Comparative static and computational methods for an empirical one-to-one transferable utility matching model

Bryan S. Graham*

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Abstract

I show that the equilibrium distribution of matches associated with the empirical transferable utility (TU) one-to-one matching model introduced by Choo and Siow (2006a,b) corresponds to the fixed point of system of $K + L$ nonlinear equations; with K and L respectively equal to the number of discrete types of women and men. I use this representation to derive new comparative static results, showing how the match distribution varies with match surplus and the marginal distributions of agent types.

In the context of a single agent discrete choice problem, the assumption of utility maximization provides a tight link between the observed population distribution of choices and the unobserved population distribution of preferences (McFadden, 1974; Manski, 1975; Matzkin, 2007). In contrast the mapping from an observed distribution of matches between *two sets of heterogenous agents*, say men and women in a marriage market, and total match utility or surplus is less well-understood. In one-to-one matching problems two agents must agree to form a match. Rivalry is important, constraining choice: an individual's utility maximizing match partner may be unavailable, herself preferring to match with someone else. Rivalry, a consequence of the two-sided aspect of the problem, makes the problem of inferring the distribution of match surplus from information of who matches with whom difficult (Fox, 2010; Graham, 2011; Echenique, Lee, Shum and Yenmez, 2013).

I study a variant of the structural matching model introduced by Choo and Siow (2006,a,b) (henceforth the 'CS model').[1] Abstractly the CS model is a two-sided model of discrete choice subject to a market clearing, or adding-up, restriction (cf., Graham, 2011). The general equilibrium nature of the CS model makes a complete understanding of its economic properties difficult. In his

*Department of Economics, University of California - Berkeley, 530 Evans Hall #3380, Berkeley, CA 94720-3888, e-mail: bgraham@econ.berkeley.edu, web: http://emlab.berkeley.edu/ bgraham/. I would like to thank Gary Chamberlain, Eugene Choo, Alfred Galichon, Michael Jansson, Konrad Menzel, Ariel Pakes, and numerous participants in the 2013 Advances in Econometrics Conference at Louisiana State University and a seminar at Harvard-MIT for helpful discussions. A special thanks to Gary Chamberlain, Eugene Choo and Alfred Galichon for written feedback on an earlier draft. All the usual disclaimers apply.

[1]See Dagsvik (2000) for an earlier, and related, contribution.

Canadian Economics Association Presidential Address, Siow (2008) noted that (i) whether an equilibrium matching was globally unique was an open question and (ii) that the substitution patterns generated by the model were poorly understood. Decker (2010) and Decker, Lieb, McCann and Stephens (2013) subsequently proved uniqueness of the CS matching and also derived some limited qualitative comparative static results. In independent work, Galichon and Salanie (2010, 2012) also showed uniqueness of the CS matching, but did not present comparative static results.

Let $K$and L denote the number of discrete types of men and women respectively. A matching in the CS model consists of $K + L + KL$ terms. These terms correspond to the number of each type of woman and man who chooses to remain single in equilibrium $(K + L)$, as well as the number of each type of feasible couple (KL). I show that an equilibrium CS matching corresponds to a fixed point of a certain system of $K + L$ nonlinear equations. The solution to these equations equals the number of "singles" of each type in equilibrium. The number of each type of couple is a closed form expression of the number singles and the model parameters (e.g., Choo and Siow, 2006a,b).

I use the new equilibrium representation to derive comparative static results, showing how the equilibrium match distribution varies smoothly with model fundamentals. These results extend the qualitative ones derived by Decker, Lieb, McCann and Stephens (2013). Specifically I derive inequalities on the magnitude of, and in some cases closed-form expressions for, various elasticities. These results provide insight into the substitution patterns implied by the CS model (i.e., how the distribution of matches changes in response to changes in the availability of different types of agents and other model parameters). These results speak to the testability of the model.

Identification of the parameters indexing the CS model from an equilibrium distribution of matches was first considered by Choo and Siow (2006a,b). Additional results for their original model, as well as different extensions, can be found in Siow (2008), Galichon and Salanie (2010, 2012), Graham (2011), and Chiappori, Salanie and Weiss (2012). Here I focus on characterizing the equilibrium and comparative static properties of the CS model, a topic, as noted above, also considered by Decker (2010) and Decker, Lieb, McCann and Stephens (2013).

Section 1 outlines the version of the CS model I study. Section 2 shows that the equilibrium match distribution corresponds to the fixed point of a certain system of nonlinear equations. Section 3 presents comparative static results. Section 4 summarizes and discusses additional areas for research. Proofs and derivations are collected in Section 5.

1 The matching model

1.1 Preferences

Consider a single matching market composed of two large populations of, for concreteness, women and men. While I will often invoke language familiar from the marriage market application, there are numerous other empirically relevant examples of assignment games (cf., Koopmans and Beckmann, 1957; Shapley and Shubik, 1971; Fox, 2010). For each woman and man we respectively observe the discretely-valued characteristics $W_i \in \mathbb{W} = \{w_{1,}, \ldots, w_K\}$ and $X^j \in \mathbb{X} = \{x_1, \ldots, x_L\}$.[2] The K types of women and L types of men may encode, for example, different unique combinations of years-of-schooling and age. While K and L are assumed finite, they may be very large in practice.

Observationally identical women have heterogeneous preferences over different types of men, but are indifferent between men of the same type. Specifically female i's utility from matching with male j is given by

$$U\left(W_i, X^j, \varepsilon_i\right) \quad = \quad \alpha\left(W_i, X^j\right) + \tau\left(W_i, X^j\right) + \varepsilon_i\left(X^j\right),$$

where $\alpha(w_k, x_l)$ is the systematic utility a type $W_i = w_k$ women derives from matching with a type $X^j = x_l$ man, $\varepsilon_i\left(X^j\right) = \sum_{l=1}^{L} \mathbf{1}\left(X^j = x_l\right)\varepsilon_{il}$ captures unobserved heterogeneity in women's preferences over alternative *types* of men, and $\tau(w_k, x_l)$ is the *equilibrium* transfer that a type $X^j = x_l$ man must pay a type $W_i = w_k$ women in order to match. Transfers may be negative and their determination is discussed below. Here $\mathbf{1}(\bullet)$ denotes the indicator function. Since the stochastic component of female match utility, $\varepsilon_i\left(X^j\right)$, varies with male type alone (i.e., his specific identify does not matter), women are indifferent amongst observationally identical men. A similar restriction on male preferences ensures that the equilibrium transfer, $\tau(w_k, x_l)$, depends on agent types alone (as asserted) (cf., Galichon and Salanie, 2012).

A women may also choose to remained unmatched, or 'single', in which case her utility is given by

$$\underline{U}(W_i, \varepsilon_i) = \underline{\alpha}(W_i) + \varepsilon_{i0}.$$

Men also have heterogenous preferences. Man j's utility from matching with woman i is given by

$$V\left(W_i, X^j, \upsilon_i\right) \quad = \quad \beta\left(W_i, X^j\right) - \tau\left(W_i, X^j\right) + \upsilon^j(W_i),$$

where $\beta(w_k, x_l)$ is the systematic utility a type $X^j = x_l$ men derives from matching with a type $W_i = w_k$ woman and $\upsilon^j(W_i) = \sum_{k=1}^{K} \mathbf{1}(W_i = w_k)\upsilon_k^j$ is a heterogenous component of match utility. Here $\tau(w, x)$ enters with a negative sign as we conceptually imagine men 'paying' women (recall that transfers may be negative). The utility from remaining unmatched is

$$\underline{V}\left(X^j, \upsilon^j\right) = \underline{\beta}\left(X^j\right) + \upsilon_0^j.$$

Preference heterogeneity ensures that, for any given transfer function $\tau(w, x)$, observationally identical women will match with different types of men. If the support of the heterogeneity distribution is rich enough all types of matches will be observed in equilibrium.

Let $\underline{\varepsilon} = (\varepsilon_{i0}, \varepsilon_{i1}, \ldots, \varepsilon_{iL})'$ and $\underline{\upsilon} = \left(\upsilon_0^j, \upsilon_1^j, \ldots, \upsilon_K^j\right)'$. I assume that the components of these vectors

[2]The subscript 'i' denotes a generic random draw from the population of women, while the superscript 'j' one from men.

Table 1: Feasible matchings

W\M	Single(x_0)	x_1	$\cdots$	x_L	
Single(w_0)	-	$q_1 - \sum_{k=1}^{K} r_{k1}$	$\cdots$	$q_L - \sum_{k=1}^{K} r_{kL}$	-
w_1	$p_1 - \sum_{l=1}^{L} r_{1l}$	r_{11}	$\cdots$	r_{1L}	p_1
$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$
w_K	$p_K - \sum_{l=1}^{L} r_{Kl}$	r_{K1}	$\cdots$	r_{KL}	p_K
	-	q_1	$\cdots$	q_L	

Notes: Let $r_{kl} \geq 0$ denote the number of k-to-l matches. Feasibility of a matching imposes the $K + L$ adding up constraints $\sum_{l=1}^{L} r_{kl} \leq p_k$ for $k = 1, \ldots, K$ and $\sum_{k=1}^{K} r_{kl} \leq q_l$ for $l = 1, \ldots, L$.

are independently and identically distributed Type I extreme value random variables

$$\begin{aligned} F_{\underline{\varepsilon}|W}\left(\underline{e}|\, W = w_k\right) &= \prod_{l=0}^{L} \exp\left(-\exp\left(-\frac{e_l}{\sigma_\varepsilon}\right)\right) \\ F_{\underline{\upsilon}|X}\left(\underline{\upsilon}|\, X = x_l\right) &= \prod_{k=0}^{K} \exp\left(-\exp\left(-\frac{\upsilon_k}{\sigma_\upsilon}\right)\right). \end{aligned} \tag{1}$$

