility of reporting lower types.

1. If $p_{00} \geq p_{10}$, then it is optimal to set $x_{11} = 1$, and the optimal mechanism (by replacing in equations (1) and (2)) is

$$x_i^{DS}(\theta) = \begin{cases} 1 & \text{if } \theta_i > \theta_{-i} \\ \frac{1}{2} & \text{if } \theta_i = \theta_{-i} \\ 0 & \text{if } \theta_i < \theta_{-i} \end{cases}, \; t_i^{DS}(\theta) = \begin{cases} 1 & \text{if } \theta_i = \theta_{-i} = 1 \\ \frac{3}{2} & \text{if } \theta_i > \theta_{-i} \\ 2 & \text{if } \theta_i = \theta_{-i} = 2 \end{cases}$$

   The expected revenue is $\mathbb{E}[\Pi] = 1 + p_{11} + p_{10}$.

2. If $p_{00} < p_{10}$, then it is optimal to set $x_{11} = 0$, and the optimal mechanism (by replacing in equations (1) and (2)) is

$$x_i^{DS}(\theta) = \begin{cases} 1 & \text{if } 2 = \theta_i > \theta_{-i} \\ \frac{1}{2} & \text{if } \theta_i = \theta_{-i} = 2 \\ 0 & \text{if } \theta_i = 1 \end{cases}, \; t_i^{DS}(\theta) = \begin{cases} 2 & \text{if } \theta_i = 2 \\ 0 & \text{if } \theta_i = 1 \end{cases}$$

   Which is the second price auction with reserve price $r = 2$. The expected revenue is $\mathbb{E}[\Pi] = 1 + p_{11} + 2p_{10} - p_{00}$

(c) Suppose, that $p_{10}^2 = p_{11} \cdot p_{00}$. Derive the revenue maximizing Bayes incentive compatible mechanism, where the expected utility for each type from participating in the mechanism is greater zero. (4pts)

Answer: To build the the revenue maximizing Bayes incentive compatible mechanism we focus on transfers rule where the payment only depends on the true report. Since utilities and revenue are linear on transfers, this restriction will not affect the objective function or any constraint. To simplify notation, define x_{11}, x_{12}, x_{21}, and x_{22} as before, and

$$t_1 \triangleq t_1(1,\cdot) = t_2(\cdot,1),\; t_2 \triangleq t_1(2,\cdot) = t_2(\cdot,2)$$

Again we focus only on agent i's IC constraint corresponding to reporting a type of 1 when her real type is 2:

$$x_{22}p_{11}(2-t_2) + x_{21}p_{10}(2-t_2) \geq x_{12}p_{11}(2-t_1) + x_{11}p_{10}(2-t_1)$$

Again we make the constraint binding. As increasing t_1 relaxes the constraint, we make it as bug as possible without violating the participation constraint, i.e. $t_1 = 1$. Then

$$(2-t_2)(x_{22}p_{11} + x_{21}p_{10}) = x_{12}p_{11} + x_{11}p_{10} \Rightarrow t_2 = 2 - \frac{x_{12}p_{11} + x_{11}p_{10}}{x_{22}p_{11} + x_{21}p_{10}} \tag{3}$$

In this case the expected revenue from agent i is

$$\begin{aligned} R &= p_{00}x_{11}t_1 + p_{10}x_{12}t_1 + p_{10}x_{21}t_2 + p_{11}x_{22}t_2 \\ &= p_{00}x_{11} + p_{10}x_{12} + t_2(p_{10}x_{21} + p_{11}x_{22}) \\ &= (p_{00} - p_{10})x_{11} + (p_{10} - p_{11})x_{12} + 2x_{21}p_{10} + 2x_{22}p_{11} \end{aligned}$$

Which is exactly the same revenue obtained in the previous case! This is not surprising since for any BNIC mechanism there exists a DSIC mechanism that delivers the same interim expected utilities for all agents and the same expected revenue[3].

As the expected revenue is the same, the optimal allocation rules has to be the same, i.e.

1. If $p_{00} \geq p_{10}$, then it is optimal to set $x_{11} = 1$, and the optimal mechanism (by replacing in equation (3)) is

$$x_i^{BN}(\theta) = \begin{cases} 1 & \text{if } \theta_i > \theta_{-i} \\ \frac{1}{2} & \text{if } \theta_i = \theta_{-i} \\ 0 & \text{if } \theta_i < \theta_{-i} \end{cases}, \; t_i^{BN}(\theta) = \begin{cases} 1 & \text{if } \theta_i = 1 \\ 2 - \frac{p10}{p_{11}+2p_{10}} & \text{if } \theta_i = 2 \end{cases}$$

   As expected, $t_2 \geq \frac{3}{2}$. The expected revenue is $\mathbb{E}[\Pi] = 1 + p_{11} + p_{10}$.

