Exercise 1 - Suggested Solutions

ECON / MATH C103 - Mathematical Economics
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due Tue Jan 24, 4:59pm

Please raise questions, in the office hours, via email or at bcourses:

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bcourses: `http://bcourses.berkeley.edu`.

Each sub-exercise (a,b,..) is weighted equally and gives 4 points.

Helpful Material:

- De la Fuente 2000, Chapter 5.2 page 200 ff.
- Krishna 2009, Appendix A page 253 ff. and Appendix C page 265 ff.

Exercise 1 (Random Variables)**:** Let x be a random variable distributed according to the absolutely continuous cumulative distribution function (CDF) $F : \mathbb{R} \to [0,1]$, with density $f : \mathbb{R} \to \mathbb{R}_+$.

(a) What is the expected value and the variance of x?

Answer: By definition, the expected value is

$$\mathbb{E}[x] = \int_{\mathbb{R}} t f(t) \mathrm{d}t$$

and the variance is

$$\begin{aligned}
\mathrm{Var}(x) &= \mathbb{E}\left[(x - \mathbb{E}[x])^2\right] \\
&= \int_{\mathbb{R}} (t - \mathbb{E}[x])^2 f(t) \mathrm{d}t \\
&= \int_{\mathbb{R}} \left(t^2 - 2t\mathbb{E}[x] - \mathbb{E}[x]^2\right) f(t) \mathrm{d}t \\
&= \int_{\mathbb{R}} t^2 f(t) \mathrm{d}t - 2\mathbb{E}[x] \int_{\mathbb{R}} t f(t) \mathrm{d}t + \mathbb{E}[x]^2 \int_{\mathbb{R}} f(t) \mathrm{d}t \\
&= \mathbb{E}\left[x^2\right] - \mathbb{E}[x]^2
\end{aligned}$$

(b) Show that for every bounded, differentiable function $v : \mathbb{R} \to \mathbb{R}$ the following equality holds (hint: use integration by parts)

$$\int_{\mathbb{R}} v(x) f(x) \mathrm{d}x = \int_{\mathbb{R}} v'(x)(1 - F(x)) \mathrm{d}x + \lim_{x \to -\infty} v(x) \,.$$

Answer: Integrating by parts we have

$$\begin{aligned}
\int_{\mathbb{R}} v(x) f(x) \mathrm{d}x &= \lim_{a \to \infty} v(x) F(x)|_{x=-a}^{x=a} - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x \\
&= \lim_{x \to \infty} v(x) F(x) - \lim_{x \to \infty} v(-x) F(-x) - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x
\end{aligned}$$

Since the limits exist and $F(\cdot)$ is a CDF, we have that $\lim_{x \to \infty} v(x) F(x) = \lim_{x \to \infty} v(x) \lim_{x \to \infty} F(x) =$ $= \lim_{x \to \infty} v(x)$, and $\lim_{x \to -\infty} v(x) F(x) = \lim_{x \to -\infty} v(x) \lim_{x \to -\infty} F(x) = 0$. Thus

$$\begin{aligned}
\int_{\mathbb{R}} v(x) f(x) \mathrm{d}x &= \lim_{x \to \infty} v(x) - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x \\
&= \lim_{x \to \infty} v(x) - \lim_{x \to -\infty} v(x) - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x + \lim_{x \to -\infty} v(x) \\
&= \lim_{a \to \infty} v(x)|_{x=-a}^{x=a} - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x + \lim_{x \to -\infty} v(x) \\
&= \int_{\mathbb{R}} v'(x) \mathrm{d}x - \int_{\mathbb{R}} v'(x) F(x) \mathrm{d}x + \lim_{x \to -\infty} v(x) \\
&= \int_{\mathbb{R}} v'(x)(1 - F(x)) \mathrm{d}x + \lim_{x \to -\infty} v(x)
\end{aligned}$$

Given the Fundamental Theorem of Calculus.

