Using Invalid Instruments on Purpose: Focused Moment Selection and Averaging for GMM

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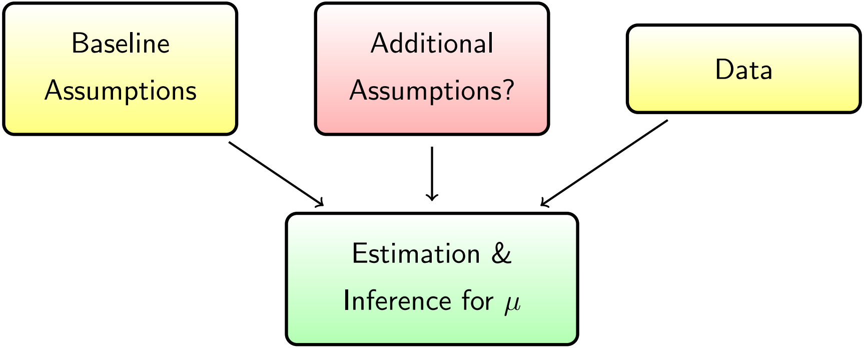
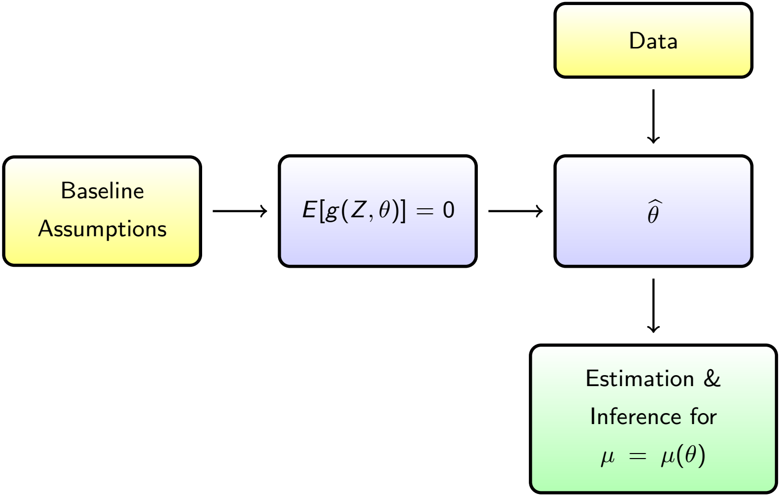
Focused Moment Selection Criterion (FMSC)

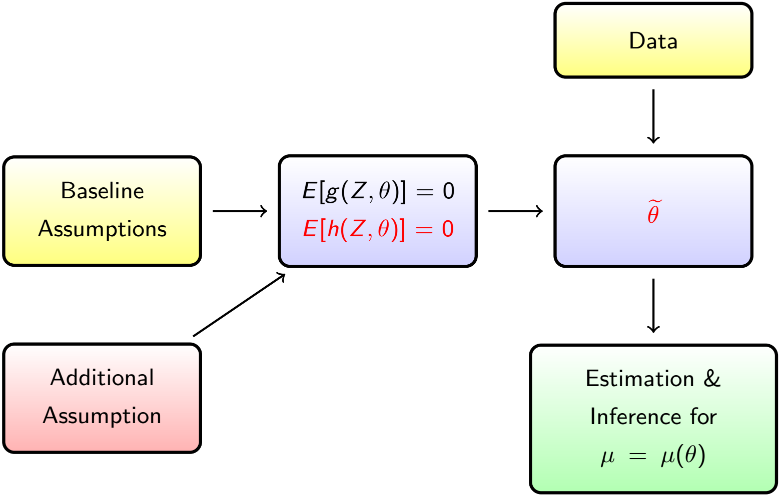


1. Choose False Assumptions on Purpose
2. Focused Choice of Assumptions
3. Local mis-specification
4. Averaging, Inference post-selection

GMM Framework



Adding Moment Conditions



Ordinary versus Two-Stage Least Squares

$$
\begin{aligned}
y_i &= \beta x_i + \epsilon_i \\
x_i &= \mathbf{z}_i' \boldsymbol{\pi} + v_i
\end{aligned}
$$

