General factorial designs

Session 7

MATH 80667A: Experimental Design and Statistical Methods
for Quantitative Research in Management
HEC Montréal

Outline

Blocking

More complex factorial designs

Custom contrasts and marginal means

Blocking

Terminology for *nuisance*

Block

Source of variation, but
of no interest
known and controllable

Example

timing
lab technician
machine

Covariates

Explanatory measured **before** the experiment

Cannot be acted upon

Example

socioeconomic variables
environmental conditions

Noise factor

Under which setting is response least affected?

Example

temperature
processing

Why blocking?

Design experiment to reduce the effect of uncontrolled variations

In general, increases the power of the F test for treatment effects.

Group units in sets as alike as possible.

(Often) compare only treatments, so interactions are not included.

Assignment to treatment

Divide subjects within each block

Randomly allocate to treatment within block

(stratified sampling)

Block-treatment design

Without interaction,

$$\underset{\text{response}}{Y_{ij}} = \mu + \underset{\text{treatment}}{\alpha_i} + \underset{\text{blocking}}{\beta_j} + \underset{\text{error}}{\varepsilon_{ij}}$$

Compromise between

- reduced variability for residuals
- loss of degrees of freedom due to estimation of β's.

Example: Resting metabolic rate

From Dean, Voss and Draguljić (2017), Example 10.4.1 (p. 311)

> experiment that was run to compare the effects of inpatient and outpatient protocols on the in-laboratory measurement of resting metabolic rate (RMR) in humans. A previous study had indicated measurements of RMR on elderly individuals to be 8% higher using an outpatient protocol than with an inpatient protocol. If the measurements depend on the protocol, then comparison of the results of studies conducted by different laboratories using different protocols would be difficult. The experimenters hoped to conclude that the effect on RMR of different protocols was negligible.

Clean Fit Plot

```
library(tidyverse)
url <- "https://edsm.rbind.io/data/resting_metabolic_rate.txt"
# Load data, transform integers to factors (categorical)
resting <- read.table(url, header = TRUE) %>%
  mutate(subject = factor(subject), #blocking factor
         protocol = factor(protocol),
         rate = rate/1000)
```

Clean | **Fit** | Plot

```r
# Force sum-to-zero parametrization for unordered factors
options(contrasts = c("contr.sum", "contr.poly"))
# Fit model with blocking
model_block <- lm(rate ~ subject + protocol, data = resting)
# One-way ANOVA (no blocking
model_raw <- lm(rate ~ protocol, data = resting)
# anova(model_block)
# anova(model_raw)
```

Clean Fit **Plot**

```
ggplot(data = resting,
       aes(x = subject,
           y = rate,
           group = protocol,
           color = protocol)) +
    geom_line() +
    labs(y = "mean resting\n metabolic rate") +
    theme(legend.position = "bottom")
```

Interaction plot

Impact of blocking

ANOVA table (with blocking) | ANOVA table (without blocking)

Analysis of variance table - with blocking

	Degrees of freedom	Sum of squares	Mean square	F statistic	p-value
subject	8	23.12	2.89	37.42	0.000
protocol	2	0.04	0.02	0.23	0.795
Residuals	16	1.24	0.08		

Impact of blocking

ANOVA table (with blocking) | ANOVA table (without blocking)

Analysis of variance table - without blocking

	Degrees of freedom	Sum of squares	Mean square	F statistic	p-value
protocol	2	0.04	0.02	0.02	0.982
Residuals	24	24.35	1.01		

Multifactorial designs

Beyond two factors

We can consider multiple factors A, B, C, $\dots$ with respectively a, b, c, $\dots$ levels and with r replications for each.

