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Estimating Dynamic Equilibrium Models using Mixed Frequency Macro and Financial Data

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1 ABSTRACT

This paper provides a unified framework that allows to combine mixed frequency data for estimation of dynamic equilibrium models. The paper is based on the idea that incorporating financial data in addition to macro data may be beneficial when estimating a DSGE model because it contains extra information about the state of the economy and can thus enhance the inference about the structural parameters of the model. Financial data usually are more accurate and are also available at higher frequency than macroeconomic indicators. In order to fully take advantage of the mixed-frequencies, the authors propose formulating DSGE models in continuous time. They also develop estimation techniques for such models based on martingale estimating functions (MEF) that uses model-based aggregation to incorporate the mixed-frequency data. Finally, they extend the estimation techniques to further explore mixed-frequency data and also account for latent variables using simulation based methods. The paper compares the suggested approach with regression-based methods and GMM for a simple AK-Vasicek model and finds that the GMM and MEF-based approaches exhibit good performance. The authors also apply the estimation method to the U.S. data and find the MEF approach demonstrates the best performance. However, the model is so simple that it is likely to be misspecified.

2 AK-VASICEK MODEL

While the framework developed in the paper is quite general, the authors mainly concentrate on a simple AK-Vasicek model with mean-reverting interest rates, so below I provide a short description of the model.
Assume that the technology is given by $Y_t = A_t K_t$ where Y_t represents output, A_t is total factor productivity and K_t is the aggregate capital stock. TFP is driven by a Brownian motion $B_t : dA_t = \mu(A_t)dt + \eta(A_t)dB_t$.
The law of motion for capital is defined by investment I_t, capital depreciation rate δ and Brownian motion Z_t: $dK_t = (I_t - \delta K_t)dt + \sigma K_t Z_t$.
Preferences take the logarithmic form as a function of consumption C_t: $u(C_t) = \ln C_t$. Under these assumptions, the rental rate of capital $r_t = Y_K = A_t$ and $K_t = Y_t / A_t = Y_t / r_t$, so unobserved state variables (A_t, K_t) can be expressed as functions of observed variables (Y_t, r_t).
The authors use Vasicek specification to pin down the process that drives the TFP and the interest rate: $\mu(r_t) = \kappa(\gamma - r_t)$ and $\eta(r_t) = \eta$. The relationship between the risk free rate and the rental rate of capital is given by $r_t^f = r_t - \delta - \sigma^2$. The natural problem that arises when estimating this model is that rental rate of capital is difficult to obtain while the risk free rate is readily available, so it's easier to find the rental rate of capital employing its relation to the risk-less rate than directly estimating it from the data.

3 ESTIMATION

Suppose that the triple (C_t, Y_t, r_t^f) is observed. We can write the system of 3 equations that characterize the equilibrium of the model above at each time t. However, this model must be discretized to match the data. Consider two time periods $t > s$ and $\Delta = t - s$. Integrating equilibrium conditions yields:

$$y_{C,t} = \ln(C_s/C_t) - \int_{t-\Delta}^{t} r_v^f dv = const(\rho,\sigma^2) + \underbrace{\sigma(Z_t - Z_{t-\Delta})}_{\epsilon_{C,t}} \tag{3.1}$$

$$\begin{aligned} y_{Y,t} = \ln(Y_s/Y_t) - \int_{t}^{t-\Delta} r_v^f dv = const(\kappa,\rho,\sigma^2) + \kappa\gamma \int_{t-\Delta}^{t} 1/r_v dv - \frac{1}{2}\eta^2 \int_{t-\Delta}^{t} 1/r_v^2 dv + \\ + \underbrace{\int_{t-\Delta}^{t} \eta/r_v dB_v + \sigma(Z_t - Z_{t-\Delta})}_{\epsilon_{Y,t}} \end{aligned} \tag{3.2}$$

$$y_{r,t} = r_t^f = const(\kappa,\gamma,\delta,\sigma^2) + e^{-\kappa\Delta} r_{t-\Delta}^f + \underbrace{\eta e^{-\kappa\Delta} \int_{t-\Delta}^{t} e^{\kappa(v-(t-\Delta))} dB_v}_{\epsilon_{r,t}} \tag{3.3}$$

where $(\epsilon_{C,t}, \epsilon_{Y,t}, \epsilon_{r,t})$ are martingale increments. There are two things to note. First, the expressions above depend on the integral of the functions the interest rate. The key idea of the paper is to approximate these integrals by taking sums of the interest rates obtained at higher frequency. Second, the system would be linear in parameters if the capital rental rate r_t was observed and thus could easily be estimated using usual techniques. The authors suggest to replace the unknown interest rate r_t with a proxy $\widehat{r}_t$ that can be obtained using its relation to the risk free interest rate r_t^f and some parameter values δ_0 and σ_0: $\widehat{r}_t = r_t^f + \delta_0 + \sigma_0$. Once the system is linear, usual regression-based methods can be applied.

3.1 THE REGRESSION-BASED APPROACHES

If we have a proxy for the capital rental rate $\widehat{r}_t$, the system is linear. The reduced form parameters can then be estimated using equation-by-equation OLS by regressing $y_t = (y_{C,t}, y_{Y,t}, y_{r,t})$ on $x_t = (x_{C,t}, x_{Y,t}, x_{r,t})$, where $x_{C,t} = 1$, $x_{Y,t} = (1, \int_{t-\Delta}^{t} 1/\widehat{r}_v dv, \int_{t-\Delta}^{t} 1/\widehat{r}_v^2 dv)$ and $x_t = (1, r_{t-\Delta}^f)$.
However, the basic OLS can be extended to take into account the peculiarities of the system at hand. First, note a common term $\sigma(Z_t - Z_{t-\Delta})$ in the errors (eq. 3.1, 3.2). This cross-correlation can be taken care of by applying seemingly-unrelated regressions (SUR) analysis. Next, there is a possibility of endogeneity of the regressors. The authors suggest using an IV estimator to account for that.
The reduced-form parameters are functions of structural parameters $\phi = (\kappa, \gamma, \eta, \rho, \delta, \sigma)$. Once the estimators of the reduced-form system are obtained, they can be used to make inference about the structural parameters using minimum distance estimation. However, only five parameter functions out of 6 parameters can be identified. In order to solve the identification problem, the authors suggest to augment the regression with higher order moments.

