Supplemental appendix to A Quantile Correlated Random Coefficients Panel Data Model

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September 23, 2015

[*]Earlier versions of this paper, with an initial draft date of March 2008, were presented under a variety of titles. We would like to thank seminar participants at Berkeley, CEMFI, Duke, University of Michigan, Université de Montréal, NYU, Northwestern and at the 2009 North American Winter Meetings of the Econometric Society, the 2009 All-California Econometrics Conference at UC - Riverside, the 2009 CEMMAP Quantile Regression Conference, the 2014 Midwest Econometrics Group and the 2nd annual IAAE Conference. Financial support from the National Science Foundation (SES #0921928) is gratefully acknowledged. All the usual disclaimers apply.

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Proof of Theorem 8. $\hat{\delta}(\tau)-\delta(\tau)$ has the following linear representation

$$\begin{aligned}\sqrt{N}\left(\hat{\delta}(\tau)-\delta(\tau)\right) &= \left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{W}_i^{*\prime}\mathbf{W}_i^{*}\mathbf{1}(D_i=0)\right)^{-1} \\ &\quad\times\frac{1}{N}\sum_{i=1}^{N}\mathbf{W}_i^{*\prime}\mathbf{X}_i^{*}\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{X}_i)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{X}_i)\right) \\ &= \left(\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}\hat{p}_l\right)^{-1}\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\hat{p}_l\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)\end{aligned} \tag{1}$$

with

$$\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}\hat{p}_l \xrightarrow{p} \sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{w}_l^{*}p_l=\mathbb{E}\left[\mathbf{W}^{*\prime}\mathbf{W}^{*}|D=0\right]\pi_0$$

and

$$\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\hat{p}_l\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)\xrightarrow{d}\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\sqrt{p_l}\mathbf{Z}_Q(\tau,\mathbf{x}_l),$$

which has asymptotic covariance equal to

$$\begin{aligned}&\mathbb{E}\left[\sum_{l=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}\sqrt{p_l}\mathbf{Z}_Q(\tau,\mathbf{x}_l)\left(\sum_{l'=L+1}^{M}\mathbf{w}_{l'}^{*\prime}\mathbf{x}_{l'}^{*}\sqrt{p_{l'}}\mathbf{Z}_Q(\tau',\mathbf{x}_{l'})\right)'\right] \\ &=\sum_{l=L+1}^{M}\sum_{l'=L+1}^{M}\mathbf{w}_l^{*\prime}\mathbf{x}_l^{*}(\min(\tau,\tau')-\tau\tau')\Lambda\left(\tau,\tau';\mathbf{x}_l\right)\cdot\mathbf{1}\left(l=l'\right)\mathbf{x}_l^{*\prime}\mathbf{w}_l^{*}p_lp_{l'} \\ &=(\min(\tau,\tau')-\tau\tau')\mathbb{E}\left[\mathbf{W}^{*\prime}\mathbf{X}^{*}\Lambda(\tau,\tau';\mathbf{X})\mathbf{X}^{*\prime}\mathbf{W}^{*}|D=0\right]\pi_0.\end{aligned}$$

To derive the asymptotic distribution of $\sqrt{N}\left(\widehat{\beta}\left(\cdot;\cdot\right)-\beta\left(\cdot;\cdot\right)\right)$ we note that

$$\begin{aligned}\sqrt{N}\left(\widehat{\beta}\left(\tau;\mathbf{x}_l\right)-\beta\left(\tau;\mathbf{x}_l\right)\right)&=\mathbf{x}_l^{-1}\sqrt{N}\left(\widehat{Q}_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)-Q_{\mathbf{Y}|\mathbf{X}}(\tau|\mathbf{x}_l)\right)+\mathbf{x}_l^{-1}\mathbf{w}_l\sqrt{N}\left(\hat{\delta}(\tau)-\delta(\tau)\right) \\ &\xrightarrow{d}\mathbf{x}_l^{-1}\mathbf{Z}_Q(\tau,\mathbf{x}_l)+\mathbf{x}_l^{-1}\mathbf{w}_l\mathbf{Z}_\delta(\tau).\end{aligned} \tag{2}$$

$\mathbf{Z}_Q(\tau,\mathbf{x}_l)$ and $\mathbf{Z}_\delta(\tau)$ are independent processes since they are computed using disjoint subpopulations: $\mathbf{x}_l$ for $l=1,\ldots,L$ are not used in the computation of $\hat{\delta}(\tau)$. Therefore, the asymptotic variance of (2) is the sum of the variance of its terms. □