2. If $p_{00} < p_{10}$, then it is optimal to set $x_{11} = 0$, and the optimal mechanism (by replacing in equation (3)) is

$$x_i^{DS}(\theta) = \begin{cases} 1 & \text{if } 2 = \theta_i > \theta_{-i} \\ \frac{1}{2} & \text{if } \theta_i = \theta_{-i} = 2 \\ 0 & \text{if } \theta_i = 1 \end{cases}, \; t_i^{DS}(\theta) = \begin{cases} 2 & \text{if } \theta_i = 2 \\ 0 & \text{if } \theta_i = 1 \end{cases}$$

   Which is the second price auction with reserve price $r = 2$. The expected revenue is $\mathbb{E}[\Pi] = 1 + p_{11} + 2p_{10} - p_{00}$

(d) Suppose, now that $p_{10}^2 \neq p_{11} \cdot p_{00}$. Derive the revenue maximizing Bayes incentive compatible mechanism, where the expected utility for each type from participating in the mechanism is greater zero. (10pts)

Answer: Given the Cremer-McLean result, we know that we can derive a mechanism that yields an allocation in which each agent has an expected utility of zero independent of her type.

Note that the expected interim utilities of the previous mechanism are $V_i(1) = 0$ and $V_i(2) = \frac{p_{10}}{2}$.

Let $\phi(\theta_i) = P(\theta_{-i}|\theta_i)$. As $p_{10}^2 \neq p_{00}p_{11}$, exists $\tau(\theta_{-i}) = (\tau(\theta_{-i} = 1) \; \tau(\theta_{-i} = 2))'$ such that $\tau(\theta_{-i})'\phi(1) = V_i(1) = 0$ and $t(\theta_{-i})'\phi(2) = V_i(2) = \frac{p_{10}}{2}$. The solution is

$$\tau(\theta_{-i}) = \begin{pmatrix} \frac{p_{10}^2(p_{10}+p_{11})}{2(p_{10}^2 - p_{00}p_{11})} \\ \frac{p_{00}p_{10}(p_{10}+p_{11})}{2(p_{00}p_{11}-p_{10}^2)} \end{pmatrix}$$

Set $x^{**}(\theta) = x^*(\theta)$ and $t^{**}(X, \theta) = t^*(X, \theta) + \tau(\theta_{-i})$. This is the the revenue maximizing Bayes incentive compatible mechanism.

[3] For more details see Gershkov, A., Goeree, J. K., Kushnir, A., Moldovanu, B., & Shi, X. (2013). On the equivalence of Bayesian and dominant strategy implementation. *Econometrica*, 81(1), 197-220.

(e) Discuss your findings in terms of information rents (3pts).

Answer: When agents types are correlated it is possible to use that correlation to extract all the information rents. The idea behind this is that the correlation allows the principal to exploit the change in each agent's expectation about other agents' types. Using this information, the principal can extract all the expected utility from each agent.

Exercise 2: Consider a situation with n agents. Each agent's type θ_i is uniformly drawn from $[0,1]$. Suppose, that there is a single object to allocate and agent i's value for getting the object is a convex combination of his own type and the average of all types

$$h_i(\theta) = \alpha\theta_i + (1-\alpha)\left(\frac{1}{n}\sum_{j=1}^{n}\theta_j\right).$$

Here, $\alpha \in [0,1]$. Suppose, that preferences are quasi-linear and given by

$$u_i = h_i(\theta)x_i - t_i.$$

(a) Derive the welfare maximizing allocation. (4 pts)

Answer: Let x_i be the amount of the object agent i obtains and $x = (x_1, \ldots, x_n)$. Note that $x_i \in [0,1]$ $\forall i$ and $\sum_{i=1}^{n} x_i = 1$. The objective function is

$$\begin{aligned}
W(\theta, x) &= \sum_{i=1}^{n} x_i h_i(\theta) \\
&= \sum_{i=1}^{n} x_i \left(\alpha\theta_i + (1-\alpha)\left(\frac{1}{n}\sum_{j=1}^{n}\theta_j\right)\right) \\
&= \alpha\sum_{i=1}^{n} x_i\theta_i + \frac{(1-\alpha)}{n}\left(\sum_{j=1}^{n}\theta_j\right)\left(\sum_{i=1}^{n} x_i\right) \\
&= \alpha\sum_{i=1}^{n} x_i\theta_i + \frac{(1-\alpha)}{n}\left(\sum_{i=1}^{n}\theta_i\right)
\end{aligned}$$

It is clear that the allocation that maximizes W is to give the object to the agent with the highest valuation.