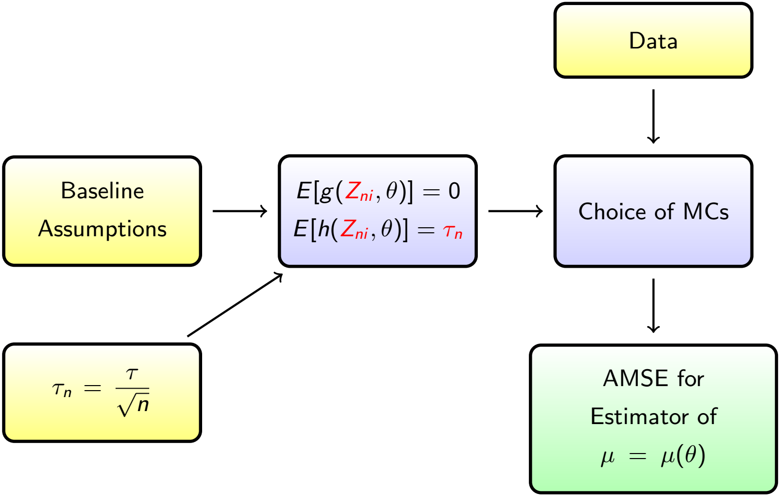
$$
\begin{aligned}
E[\mathbf{z}_i \epsilon_i] &= 0 \\
E[x_i \epsilon_i] &= ?
\end{aligned}
$$

Choosing Instrumental Variables

$$
\begin{aligned}
y_i &= \beta x_i + \epsilon_i \\
x_i &= \Pi_1' \mathbf{z}_i^{(1)} + \Pi_2' \mathbf{z}_i^{(2)} + v_i
\end{aligned}
$$

$$
\begin{aligned}
E[\mathbf{z}_i^{(1)} \epsilon_i] &= 0 \\
E[\mathbf{z}_i^{(2)} \epsilon_i] &= ?
\end{aligned}
$$

FMSC Asymptotics – Local Mis-Specification



Local Mis-Specification for OLS versus TSLS

$$
\begin{aligned}
y_i &= \beta x_i + \epsilon_i \\
x_i &= \mathbf{z}_i' \boldsymbol{\pi} + v_i
\end{aligned}
$$

$$
\begin{aligned}
E[\mathbf{z}_i \epsilon_i] &= 0 \\
E[x_i \epsilon_i] &= \tau / \sqrt{n}
\end{aligned}
$$

Local Mis-Specification for Choosing IVs

$$
\begin{aligned}
y_i &= \beta x_i + \epsilon_i \\
x_i &= \Pi_1' \mathbf{z}_i^{(1)} + \Pi_2' \mathbf{z}_i^{(2)} + v_i
\end{aligned}
$$

$$
\begin{aligned}
E[\mathbf{z}_i^{(1)} \epsilon_i] &= 0 \\
E[\mathbf{z}_i^{(1)} \epsilon_i] &= \tau / \sqrt{n}
\end{aligned}
$$

Local Mis-Specification

Triangular Array $\{Z_{ni}\colon 1 \leq i \leq n, n = 1, 2, \ldots\}$ with

(a) $E[g(Z_{ni}, \theta_0)] = 0$

(b) $E[h(Z_{ni}, \theta_0)] = n^{-1/2}\tau$

(c) $\{f(Z_{ni}, \theta_0)\colon 1 \leq i \leq n, n = 1, 2, \ldots\}$ uniformly integrable

(d) $Z_{ni} \to_d Z_i$, where the Z_i are identically distributed.

Shorthand: Write Z for Z_i

Candidate GMM Estimator

$$\widehat{\theta}_S = \underset{\theta \in \Theta}{\arg\min} \; [\Xi_S f_n(\theta)]' \, \widetilde{W}_S \; [\Xi_S f_n(\theta)]$$

$$\begin{aligned}
\Xi_S &= \text{Selection Matrix (ones and zeros)} \\
\widetilde{W}_S &= \text{Weight Matrix (p.s.d.)} \\
f_n(\theta) &= \begin{bmatrix} g_n(\theta) \\ h_n(\theta) \end{bmatrix} = \begin{bmatrix} n^{-1} \sum_{i=1}^{n} g(Z_{ni}, \theta) \\ n^{-1} \sum_{i=1}^{n} h(Z_{ni}, \theta) \end{bmatrix}
\end{aligned}$$