The total number of treatment combinations is

$$a \times b \times c \times \cdots$$

Curse of dimensionality

Full three-way ANOVA model

Each cell of the cube is allowed to have a different mean

$$\underset{\text{response}}{Y_{ijkr}} = \underset{\text{cell mean}}{\mu_{ijk}} + \underset{\text{error}}{\varepsilon_{ijkr}}$$

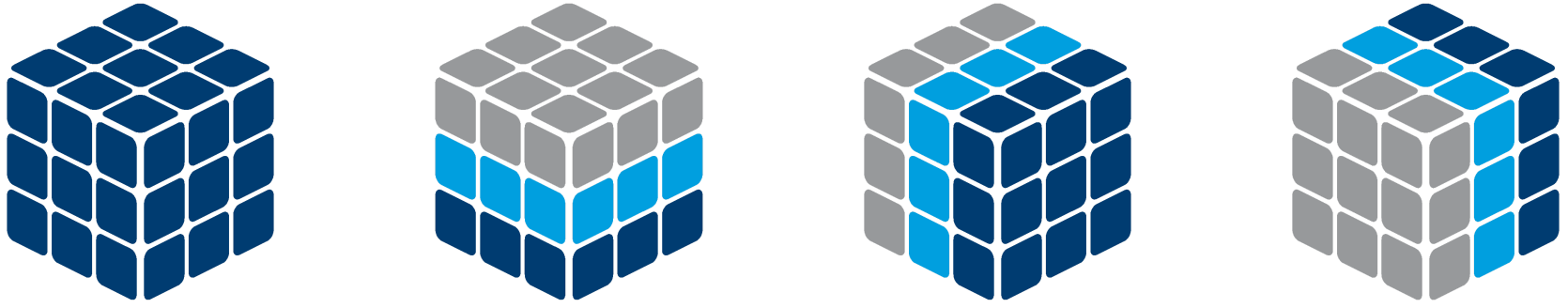
with ε_{ijkt} an independent $\text{No}(0, \sigma^2)$ error term for

- row i
- column j
- depth k
- replication r

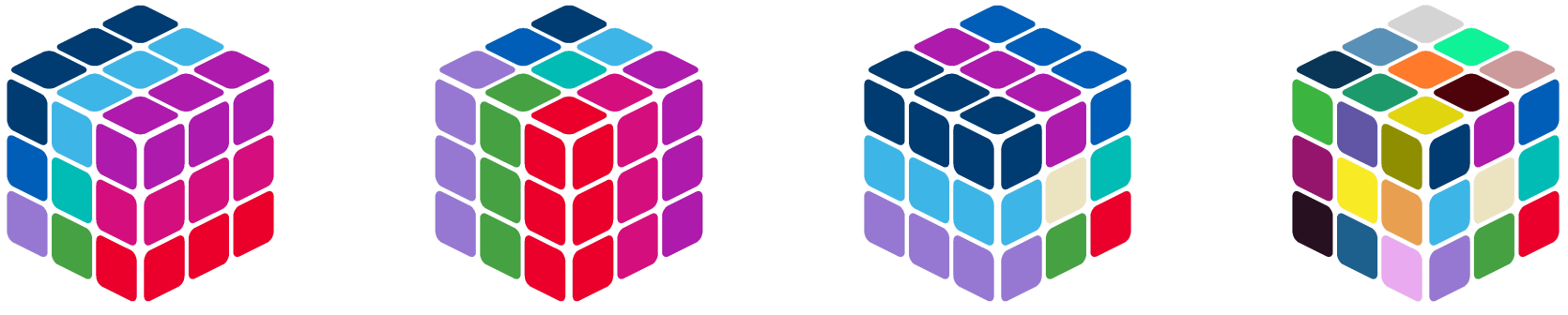
Parametrization of a three-way ANOVA model

With the **sum-to-zero** parametrization with factors A, B and C, write the response as

$$\underbrace{\mathsf{E}(Y_{ijkr})}_{\text{theoretical average}} = \underbrace{\mu}_{\text{global mean}} + \underbrace{\alpha_i + \beta_j + \gamma_k}_{\text{main effects}} + \underbrace{(\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}}_{\text{two-way interactions}} + \underbrace{(\alpha\beta\gamma)_{ijk}}_{\text{three-way interaction}}$$



global mean, row, column and depth main effects



row/col, row/depth and col/depth interactions and three-way interaction.

Example of three-way design

Petty, Cacioppo and Heesacker (1981). Effects of rhetorical questions on persuasion: A cognitive response analysis. Journal of Personality and Social Psychology.