Assumption (1) is slightly more general than that maintained by Choo and Siow (2006a,b) who additionally impose the restriction $\sigma_\varepsilon = \sigma_\upsilon$. Chiappori, Salanie and Weiss (2012) allow the scale parameters, σ_ε and σ_υ, to vary with, respectively, k and l.[3]

1.2 Equilibrium

Let $\alpha_{kl} = \alpha\left(w_k, x_l\right)$, $\underline{\alpha}_{k0} = \underline{\alpha}\left(w_k\right)$, $\beta_{kl} = \beta\left(w_k, x_l\right)$, $\beta_{0l} = \underline{\beta}\left(x_l\right)$ and $\tau_{kl} = \tau\left(w_k, x_l\right)$. Let θ be a vector of model parameters – to be more precisely specified below – and $\underline{\tau}$ a $KL \times 1$ vector of transfers. The total number of type k women is given by p_k, that of type l men by q_l. Let $\mathbf{p} = \left(p_1, \ldots, p_K\right)'$ and $\mathbf{q} = \left(q_1, \ldots, q_L\right)'$. Denote the probability, given a hypothetical transfer vector $\underline{\tau}$, that a type k woman matches with a type l man by $e_{kl}^{D}\left(\theta; \underline{\tau}\right)$. The probability of remaining unmatched is $e_{k0}^{D}\left(\theta, \underline{\tau}\right) = 1 - \sum_{l=1}^{L} e_{kl}^{D}\left(\theta, \underline{\tau}\right)$. Under the Type I extreme value assumption we have for $k = 1, \ldots, K$ (McFadden, 1974):

$$\begin{aligned} e_{k0}^{D}\left(\theta, \underline{\tau}\right) &= \frac{1}{1 + \sum_{n=1}^{L} \exp\left(\sigma_\varepsilon^{-1}\left[\alpha_{kn} - \underline{\alpha}_{k0} + \tau_{kn}\right]\right)} \\ e_{kl}^{D}\left(\theta, \underline{\tau}\right) &= \frac{\exp\left(\sigma_{\varepsilon k}^{-1}\left[\alpha_{kl} - \underline{\alpha}_{k0} + \tau_{kl}\right]\right)}{1 + \sum_{n=1}^{L} \exp\left(\sigma_\varepsilon^{-1}\left[\alpha_{kn} - \underline{\alpha}_{k0} + \tau_{kn}\right]\right)}, \; l = 1, \ldots, L. \end{aligned}$$

[3] Galichon and Salanie (2012) consider other parametric families of distributions. Graham (2011) considers the case where $F_{\underline{\varepsilon}|W}$ and $F_{\underline{\upsilon}|X}$ are left nonparametric.

Total 'demand' for type l men by type k women is therefore

$$\begin{aligned} r_{k0}^{D}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right) &\overset{def}{=} p_k e_{k0}^{D}\left(\theta,\underline{\tau}\right) \\ r_{kl}^{D}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right) &\overset{def}{=} p_k e_{kl}^{D}\left(\theta,\underline{\tau}\right), \end{aligned} \tag{2}$$

which, after some manipulation, gives

$$\sigma_{\varepsilon}\ln\left(\frac{r_{kl}^{D}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right)}{r_{k0}^{D}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right)}\right)=\alpha_{kl}-\underline{\alpha}_{k0}+\tau_{kl}. \tag{3}$$

For $l=1,\ldots,L$ we get a conditional 'supply' of type l men to each of the $k=1,\ldots,K$ types of women equal to

$$\begin{aligned} r_{0l}^{S}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right) &\overset{def}{=} q_l g_{0l}^{S}\left(\theta,\underline{\tau}\right) \\ r_{kl}^{S}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right) &\overset{def}{=} q_l g_{kl}^{S}\left(\theta,\underline{\tau}\right), \end{aligned} \tag{4}$$

where

$$\begin{aligned} g_{0l}^{S}\left(\theta,\underline{\tau}\right) &= \frac{1}{1+\sum_{m=1}^{K}\exp\left(\sigma_{\upsilon}^{-1}\left[\beta_{ml}-\underline{\beta}_{0l}-\tau_{ml}\right]\right)} \\ g_{kl}^{S}\left(\theta,\underline{\tau}\right) &= \frac{\exp\left(\sigma_{\upsilon}^{-1}\left[\beta_{kl}-\underline{\beta}_{0l}-\tau_{kl}\right]\right)}{1+\sum_{m=1}^{K}\exp\left(\sigma_{\upsilon}^{-1}\left[\beta_{ml}-\underline{\beta}_{0l}-\tau_{ml}\right]\right)},\ k=1,\ldots,K, \end{aligned}$$

so that

$$\sigma_{\upsilon}\ln\left(\frac{r_{kl}^{S}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right)}{r_{0l}^{S}\left(\theta,\underline{\tau},\mathbf{p},\mathbf{q}\right)}\right)=\beta_{kl}-\underline{\beta}_{0l}-\tau_{kl}. \tag{5}$$

The transfer vector, $\underline{\tau}$, adjusts to equate the KL female 'demands' with the KL male 'supplies' so that in equilibrium

$$r_{kl}^{\text{eq}}\left(\theta,\underline{\tau}^{\text{eq}},\mathbf{p},\mathbf{q}\right)=r_{kl}^{D}\left(\theta,\underline{\tau}^{\text{eq}},\mathbf{p},\mathbf{q}\right)=r_{kl}^{S}\left(\theta,\underline{\tau}^{\text{eq}},\mathbf{p},\mathbf{q}\right),\ k=1,\ldots,K,\ l=1,\ldots,L, \tag{6}$$

with the 'eq' superscript denoting an equilibrium quantity.

Let

$$\gamma_{kl}=\frac{\alpha_{kl}+\beta_{kl}-\underline{\alpha}_{k0}-\underline{\beta}_{0l}}{\sigma_{\varepsilon}+\sigma_{\upsilon}},\ \lambda=\frac{\sigma_{\upsilon}}{\sigma_{\varepsilon}+\sigma_{\upsilon}}.$$

Imposing (6), adding (3) and (5), exponentiating and rearranging yields

$$r_{kl}^{\text{eq}}=\left(r_{k0}^{\text{eq}}\right)^{1-\lambda}\left(r_{0l}^{\text{eq}}\right)^{\lambda}\exp\left(\gamma_{kl}\right) \tag{7}$$

where I let $r_{kl}^{\text{eq}}=r_{kl}^{\text{eq}}\left(\theta,\underline{\tau}^{\text{eq}},\mathbf{p},\mathbf{q}\right)$ to economize on notation.

2 Fixed point representation of the equilibrium matching

This section develops a fixed point representation of the equilibrium match distribution or 'matching'. Taking the logarithm (7) and manipulating yields the following two equalities that hold in equilibrium for all (k, l) pairs:

$$\ln\left(\frac{r_{kl}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right) = \gamma_{kl} + \lambda \ln\left(\frac{r_{0l}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right) \tag{8}$$

$$\ln\left(\frac{r_{kl}^{\text{eq}}}{r_{0l}^{\text{eq}}}\right) = \gamma_{kl} - (1-\lambda) \ln\left(\frac{r_{0l}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right). \tag{9}$$

Exponentiating both sides of equations (8) and (9) and summing over, respectively, l and k yields

$$\left(\frac{\sum_{n=1}^{L} r_{kn}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right) = \sum_{n=1}^{L} \exp\left[\gamma_{kn} + \lambda \ln\left(\frac{r_{0n}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right)\right]$$

$$\left(\frac{\sum_{m=1}^{K} r_{ml}^{\text{eq}}}{r_{0l}^{\text{eq}}}\right) = \sum_{m=1}^{K} \exp\left[\gamma_{ml} - (1-\lambda) \ln\left(\frac{r_{0n}^{\text{eq}}}{r_{m0}^{\text{eq}}}\right)\right]$$

which, since $\sum_{n=1}^{L} r_{kn}^{\text{eq}} = p_k - r_{k0}^{\text{eq}}$ and $\sum_{m=1}^{K} r_{ml}^{\text{eq}} = q_l - r_{0l}^{\text{eq}}$, implies that the equilibrium number of *unmatched* agents of each type satisfies the $K + L$ implicit equations

$$r_{k0}^{\text{eq}} = \frac{p_k}{1 + \sum_{n=1}^{L} \exp\left[\gamma_{kn} + \lambda \ln\left(\frac{r_{0n}^{\text{eq}}}{r_{k0}^{\text{eq}}}\right)\right]}, \ k = 1, \ldots, K$$

$$r_{0l}^{\text{eq}} = \frac{q_l}{1 + \sum_{m=1}^{K} \exp\left[\gamma_{ml} - (1-\lambda) \ln\left(\frac{r_{0l}^{\text{eq}}}{r_{m0}^{\text{eq}}}\right)\right]}, \ l = 1, \ldots, L,$$

for $\mathbf{r}_0 = (r_{10}, \ldots, r_{K0}, r_{01}, \ldots, r_{0L})'$ and $\theta = (\gamma', \lambda)'$ with

$$\gamma = (\gamma_{11}, \ldots, \gamma_{1L}, \ldots, \gamma_{K1}, \ldots, \gamma_{KL})'.$$

Let $B_{k0}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta) \stackrel{def}{=} p_k \left[1 + \sum_{n=1}^{L} \exp\left[\gamma_{kn} + \lambda \ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]\right]^{-1}$ for $k = 1, \ldots, K$ and $B_{0l}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta) \stackrel{def}{=} q_l \left[1 + \sum_{m=1}^{K} \exp\left[\gamma_{ml} - (1-\lambda) \ln\left(\frac{r_{0l}^{\text{eq}}}{r_{m0}^{\text{eq}}}\right)\right]\right]^{-1}$ for $l = 1, \ldots, L$.
We have shown that $\mathbf{r}_0^{\text{eq}}$ – the $(K + L) \times 1$ vector giving the equilibrium number of agents of each type who choose not to match – is a solution to the $(K + L) \times 1$ vector of implicit functions

$$\mathbf{r}_0^{\text{eq}} - \mathbf{B}\left(\mathbf{r}_\mathbf{0}^{\text{eq}}; \mathbf{p}, \mathbf{q}, \theta\right) = 0 \tag{10}$$

with $\mathbf{B}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta) = (B_{10}(\bullet), \ldots, B_{K0}(\bullet), B_{01}(\bullet), \ldots, B_{0L}(\bullet))'$. Given a solution to (10) we can solve for the number of each of the $K \times L$ types of matches in closed form using (7) above:

$$r_{kl}^{\text{eq}} = \left(r_{k0}^{\text{eq}}\right)^{1-\lambda} \left(r_{0l}^{\text{eq}}\right)^{\lambda} \exp\left(\gamma_{kl}\right), \tag{11}$$

for $k=1,\ldots,K$ and $l=1,\ldots,L$.