Notation: Limit Quantities

$$G = E\left[\nabla_\theta\, g(Z, \theta_0)\right], \quad H = E\left[\nabla_\theta\, h(Z, \theta_0)\right], \quad F = \begin{bmatrix} G \\ H \end{bmatrix}$$

$$\Omega = Var\left[f(Z, \theta_0)\right] = \begin{bmatrix} \Omega_{gg} & \Omega_{gh} \\ \Omega_{hg} & \Omega_{hh} \end{bmatrix}$$

$$\widetilde{W}_S \to_p W_S \text{ (p.d.)}$$

Local Mis-Specification + Standard Regularity Conditions

Every candidate estimator is consistent for θ_0 and

$$\sqrt{n}(\widehat{\theta}_S - \theta_0) \rightarrow_d -K_S \Xi_S \left(\begin{bmatrix} M_g \\ M_h \end{bmatrix} + \begin{bmatrix} 0 \\ \tau \end{bmatrix} \right)$$

$$\begin{aligned} K_S &= [F_S' W_S F_S]^{-1} F_S' W_S \\ M &= (M_g', M_h')' \\ M &\sim N(0, \Omega) \end{aligned}$$

Scalar Target Parameter μ

$$
\begin{aligned}
\mu &= \mu(\theta) && \text{Z-a.s. continuous function} \\
\mu_0 &= \mu(\theta_0) && \text{true value} \\
\widehat{\mu} &= \mu(\widehat{\theta}_S) && \text{estimator}
\end{aligned}
$$

Delta Method

$$
\sqrt{n}\left(\widehat{\mu}_S - \mu_0\right) \to_d -\nabla_\theta \mu(\theta_0)' K_S \Xi_S \left(M + \left[\begin{array}{c} 0 \\ \tau \end{array} \right] \right)
$$

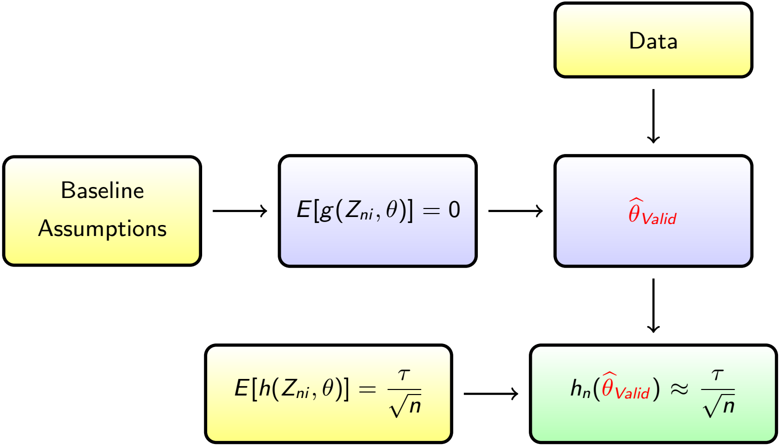
FMSC: Estimate AMSE($\widehat{\mu}_S$) and minimize over S

$$\text{AMSE}\left(\widehat{\mu}_S\right) = \nabla_\theta \mu(\theta_0)' K_S \Xi_S \left\{ \left[\begin{array}{cc} 0 & 0 \\ 0 & \tau\tau' \end{array} \right] + \Omega \right\} \Xi_S' K_S' \nabla_\theta \mu(\theta_0)$$

Estimating the unknowns

No consistent estimator of τ exists! (But everything else is easy)

A Plug-in Estimator of τ



An Asymptotically Unbiased Estimator of $\tau\tau'$

$$\sqrt{n}h_n(\widehat{\theta}_v) = \widehat{\tau} \rightarrow_d (\Psi M + \tau) \sim N_q(\tau, \Psi\Omega\Psi')$$

$$\Psi = \left[\begin{array}{cc} -HK_v & \mathbf{I}_q \end{array} \right]$$

$\widehat{\tau}\widehat{\tau}' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}$ is an asymptotically unbiased estimator of $\tau\tau'$.

FMSC: Asymptotically Unbiased Estimator of AMSE

$$\text{FMSC}_n(S) = \nabla_\theta \mu(\widehat{\theta})' \widehat{K}_S \Xi_S \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \widehat{B} \end{bmatrix} + \widehat{\Omega} \right\} \Xi_S' \widehat{K}_S' \nabla_\theta \mu(\widehat{\theta})$$

$$\widehat{B} = \widehat{\tau}\widehat{\tau}' - \widehat{\Psi}\widehat{\Omega}\widehat{\Psi}'$$

Choose S to minimize $\text{FMSC}_n(S)$ over the set of candidates $\mathscr{S}$.