(c) Suppose x_1 and x_2 are independently drawn from F. Derive the distribution of

$$y_1 \triangleq \max\{x_1, x_2\} \text{ and } y_2 \triangleq \min\{x_1, x_2\} \,.$$

Answer: For y_1 we have that

$$F_{y_1}(y) = P(y_1 \leq y) = P(x_1 \leq y \wedge x_2 \leq y) = P(x_1 \leq y) P(x_2 \leq y) = F(y) F(y) = F(y)^2$$

since x_1 and x_2 are independent. Similarly for y_2 we have that

$$\begin{aligned}
F_{y_2}(y) &= P(y_2 \leq y) = 1 - P(y_2 > y) = 1 - P(x_1 > y \wedge x_2 > y) = 1 - P(x_1 > y) P(x_2 > y) \\
&= 1 - (1 - F(y))(1 - F(y)) = 1 - (1 - F(y))^2
\end{aligned}$$

(d) Derive the expected value of the maximum y_1, and the minimum y_2.

Answer: Since x_1 and x_2 come from a continuous distribution both y_1 and y_2 are continuous random variables and their probability density functions are

$$\begin{aligned} f_{y_1}(y) &= \frac{\mathrm{d}F_{y_1}(y)}{\mathrm{d}y} = 2F(y)f(y) \\ f_{y_2}(y) &= \frac{\mathrm{d}F_{y_2}(y)}{\mathrm{d}y} = 2(1-F(y))f(y) \end{aligned}$$

So the expected values are

$$\begin{aligned} \mathbb{E}[y_1] &= \int_{\mathbb{R}} y f_{y_1}(y)\mathrm{d}y = 2\int_{\mathbb{R}} yF(y)f(y)\mathrm{d}y \\ \mathbb{E}[y_2] &= \int_{\mathbb{R}} y f_{y_2}(y)\mathrm{d}y = 2\int_{\mathbb{R}} y(1-F(y))f(y)\mathrm{d}y = 2\int_{\mathbb{R}} yf(y)\mathrm{d}y - 2\int_{\mathbb{R}} yF(y)f(y)\mathrm{d}y \\ &= 2\mathbb{E}[x] - \mathbb{E}[y_1] \end{aligned}$$

(e) Derive the expected value of the maximum y_1 conditional on it being above a constant threshold $z \in \mathbb{R}$,

$$\mathbb{E}[y_1 \mid y_1 \geq z]\,.$$

Answer: The probability density function of y_1 conditional on $y_1 \geq z$ is

$$f_{y_1|y_1\geq z}(y) = \begin{cases} 0 & \text{if } y < z \\ \frac{f_{y_1}(y)}{1-F_{y_1}(z)} & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}[y_1|y_1 \geq z] = \int_{\mathbb{R}} y f_{y_1|y_1\geq z}(y)\mathrm{d}y = \frac{\int_z^\infty y f_{y_1}(y)\mathrm{d}y}{1-F_{y_1}(z)} = 2\frac{\int_z^\infty yF(y)f(y)\mathrm{d}y}{1-F(z)^2}$$

(f) Derive the expectation of y_1 conditional on y_2 being equal to $z \in \mathbb{R}$,

$$\mathbb{E}[y_1 \mid y_2 = z]\,.$$

Answer: Note that given two values w_1 and w_2 with $w_1 \geq w_2$ there are two different events in which $y_1 = w_1$ and $y_2 = w_2$. The first event is $x_1 = w_1$ and $x_2 = w_2$, and the second is $x_1 = w_2$ and $x_2 = w_1$. Given that x_1 and x_2 are independent the joint pdf of y_1 and y_2 is

$$f_{y_1,y_2}(w_1,w_2) = f(w_1)f(w_2) + f(w_2)f(w_1) = 2f(w_1)f(w_2)$$

Then, the density function of y_1 given $y_2 = z$ is

$$f_{y_1|y_2=z}(y) = \begin{cases} 0 & \text{if } y_1 < z \\ \frac{f_{y_1,y_2}(y,z)}{f_{y_2}(z)} = \frac{2f(y)f(z)}{2(1-F(z))f(z)} = \frac{f(y)}{1-F(z)} & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}[y_1|y_2 = z] = \frac{\int_z^\infty yf(y)\mathrm{d}y}{1-F(z)}$$

Exercise 2 (Optimization and Maximizers)**:** Let $w : X \times \Theta \to \mathbb{R}$, where $X \triangleq [\underline{x}, \overline{x}], \Theta \triangleq [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$ are compact bounded intervals. Assume that w is differentiable in both arguments and strictly concave in the first.