The representation of $\mathbf{r}_0^{\text{eq}}$ as a solution to (10) is, to my knowledge, a new result; one with, as I argue below, useful implications for estimation and inference.

From the prior work of Decker (2010), Decker, Lieb, McCann and Stephens (2013) and Galichon and Salanie (2010, 2012) we know that the solution to (10) must be unique. Let $\mathbb{T}_\epsilon = \{\mathbf{r}_0 : \epsilon \leq r_{k0} \leq p_k - \epsilon,\ \epsilon \leq r_{0l} \leq q_l - \epsilon,\ (k=1,\ldots,K,\ l=1,\ldots,L)\}$ be a closed rectangular region with ϵ some arbitrarily small positive constant and Γ be a closed and bounded subset of $\mathbb{R}^{KL}$. Let $J(\mathbf{r}_0) = I_{K+L} - \nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0;\mathbf{p},\mathbf{q},\theta)$, with $\nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0;\mathbf{p},\mathbf{q},\theta) = \frac{\partial \mathbf{B}(\mathbf{r}_0;\mathbf{p},\mathbf{q},\theta)}{\partial \mathbf{r}_0'}$, be the $(K+L)\times(K+L)$ Jacobian matrix associated with (10). If $J(\mathbf{r}_0)$ is a P-matrix on $\mathbb{T}_\epsilon$, meaning all its principal minors are positive, then Theorem 4 of Gale and Nikaido (1965) implies uniqueness of $\mathbf{r}_0^{\text{eq}}$ (on $\mathbb{T}_\epsilon$) for all $(\gamma,\lambda) \in \Gamma \times (0,1)$. Thus showing that $J(\mathbf{r}_0)$ is a P-matrix on $\mathbb{T}_\epsilon$ would provide an alternative proof of equilibrium uniqueness in the CS model. Unfortunately verifying the P-matrix property can be computationally hard in practice, especially when the Jacobian is large, non-symmetric and otherwise complicated, as is the case here (cf., Coxson, 1994).[4] In results not reported here, I have shown that $J(\mathbf{r}_0^{\text{eq}})$ is a P-matrix (indeed a diagonally dominant matrix in the sense of McKenzie (1960)). However, I have been unable to show that the result holds for $\mathbf{r}_0 \neq \mathbf{r}_0^{\text{eq}}$ (although I conjecture that it does). It is also possible show that $\mathbf{B}(\mathbf{r_0};\mathbf{p},\mathbf{q},\theta)$ is a contraction mapping in the neighborhood of $\mathbf{r}_0^{\text{eq}}$. This provides some justification for using fixed point iteration to find an equilibrium (as was done for the numerical examples reported below). However, proving that $\mathbf{B}(\mathbf{r_0};\mathbf{p},\mathbf{q},\theta)$ is a contraction on all of $\mathbb{T}_\epsilon$ remains a goal for future research.

Representation (10) suggest the an interpretation of the CS model as an 'as if' incomplete information game (e.g., Aradillas-Lopez, 2010; Bajari, Hong, Krainer and Nekipelov, 2010). Define the inclusive values

$$\begin{aligned} v_k^f(\mathbf{s}_0;\theta) &= \ln\left(\sum_{n=1}^{L}\exp\left[\gamma_{kn} + \lambda\ln\left(\frac{q_n(1-s_{0n})}{p_k(1-s_{k0})}\right)\right]\right),\ k=1,\ldots,K \\ v_l^m(\mathbf{s}_0;\theta) &= \ln\left(\sum_{m=1}^{K}\exp\left[\gamma_{ml} - (1-\lambda)\ln\left(\frac{q_l(1-s_{0l})}{p_m(1-s_{m0})}\right)\right]\right),\ l=1,\ldots,L, \end{aligned} \tag{12}$$

where $s_{k0} = (p_k - r_{k0})/p_k$ for $k=1,\ldots,K$ and $s_{0l} = (q_l - r_{0l})/q_l$ for $l=1,\ldots,L$. The $K+L$ vector of matching market entrance probabilities for each type of women and man is $\mathbf{s}_0 = (s_{10},\ldots,s_{K0},s_{01},\ldots,s_{0L})'$. These probabilities are the unique solution to the $K+L$ sys-

[4] The difficulty of determining whether a Jacobian matrix is a P-matrix is one motivation for deriving alternative conditions which ensure invertibility (e.g., Berry, Gandhi and Haile, 2012).

tem of nonlinear equations

$$
\begin{aligned}
s_{k0} &= \frac{\exp\left(v_k^f\left(\mathbf{s}_0;\theta\right)\right)}{1+\exp\left(v_k^f\left(\mathbf{s}_0;\theta\right)\right)},\ k=1,\ldots,K \\
s_{0l} &= \frac{\exp\left(v_l^m\left(\mathbf{s}_0;\theta\right)\right)}{1+\exp\left(v_l^m\left(\mathbf{s}_0;\theta\right)\right)},\ l=1,\ldots,L.
\end{aligned}
\tag{13}
$$

Equation (13) is identical in form to an incomplete information entry game with $K+L$ players. Multiplicity of equilibria in such games is frequent in practice (e.g., Bajari, Hahn, Hong and Ridder, 2011). Here uniqueness evidently follows from the specific form of the inclusive values.

In an initial stage each agent decides whether to enter the matching market or remain unmatched. The probability of entry depends on the surplus associated with the different types of matches available to an agent (e.g., $\gamma_{k1},\ldots,\gamma_{kL}$ for a type k female). It also depends on the participation rate of all other types of agents as well as their relative population sizes. The influence of these various factors is summarized by the inclusive value terms (12).

Once the 'entry' decision has been made agents match with different types of partners according to the closed form rule (11).

3 Comparative statics

An attraction of the CS model, compared to reduced form models of the marriage market (e.g., Angrist, 2002; Schwartz, 2010), is that it allows the researcher to undertake counterfactual analysis. What would happen to the distribution of marriage if the number of college educated females increased? How would the marriage rate change in response to a decline in match surplus, induced by, for example, changes in tax policy or divorce laws? By comparing the observed matching with a counterfactual one computed under a different distribution of types or model parameters, these questions may be answered numerically. Representation (10) suggests that we may compute such counterfactuals by fixed point iteration, using the initial assignment for starting values.

Deriving analytic results on the substitution patterns implied by the CS model is more difficult. Such results are useful because they provide insight into the economic structure of the model. In a nonlinear system of equations comparative static analysis generally involves an application of the Implicit Function Theorem. To derive precise comparative static results requires a closed form expression for the inverse Jacobian matrix, $J\left(\mathbf{r}_0^{\mathrm{eq}}\right)^{-1}$. In the CS model explicit calculation of $J\left(\mathbf{r}_0^{\mathrm{eq}}\right)^{-1}$, as in other large non-linear fixed point problems, appears difficult, however certain features of this matrix can be derived. Specifically I show that $J\left(\mathbf{r}_0^{\mathrm{eq}}\right)$ coincides with the similarity transform of a row stochastic matrix with certain diagonal dominance properties. These properties are sufficient to bound every element of $J\left(\mathbf{r}_0^{\mathrm{eq}}\right)^{-1}$. Deriving these bounds involves linear algebra results on M-matrices and inverse M-matrices (e.g., Fielder and Ptak, 1962; Carlson and Markham, 1979; Johnson, 1982).

My main result is:

Theorem 1. *(Comparative Statics). Let $\mathbf{r_0} = \mathbf{B}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta)$ be the equilibrium prevalence of single-hood, then*
(i) Type-specific elasticities of single-hood: for $m = 1, \ldots, K$

$$\frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k}\frac{p_k}{r_{m0}} \geq \begin{cases} \frac{1}{(1-\lambda)p_m+\lambda r_{m0}}\frac{p_k}{(1-\lambda)p_k+\lambda r_{k0}}\sum_{n=1}^{L}\frac{r_{mn}r_{kn}}{\lambda q_n+(1-)r_{0n}} > 0 & i \neq k \\ \frac{p_k}{(1-\lambda)p_k+\lambda r_{k0}}\left[1+\frac{1}{(1-\lambda)p_k+\lambda r_{k0}}\sum_{n=1}^{L}\frac{r_{1n}r_{kn}}{\lambda q_n+(1-)r_{0n}}\right] > 1 & i = k \end{cases},$$

while for $l = 1, \ldots, L$

$$\frac{\mathrm{d}r_{0l}}{\mathrm{d}p_k}\frac{p_k}{r_{0l}} \leq -\frac{(1-\lambda)\, r_{kl}}{\lambda q_l + (1-\lambda)\, r_{0l}}\frac{p_k}{(1-\lambda)\, p_k + \lambda r_{k0}} < 0,$$

and an analogous result holding for $\frac{\mathrm{d}\mathbf{r_0}}{\mathrm{d}q_l}$;
(ii) Partial symmetry of semi-elasticities:

$$\begin{aligned} \frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k}\frac{1}{r_{m0}} &= \frac{\mathrm{d}r_{k0}}{\mathrm{d}p_m}\frac{1}{r_{k0}},\ k, m = 1, \ldots, K \\ \frac{\mathrm{d}r_{0n}}{\mathrm{d}q_l}\frac{1}{r_{0n}} &= \frac{\mathrm{d}r_{0l}}{\mathrm{d}q_n}\frac{1}{r_{0l}},\ l, n = 1, \ldots, L \\ \lambda\frac{\mathrm{d}r_{k0}}{\mathrm{d}q_l}\frac{1}{r_{k0}} &= (1-\lambda)\frac{\mathrm{d}r_{0l}}{\mathrm{d}p_k}\frac{1}{r_{0k}},\ k = 1, \ldots, K,\ l = 1, \ldots, L; \end{aligned}$$

