Lab #9 - Logistic Regression Part I

Econ 224

September 25th, 2018

Notation

In this lab we'll be looking at logistic regression, a commonly-used binary classification method. To make things simpler, we'll use some slightly different notation and terminology than ISL. First we'll define the *column vectors X and β* as follows:

$$X = \begin{bmatrix} 1 \\ X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix}$$

Notice that the first element of X is *not* X_1: it is simply the number 1. There's an important reason for this that you'll see in a moment. From the reading, we know that logistic regression is a *linear model* for the *log odds*, namely

$$\log\left[\frac{P(X)}{1 - P(X)}\right] = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

where $P(X)$ is shorthand for $\mathbb{P}(Y = 1|X)$. Note that when I write log I **always** mean the natural logarithm. Also note that when I write $\exp(z)$ I mean e^z. This comes in handy if z is a complicated expression.

Using the vector notation introduced above, we can express this more compactly as

$$\log\left[\frac{P(X)}{1 - P(X)}\right] = X'\beta$$

since

$$X'\beta = \begin{bmatrix} 1 & X_1 & X_2 & \cdots & X_p \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \beta_0 + \beta_1 X_2 + \beta_2 X_2 + \cdots + \beta_p X_p$$

I will call $X'\beta$ the *linear predictor* since it is the linear function of X that we use to predict Y. By exponentiating both sides of the log-odds expression from above and re-arranging, obtain the following:

$$\frac{P(X)}{1 - P(X)} = \exp(X'\beta)$$

$$P(X) = [1 - P(X)]\exp(X'\beta)$$

$$P(X) + P(X)\exp(X'\beta) = \exp(X'\beta)$$

$$P(X)[1 + \exp(X'\beta)] = \exp(X'\beta)$$

$$P(X) = \frac{\exp(X'\beta)}{1 + \exp(X'\beta)}$$

$$P(X) = \Lambda(X'\beta)$$

where the function Λ is defined as follows

$$\Lambda(z) = \frac{e^z}{1 + e^z}$$

Exercise #1

(a) Verify that $\Lambda(z) = \dfrac{1}{1 + e^{-z}}$.

(b) Using (b), write an alternative expression for $P(X)$.

Solution to Exercise #1

The function Λ takes values between 0 and 1. When $z = 0$, $\Lambda(z) = e^0/(1 + e^0) = 1/2$. As $z \to \infty$, $\Lambda(z) \to 1$ and as $z \to -\infty$, $\Lambda(z) \to 0$.

Interpreting β in a Logistic Regression

From the expression above, we see that β_j is the partial derivative of the log-odds with respect to X_j. But it's difficult to think in terms of log-odds. By doing some calculus (see the exercises below), we can work out an interpretation of β_j as the partial derivative of $p(X)$ with respect to X_j. **However, $P(X)$ is *not* a linear function of X so the derivative varies with X!** There are two solutions to this problem. One is to evaluate the derivative at a "typical" value of X such as the sample mean. Another is to use the "divide by 4 rule." This rule says that if we increase X_j by one unit, $P(X)$ will change by *no more than* $\beta_j/4$. In the following exercise, you'll derive this rule.

Exercise #2

(a) Analyze the function $\Lambda(z)$: calculate its derivative, and its limits as $z \to -\infty$ and $+\infty$. What values can this function take? Is it increasing? Decreasing? Explain.

(b) What is the maximum value of the *derivative* of $\Lambda(z)$? At what value of z does it occur?

(c) Use the chain rule and your answer to (a) to find the partial derivative of $\Lambda(X'\beta)$ with respect to β_j.

(d) Use your answers to (b) and (c) to explain the "divide by 4 rule."

Solution to Exercise #2

The function Λ takes values between 0 and 1. When $z = 0$, $\Lambda(z) = e^0/(1 + e^0) = 1/2$. As $z \to \infty$, $\Lambda(z) \to 1$ and as $z \to -\infty$, $\Lambda(z) \to 0$. We calculate its derivative using the quotient rule as follows

$$\Lambda'(z) = \frac{e^z(1 + e^z) - e^z e^z}{(1 + e^z)^2} = \frac{e^z}{(1 + e^z)^2}$$

Since e^z is always greater than zero, the derivative is always positive so $\Lambda(z)$ is strictly increasing. To find the value of z that maximizes the first derivative, we take the *second* derivative of Λ as follows