A (Very) Special Case of the FMSC

Under homoskedasticity, FMSC selection in the OLS versus TSLS example is *identical* to a Durbin-Hausman-Wu test with $\alpha \approx 0.16$

$$\widehat{\tau} = n^{-1/2}\mathbf{x}'(\mathbf{y} - \mathbf{x}\widetilde{\beta}_{TSLS})$$

OLS gets benefit of the doubt, but not as much as $\alpha = 0.05,\ 0.1$

Limit Distribution of FMSC

$FMSC_n(S) \rightarrow_d FMSC_S$, where

$$
\begin{aligned}
\text{FMSC}_S &= \nabla_\theta \mu(\theta_0)' K_S \Xi_S \left\{ \left[\begin{array}{cc} 0 & 0 \\ 0 & B \end{array} \right] + \Omega \right\} \Xi_S' K_S' \nabla_\theta \mu(\theta_0) \\
B &= (\Psi M + \tau)(\Psi M + \tau)' - \Psi \Omega \Psi'
\end{aligned}
$$

Conservative criterion: random even in the limit.

Moment Average Estimators

$$\widehat{\mu} = \sum_{S \in \mathscr{S}} \widehat{\omega}_S \widehat{\mu}_S$$

Additional Notation

- $\widehat{\mu}$ Moment-average Estimator
- $\widehat{\mu}_S$ Estimator of target parameter under moment set S
- $\widehat{\omega}_S$ Data-dependent weight function
- $\mathscr{S}$ Collection of moment sets under consideration

Examples of Moment-Averaging Weights

Post-Moment Selection Weights

$\widehat{\omega}_S = \mathbf{1}\left\{\text{MSC}_n(S) = \min_{S' \in \mathscr{S}} \text{MSC}_n(S')\right\}$

Exponential Weights

$$\widehat{\omega}_S = \exp\left\{-\frac{\kappa}{2}\text{MSC}(S)\right\} \Big/ \sum_{S' \in \mathscr{S}} \exp\left\{-\frac{\kappa}{2}\text{MSC}(S')\right\}$$

Minimum-AMSE Weights...

Minimum AMSE-Averaging Estimator: OLS vs. TSLS

$$\widetilde{\beta}(\omega) = \omega\widehat{\beta}_{OLS} + (1-\omega)\widetilde{\beta}_{TSLS}$$

Under homoskedasticity:

$$\omega^* = \left[1 + \frac{\text{ABIAS(OLS)}^2}{\text{AVAR(TSLS)} - \text{AVAR(OLS)}}\right]^{-1}$$

Estimate by:

$$\widehat{\omega}^* = \left[1 + \frac{\max\left\{0,\ \left(\widehat{\tau}^2 - \widehat{\sigma}_\epsilon^2\widehat{\sigma}_x^2\left(\widehat{\sigma}_x^2/\widehat{\gamma}^2 - 1\right)\right)\ /\ \widehat{\sigma}_x^4\right\}}{\widehat{\sigma}_\epsilon^2(1/\widehat{\gamma}^2 - 1/\widehat{\sigma}_x^2)}\right]^{-1}$$

Where $\widehat{\gamma}^2 = n^{-1}\mathbf{x}'Z(Z'Z)^{-1}Z'\mathbf{x}$

Limit Distribution of Moment-Average Estimators

$$\widehat{\mu} = \sum_{S \in \mathscr{S}} \widehat{\omega}_S \widehat{\mu}_S$$

(i) $\sum_{S \in \mathscr{S}} \widehat{\omega}_S = 1$ a.s.