(iii) Aggregate elasticity of single-hood:

$$\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}p_k}\frac{p_k}{\mathbf{r}_0'\iota_{K+L}} = \frac{r_{k0}}{\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}}$$

for $k = 1, \ldots, K$ and an analogous result holding for $\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}q_l}$;
(iv) Constant returns to scale:

$$\begin{aligned} \sum_{k=1}^{K}\frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k}\frac{p_k}{r_{m0}} + \sum_{l=1}^{L}\frac{\mathrm{d}r_{m0}}{\mathrm{d}q_l}\frac{q_l}{r_{m0}} &= 1,\ m = 1, \ldots, K \\ \sum_{k=1}^{K}\frac{\mathrm{d}r_{0n}}{\mathrm{d}p_k}\frac{p_k}{r_{0n}} + \sum_{l=1}^{L}\frac{\mathrm{d}r_{0n}}{\mathrm{d}q_l}\frac{q_l}{r_{0n}} &= 1,\ n = 1, \ldots, L. \end{aligned}$$

Proof. See Appendix 5.1. □

The first two implications of Theorem 1 generalize Theorem 2 of Decker, Lieb, McCann and Stephens (2013). Implication (i) states that if the k^{th} type of women becomes more abundant then (a) the prevalence of single-hood across all types of women increases (with the response being strongest among type k women) and (b) the prevalence of marriage increases across all types of men. These

Table 2: Substitution Patterns Implied by CS Model, Part 1

W\M	Single(x_0)	x_1	
Single(w_0)	-	0.4951	-
w_1	0.3157	0.2398	$p_1 = 5/9$
w_2	0.1617	0.1716	$p_2 = 3/9$
w_3	0.0176	0.0935	$p_3 = 1/9$
	-	$q_1 = 1$	

Notes: In the above simulated market $\gamma_{11} = \gamma_{21} = 0$ and $\gamma_{31} = 0.25$ with $\lambda = 1/2$.

results are a consequence of more women "competing" for the same number of men. Implication (ii) extends the symmetry result Decker, Lieb, McCann and Stephens (2013) to the case where $\lambda \neq 1/2$.

Implication (iii), which is new, shows that the special structure of the CS model makes it possible to derive some equilibrium elasticities in closed form. Consider the probability that a random draw from the population, women *and* men, is single. How does this probability change in response to an increase in the size of a specific subgroup, say type k, women? Implication (i) of the Theorem suggests that single-hood rises across all women, and especially type k women, and declines across all men. Implication (iii) shows that the net effect is an overall increase in single-hood with the elasticity being equal to type k's share of all singles.

Implication (iv) confirms that the CS model exhibits constant results to scale. Holding the type distributions of men and women fixed, increasing market size has no effect on the probability of marriage.

It is well known that the independence of irrelevant alternatives (IIA) property of the conditional logit model structure places strong restrictions on substitution patterns in single agent discrete choice problems (McFadden, 1974). A concern is that similar problems arise here. Consider a model with three types of women: high school dropouts (type 1), high school graduates (type 2) and those with a 4-year degree (type 3). For simplicity assume there is only one type of man. Not all women secure a husband and vice versa. I assume that high school dropouts and high school graduates generate similar systematic match surpluses with $\gamma_{11} = \gamma_{21} = -1/2$. College graduates generate a systematic match surplus of $\gamma_{31} = 0$.

Table 2 gives the equilibrium matching associated with this marriage market when $p_1 = 5/9$, $p_2 = 3/9$, and $p_3 = 1/9$. I set $q_1 = 1$ so that, initially, there are a sufficient number of men for all women. As expected, given the form of γ, high school dropouts and high school graduates are more likely than college graduates to remain unmarried.

Now consider a doubling of the population of high school dropout women. This influx of women increases the pools of unmarried among all types of women. We might also expect that the response is stronger for high school graduates than it is for college graduates. Since high school graduates are 'similar' to dropouts in terms of the marriage surplus they generate, this group should be disproportionally affected. It turns out that this predicted elasticity ranking is incorrect. The number of unmarried high school graduate women increases by $100 \times (0.1724 - 0.1617)/0.1617 \approx 6.6$

Table 3: Substitution Patterns Implied by CS Model, Part 2

W\M	Single(x_0)	x_1	
Single(w_0)	-	0.4080	-
w_1	0.7709	0.3402	$p_1 = 10/9$
w_2	0.1724	0.1609	$p_2 = 3/9$
w_3	0.0202	0.0909	$p_3 = 1/9$
	-	$q_1 = 1$	

Notes: In the above simulated market $\gamma_{11} = \gamma_{21} = 0$ and $\gamma_{31} = 0.25$ with $\lambda = 1/2$.

percent, while the ranks of unmarried college graduates by $100 \times (0.0202 - 0.0176)/0.0176 \approx 15.8$ percent. As in the standard single agent discrete choice model, the source of these counter-intuitive substitution patterns is the assumed independence of v_1^j and v_2^j.

3.1 Derived results

It is possible to use Theorem 1 to derive some additional comparative static results.

First, consider the effect of a percent increase in the population size of women, holding is type composition fixed. We have, using (iii) of Theorem 1

$$\left.\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}\left(\mathbf{p}'\iota_K\right)}\frac{\left(\mathbf{p}'\iota_K\right)}{\left(\mathbf{r}_0'\iota_{K+L}\right)}\right|_{\frac{\frac{\mathrm{d}p_k}{p_k}}{\frac{\mathrm{d}(\mathbf{p}'\iota_K)}{(\mathbf{p}'\iota_K)}}=1,\, k=1,\ldots,K} = \frac{\sum_{k=1}^K r_{k0}}{\sum_{k=1}^K r_{k0} + \sum_{l=1}^L r_{0l}}. \tag{14}$$

Equation (14) shows that CS model generates sharp implications for the effect of changes in the sex ratio on the marriage rate (cf., Angrist, 2002).

Second, the semi-elasticity of single-hood for type m females given an unit increase in the surplus parameter γ_{kl} is

$$\frac{\mathrm{d}r_{m0}}{\mathrm{d}\gamma_{kl}}\frac{1}{r_{m0}} = \frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k}\frac{r_{kl}}{r_{m0}} + \frac{\mathrm{d}r_{m0}}{\mathrm{d}q_l}\frac{r_{kl}}{r_{m0}}.$$

Since this is the sum of opposite-signed terms the sign of this effect is ambiguous (cf., Decker, Lieb, McCann and Stephens, 2013). However it turns out that we can get a sharper result if we consider the effect on *aggregate* single-hood of a unit increase in *all* γ_{kl} for $k = 1, \ldots, K$ and $l = 1, \ldots, L$. To motivate this counterfactual we can think of an economy-wide change in tax policy, divorce law, or contraceptive availability which changes the utility of marriage relative to single-hood. Calculations in the proof of Theorem 1 show that this semi-elasticity is a weighted average of type-specific population shares. Weights are greatest for types that are neither very likely, nor very unlikely, to be single under the status quo. Specifically we get the closed form expression

$$\left.\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}\left(\gamma'\iota_{KL}\right)}\frac{\left(\gamma'\iota_{KL}\right)}{\left(\mathbf{r}_0'\iota_{K+L}\right)}\right|_{\frac{\frac{\mathrm{d}\gamma_{kl}}{\gamma_{kl}}}{\frac{\mathrm{d}(\gamma'\iota_{KL})}{(\gamma'\iota_{KL})}}=1,\, k=1,\ldots,K,\, l=1,\ldots,L} = -\frac{\sum_{k=1}^{K} p_k e_{k0}\left(1-e_{k0}\right)+\sum_{l=1}^{L} q_l e_{0l}\left(1-e_{0l}\right)}{\sum_{k=1}^{K} p_k+\sum_{l=1}^{L} q_l}. \tag{15}$$

This semi-elasticity measures the effect – on the unconditional probability of matching – of an economy-wide increase in the returns from doing so.

Third, observe that

$$\frac{\mathrm{d}\ln r_{kl}}{\mathrm{d}\ln p_o}=(1-\lambda)\frac{\mathrm{d}\ln r_{k0}}{\mathrm{d}\ln p_o}+\lambda\frac{\mathrm{d}\ln r_{0l}}{\mathrm{d}\ln p_o}$$

and hence that

$$\frac{\mathrm{d}\ln r_{mn}}{\mathrm{d}\ln p_o}-\frac{\mathrm{d}\ln r_{ml}}{\mathrm{d}\ln p_o}-\left[\frac{\mathrm{d}\ln r_{kn}}{\mathrm{d}\ln p_o}-\frac{\mathrm{d}\ln r_{kl}}{\mathrm{d}\ln p_o}\right]=0,$$

which implies the zero restriction

$$\frac{\mathrm{d}\ln\left(\frac{r_{mn}}{r_{ml}}\frac{r_{kl}}{r_{kn}}\right)}{\mathrm{d}\ln p_o}=0$$

for $o=1,\ldots,K$. The local assortativeness properties of the CS matching are unaffected by the type distribution of the population.