(b) Argue that the welfare maximizing allocation implementable? (2 pts)

Answer: Since the efficient allocation rule is monotonic and payoffs satisfy *strictly increasing differences*, then the efficient allocation is implementable in ex-post strategies using a generalized VCG mechanism (see Välimäki's Lecture Notes, pp. 33-37).

(c) Apply the envelope theorem to derive the derivative of agent i's utility in the mechanism with respect to her type θ_i for any $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. (4pts)

Answer: Let $X_i(\theta)$ be the probability that agent i gets the object given θ and $t_i(\theta)$ be the transfer. Then the Envelope Theorem implies

$$\begin{aligned}
X_i(\theta)h_i(\theta) - t_i(\theta) &= X_i(0,\theta_{-i})h_i(0,\theta_{-i}) - t_i(0,\theta_{-i}) + \int_0^{\theta_i}\left(\alpha + \frac{1-\alpha}{n}\right)X_i(s,\theta_{-i})ds \\
X_i(\theta)h_i(\theta) - t_i(\theta) &= X_i(0,\theta_{-i})\frac{1-\alpha}{n}\sum_{j\neq i}\theta_j - t_i(0,\theta_{-i}) + \int_0^{\theta_i}\left(\alpha + \frac{1-\alpha}{n}\right)X_i(s,\theta_{-i})ds
\end{aligned}$$

(d) Use the condition obtained in (c) to characterize the transfer in any dominant strategy incentive compatible mechanism. (4pts)

Answer: The transfer in any DSIC mechanism needs to satisfy

$$t_i(\theta) = X_i(\theta)h_i(\theta) - X_i(0,\theta_{-i})\frac{1-\alpha}{n}\sum_{j\neq i}\theta_j + t_i(0,\theta_{-i}) - \int_0^{\theta_i}\left(\alpha + \frac{1-\alpha}{n}\right)X_i(s,\theta_{-i})ds$$

for all $\theta_{-i} \in [0,1]^{n-1}$

(e) Construct a Mechanisms that maximizes welfare. (8 pts)

Answer: Let $N = \{1,2,\dots,n\}$ and define

$$x_i^*(\theta) = \begin{cases} 1 & \text{if } \theta_i = \max\limits_{j\in N}\theta_j \\ 0 & \text{otherwise} \end{cases}$$

Note that the probability of agent i getting the object conditional on her type is θ_i^{n-1}. In this case I set the auction to be an all-pay auction, which requires the equilibrium to be a BNIC one. Then the envelope condition applied to the expected utility implies that the transfer is

$$\begin{aligned} t_i^*(\theta_i) &= \mathbb{E}\left[x_i(\theta)h_i(\theta)|\theta_i\right] - \int_0^{\theta}\left(\alpha + \frac{1-\alpha}{n}\right)\mathbb{E}[x_i(s,\theta_{-i})|s]ds \\ &= \mathbb{E}\left[x_i(\theta)\left(\alpha\theta_i + \frac{1-\alpha}{n}\sum_{j=1}^{n}\theta_j\right)\middle|\theta_i\right] - \left(\alpha + \frac{1-\alpha}{n}\right)\int_0^{\theta_i}\mathbb{E}[x_i(s,\theta_{-i})|s]ds \\ &= \left(\alpha + \frac{1-\alpha}{n}\right)\theta_i\mathbb{E}[x_i(\theta)|\theta_i] + \frac{1-\alpha}{n}\sum_{j\neq i}\mathbb{E}\left[x_i(\theta)\theta_j|\theta_i\right] - \\ &\quad - \left(\alpha + \frac{1-\alpha}{n}\right)\int_0^{\theta_i}\mathbb{E}[x_i(s,\theta_{-i})|s]ds \end{aligned}$$

Given that $x_i \in \{0,1\}$ we have that $\mathbb{E}[x_i(\theta)|\theta_i] = \mathbb{P}(x_i(\theta) = 1|\theta_i) = \theta_i^{n-1}$. Also we have that $\mathbb{E}\left[x_i(\theta)\theta_j|\theta_i\right]$ is equal to the probability that θ_i is greater that the maximum among $n-2$ valuations (all agents besides i and j) times the expected value of θ_j conditional on $\theta_j < \theta_i$

$$\mathbb{E}\left[x_i(\theta)\theta_j|\theta_i\right] = \int_0^{\theta_i}\theta_j\theta_i^{n-2}dF(\theta_j) = \theta_i^{n-2}\int_0^{\theta_i}\theta_j d\theta_j = \frac{1}{2}\theta_i^n$$

Replacing we have

$$t_i^*(\theta) = \left(\alpha + \frac{1-\alpha}{n}\right)\theta_i^n + (1-\alpha)\frac{n-1}{2n}\theta_i^n - \left(\alpha + \frac{1-\alpha}{n}\right)\frac{1}{n}\theta_i^n$$