(ii) $\widehat{\omega}(S) \to_d \varphi_S(\tau, M)$ a.s.-continuous function of τ, M and consistently-estimable constants only

$$\sqrt{n}\,(\widehat{\mu} - \mu_0) \to_d \Lambda(\tau)$$

$$\Lambda(\tau) = -\nabla_\theta \mu(\theta_0)' \left[\sum_{S \in \mathscr{S}} \varphi_S(\tau, M) K_S \Xi_S \right] \left(M + \left[\begin{array}{c} 0 \\ \tau \end{array} \right] \right)$$

Simulating from the Limit Experiment

Suppose τ Known, Consistent Estimators of Everything Else

1. `for` $j \in \{1, 2, \ldots, J\}$
   (i) $M_j \overset{iid}{\sim} N_{p+q}\left(0, \widehat{\Omega}\right)$
   (ii) $\Lambda_j(\tau) = -\nabla_\theta \mu(\widehat{\theta})' \left[\sum_{S \in \mathscr{S}} \widehat{\varphi}_S(M_j + \tau)\widehat{K}_S \Xi_S\right](M_j + \tau)$
2. Using $\{\Lambda_j(\tau)\}_{j=1}^J$ calculate $\widehat{a}(\tau)$, $\widehat{b}(\tau)$ such that
$$P\left[\widehat{a}(\tau) \leq \Lambda(\tau) \leq \widehat{b}(\tau)\right] = 1 - \alpha$$
3. $P\left[\widehat{\mu} - \widehat{b}(\tau)/\sqrt{n} \leq \mu_0 \leq \widehat{\mu} - \widehat{a}(\tau)/\sqrt{n}\right] \approx 1 - \alpha$

Two-step Procedure for Conservative Intervals

1. Construct $1-\delta$ confidence region $\mathscr{T}(\widehat{\tau},\delta)$ for τ
2. For each $\tau^* \in \mathscr{T}(\widehat{\tau},\delta)$ calculate $1-\alpha$ confidence interval $\left[\widehat{a}(\tau^*), \widehat{b}(\tau^*)\right]$ for $\Lambda(\tau^*)$ as descibed on previous slide.
3. Take the lower and upper bound over the resulting intervals: $\widehat{a}_{min}(\widehat{\tau}) = \min_{\tau^* \in \mathscr{T}} \widehat{a}(\tau^*), \quad \widehat{b}_{max}(\widehat{\tau^*}) = \max_{\tau^* \in \mathscr{T}} \widehat{b}(\tau)$
4. The interval

$$\mathrm{CI}_{sim} = \left[\widehat{\mu} - \frac{\widehat{b}_{max}(\widehat{\tau})}{\sqrt{n}}, \quad \widehat{\mu} - \frac{\widehat{a}_{min}(\widehat{\tau})}{\sqrt{n}}\right]$$