4 Further areas for research

This paper has presented a systematic analysis of the equilibrium properties of the CS model, building on prior work by others. Theorem 1 indicates, perhaps unsurprisingly given its conditional logit foundation, that the CS model has strong comparative static implications. These implications could be used to test the empirical usefulness of the CS model, and also to guide the development of more general models. As in single agent discrete choice models, it seems fruitful to explore modeling frameworks that allow for correlation in the unobserved component of match utility associated with different types of partners.

Fixed point iteration could be useful for researchers interested in estimating models where K and/or L are large, but where $\gamma=\gamma(\eta)$ is parameterized in terms of a smaller dimensional η. In such a situation the researcher could use iteration to solve for the equilibrium matching associated with a particular choice of η and λ in order to evaluate a log likelihood function (Rust, (1987). [5] This method of computation would be especially attractive if it could be shown that $\mathbf{B}\left(\mathbf{r_0};\mathbf{p},\mathbf{q},\theta\right)$ is a contraction on all of $\mathbb{T}_\epsilon$.

[5] Alternatively other, more recent, approaches to parameter estimation in fixed point problems may be utilized (e.g., Aguirregabiria, 2004; Aguirregabiria and Mira, 2002, 2007, 2010; Su and Judd, 2012).

5 Proofs and derivations

In what follows we write, for example, $\mathbf{B}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta) = \mathbf{B}(\mathbf{r}_0)$ when dependence on $\mathbf{p}$, $\mathbf{q}$ and θ is not germane.

5.1 Proof of Theorem 1 and Corollaries

The proof of Theorem 1 involves four steps. First the Jacobian matrix $J(\mathbf{r}_0)$ is calculated and evaluated at $\mathbf{r}_0 = \mathbf{r}_0^{\mathrm{eq}}$. Second, $J(\mathbf{r}_0^{\mathrm{eq}})$ is factored into the similarity transform of a row stochastic matrix $H(\mathbf{r}_0)$. Third, $H(\mathbf{r}_0)$ is partitioned and the Schur complements of its upper $K \times K$ and lower $L \times L$ diagonal blocks are shown to be row diagonal dominant Z-matrices (i.e., matrices with non-positive off diagonal elements). This is sufficient to show that the upper $K \times K$ and lower $L \times L$ diagonal blocks of $H(\mathbf{r}_0)^{-1}$ are positive matrices while the two off-diagonal blocks are non-positive. This identifies the sign structure of the inverse Jacobian. Fourth, the partitioned inverse formula is used to establish various bounds on the elements of the inverse Jacobian which in turn provide bounds for various elasticities. In what follows I drop the 'eq' superscript from $\mathbf{r}_0$ to simplfy the notation when doing so does not cause confusion.

Step 1: Calculation of Jacobian

The Jacobian, recalling that we set $\mathbf{r}_0 = \mathbf{r}_0^{eq}$ in what follows, is given by $J(\mathbf{r}_0) = I_{K+L} - \nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta)$ where

$$\nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0; \mathbf{p}, \mathbf{q}, \theta) = \nabla_{\mathbf{r}_0}\mathbf{B}(\mathbf{r}_0) = \begin{pmatrix} E_{11}(\mathbf{r}_0) & E_{12}(\mathbf{r}_0) \\ E_{21}(\mathbf{r}_0) & E_{22}(\mathbf{r}_0) \end{pmatrix},$$

with

$$\begin{aligned}
E_{11}(\mathbf{r}_0) &= \lambda \cdot \mathrm{diag}\left\{ \sum_{n=1}^{L} e_{n|1}(\mathbf{r}_0), \ldots, \sum_{n=1}^{L} e_{n|K}(\mathbf{r}_0) \right\} \\
E_{12}(\mathbf{r}_0) &= -\lambda \begin{pmatrix} \frac{r_{10}}{r_{01}} e_{1|1}(\mathbf{r}_0) & \cdots & \frac{r_{10}}{r_{0L}} e_{L|1}(\mathbf{r}_0) \\ \vdots & \ddots & \vdots \\ \frac{r_{K0}}{r_{01}} e_{1|K}(\mathbf{r}_0) & \cdots & \frac{r_{K0}}{r_{0L}} e_{L|K}(\mathbf{r}_0) \end{pmatrix} \\
E_{21}(\mathbf{r}_0) &= -(1-\lambda) \begin{pmatrix} \frac{r_{0l}}{r_{10}} g_{1|1}(\mathbf{r}_0) & \cdots & \frac{r_{0l}}{r_{K0}} g_{K|1}(\mathbf{r}_0) \\ \vdots & \ddots & \vdots \\ \frac{r_{0L}}{r_{10}} g_{1|L}(\mathbf{r}_0) & \cdots & \frac{r_{0L}}{r_{K0}} g_{K|L}(\mathbf{r}_0) \end{pmatrix} \\
E_{22}(\mathbf{r}_0) &= (1-\lambda) \cdot diag\left\{ \sum_{m=1}^{K} g_{m|1}(\mathbf{r}_0), \ldots, \sum_{m=1}^{K} g_{m|L}(\mathbf{r}_0) \right\},
\end{aligned}$$

where we define

$$
\begin{aligned}
e_{l|k}(\mathbf{r}_0) &\overset{def}{=} \frac{\exp\left[\gamma_{kl} + \lambda \ln\left(\frac{r_{0l}}{r_{k0}}\right)\right]}{1 + \sum_{n=1}^{L} \exp\left[\gamma_{kn} + \lambda \ln\left(\frac{r_{0n}}{r_{k0}}\right)\right]} \\
g_{k|l}(\mathbf{r}_0) &\overset{def}{=} \frac{\exp\left[\gamma_{kl} - (1-\lambda) \ln\left(\frac{r_{0l}}{r_{k0}}\right)\right]}{1 + \sum_{m=1}^{K} \exp\left[\gamma_{ml} - (1-\lambda) \ln\left(\frac{r_{0l}}{r_{m0}}\right)\right]}.
\end{aligned}
$$

Note that for all $\mathbf{r}_0 \in \mathbb{T}_\epsilon$ and $k = 1, \ldots, K$ and $l = 1, \ldots, L$ we have $\sum_{l=1}^{L} e_{l|k}(\mathbf{r}_0) < 1$ and $\sum_{k=1}^{K} g_{k|l}(\mathbf{r}_0) < 1$. In what follows I will use the abbreviated notation $e_{l|k} = e_{l|k}(\mathbf{r}_0)$ and $g_{k|l} = g_{k|l}(\mathbf{r}_0)$.

Step 2: Factorization of Jacobian matrix

By the Implicit Function Theorem we have

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}p_k} &= J(\mathbf{r}_0)^{-1} \frac{\partial \mathbf{B}}{\partial p_k} \\
\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}q_l} &= J(\mathbf{r}_0)^{-1} \frac{\partial \mathbf{B}}{\partial q_l} \\
\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\gamma_{kl}} &= J(\mathbf{r}_0)^{-1} \frac{\partial \mathbf{B}}{\partial \gamma_{kl}}.
\end{aligned}
$$

The $K + L \times 1$ vector of partial derivatives, $\frac{\partial \mathbf{B}}{\partial p_k}$, consists of a zero vector with the exception of the k^{th} element which equals r_{k0}/p_k. The form of $\frac{\partial \mathbf{B}}{\partial q_l}$ is similar with $(K+l)^{th}$ element r_{0l}/q_l. The form of $\frac{\partial \mathbf{B}}{\partial \gamma_{kl}}$ consists of a zero vector with the exception of the k^{th} and $(K+l)^{th}$ elements, which equal $-r_{k0}(1 - r_{kl}/p_k)$ and $-r_{0l}(1 - r_{kl}/q_l)$ respectively.

We begin by rewriting the Jacobian matrix in the form

$$
\begin{aligned}
J(\mathbf{r}_0) &= C(\mathbf{r}_0)^{-1} \left(A(\mathbf{r}_0) + U(\mathbf{r}_0) B(\mathbf{r}_0) U(\mathbf{r}_0)^{-1}\right) && (16) \\
&= U(\mathbf{r}_0) C(\mathbf{r}_0)^{-1} \left(A(\mathbf{r}_0) + B(\mathbf{r}_0)\right) U(\mathbf{r}_0)^{-1}, && (17)
\end{aligned}
$$

where the second equality exploits the diagonality of $A(\mathbf{r}_0)$, $C(\mathbf{r}_0)$, and $U(\mathbf{r}_0)$, defined as:

$$\begin{aligned}
A(\mathbf{r}_0) &= \begin{pmatrix} \operatorname{diag}\{\mathbf{p}-\lambda\mathbf{R}\iota_L\} & 0 \\ 0 & \operatorname{diag}\{\mathbf{q}-(1-\lambda)\mathbf{R}'\iota_K\} \end{pmatrix} \\
B(\mathbf{r}_0) &= \begin{pmatrix} 0 & \lambda\mathbf{R} \\ (1-\lambda)\mathbf{R}' & 0 \end{pmatrix} \\
C(\mathbf{r}_0) &= \begin{pmatrix} \operatorname{diag}\{\mathbf{p}\} & 0 \\ 0 & \operatorname{diag}\{\mathbf{q}\} \end{pmatrix} \\
U(\mathbf{r}_0) &= \begin{pmatrix} \operatorname{diag}\{\mathbf{r}_{.0}\} & 0 \\ 0 & \operatorname{diag}\{\mathbf{r}_{0.}\} \end{pmatrix}.
\end{aligned}$$