3.2 GMM AND MEF

GMM estimator can be constructed using the properties of the martingale-difference sequence $m_t = (\epsilon_{C,t}, \epsilon_{Y,t}, \epsilon_{r,t})$: $\mathbb{E}_{t-\Delta}[m_t] = 0$. For a set of instruments z_t known at period $t - \Delta$(the authors suggest to use lagged right-hand side variables) the moment conditions are: $\mathbb{E}_{t-\Delta}(z_t \otimes m_t) = \mathbb{E}_{t-\Delta} h_t = 0$. Note that the set of instruments depends on the structural parameters of the model ϕ. Optimal GMM estimator $\widehat{\phi}_{GMM}$ can then be constructed in a usual fashion either for exact- or over-identified case by minimizing $H_T(\phi)' W H_T(\phi)$ where $H_T = \sum_{t=1}^{T} h_t(\phi)/T$ in two steps in order to compute the optimal weight matrix W.

Note that the first order conditions for GMM take the following form: $G\sum_{t=1}^{T} h_t(\phi) = 0$ for some matrix $G = \partial H_T \phi' / \partial \phi W$. We could estimate ϕ by using some first stage estimator of G instead of W and solving the equation $G(\widehat{\phi}_0)\sum_{t=1}^{T} h_t(\phi) = 0$ which will result in an estimator that is asymptotically equivalent to GMM. The central idea of MEF is to use a more general form $\sum_{t=1}^{T} g_t(\widehat{\phi}_0) h_t(\phi) = 0$. The problem that MEF solves is formulated as a weighted sum of the matringale differences: $M_T = \sum_{t=1}^{T} w_t m_t$ and at the true parameter values $M_T(\phi) = 0$. The MEF estimator $\widehat{\phi}_{MEF}$ solves $\sum_{t=1}^{T} w_t(\phi) m_t(\phi) = 0$ for some optimal set of weights w_t. Optimal weights depend on the conditional variance of the vector martingale increment and its partial derivative with respect to parameters ϕ and can be estimated using two step approach. It can be shown that the MEF method produces strictly more efficient estimators than GMM (except in the case when they coincide).
MEF approach can be further extended to account for mixed frequency data and latent variables. Data at lower frequency (e.g. output observed quarterly instead of monthly) can be treated as if some observations are missing. The authors suggest to replace the missing values by conditional predictions under the model at hand given the actual observations. Latent variables (e.g. daily rental rate of capital) can be simulated using the process implied by the model. The simulated paths can later be used to compute the conditional expectation needed for the MEF approach.

4 Simulation Study

The authors compare performance of the OLS, FGLS-SUR-IV (IV augmented regression with SUR that relies on feasible GLS), GMM and MEF approaches. The structural parameters for reduced form methods are estimated using minimum distance method. In order to solve the identification issue, the authors set $\delta = 0.05$ equal to its true value for regression-based approaches and GMM. As can be seen from the table, FGLS-SUR-IV, GMM and MEF demonstrate very accurate estimates of the true DGP parameters for γ, η and ρ but all of the have troubles correctly estimating κ.
When taking the model to empirical data, the authors encountered some numerical optimization

Table 1: Simulation Study – Monthly and Quarterly Data
The table reports output of a simulation study of the accuracy of the structural model parameters estimated using the OLS, FGLS-SUR-IV, GMM and MEF approaches for the AK-Vasicek model. For 1,000 replications, we generate 25 years of data from the underlying data generating process (DGP) and apply our estimation strategy. We show the median estimate, and provide the interquartile range below it.

Parameter Estimates from Simulation Study – Monthly & Quarterly Data									
		Monthly Data				Quarterly Data			
	DGP	OLS	FGLS-SUR-IV	GMM	MEF	OLS	FGLS-SUR-IV	GMM	MEF
κ	0.2	0.349	0.299	0.345	0.354	0.354	0.225	0.287	0.353
		0.286	0.134	0.345	0.284	0.290	0.119	0.319	0.305
γ	0.1	0.201	0.101	0.100	0.099	0.198	0.100	0.101	0.099
		0.036	0.013	0.014	0.013	0.036	0.014	0.015	0.013
η	0.01	0.083	0.008	0.010	0.010	0.083	0.007	0.010	0.010
		0.036	0.004	0.001	0.001	0.035	0.003	0.002	0.001
ρ	0.03	0.080	0.030	0.030	0.030	0.079	0.030	0.031	0.030
		0.015	0.006	0.007	0.006	0.015	0.006	0.007	0.006
δ	0.05	0.05	0.05	0.05	0.050	0.05	0.05	0.05	0.050
					0.002				0.003
σ	0.02	0.317	0.000	0.027	0.023	0.312	0.000	0.040	0.025
		0.040	<0.001	0.047	0.005	0.044	<0.001	0.064	0.010

problems that may be due to possible model misspecification that arise , so they pre-fixed values of δ and σ for regression-based methods and GMM. The MEF approach may thus be preferable as it doesn't have these identification issues. However, the results they obtained look reasonable and suggest that the MEF is a promising approach.