has asymptotic coverage of at least $1-(\alpha+\delta)$

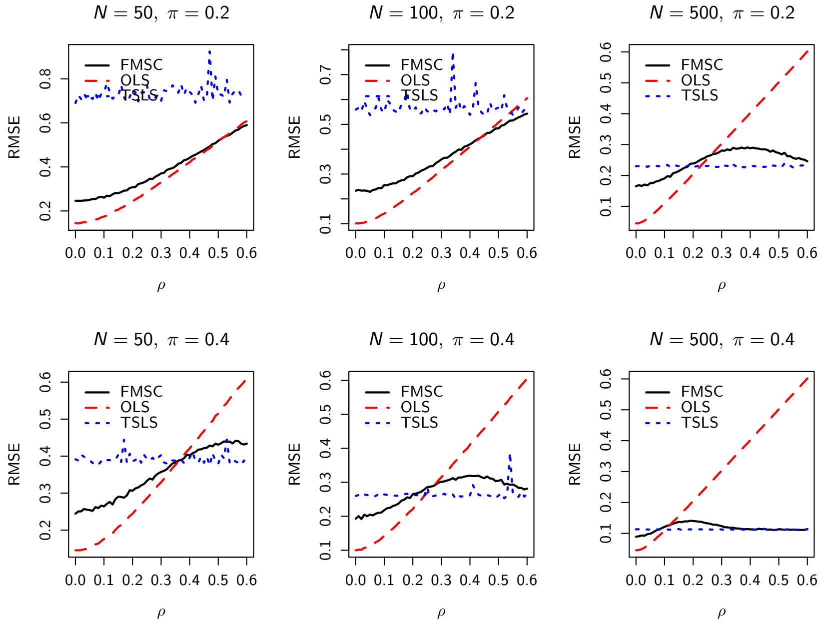
OLS versus TSLS Simulation

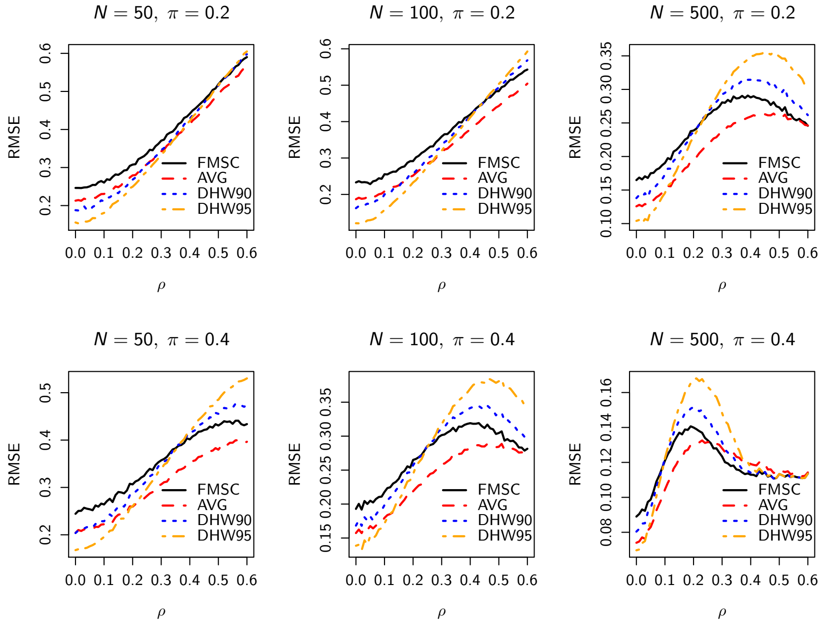
$$
\begin{aligned}
y_i &= 0.5x_i + \epsilon_i \\
x_i &= \pi(z_{1i} + z_{2i} + z_{3i}) + v_i
\end{aligned}
$$

$$(\epsilon_i, v_i, z_{1i}, z_{2i}, z_{3i}) \sim \text{ iid } N(0, \mathcal{S})$$

$$
\mathcal{S} = \begin{bmatrix}
1 & \rho & 0 & 0 & 0 \\
\rho & 1-\pi^2 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 1/3
\end{bmatrix}
$$

$$Var(x) = 1, \qquad \rho = Cor(x, \epsilon), \qquad \pi^2 = \text{First-Stage } R^2$$


N = 50, π = 0.2
N = 100, π = 0.2
N = 500, π = 0.2
N = 50, π = 0.4
N = 100, π = 0.4
N = 500, π = 0.4
FMSC
OLS
TSLS
RMSE
ρ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$N = 100,\ \pi = 0.4$
$N = 500,\ \pi = 0.4$
RMSE
ρ
FMSC
AVG
DHW90
DHW95

Choosing Instrumental Variables Simulation

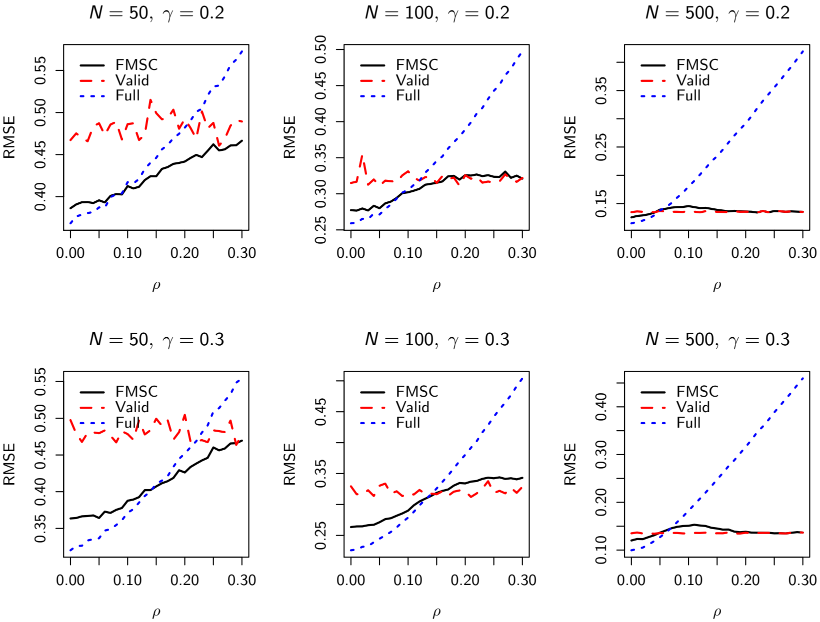
$$\begin{aligned} y_i &= 0.5x_i + \epsilon_i \\ x_i &= (z_{1i} + z_{2i} + z_{3i})/3 + \gamma w_i + v_i \end{aligned}$$