Let $H(\mathbf{r}_0) = C(\mathbf{r}_0)^{-1}(A(\mathbf{r}_0)+B(\mathbf{r}_0))$. Writing out $H(\mathbf{r}_0)$ we get

$$\begin{pmatrix}
\frac{(1-\lambda)p_1+\lambda r_{10}}{p_1} & \cdots & 0 & \lambda\frac{r_{11}}{p_1} & \cdots & \lambda\frac{r_{1L}}{p_1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \frac{(1-\lambda)p_K+\lambda r_{K0}}{p_K} & \lambda\frac{r_{K1}}{p_K} & \cdots & \lambda\frac{r_{KL}}{p_K} \\
\frac{(1-\lambda)r_{11}}{q_1} & \cdots & \frac{(1-\lambda)r_{K1}}{q_1} & \frac{\lambda q_1+(1-\lambda)r_{01}}{q_1} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{(1-\lambda)r_{1L}}{q_L} & \cdots & \frac{(1-\lambda)r_{KL}}{q_L} & 0 & \cdots & \frac{\lambda q_L+(1-\lambda)r_{0L}}{q_L}
\end{pmatrix}.$$

Observe that all the elements of $H(\mathbf{r}_0)$ are non-negative and that its rows sum to one (i.e., $H(\mathbf{r}_0)\,\iota_{K+L} = \iota_{K+L}$). Therefore $H(\mathbf{r}_0)$ is a row stochastic matrix (cf., Horn and Johnson, 2013, p. 547) with an inverse whose rows also sum to one since:

$$\begin{aligned}
H(\mathbf{r}_0)^{-1}\iota_{K+L} &= H(\mathbf{r}_0)^{-1}H(\mathbf{r}_0)\iota_{K+L} \\
&= I_{K+L}\iota_{K+L} \\
&= \iota_{K+L}. \qquad (18)
\end{aligned}$$

Step 3: Derivation of Z-matrix property

Let $D = \operatorname{diag}\{\lambda I_K, (1-\lambda) I_L\}$ and partition of $H^*(\mathbf{r}_0) = H(\mathbf{r}_0)D$ as

$$\begin{aligned}
H^*(\mathbf{r}_0) &= \begin{pmatrix} \lambda\operatorname{diag}(\mathbf{p})^{-1}\operatorname{diag}((1-\lambda)\mathbf{p}+\lambda\mathbf{r}_{.\mathbf{0}}) & \lambda(1-\lambda)\operatorname{diag}(\mathbf{p})^{-1}\mathbf{R} \\ \lambda(1-\lambda)\operatorname{diag}(\mathbf{q})^{-1}\mathbf{R}' & (1-\lambda)\operatorname{diag}(\mathbf{q})^{-1}\operatorname{diag}(\lambda\mathbf{q}+(1-\lambda)\mathbf{r}_{\mathbf{0}.}) \end{pmatrix} \\
&= \begin{pmatrix} H^*_{11}(\mathbf{r}_0) & H^*_{12}(\mathbf{r}_0) \\ H^*_{21}(\mathbf{r}_0) & H^*_{22}(\mathbf{r}_0) \end{pmatrix}.
\end{aligned}$$

Observe that $H^*(\mathbf{r}_0)$ is strictly diagonally dominant since, for $i = 1, \ldots, K$,

$$\sum_{j=1, j\neq i}^{K+L} \left|h_{ij}^*\right| = \lambda\left(1-\lambda\right)\sum_{n=1}^{L}\frac{r_{in}}{p_i} = \frac{\lambda\left(1-\lambda\right)\left(p_i - r_{i0}\right)}{p_i} < \frac{\lambda r_{i0} + \lambda\left(1-\lambda\right)\left(p_i - r_{i0}\right)}{p_i} = \left|h_{ii}^*\right|,$$

and, for $i = K+1, \ldots, K+L$,

$$\begin{aligned}
\sum_{j=1, j\neq i}^{K+L} \left|h_{ij}^*\right| &= \lambda\left(1-\lambda\right)\sum_{m=1}^{K}\frac{r_{mi-K}}{q_{i-K}} = \frac{\lambda\left(1-\lambda\right)\left(q_{i-K} - r_{0i-K}\right)}{q_{i-K}} \\
&< \frac{\left(1-\lambda\right) r_{0i-K} + \lambda\left(1-\lambda\right)\left(q_{i-K} - r_{0i-K}\right)}{q_{i-K}} = \left|h_{ii}^*\right|.
\end{aligned}$$

The Schur complement of H_{11}^* in H^* equals $H_{22}^* - H_{21}^*\left(H_{11}^*\right)^{-1}H_{12}^*$ and that of H_{22}^* in H^* equals $H_{11}^* - H_{12}^*\left(H_{22}^*\right)^{-1}H_{21}^*$ (we suppress the dependence of these matrices on $\mathbf{r}_0$ in what immediately follows). Theorem 1 of Carlson and Markham (1979, p. 248) implies that both of these Schur complements are diagonally dominant (this also easily shown directly). Observe that

$$\begin{aligned}
H_{22}^* - H_{21}^*\left(H_{11}^*\right)^{-1}H_{12}^* &= \left(1-\lambda\right)H_{22} - \lambda H_{21}\left(\lambda H_{11}\right)^{-1}\left(1-\lambda\right)H_{12} \\
&= \left(1-\lambda\right)\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)
\end{aligned}$$

and also

$$\begin{aligned}
H_{11}^* - H_{12}^*\left(H_{22}^*\right)^{-1}H_{21}^* &= \lambda H_{11} - \left(1-\lambda\right)H_{12}\left(\left(1-\lambda\right)H_{22}\right)^{-1}\lambda H_{21} \\
&= \lambda\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right).
\end{aligned}$$

Partitioning $H\left(\mathbf{r}_0\right)$ conformably to the $H^*\left(\mathbf{r}_0\right)$ partition defined above we therefore have that the Schur complements of H_{11} and H_{22} in H are row diagonally dominant. Diagonality of H_{11} and H_{22}, positivity of the elements of $H_{21}H_{11}^{-1}H_{12}$ and $H_{12}H_{22}^{-1}H_{21}$, and diagonal dominance further imply that these matrices are Z-matrices (i.e., members of the class of real matrices with non-positive off-diagonal elements). By applying Theorem 4.3 of Fiedler and Ptak (1962) it follows that $H_{22}-H_{21}H_{11}^{-1}H_{12}$ and $H_{11}-H_{12}H_{22}^{-1}H_{21}$ are M-matrices and hence that $\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1} \geqq 0$ and $\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1} \geqq 0$. These results are sufficient to establish the sign structure of $H(\mathbf{r}_0)^{-1}$. Specifically we have, using the partitioned inverse formula,

$$H(\mathbf{r}_0)^{-1} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} + & \vdots & - \\ \cdots & & \cdots \\ - & \vdots & + \end{pmatrix}$$

where

$$
\begin{aligned}
W_{11} &= \left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1} \\
&= H_{11}^{-1} + H_{11}^{-1}H_{12}\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1}H_{21}H_{11}^{-1} \\
W_{12} &= -\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1}H_{12}H_{22}^{-1} \\
&= -H_{11}^{-1}H_{12}\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1} \\
W_{21} &= -H_{22}^{-1}H_{21}\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1} \\
&= -\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1}H_{21}H_{11}^{-1} \\
W_{22} &= \left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1} \\
&= H_{22}^{-1} + H_{22}^{-1}H_{21}\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1}H_{12}H_{22}^{-1}.
\end{aligned}
$$

Step 4: Derivation of the structure of $H\left(\mathbf{r}_0\right)^{-1}$

The known sign structures of $\left(H_{22} - H_{21}H_{11}^{-1}H_{12}\right)^{-1}$ and $\left(H_{11} - H_{12}H_{22}^{-1}H_{21}\right)^{-1}$ and the partitioned inverse formula immediately yield the inequality relationships:

$$
\begin{aligned}
W_{11} &\geq H_{11}^{-1} \\
W_{22} &\geq H_{22}^{-1}
\end{aligned}
$$

and hence the refined inequalities

$$
\begin{aligned}
W_{11} &= H_{11}^{-1} + H_{11}^{-1}H_{12}W_{22}H_{21}H_{11}^{-1} \\
&\geq H_{11}^{-1} + H_{11}^{-1}H_{12}H_{22}^{-1}H_{21}H_{11}^{-1} \\
W_{22} &= H_{22}^{-1} + H_{22}^{-1}H_{21}W_{11}H_{12}H_{22}^{-1} \\
&\geq H_{22}^{-1} + H_{22}^{-1}H_{21}H_{11}^{-1}H_{12}H_{22}^{-1}
\end{aligned}
$$