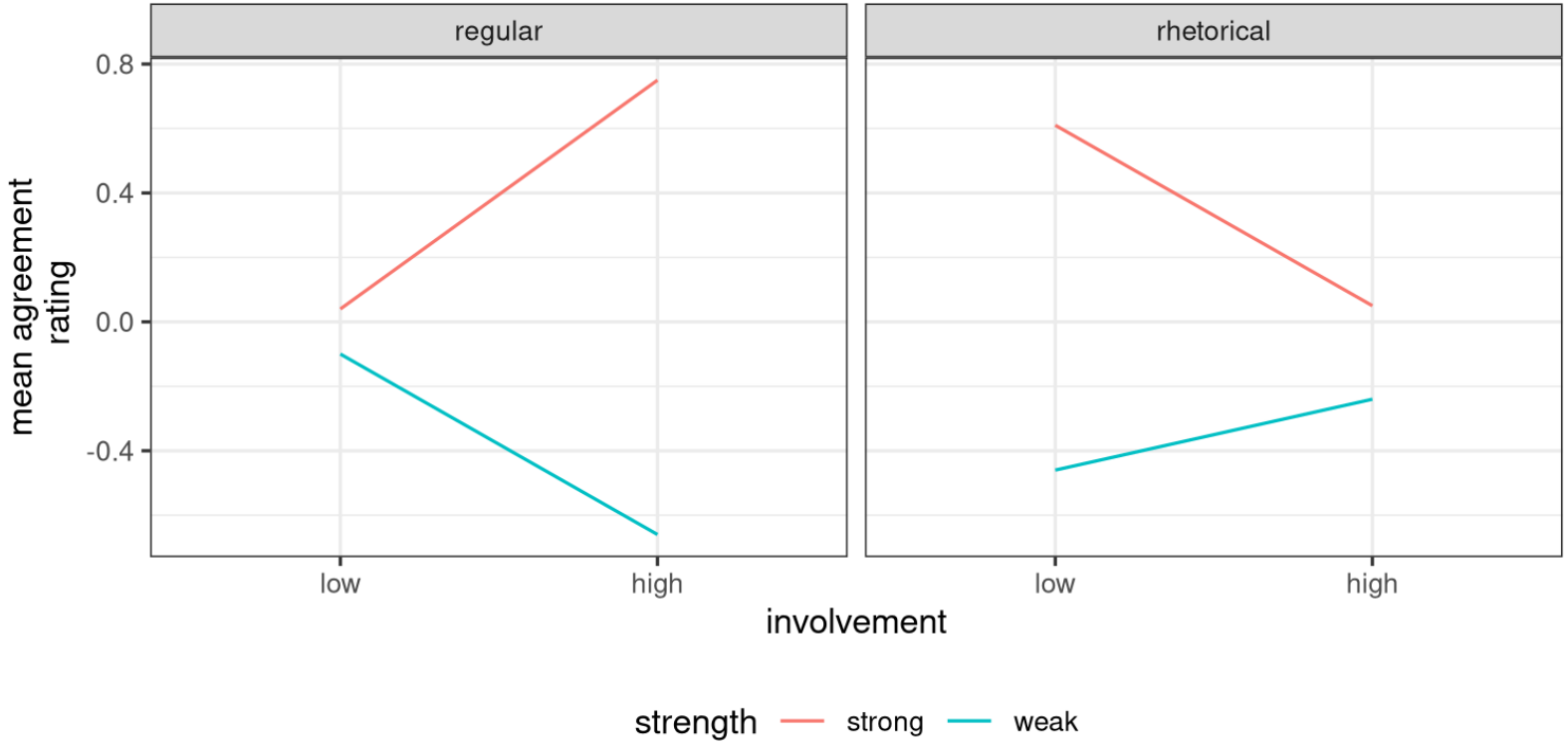
A $2 \times 2 \times 2$ factorial design with 8 treatments groups and $n = 160$ undergraduates.

Setup: should a comprehensive exam be administered to bachelor students in their final year?

- **Response** Likert scale on -5 (do not agree at all) to 5 (completely agree)
- **Factors**
  - A: strength of the argument (`strong` or `weak`)
  - B: involvement of students `low` (far away, in a long time) or `high` (next year, at their university)
  - C: style of argument, either `regular` form or `rhetorical` (Don't you think?, ...)