(a) For all $\theta \in \Theta$ let

$$m(\theta) \triangleq \arg\max_{x \in X} w(x, \theta).$$

What is the derivative of m?

Answer: There are two possible cases:

- $m(\theta) \in (\underline{x}, \overline{x})$. In this case we have that the first order condition for optimality is

$$\left.\frac{\partial w(x,\theta)}{\partial x}\right|_{x=m(\theta)} = 0$$

  and the second order condition is always satisfies since w is strictly concave in the first argument. Since the first order condition is satisfied $\forall\, \theta \in [\underline{\theta}, \overline{\theta}]$, the derivatives of both sides with respect to θ have to be equal. Differentiating with respect to θ

$$\left.\frac{\partial^2 w(x,\theta)}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}\theta}\right|_{x=m(\theta)} + \left.\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}\right|_{x=m(\theta)} = 0$$

$$\left.\frac{\partial^2 w(x,\theta)}{\partial x^2}\right|_{x=m(\theta)} \cdot \frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} + \left.\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}\right|_{x=m(\theta)} = 0$$

  Solving for $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta}$

$$\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = -\left.\frac{\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}}{\frac{\partial^2 w(x,\theta)}{\partial x^2}}\right|_{x=m(\theta)}$$

- $m(\theta) \in \{\underline{x}, \overline{x}\}$. If $\left.\frac{\partial w(x,\theta)}{\partial x}\right|_{x=m(\theta)} \neq 0$ we have that $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = 0$ since w is differentiable in θ.
  If $\left.\frac{\partial w(x,\theta)}{\partial x}\right|_{x=m(\theta)} = 0$ we have several cases:

  - If $m(\theta) = \underline{x}$ and $-\left.\frac{\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}}{\frac{\partial^2 w(x,\theta)}{\partial x^2}}\right|_{x=m(\theta)} > 0$, or $m(\theta) = \overline{x}$ and $-\left.\frac{\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}}{\frac{\partial^2 w(x,\theta)}{\partial x^2}}\right|_{x=m(\theta)} < 0$, then

$$\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = -\left.\frac{\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}}{\frac{\partial^2 w(x,\theta)}{\partial x^2}}\right|_{x=m(\theta)}$$

  - Otherwise, $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = 0$

(b) When is $m(\theta)$ increasing (decreasing) in θ for all $\theta \in \Theta$?

Answer: Note that since w is strictly convex in the first argument, we have that $\frac{\partial^2 w(x,\theta)}{\partial x^2} < 0$. Then, the sign of $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta}$ is the same sign than the one of $\left.\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}\right|_{x=m(\theta)}$. Therefore, $m(\theta)$ is increasing (decreasing) in θ for all $\theta \in \Theta$ if $\left.\frac{\partial^2 w(x,\theta)}{\partial x \partial \theta}\right|_{x=m(\theta)}$ is positive (negative) for all $\theta \in \Theta$. We also need to include the cases in which $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = 0$

(c) For all $\theta \in \Theta$ let

$$v(\theta) \triangleq \max_{x \in X} w(x, \theta).$$

Denote by w_θ the partial derivative of w with respect to the second argument. Show that

$$v'(\theta) = w_\theta(m(\theta), \theta).$$

Answer: Note that

$$v(\theta) = w(m(\theta), \theta)$$

Denote w_x the partial derivative of w with respect to the first argument. Then

$$v'(\theta) = w_x(m(\theta), \theta)\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} + w_\theta(m(\theta), \theta)$$

But in part (a) we shown that either $w_x(m(\theta), \theta) = 0$ or $\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = 0$. Then

$$v'(\theta) = v_\theta(m(\theta), \theta)$$

References

De la Fuente, A. (2000). *Mathematical methods and models for economists*. Cambridge University Press.

Krishna, V. (2009). *Auction theory*. Academic press.