where

$$
\begin{aligned}
W_{11} &\geq H_{11}^{-1} + H_{11}^{-1} H_{12} H_{22}^{-1} H_{21} H_{11}^{-1} \\
&= \operatorname{diag}\left\{ \frac{p_1}{(1-\lambda)\, p_1 + \lambda r_{10}}, \dots, \frac{p_K}{(1-\lambda)\, p_K + \lambda r_{K0}} \right\} \\
&\quad + \lambda (1-\lambda) \begin{pmatrix}
\frac{1}{(1-\lambda)p_1+\lambda r_{10}} \frac{p_1}{(1-\lambda)p_1+\lambda r_{10}} \sum_{n=1}^{L} \frac{r_{1n} r_{1n}}{\lambda q_n + (1-\lambda) r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_1+\lambda r_{10}} \frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}} \sum_{n=1}^{L} \frac{r_{1n} r_{Kn}}{\lambda q_n + (1-\lambda) r_{0n}} \\
\vdots & \ddots & \vdots \\
\frac{1}{(1-\lambda)p_K+\lambda r_{K0}} \frac{p_1}{(1-\lambda)p_1+\lambda r_{10}} \sum_{n=1}^{L} \frac{r_{Kn} r_{1n}}{\lambda q_n + (1-\lambda) r_{0n}} & \cdots & \frac{1}{(1-\lambda)p_K+\lambda r_{K0}} \frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}} \sum_{n=1}^{L} \frac{r_{Kn} r_{Kn}}{\lambda q_n + (1-\lambda) r_{0n}}
\end{pmatrix} \\
&> I_K \\
W_{22} &\geq H_{22}^{-1} + H_{22}^{-1} H_{21} H_{11}^{-1} H_{12} H_{22}^{-1} \\
&= \operatorname{diag}\left\{ \frac{q_1}{\lambda q_1 + (1-\lambda)\, r_{01}}, \dots, \frac{q_L}{\lambda q_L + (1-\lambda)\, r_{0L}} \right\} \\
&\quad + \lambda (1-\lambda) \begin{pmatrix}
\frac{1}{\lambda q_1+(1-\lambda) r_{01}} \frac{q_1}{\lambda q_1+(1-\lambda) r_{01}} \sum_{m=1}^{K} \frac{r_{m1} r_{m1}}{(1-\lambda) p_m + \lambda r_{m0}} & \cdots & \frac{1}{\lambda q_1+(1-\lambda) r_{01}} \frac{q_L}{\lambda q_L+(1-\lambda) r_{0L}} \sum_{m=1}^{K} \frac{r_{m1} r_{mL}}{(1-\lambda) p_m + \lambda r_{m0}} \\
\vdots & \ddots & \vdots \\
\frac{1}{\lambda q_L+(1-\lambda) r_{0L}} \frac{q_1}{\lambda q_1+(1-\lambda) r_{01}} \sum_{m=1}^{K} \frac{r_{mL} r_{m1}}{(1-\lambda) p_m + \lambda r_{m0}} & \cdots & \frac{1}{\lambda q_L+(1-\lambda) r_{0L}} \frac{q_L}{\lambda q_L+(1-\lambda) r_{0L}} \sum_{m=1}^{K} \frac{r_{mL} r_{mL}}{(1-\lambda) p_m + \lambda r_{m0}}
\end{pmatrix} \\
&> I_L.
\end{aligned}
$$

This implies that the diagonal elements of $H(\mathbf{r}_0)^{-1}$ exceed one.

For the off-diagonal blocks of $H(\mathbf{r}_0)^{-1}$ we have

$$
\begin{aligned}
W_{12} &\leq -H_{11}^{-1} H_{12} H_{22}^{-1} \\
W_{21} &\leq -H_{22}^{-1} H_{21} H_{11}^{-1}.
\end{aligned}
$$

Evaluating $-H_{11}^{-1} H_{12} H_{22}^{-1}$ and $-H_{22}^{-1} H_{21} H_{11}^{-1}$ yields

$$
\begin{aligned}
-H_{11}^{-1} H_{12} H_{22}^{-1} &= -\begin{pmatrix}
\frac{\lambda r_{11}}{(1-\lambda)p_1+\lambda r_{10}} \frac{q_1}{\lambda q_1+(1-\lambda) r_{01}} & \cdots & \frac{\lambda r_{1L}}{(1-\lambda)p_1+\lambda r_{10}} \frac{q_L}{\lambda q_L+(1-\lambda) r_{0L}} \\
\vdots & \ddots & \vdots \\
\frac{\lambda r_{K1}}{(1-\lambda)p_K+\lambda r_{K0}} \frac{q_1}{\lambda q_1+(1-\lambda) r_{01}} & \cdots & \frac{\lambda r_{KL}}{(1-\lambda)p_K+\lambda r_{K0}} \frac{q_L}{\lambda q_L+(1-\lambda) r_{0L}}
\end{pmatrix} \\
-H_{22}^{-1} H_{21} H_{11}^{-1} &= -\begin{pmatrix}
\frac{(1-\lambda) r_{11}}{\lambda q_1+(1-\lambda) r_{01}} \frac{p_1}{(1-\lambda)p_1+\lambda r_{10}} & \cdots & \frac{(1-\lambda) r_{K1}}{\lambda q_1+(1-\lambda) r_{01}} \frac{p_K}{(1-\lambda)p_K+\lambda r_{K0}} \\
\vdots & \ddots & \vdots \\
\frac{(1-\lambda) r_{1L}}{\lambda q_L+(1-\lambda) r_{0L}} \frac{p_1}{(1-\lambda)p_1+\lambda r_{10}} & \cdots & \frac{(1-\lambda) r_{KL}}{\lambda q_L+(1-\lambda) r_{0L}} \frac{P_K}{(1-\lambda)p_K+\lambda r_{K0}}
\end{pmatrix}.
\end{aligned}
$$

Theorem 2.5.12 of Horn and Johnson (1991, p. 125) further implies that the diagonal elements of $H^*(\mathbf{r}_0)^{-1}$ are larger in absolute value that any off diagonal element in the same column (i.e. $H^*(\mathbf{r}_0)^{-1}$ is strictly dominant in its column entries). Since $H^*(\mathbf{r}_0)^{-1} = D^{-1}H(\mathbf{r}_0)^{-1}$ with $D = \operatorname{diag}\{\lambda I_K, (1-\lambda) I_L\}$ we have the the upper-left $K \times K$ and lower-right $L \times L$ blocks of $H(\mathbf{r}_0)^{-1}$ are strictly dominant in their column entries.

Step 5a: Type-specific elasticities of single-hood with respect to type availability

Let $\mathbf{h}_k$ be a $K+L$ column vector of zeros with the exception of a one in the k^{th} row. The $K+L$ elasticities of single-hood with respect to p_k are given by

$$
\begin{aligned}
U(\mathbf{r}_0)^{-1} \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}p_k} p_k &= U(\mathbf{r}_0)^{-1} J(\mathbf{r}_0)^{-1} \frac{\partial \mathbf{B}}{\partial p_k} p_k \\
&= U(\mathbf{r}_0)^{-1} U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1} U(\mathbf{r}_0)^{-1} \mathbf{h}_k (r_{k0}/p_k) p_k \\
&= H(\mathbf{r}_0)^{-1} \mathbf{h}_k
\end{aligned}
$$

The sign structure of $H(\mathbf{r}_0)^{-1}$ as well as strict dominance of its two diagonal blocks in their column entries gives part 1 of Theorem 1. Specifically, tedious calculation yields

$$
\begin{aligned}
\frac{\mathrm{d}r_{10}}{\mathrm{d}p_k}\frac{p_k}{r_{10}} &= W_{11}[1,k] \geq \frac{1}{(1-\lambda)p_1 + \lambda r_{10}} \frac{p_k}{(1-\lambda)p_k + \lambda r_{k0}} \sum_{n=1}^{L} \frac{r_{1n} r_{kn}}{\lambda q_n + (1-\lambda) r_{0n}} > 0 \\
&\vdots \\
\frac{\mathrm{d}r_{k0}}{\mathrm{d}p_k}\frac{p_k}{r_{k0}} &= W_{11}[k,k] \geq \frac{p_k}{(1-\lambda)p_k + \lambda r_{k0}} \left(1 + \frac{1}{(1-\lambda)p_k + \lambda r_{k0}} \sum_{n=1}^{L} \frac{r_{kn} r_{kn}}{\lambda q_n + (1-\lambda) r_{0n}}\right) > 1 \\
&\vdots \\
\frac{\mathrm{d}r_{K0}}{\mathrm{d}p_k}\frac{p_k}{r_{K0}} &= W_{11}[K,k] \geq \frac{1}{(1-\lambda)p_K + \lambda r_{K0}} \frac{p_k}{(1-\lambda)p_k + \lambda r_{k0}} \sum_{n=1}^{L} \frac{r_{Kn} r_{kn}}{\lambda q_n + (1-\lambda) r_{0n}} > 0 \\
\frac{\mathrm{d}r_{01}}{\mathrm{d}p_k}\frac{p_k}{r_{01}} &= W_{21}[1,k] \leq -\frac{(1-\lambda) r_{k1}}{\lambda q_1 + (1-\lambda) r_{01}} \frac{p_k}{(1-\lambda)p_k + \lambda r_{k0}} < 0 \\
&\vdots \\
\frac{\mathrm{d}r_{0L}}{\mathrm{d}p_k}\frac{p_k}{r_{0L}} &= W_{21}[L,k] \leq -\frac{(1-\lambda) r_{kL}}{\lambda q_L + (1-\lambda) r_{0L}} \frac{p_k}{(1-\lambda)p_k + \lambda r_{k0}} < 0.
\end{aligned}
$$

Step 5b: Limited symmetry

The entire matrix of semi-elasticities can be written as

$$
\begin{aligned}
U(\mathbf{r}_0)^{-1} \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}'} &= U(\mathbf{r}_0)^{-1} U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1} U(\mathbf{r}_0)^{-1} \operatorname{diag}\left\{\frac{r_{10}}{p_1}, \ldots, \frac{r_{K0}}{p_K}, \frac{r_{01}}{q_1}, \ldots, \frac{r_{0L}}{q_L}\right\} \\
&= H(\mathbf{r}_0)^{-1} \operatorname{diag}\left\{\mathbf{p}', \mathbf{q}'\right\}^{-1}.
\end{aligned}
$$

Now observe that $\operatorname{diag}\left\{\frac{\mathbf{p}'}{\lambda}, \frac{\mathbf{q}'}{1-\lambda}\right\} H(\mathbf{r}_0)$ is symmetric and hence, also using the calculation immediately above,

$$
\begin{aligned}
U^{-1}\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}'} &= \left[\operatorname{diag}\left\{\frac{\mathbf{p}'}{\lambda}, \frac{\mathbf{q}'}{1-\lambda}\right\}^{-1}\operatorname{diag}\left\{\frac{\mathbf{p}'}{\lambda}, \frac{\mathbf{q}'}{1-\lambda}\right\} H(\mathbf{r}_0)\right]^{-1}\operatorname{diag}\left\{\mathbf{p}', \mathbf{q}'\right\}^{-1} \\
&= \left[\operatorname{diag}\left\{\frac{\mathbf{p}'}{\lambda}, \frac{\mathbf{q}'}{1-\lambda}\right\} H(\mathbf{r}_0)\right]^{-1}\operatorname{diag}\left\{\frac{\mathbf{p}'}{\lambda}, \frac{\mathbf{q}'}{1-\lambda}\right\}\operatorname{diag}\left\{\mathbf{p}', \mathbf{q}'\right\}^{-1} \\
&= \underbrace{\begin{pmatrix} + & \vdots & - \\ \cdots & & \cdots \\ - & \vdots & + \end{pmatrix}}_{\text{symmetric}}\begin{pmatrix}\lambda^{-1}I_K & 0 \\ 0 & (1-\lambda)^{-1}I_L\end{pmatrix},
\end{aligned}
$$

from which the result follows.