Interaction plot

An interaction plot for a $2 \times 2 \times 2$ factorial design from Petty, Cacioppo and Heesacker (1981)



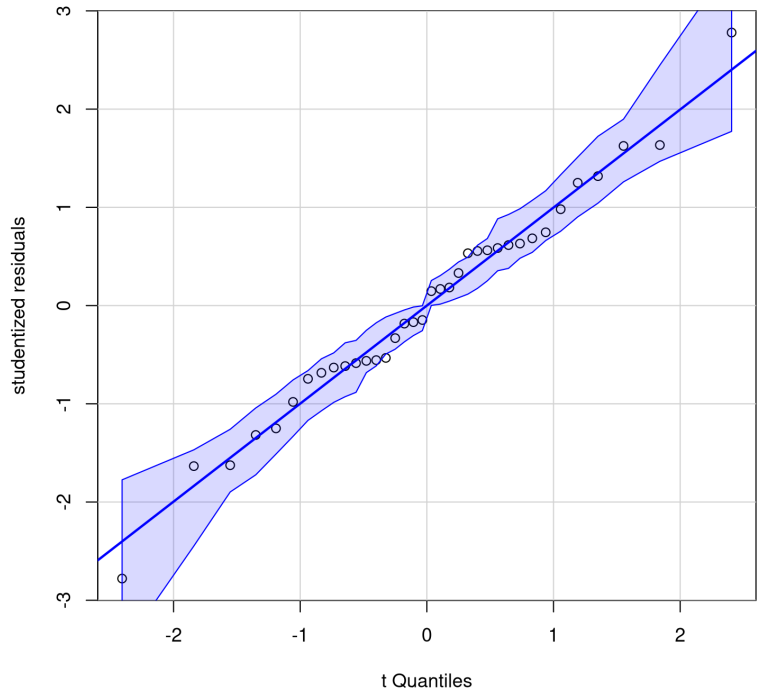
The microwave popcorn experiment

What is the best brand of microwave popcorn?

- **Factors**
  - brand (two national, one local)
  - power: 500W and 600W
  - time: 4, 4.5 and 5 minutes
- **Response**: ~~weight~~, ~~volume~~, ~~number~~, percentage of popped kernels.
  - Pilot study showed average of 70% overall popped kernels (10% standard dev), timing values reasonable
  - Power calculation suggested at least $r = 4$ replicates, but researchers proceeded with $r = 2$...

