ECON206

PS 4 - suggested solutions

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Exercise 1

(a) The mechanism $(M, (x, t))$ is

- $M = \mathbb{R}_+$
- $x(m_i) = \begin{cases} 1 & \text{if } i = \underset{1 \leq j \leq n}{\operatorname{argmax}}\, m_j \\ 0 & \text{otherwise} \end{cases}$
- $t(m_i) = \begin{cases} \max\limits_{j \neq i} m_j & \text{if } m = \underset{1 \leq i \leq n}{\operatorname{argmax}}\, m_i \\ 0 & \text{otherwise} \end{cases}$

(b) Define $p_{-i} = \max m_{-i}$, i.e., the highest bid from any player other than i

Case 1: $\theta_i < p_{-i}$. The agent has no chance of getting the object unless he lies about his true valuation, and bids an amount $m_i > p_{-i}$. But if he does so, his transfer would be p_{-i} (the second highest bid) and his utility strictly negative. On the other hand, for any bid $m_i \leq p_{-i}$, the agent will have a net utility of 0. Thus, the optimal strategy in this case is to quote any $m_i < p_{-i}$.

Case 2: $\theta_i > p_{-i}$. The agent will win the object with any bid $m_i > p_{-i}$, and pay only the second highest price, i.e., p_{-i}. Thus, the optimal strategy is to quote any $m_i > p_{-i}$.

Case 3: $\theta_i = p_{-i}$. In this case, if the agent quotes any bid $m_i > p_{-i}$, he gets the object; and his utility is exactly 0 (his valuation and cost cancel each other). If he quotes a bid $m_i < p_{-i}$, he doesn't get the object and pays nothing; his utility is 0 again. Finally, if he quotes $m_i = p_{-i}$, he may or may not get the object; (assume ties are broken arbitrarily) but his utility is 0 either way. Thus the agent's payoff is exactly 0 irrespective of his bid; and the optimal strategy is to quote any $m_i \in \mathbb{R}_+$

(c) In all 3 cases discussed above, $m_i = \theta_i$ lies in the set of optimal choices. That is, if the agent quotes his true type/valuation, he is guaranteed to achieve the optimal payoff, even without knowing others' bids. Thus, the agent has a dominant strategy, which is to quote his value truthfully.

(d) *Part 1:* Since every agent has a weakly dominant strategy, there's a dominant strategy equilibrium. Now let's prove uniqueness.

Proof. Suppose there is another dominant strategy equilibrium different from reporting truthfully. Call this equilibrium $\sigma = \{\sigma_1, \ldots, \sigma_n\}$, where exists i such that $\sigma_i \neq \theta_i$. As σ is a dominant strategy equilibrium, we have that $\forall m \in \mathbb{R}_+$, $m_{-i} \in \mathbb{R}_+^{n-1}$

$$u(x(\sigma_i, m_{-i}), t(\sigma_i, m_{-i}), \theta_i) \geq u(x(m, m_{-i}), t(m, m_{-i}), \theta_i)$$

As before, let $p_{-i} = \max m_{-i}$

- If $\sigma_i < \theta_i$: Suppose $\sigma_i < p_{-i} < \theta_i$. Then the utility from sending the message σ_i is $\theta_i - p_{-i} < 0$, and the utility from sending the message θ_i is zero.
- If $\sigma_i > \theta_i$: Suppose $\sigma_i > p_{-i} > \theta_i$. Then the utility from sending the message σ_i is zero, and the utility from sending the message θ_i is $\theta_i - p_{-i} > 0$.

Therefore, the message σ_i is not optimal for every m_{-i}, which contradicts the fact that σ is a dominant strategy equilibrium. $\square$

Part 2: The expected revenue in this equilibrium is the expected value of the second highest value of θ_i. Let X be the second highest value among the n draws of θ. The CDF of X is:

$$G(x) = n(1 - F(x))F(x)^{n-1}\,.$$

See for example the chapter on order statistics in the book by Vijai Krishna.

And the expected revenue is

$$\int_0^{\overline{\theta}} (1 - G(x))dx = \int_0^{\overline{\theta}} (1 - n(1 - F(x))F(x)^{n-1})dx\,.$$

(e) As proven in question d) the only dominant strategy equilibrium is to report truthfully. Therefore this strategy profile is not dominant strategy equilibrium.

(f)
- If agent 1 changes her bid to $m \in (0, \overline{\theta})$ her utility does not change. If she changes her bid to $m = 0$ her utility decreases to $\theta_1/n \leq \theta_1$.
- If agent $i > 1$ changes her bid to $m \in (0, \overline{\theta})$ her utility does not change. If she changes her bid to $m = \overline{\theta}$ her utility decreases to $1/2(\theta_i - \overline{\theta}) \leq 0$.

Therefore all strategies are optimal, the strategy profile is a Bayes Nash equilibrium.

(g) The second highest bid is zero, and therefore expected revenue is zero.

Exercise 2

(a) A vector σ is a dominant strategy equilibrium if for all i, for all θ_i, θ_{-i}, m_i, m_{-i}

$$u_i\left(\phi\left(\sigma_i\left(\theta_i\right), m_{-i}\right), \theta_i\right) \geq u_i\left(\phi\left(m_i, m_{-i}\right), \theta_i\right) \quad \forall i,\ \forall \theta_i, \theta_{-i}, m_i, m_{-i}$$

Where $\phi(\cdot)$ is the allocation function. In particular, this is true when $m_{-i} = \sigma_{-i}\left(\theta_{-i}\right)$

$$u_i\left(\phi\left(\sigma_i\left(\theta_i\right), \sigma_{-i}\left(\theta_{-i}\right)\right), \theta_i\right) \geq u_i\left(\phi\left(m_i, \sigma_{-i}\left(\theta_{-i}\right)\right), \theta_i\right) \quad \forall i,\ \forall \theta_i, \theta_{-i}, m_i$$

Given that the previous inequality is true for every realization of θ_{-i}, its expected value when θ_i is fixed also has to be true

$$E\left[u_i\left(\phi\left(\sigma_i\left(\theta_i\right), \sigma_{-i}\left(\theta_{-i}\right)\right), \theta_i\right) | \theta_i\right] \geq E\left[u_i\left(\phi\left(m_i, \sigma_{-i}\left(\theta_{-i}\right)\right), \theta_i\right) | \theta_i\right] \quad \forall i,\ \forall \theta_i, m_i$$

Therefore σ is a Bayes Nash equilibrium.

(b) Consider for example a first price auction where the agent who makes the highest bid receives the object and the winning agent pays his bid. In such an auction the winning agent always has an incentive to lower his bid to a smaller value above the second highest bid as such a change in his bid decreases the price he pays. Consequently, the optimal bid in the first price auction depends on the bids made by other agents and thus, the first price auction has no equilibrium in dominant strategies.