Proof of Theorem 9. We see that

$$\sqrt{N}\left(\widehat{\bar{\beta}}^{M}(\tau)-\bar{\beta}^{M}(\tau)\right)=\sum_{l=1}^{L}\beta(\tau;\mathbf{x}_l)\sqrt{N}\left(\hat{q}_l^{M}-q_l^{M}\right) \tag{3}$$

$$+\sum_{l=1}^{L}\sqrt{N}\left(\hat{\beta}(\tau;\mathbf{x}_l)-\beta(\tau;\mathbf{x}_l)\right)\hat{q}_l^{M}. \tag{4}$$

By a result similar to that in (76) in the main text, term (3) converges to a mean zero Gaussian process with covariance equal to $\frac{\mathbb{C}\left(\beta(\tau,\mathbf{X}),\beta(\tau',\mathbf{X})|X\in\mathbb{X}^{M}\right)}{\Pr(\mathbf{X}\in\mathbb{X}^{M})}$. Term (4) converges to

$$\sum_{l=1}^{L}\sqrt{N}\left(\hat{\beta}(\tau;\mathbf{x}_l)-\beta(\tau;\mathbf{x}_l)\right)\hat{q}_l^{M}\xrightarrow{d}\sum_{l=1}^{L}\mathbf{Z}(\tau,\mathbf{x}_l)q_l^{M} \tag{5}$$

which has a covariance kernel equal to

$$\begin{aligned}
&\mathbb{E}\left[\sum_{l=1}^{L}\sum_{l'=1}^{L}\mathbf{Z}(\tau,\mathbf{x}_l)\mathbf{Z}(\tau,\mathbf{x}_{l'})q_l^{M}q_{l'}^{M}\right]\\
&=\mathbb{E}\left[\mathbf{Z}(\tau,\mathbf{x}_l)\mathbf{Z}(\tau,\mathbf{x}_{l'})\right]q_l^{M}q_{l'}^{M}\\
&=\left(\min\left(\tau,\tau'\right)-\tau\tau'\right)\sum_{l=1}^{L}\sum_{l'=1}^{L}\frac{\mathbf{x}_l^{-1}\Lambda\left(\tau,\tau';\mathbf{x}_l\right)\mathbf{x}_l^{-1\prime}}{p_l}\cdot\mathbf{1}\left(l=l'\right)q_l^{M}q_{l'}^{M}\\
&+\sum_{l=1}^{L}\sum_{l'=1}^{L}\mathbf{x}_l^{-1}\mathbf{w}_l\Sigma_{\delta}(\tau,\tau')\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime}q_l^{M}q_{l'}^{M}\\
&=\frac{\min\left(\tau,\tau'\right)-\tau\tau'}{\Pr\left(\mathbf{X}\in\mathbb{X}^{M}\right)}\mathbb{E}\left[\mathbf{X}^{-1}\Lambda(\tau,\tau',\mathbf{X})\mathbf{X}^{-1\prime}|\mathbf{X}\in\mathbb{X}^{M}\right]+\sum_{l=1}^{L}\mathbf{x}_l^{-1}\mathbf{w}_l q_l^{M}\Sigma_{\delta}(\tau,\tau')\sum_{l'=1}^{L}\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime}q_{l'}^{M}\\
&=\Upsilon_1(\tau,\tau')+\Xi_0\Sigma_{\delta}(\tau,\tau')\Xi_0'.
\end{aligned}$$

Since terms (3) and (4) are uncorrelated, the asymptotic covariance of $\sqrt{N}\left(\widehat{\bar{\beta}}^{M}(\tau)-\bar{\beta}^{M}(\tau)\right)$ is equal to the sum of the covariance of its two terms. $\square$

Proof of Theorem 10. We start by deriving the asymptotic distribution of the sample cumulative distribution function of $\widehat{\beta}_p(U;\mathbf{X})$ with U distributed uniformly on $[0,1]$ independently from $\mathbf{X}$, while conditioning on $\mathbf{X}\in\mathbb{X}^{M}$. The CDF estimand at $c\in\mathbb{R}$ is denoted as

$F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c)$ and the estimator is

$$\begin{aligned}
\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) &= \frac{\frac{1}{N}\sum_{i=1}^{N}\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{X}_i)\leq c)du\mathbf{1}(\mathbf{X}_i\in\mathbb{X}^M)}{\frac{1}{N}\sum_{i=1}^{N}\mathbf{1}(\mathbf{X}_i\in\mathbb{X}^M)} \\
&= \sum_{l=1}^{L}\left(\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{x}_l)\leq c)du\right)\widehat{q}_l^M. \qquad (6)
\end{aligned}$$