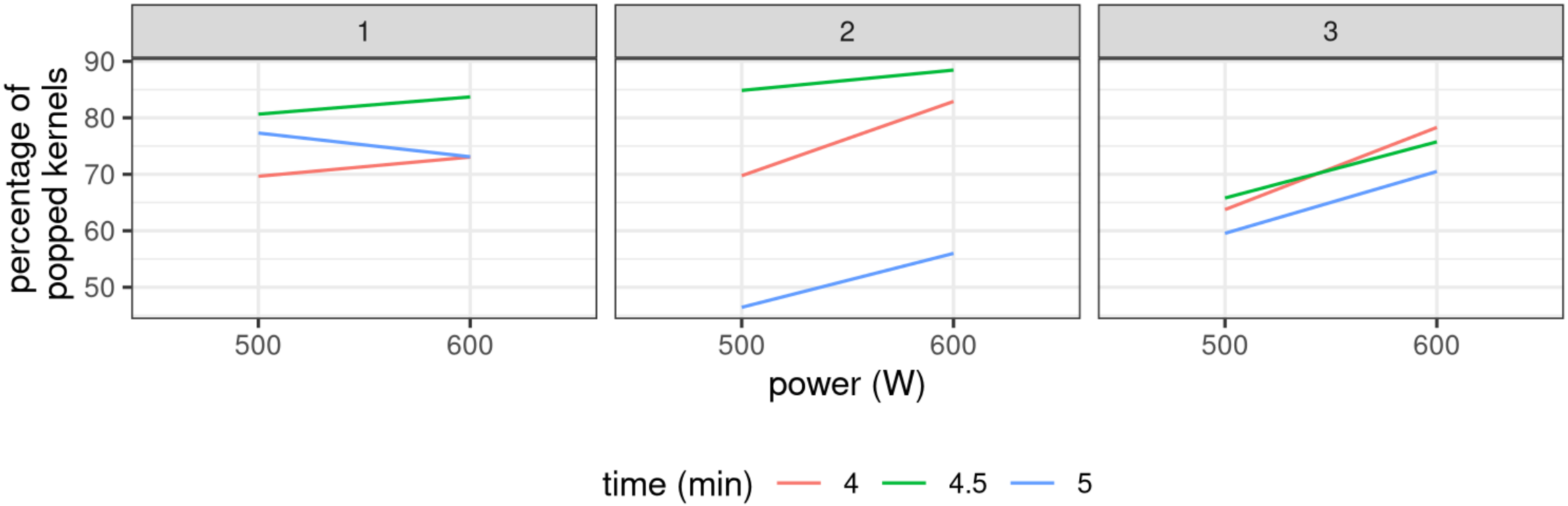
```r
library(tidyverse)
url <- "https://edsm.rbind.io/data/popcorn.txt"
popcorn <- read.table(url,
                      header = TRUE,
                      stringsAsFactor = TRUE) %>%
   mutate(brand = factor(brand),
      power = factor(power, labels = c("500","600")),
      time = factor(time, labels = c("4","4.5","5")))
```

```r
# Sum-to-zero parametrization
options(contrast = "contr.sum")
# Fit model with three-way interaction
model <- lm(percentage ~ brand*power*time, data = popcorn)
# ANOVA table
anova_table <- anova(model) # 'anova' is for balanced designs
# Check model assumptions
# plots + tests meaningless with r=2 replications per group...
# except for quantile-quantile plot
car::qqPlot(model, # points should be on straight line!
        id = FALSE,
        ylab = 'studentized residuals')
```



All points fall roughly on a straight line.

```
popcorn %>%
 group_by(brand, time, power) %>%
 summarize(mean_percentage = mean(percentage)) %>%
 ggplot(aes(x = power,
            y = mean_percentage,
            col = time,
            group = time)) +
    geom_line() +
    facet_wrap(~brand) +
    labs(y = "percentage of\n popped kernels",
         col = "time (min)",
         x = "power (W)") +
    theme_bw() +
    theme(legend.position = "bottom")
```



No three-way interaction (hard to tell with $r = 2$ replications).

Analysis of variance table

terms	degrees of freedom	sum of squares
A	$a-1$	$[A]-[T]$
B	$b-1$	$[B]-[T]$
C	$c-1$	$[C]-[T]$
AB	$(a-1)(b-1)$	$[AB]-[A]-[B]+[T]$
AC	$(a-1)(c-1)$	$[AC]-[A]-[C]+[T]$
BC	$(b-1)(c-1)$	$[BC]-[B]-[C]+[T]$
ABC	$(a-1)(b-1)(c-1)$	$[ABC]-[AB]-[AC]-[BC]+[A]+[B]+[C]-[T]$
residual	$abc(r-1)$	$[Y]-[ABC]$
total	$abcr-1$	$[Y]-[T]$

Analysis of variance table for microwave-popcorn

	Degrees of freedom	Sum of squares	Mean square	F statistic	p-value
brand	2	331.10	165.55	1.89	0.180
power	1	455.11	455.11	5.19	0.035
time	2	1554.58	777.29	8.87	0.002
brand:power	2	196.04	98.02	1.12	0.349
brand:time	4	1433.86	358.46	4.09	0.016
power:time	2	47.71	23.85	0.27	0.765
brand:power:time	4	47.33	11.83	0.13	0.967
Residuals	18	1577.87	87.66		

Omitting terms in a factorial design

The more levels and factors, the more parameters to estimate (and replications needed)

- Costly to get enough observations / power
- The assumption of normality becomes more critical when $r = 2$!

It may be useful not to consider some interactions if they are known or (strongly) suspected not to be present

- If important interactions are omitted from the model, biased estimates/output!

Latin square

A latin square is a blocked design with one treatment with T levels and two blocking factors with also T levels each.