Step 5c: Aggregate elasticity of single-hood with respect to type availability

The elasticity of single-hood with respect to p_k is given by

$$
\begin{aligned}
\left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\left\{\sum_{m=1}^{K}\frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k} + \sum_{n=1}^{L}\frac{\mathrm{d}r_{0l}}{\mathrm{d}p_k}\right\} p_k &= \left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\iota_{K+L}'\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}p_k}p_k \\
&= \left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\iota_{K+L}' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1} U(\mathbf{r}_0)^{-1} \\
&\quad \times \frac{\partial \mathbf{B}}{\partial p_k}p_k \\
&= \left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\iota_{K+L}' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1} U(\mathbf{r}_0)^{-1} \\
&\quad \times \mathbf{h}_k (r_{k0}/p_k) p_k \\
&= \left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\mathbf{h}_k' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1}\iota_{K+L} \\
&= \left(\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}\right)^{-1}\mathbf{h}_k' U(\mathbf{r}_0)\iota_{K+L} \\
&= \frac{r_{k0}}{\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}},
\end{aligned}
$$

where I use the fact, shown above, that $H(\mathbf{r}_0)^{-1}\iota_{K+L} = \iota_{K+L}$.

Now consider the effect of an increase in the population size of women, holding its type distribution fixed (i.e, we increase $\mathrm{d}\left(\mathbf{p}'\iota_K\right)$ such that $\frac{\mathrm{d}p_k}{\mathrm{d}(\mathbf{p}'\iota_K)} = p_k / \left(\mathbf{p}'\iota_K\right)$ for $k = 1, \ldots, K$). This yields

$$\begin{aligned}
\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}\left(\mathbf{p}'\iota_K\right)} = \frac{1}{\mathbf{p}'\iota_K}\sum_{k=1}^{K}\left\{\sum_{m=1}^{K}\frac{\mathrm{d}r_{m0}}{\mathrm{d}p_k} + \sum_{n=1}^{L}\frac{\mathrm{d}r_{0l}}{\mathrm{d}p_k}\right\}p_k &= \frac{1}{\mathbf{p}'\iota_K}\sum_{k=1}^{K}\iota_{K+L}' U\left(\mathbf{r}_0\right) H\left(\mathbf{r}_0\right)^{-1} U\left(\mathbf{r}_0\right)^{-1}\mathbf{h}_k\left(r_{k0}/p_k\right)p_k \\
&= \frac{1}{\mathbf{p}'\iota_K}\sum_{k=1}^{K}\mathbf{h}_k' U\left(\mathbf{r}_0\right)\iota_{K+L} \\
&= \frac{1}{\mathbf{p}'\iota_K}\sum_{k=1}^{K} r_{k0}
\end{aligned}$$

and hence

$$\left.\frac{\mathrm{d}\left(\mathbf{r}_0'\iota_{K+L}\right)}{\mathrm{d}\left(\mathbf{p}'\iota_K\right)}\frac{\left(\mathbf{p}'\iota_K\right)}{\left(\mathbf{r}_0'\iota_{K+L}\right)}\right|_{\frac{\frac{\mathrm{d}p_k}{p_k}}{\frac{\mathrm{d}(\mathbf{p}'\iota_K)}{(\mathbf{p}'\iota_K)}}=1,\,k=1,\ldots,K} = \frac{\sum_{k=1}^{K} r_{k0}}{\sum_{k=1}^{K} r_{k0} + \sum_{l=1}^{L} r_{0l}}.$$

This is reported in equation (14) of the main text.

Step 5d: Constant Returns to Scale

The demand functions for single-hood are homogenous of degree one since

$$\begin{aligned}
\sum_{k=1}^{K} U\left(\mathbf{r}_0\right)^{-1}\frac{\mathrm{d}\mathbf{r_0}}{\mathrm{d}p_k}p_k + \sum_{l=1}^{L} U\left(\mathbf{r}_0\right)^{-1}\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}q_l}q_l &= U\left(\mathbf{r}_0\right)^{-1} U\left(\mathbf{r}_0\right) H\left(\mathbf{r}_0\right)^{-1} U\left(\mathbf{r}_0\right)^{-1}\mathbf{r}_0 \\
&= H\left(\mathbf{r}_0\right)^{-1}\iota_{K+L} \\
&= \iota_{K+L},
\end{aligned}$$

with the result following from Euler's Theorem.

Step 5e Semi-elasticity of single-hood with respect to γ_{kl}

The approximate percent increase in single-hood for each of the $K + L$ types of agents given a unit increase in γ_{kl} is given by

$$\begin{aligned}
U\left(\mathbf{r}_0\right)^{-1}\frac{\mathrm{d}\mathbf{r_0}}{\mathrm{d}\gamma_{kl}} &= U\left(\mathbf{r}_0\right)^{-1} J\left(\mathbf{r}_0\right)^{-1}\frac{\partial\mathbf{B}}{\partial\gamma_{kl}} \\
&= -H\left(\mathbf{r}_0\right)^{-1}\operatorname{diag}\left\{\mathbf{p}',\mathbf{q}'\right\}^{-1}\left(\mathbf{h}_k + \mathbf{h}_{K+l}\right)r_{kl} \\
&= -\begin{pmatrix}
\frac{\mathrm{d}r_{10}}{\mathrm{d}p_k}\frac{1}{r_{10}} + \frac{\mathrm{d}r_{10}}{\mathrm{d}q_l}\frac{1}{r_{10}} \\
\vdots \\
\frac{\mathrm{d}r_{K0}}{\mathrm{d}p_k}\frac{1}{r_{K0}} + \frac{\mathrm{d}r_{K0}}{\mathrm{d}q_l}\frac{1}{r_{K0}} \\
\frac{\mathrm{d}r_{01}}{\mathrm{d}p_k}\frac{1}{r_{01}} + \frac{\mathrm{d}r_{01}}{\mathrm{d}q_l}\frac{1}{r_{01}} \\
\vdots \\
\frac{\mathrm{d}r_{0L}}{\mathrm{d}p_k}\frac{1}{r_{0L}} + \frac{\mathrm{d}r_{0L}}{\mathrm{d}q_l}\frac{1}{r_{0L}}
\end{pmatrix} r_{kl},
\end{aligned}$$

where the final equality follows from the expression for $U(\mathbf{r}_0)^{-1}\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\begin{pmatrix} p' & q' \end{pmatrix}}$ derived above.

To derive the effect of a unit increase in net marriage surplus – for all types of marriages – on the aggregate prevalence of single-hood we begin by noting that

$$\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{\partial \mathbf{B}}{\partial \gamma_{kl}} = -\begin{pmatrix} r_{10}\left(1-r_{01}/p_1\right) \\ \vdots \\ r_{K0}\left(1-r_{K0}/p_K\right) \\ r_{01}\left(1-r_{01}/q_1\right) \\ \vdots \\ r_{0L}\left(1-r_{0L}/q_L\right) \end{pmatrix}.$$

The required semi-elasticity is then, making use of the row stochastic structure of $H(\mathbf{r}_0)$,

$$\begin{aligned}
\iota_{K+L}'\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\overline{\gamma}} &= \iota_{K+L}' J(\mathbf{r}_0)^{-1}\sum_{k=1}^{K}\sum_{l=1}^{L}\frac{\partial \mathbf{B}}{\partial \gamma_{kl}} \\
&= -\iota_{K+L}' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1} U(\mathbf{r}_0)^{-1}\begin{pmatrix} r_{10}\left(1-r_{01}/p_1\right) \\ \vdots \\ r_{K0}\left(1-r_{K0}/p_K\right) \\ r_{01}\left(1-r_{01}/q_1\right) \\ \vdots \\ r_{0L}\left(1-r_{0L}/q_L\right) \end{pmatrix} \\
&= -\iota_{K+L}' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1}\begin{pmatrix} 1-r_{01}/p_1 \\ \vdots \\ 1-r_{K0}/p_K \\ 1-r_{01}/q_1 \\ \vdots \\ 1-r_{0L}/q_L \end{pmatrix} \\
&= -\begin{pmatrix} 1-r_{01}/p_1 \\ \vdots \\ 1-r_{K0}/p_K \\ 1-r_{01}/q_1 \\ \vdots \\ 1-r_{0L}/q_L \end{pmatrix}' U(\mathbf{r}_0) H(\mathbf{r}_0)^{-1}\iota_{K+L} \\
&= -\sum_{k=1}^{K} r_{k0}\left(1-\frac{r_{k0}}{p_k}\right) - \sum_{l=1}^{L} r_{0l}\left(1-\frac{r_{0l}}{q_l}\right).
\end{aligned}$$

Hence the change in the fraction of the population that is single equals

$$-\frac{\sum_{k=1}^{K} p_k e_{k0} (1 - e_{k0}) + \sum_{l=1}^{L} q_l e_{0l} (1 - e_{0l})}{\sum_{k=1}^{K} p_k + \sum_{l=1}^{L} q_l},$$

as reported in equation (15) of the main text.

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