The integration over $u\in(0,1)$ can be done exactly since $\widehat{\beta}_p(u,\mathbf{x}_l)$ is piecewise linear for each $l\in\{1,\ldots,L\}$ with finitely many pieces. This asymptotic distribution can be written as the sum of two terms:

$$\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) - F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c) = \sum_{l=1}^{L}\left(\int_0^1 \mathbf{1}(\widehat{\beta}_p(u,\mathbf{x}_l)\leq c)du - \int_0^1 \mathbf{1}(\beta_p(u,\mathbf{x}_l)\leq c)du\right)\widehat{q}_l^M \qquad (7)$$

$$+\sum_{l=1}^{L}\int_0^1 \mathbf{1}(\beta_p(u,\mathbf{x}_l)\leq c)du\left(\widehat{q}_l^M - q_l^M\right). \qquad (8)$$

We will show that these two terms both converge in uniformly over $c\in\mathbb{R}$. For term (7), we have that $\sqrt{N}\left(\hat{\beta}_p(\tau;\mathbf{x}_l)-\hat{\beta}_p(\tau;\mathbf{x}_l)\right)\xrightarrow{d}(\mathbf{Z}(\tau,\mathbf{x}_l))_p = \mathbf{Z}_p(\tau,\mathbf{x}_l)$ over $\tau\in(0,1)$ and all $l=1,\ldots,L$, and $(\cdot)_p$ denotes the p^{th} element of the vector. By the same argument as in (79), we have

$$\begin{aligned}
&\sqrt{N}\left(\int_0^1 \mathbf{1}(\hat{\beta}_p(u;\mathbf{x}_l)\leq c)du - \int_0^1 \mathbf{1}(\beta_p(u;\mathbf{x}_l)\leq c)du\right) \\
&= \sqrt{N}\left(\hat{\beta}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l);\mathbf{x}_l) - \hat{\beta}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l);\mathbf{x}_l)\right)f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) + o_p(1) \\
&\xrightarrow{d} \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l)f_{B_p|\mathbf{X}}(c|\mathbf{x}_l).
\end{aligned}$$

This convergence is uniform in $c\in\mathbb{R}$ since $F_{B_p|\mathbf{X}}(c|\mathbf{x}_l)$ ranges between 0 and 1, and uniform in $\mathbf{x}_l$ since its support is finite. Therefore,

$$\sum_{l=1}^{L}\sqrt{N}\left(\int_0^1 \mathbf{1}(\hat{\beta}_p(u;\mathbf{x}_l)\leq c)du - \int_0^1 \mathbf{1}(\beta_p(u;\mathbf{x}_l)\leq c)du\right)\hat{q}_l^M \xrightarrow{d} \sum_{l=1}^{L}\mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l),\mathbf{x}_l)f_{B_p|\mathbf{X}}(c|\mathbf{x}_l)q_l^M \qquad (9)$$

for $c\in\mathbb{R}$. Also, (8) will converge over $c\in\mathbb{R}$ to a Gaussian process $\mathbf{Z}_{2p}(c)$ with asymptotic covariance of

$$\mathbb{E}\left[\mathbf{Z}_{2p}(c)\mathbf{Z}_{2p}(c')'\right] = \frac{\mathbb{C}\left(F_{B_p|\mathbf{X}}(c|\mathbf{X}), F_{B_p|\mathbf{X}}(c'|\mathbf{X})|\mathbf{X} \in \mathbb{X}^M\right)}{\Pr\left(\mathbf{X} \in \mathbb{X}^M\right)}.$$

Note that $\mathbf{Z}_{2p}(c)$ and $\sum_{l=1}^{L} \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l), \mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) q_l^M$ are uncorrelated since the variation in the latter is conditional on $\mathbf{X}$ while that in the former depends on $\mathbf{X}$ only. Therefore,

$$\widehat{F}_{\widehat{\beta}_p(U;\mathbf{X})|\mathbf{X}\in\mathbb{X}^M}(c) - F_{B_p|\mathbf{X}\in\mathbb{X}^M}(c) \xrightarrow{d} \sum_{l=1}^{L} \mathbf{Z}_p(F_{B_p|\mathbf{X}}(c|\mathbf{x}_l), \mathbf{x}_l) f_{B_p|\mathbf{X}}(c|\mathbf{x}_l) q_l^M + \mathbf{Z}_{2p}(c) \tag{10}$$

for $c \in \mathbb{R}$.