$$(\epsilon_i, v_i, w_i, z_{i1}, z_{2i}, z_{3i})' \sim \text{ iid } N(0, \mathcal{V})$$

$$\mathcal{V} = \begin{bmatrix} 1 & (0.5 - \gamma\rho) & \rho & 0 & 0 & 0 \\ (0.5 - \gamma\rho) & (8/9 - \gamma^2) & 0 & 0 & 0 & 0 \\ \rho & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix}$$

$$\gamma = Cor(x, w), \quad \rho = Cor(w, \epsilon), \quad \text{First-Stage } R^2 = 1/9 + \gamma^2$$

$$Var(x) = 1, \quad Cor(x, \epsilon) = 0.5$$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$N = 100,\ \gamma = 0.3$
$N = 500,\ \gamma = 0.3$
FMSC
Valid
Full
RMSE
ρ
0.00
0.10
0.20
0.30

Alternative Moment Selection Procedures

Downward J-test

Use Full instrument set unless J-test rejects.

Andrews (1999) – GMM Moment Selection Criteria

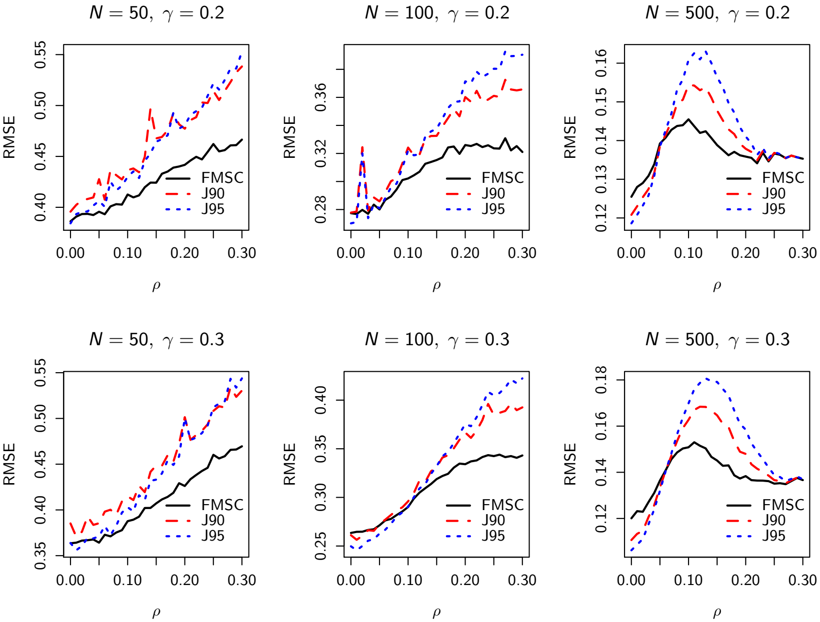
$\text{GMM-MSC}(S) = J_n(S) - \text{Bonus}$

