

One way ANOVA

Session 3

MATH 80667A: Experimental Design and Statistical Methods
for Quantitative Research in Management
HEC Montréal

Outline

Recap

F tests

Parametrizations and interpretation

Planned comparisons and *post-hoc* tests

Refresher on hypothesis tests

General recipe of hypothesis testing

(1) Define variables

(2) Write down hypotheses

(3) Choose/compute a test statistic

(4) Benchmark

(5) Compute the p -value

(6) Conclude (reject/fail to reject)

(7) Report findings

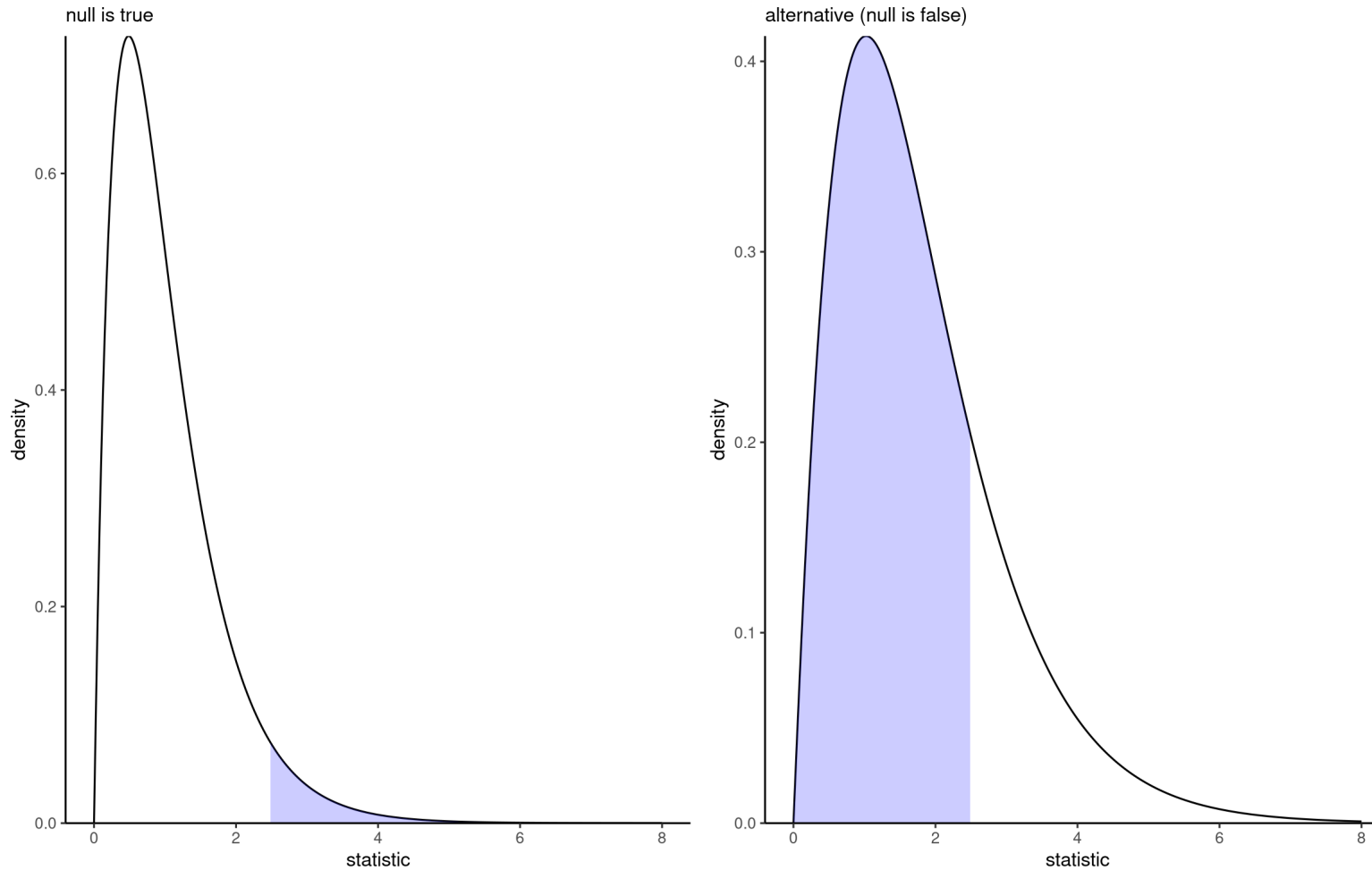
Level

Level = probability of judicial error
Analyst fixes level α before the experiment.

**Choose α as small as possible (don't condemn the innocent!)
(typical value is 5%)**

Reject \mathcal{H}_0 if p-value less than α

Errors



Confidence intervals

Test statistics are standardized,

- Good for comparisons with benchmark
- typically meaningless (standardized = unitless quantities)

Two options for reporting:

- p -value: probability of more extreme outcome if no mean difference
- confidence interval: set of all values for which we fail to reject the null hypothesis at level α for the given sample

Interpretation of confidence intervals

confidence interval = [lower, upper] units

Interpretation: under the null, if we repeat the experiments 95%, of intervals will contain the true value (if model correctly calibrated)

Analogy: coin toss

One way analysis of variance

F-test for one way ANOVA

Global null hypothesis

No difference between treatments

- \mathcal{H}_0 (null): all of the K treatment groups have the same average μ
- \mathcal{H}_a (alternative): at least two treatments have different averages

F-test statistic

Omnibus test

With K groups and n observations, the statistic is

$$F = \frac{\text{between sum of squares} / (K - 1)}{\text{within sum of squares} / (n - K)}$$

The null distribution (benchmark) is $F(K - 1, n - K)$.

Why does it work?