Experiment arranged so that treatment is assigned once to each row/column. Letters correspond to the different treatments:

	col 1	col 2	col 3	col 4
row 1	A	B	C	D
row 2	B	C	D	A
row 3	C	D	A	B
row 4	D	A	B	C

Example of a 4×4 latin square

The model takes the form

$$\underset{\text{response}}{Y_{ijk}} = \mu + \underset{\text{treatment}}{\alpha_i} + \underset{\text{blocking (row)}}{\beta_j} + \underset{\text{blocking (col)}}{\gamma_k} + \underset{\text{error}}{\varepsilon_{ijk}}$$

There are $1 + 3 \times (K - 1)$ mean parameters to estimate, and **no interaction**.

Only look at treatment effect α_i.

Custom contrasts and marginal means

Guidelines for the interpretation of effects

Start with the most complicated term (top down)

- If the three-way interaction $_{ABC}$ is significative:
  - don't interpret main effects or two-way interactions!
  - comparison is done cell by cell within each level
- If the $_{ABC}$ term isn't significative:
  - can marginalize and interpret lower order terms

Analytical comparisons

- Preplanned
- *Post-hoc*: after seeing that the three-way interaction isn't significative, compare all pairwise differences within two-way.

What contrasts are of interest?

- Can view a three-way ANOVA as a series of one-way ANOVA or two-way ANOVAs...

Depending on the goal, could compare for variable A

- marginal contrast ψ_A (averaging over B and C)
- marginal conditional contrast for particular subgroup: ψ_A within c_1
- contrast involving two variables: ψ_{AB}
- contrast differences between treatment at $\psi_A \times B$, averaging over C.
- etc.

See helper code and chapter 22 of Keppel & Wickens (2004) for a detailed example.

Effects and contrasts for microwave-popcorn

Following preplanned comparisons

- Which combo (brand, power, time) gives highest popping rate? (pairwise comparisons of all combos)
- Best brand overall (marginal means marginalizing over power and time, assuming no interaction)
- Effect of time and power on percentage of popped kernels
  - pairwise comparison of time $\times$ power
  - main effect of power
  - main effect of time

Preplanned comparisons using `emmeans`

Let $_A$=brand, $_B$=power, $_C$=time

Compare difference between percentage of popped kernels for 4.5 versus 5 minutes, for brands 1 and 2

$$\mu_{1.2} - \mu_{1.3} - \mu_{2.2} - \mu_{2.3}$$

```
library(emmeans)
# marginal means
emm_popcorn_AC <- emmeans(model, specs = c("brand","time"))
contrast_list <- list(brand12with4.5vs5min = c(0, 0, 0, 1, -1, 0, -1, 1,0))
contrast(emm_popcorn_AC,  # marginal mean (no time)
      method = contrast_list) # list of contrasts
```

Preplanned comparisons

At level 99% with Tukey's method, compare all three times (4, 4.5 and 5 minutes)

- Careful! Potentially misleading because there is a `brand * time` interaction present.

```
# List of variables to keep go in `specs`: keep only time
emm_popcorn_C <- emmeans(model, specs = "time")
pairs(emm_popcorn_C, adjust = "tukey", level = 0.99, infer = TRUE)
```

Semipartial effect sizes

Normally, partial effect sizes are of the form

- if only experimental factors, $\sigma^2_{\text{effect}}/(\sigma^2_{\text{effect}} + \sigma^2_{\text{res}})$ for the given effect
- if only blocking factors, use total effect size $\sigma^2_{\text{effect}}/\sigma^2_{\text{total}}$
- if a mix, include variance of all blocking factors and interactions (only with effect!) in denominator.

If A is effect of interest, B is a blocking factor and C is another experimental factor, use

$$\text{semipartial } \omega^2_A = \frac{\sigma^2_A}{\sigma^2_A + \sigma^2_B + \sigma^2_{AB} + \sigma^2_{\text{res}}}.$$

Power for complex studies

One can estimate (semi)partial effects and proceed as if you had a one-way ANOVA by specifying the missing ingredients.

- In more general settings, it may be easier to simulate from the fitted model directly and calculate the power as you change the number of observations per group.