Hall & Peixe (2003) – Canonical Correlations Info. Criterion

$\text{CCIC}(S) = n \log\left[1 - R_n^2(S)\right] + \textit{Penalty}$

Penalty/Bonus Terms

Analogies to AIC, BIC, and Hannan-Quinn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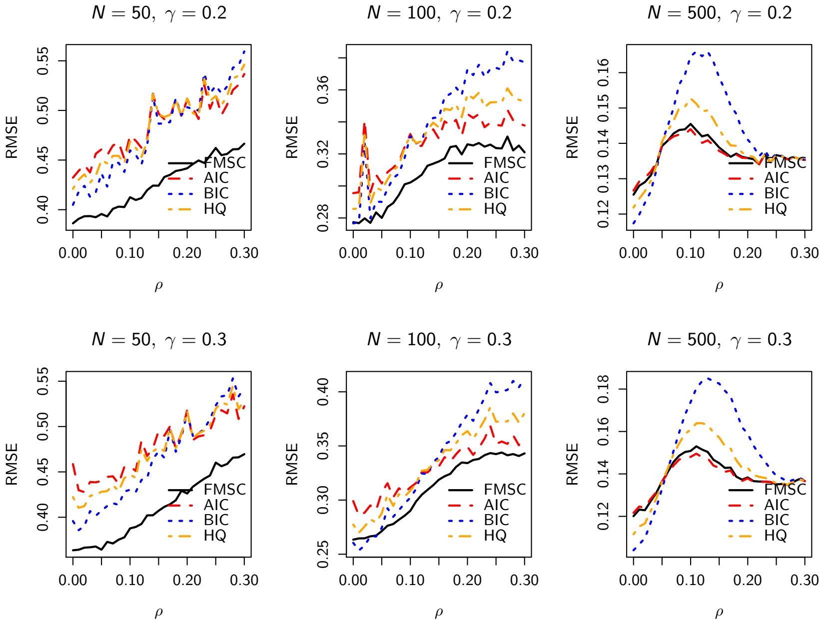
$N = 100,\ \gamma = 0.3$
$N = 500,\ \gamma = 0.3$
RMSE
ρ
FMSC
J90
J95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$N = 50,\ \gamma = 0.3$
$N = 100,\ \gamma = 0.3$
$N = 500,\ \gamma = 0.3$
RMSE
ρ
FMSC
AIC
BIC
HQ

Empirical Example: Geography or Institutions?

Institutions Rule

Acemoglu et al. (2001), Rodrik et al. (2004), Easterly & Levine (2003) – zero or negligible effects of "tropics, germs, and crops" in income per capita, controlling for institutions.

Institutions *Don't* Rule

Sachs (2003) – Large negative direct effect of malaria transmission on income.

Carstensen & Gundlach (2006)

How robust is Sachs's result?

Carstensen & Gundlach (2006)

Both Regressors Endogenous

$\ln GDPC_i = \beta_1 + \beta_2 \cdot INSTITUTIONS_i + \beta_3 \cdot MALARIA_i + \epsilon_i$

Robustness

- Various measures of *INSTITUTIONS*, *MALARIA*
- Various instrument sets
- β_3 remains large, negative and significant.

2SLS for All Results That Follow

Expand on Instrument Selection Exercise

FMSC and Corrected Confidence Intervals

1. FMSC – which instruments to estimate effect of malaria?
2. Correct CIs for Instrument Selection – effect of malaria still negative and significant?

Measures of *INSTITUTIONS* and *MALARIA*

- *rule* – Average governance indicator (Kaufmann, Kray and Mastruzzi; 2004)
- *malfal* – Proportion of population at risk of malaria transmission in 1994 (Sachs, 2001)

Instrument Sets

Baseline Instruments – Assumed Valid

- ln*mort* – Log settler mortality (per 1000), early 19th century
- *maleco* – Index of stability of malaria transmission

Further Instrument Blocks

Climate	*frost*, *humid*, *latitude*
Europe	*eurfrac*, *engfrac*
Openness	*coast*, *trade*

	μ = *malfal*			μ = *rule*		
	FMSC	posFMSC	$\widehat{\mu}$	FMSC	posFMSC	$\widehat{\mu}$
(1) Valid	3.0	3.0	-1.0	1.3	1.3	0.9
(2) Climate	3.1	3.1	-0.9	1.0	1.0	1.0
(3) Open	2.3	2.4	-1.1	1.2	1.2	0.8
(4) Eur	1.8	2.2	-1.1	0.5	0.7	0.9
(5) Climate, Eur	0.9	2.0	-1.0	0.3	0.6	0.9
(6) Climate, Open	1.9	2.3	-1.0	0.5	0.8	0.9
(7) Open, Eur	1.6	1.8	-1.2	0.8	0.8	0.8
(8) Full	0.5	1.7	-1.1	0.2	0.6	0.8
$>90\%$ CI FMSC		$(-1.6, -0.6)$			$(0.5, 1.2)$	
$>90\%$ CI posFMSC		$(-1.6, -0.6)$			$(0.6, 1.3)$	

Extensions & Future Work

- Simultaneous Model and Moment Selection – Dynamic Panel
- Welfare-Max. Treament Assignment under Budget Constraints
- Individual Heterogeneity: choosing the number of groups