Denote

- y_{ik} is observation i of group k
- $\hat{\mu}_1, \dots, \hat{\mu}_K$ the sample average of groups $1, \dots, K$
- $\hat{\mu}$ is overall sample mean

Decomposing variability into bits

$$\sum_i \sum_k (y_{ik} - \hat{\mu})^2 = \sum_i \sum_k (y_{ik} - \hat{\mu}_k)^2 + \sum_k n_i (\hat{\mu}_k - \hat{\mu})^2 .$$

total sum of squares within sum of squares between sum of squares

null model

alternative model

added variability

Degrees of freedom

The parameters of the null distribution are called **degrees of freedom**

- $K - 1$ is the number of constraints imposed by the null
- $n - K$ is the number of observations minus number of mean parameters estimated under alternative

Idea of F -statistic: under the null, both numerator and denominator are variance estimators.

- but the numerator is more variable...
- the F ratio should be approximately one on average

Parametrizations and interpretation

Parametrization 1: sample averages

Most natural parametrization, not useful for test

- Sample sizes in each group: n_1, \dots, n_K , are known.
- sample average of each treatment group: $\hat{\mu}_1, \dots, \hat{\mu}_K$.

K means = K parameters

Overall mean is

$$n\hat{\mu} = n_1\hat{\mu}_1 + \dots + n_K\hat{\mu}_K$$

Parametrization 2: contrasts

In terms of differences, relative to a baseline category j

- Intercept = sample mean $\hat{\mu}_j$
- Coefficient for group $k \neq j$: $\hat{\mu}_k - \hat{\mu}_j$
 - difference between averages of group k and baseline

In **R**, the baseline is the smallest alphanumerical value.

```
lm(response ~ group)
```


Parametrization 3: sum-to-zero

In terms of differences, relative to average of $\hat{\mu}_1, \dots, \hat{\mu}_K$

- Intercept = $(\hat{\mu}_1 + \dots + \hat{\mu}_K) / K$
- Coefficient for group k : $\hat{\mu}_k$ minus intercept

In **R**, the last factor level is dropped by default.

```
lm(response ~ group, contrasts = contr.sum(group))
```

Warning: Intercept $\neq \hat{\mu}$ unless the sample is balanced

Comparison for the arithmetic example

group	mean	contrasts	sum-to-zero
intercept		19.66	21.00
control 1	19.66		-1.33
control 2	18.33	-1.33	-2.66
praised	27.44	7.77	6.44
reproved	23.44	3.77	2.44
ignored	16.11	-3.55	

Planned comparisons and post-hoc tests

Planned comparisons

Oftentimes, we are not interested in the global null hypothesis.

- Can formulate planned comparisons *at registration time* for effects of interest

What is the scientific question of interest?

Arithmetic example

Setup

group 1

(control)

group 2

(control)

group 3

(praise, reprove, ignore)

Hypothesis of interest

- $\mathcal{H}_{01}: \mu_{\text{praise}} = \mu_{\text{reprove}}$ (attention)
- $\mathcal{H}_{02}: \frac{1}{2} (\mu_{\text{control}_1} + \mu_{\text{control}_2}) = \mu_{\text{praise}}$ (encouragement)

Contrasts

With placeholders for each group, write $\mathcal{H}_{01} : \mu_{\text{praise}} = \mu_{\text{reprove}}$ as

$$0 \cdot \mu_{\text{control}_1} + 0 \cdot \mu_{\text{control}_2} + 1 \cdot \mu_{\text{praise}} - 1 \cdot \mu_{\text{reprove}} + 0 \cdot \mu_{\text{ignore}}$$

The sum of the coefficients, $(0, 0, 1, -1, 0)$, is zero.

Contrast = sum-to-zero constraint

Similarly, for $\mathcal{H}_{02} : \frac{1}{2}(\mu_{\text{control}_1} + \mu_{\text{control}_2}) = \mu_{\text{praise}}$

$$\frac{1}{2} \cdot \mu_{\text{control}_1} + \frac{1}{2} \cdot \mu_{\text{control}_2} - 1 \cdot \mu_{\text{praise}} + 0 \cdot \mu_{\text{reprimand}} + 0 \cdot \mu_{\text{ignore}}$$

The contrast vector $(\frac{1}{2}, \frac{1}{2}, -1, 0, 0)$ sums to zero.

Equivalent formulation is obtained by picking $(1, 1, -2, 0, 0)$

Contrasts in R

```
library(emmeans)
linmod <- lm(score ~ group, data = arithmetic)
linmod_emm <- emmeans(linmod, specs = 'group')
contrast_specif <- list(
  controlvspraised = c(0.5, 0.5, -1, 0, 0),
  praisedvsreproved = c(0, 0, 1, -1, 0)
)
contrasts_res <-
  contrast(object = linmod_emm,
           method = contrast_specif)
# Obtain confidence intervals instead of p-values
confint(contrasts_res)
```

Post-hoc tests

Maybe there is some difference between groups?

Unplanned comparisons: go fishing...

Comparing all pairwise differences = $\binom{K}{2}$ tests

With $K = 5$ groups, we get 10 pairwise comparisons.

```
emmeans(modlin, pairwise ~ group)
```

If there were no differences between the groups, how many do we expect to find significant by chance with $\alpha = 0.1$?

Pairwise differences and t -tests

Technical aside

The pairwise differences (p -values) and confidence intervals for groups j and k are based on the t -statistic:

$$t = \frac{(\hat{\mu}_j - \hat{\mu}_k) - 0}{\text{se}(\hat{\mu}_j - \hat{\mu}_k)}$$

which has a null $\mathcal{T}(n - k)$ distribution.

The standard error $\text{se}(\hat{\mu}_j - \hat{\mu}_k)$ uses the pooled variance estimate, i.e., the within sum of squares divided by $n - K$