Using the same invertibility argument as in (82), we see that

$$\begin{aligned}\sqrt{N}\left(\hat{\beta}_p^M(\tau) - \beta_p^M(\tau)\right) &\xrightarrow{d} \frac{\sum_{l=1}^{L} \mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), \mathbf{x}_l) f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l) q_l^M + \mathbf{Z}_{2p}(\beta_p^M(\tau))}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau))} \\ &= \mathbf{Z}_{\beta_p}(\tau) \end{aligned} \tag{11}$$

uniformly over $\tau \in (0, 1)$.

To conclude this proof, we evaluate $\mathbb{E}\left[\mathbf{Z}_{\beta_p}(\tau)\,\mathbf{Z}_{\beta_p}(\tau')'\right]$, the asymptotic covariance of (11):

$$\begin{aligned}\mathbb{E}\left[\mathbf{Z}_{\beta_p}(\tau)\,\mathbf{Z}_{\beta_p}(\tau')'\right] &= \frac{\sum_{l=1}^{L}\sum_{l'=1}^{L} f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l) f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}) q_l^M q_{l'}^M}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau)) f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau'))} \\ &\quad \times \mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), \mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}), \mathbf{x}_{l'})\right] \\ &\quad + \frac{\mathbb{E}\left[\mathbf{Z}_{2p}(\beta_p^M(\tau))\mathbf{Z}_{2p}(\beta_p^M(\tau'))\right]}{f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau)) f_{B_p|\mathbf{X}\in\mathbb{X}^M}(\beta_p^M(\tau'))}\end{aligned}$$

where

$$\begin{aligned}&\mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), \mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}), \mathbf{x}_{l'})\right] \\ &= \left(\min\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})\right) - F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l) F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})\right) \\ &\times e_p' \frac{\mathbf{x}_l^{-1}\Lambda\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}); \mathbf{x}_l\right)\mathbf{x}_l^{-1\prime}}{p_l} e_p \cdot \mathbf{1}\left(l = l'\right) \\ &+ e_p'\mathbf{x}_l^{-1}\mathbf{w}_l\Sigma_\delta(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}))\mathbf{w}_{l'}'\mathbf{x}_{l'}^{-1\prime} e_p\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{l=1}^{L}\sum_{l'=1}^{L} f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l)q_l^M \mathbb{E}\left[\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{x}_l),\mathbf{x}_l)\mathbf{Z}_p(F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'}),\mathbf{x}_{l'})\right] f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{x}_{l'})q_{l'}^M \\
&= \mathbb{E}\Bigg[\frac{\left(\min\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\right) - F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X})F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\right)}{\Pr\left(\mathbf{X}\in\mathbb{X}^M\right)} \\
&\times e_p'\ \mathbf{X}^{-1}\Lambda\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X});\mathbf{X}\right)\mathbf{X}^{-1\prime}e_p f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}) f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})\Big|\,\mathbf{X}\in\mathbb{X}^M\Big] \\
&+ e_p'\mathbb{E}\Big[f_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}) f_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\tilde{\mathbf{X}})\mathbf{X}^{-1}\mathbf{W}\Sigma_\delta(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\tilde{\mathbf{X}})) \\
&\times \tilde{\mathbf{W}}'\tilde{\mathbf{X}}^{-1\prime}|\mathbf{X}\in\mathbb{X}^M, \tilde{\mathbf{X}}\in\mathbb{X}^M\Big]\, e_p \\
&= \Upsilon_3(\tau,\tau') + \Upsilon_4(\tau,\tau'),
\end{aligned} \tag{12}
$$

where $\tilde{\mathbf{X}}$ is an independent copy of $\mathbf{X}$. Finally,

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{Z}_{2p}(\beta_p^M(\tau))\mathbf{Z}_{2p}(\beta_p^M(\tau'))\right] &= \frac{\mathbb{C}\left(F_{B_p|\mathbf{X}}(\beta_p^M(\tau)|\mathbf{X}), F_{B_p|\mathbf{X}}(\beta_p^M(\tau')|\mathbf{X})|\mathbf{X}\in\mathbb{X}^M\right)}{\Pr\left(\mathbf{X}\in\mathbb{X}^M\right)} \\
&= \Upsilon_2(\tau,\tau').
\end{aligned}
$$

$\square$