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Math Methods for Political Science

Lecture 5: Eigenvectors and Eigenvalues

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Outline

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Definition 1 (Eigenvectors and eigenvalues)

Let A be an $n \times n$ matrix.

- $\mathbf{x} \in \mathbb{R}^n$ is an **eigenvector** if $\mathbf{x} \neq 0$ and $\exists \lambda \in \mathbb{R}$ s.t. $A\mathbf{x} = \lambda \mathbf{x}$.
- $\lambda \in \mathbb{R}$ is an **eigenvalue** if $\exists \mathbf{x} \in \mathbb{R}^n$ s.t. $\mathbf{x} \neq 0$ and $A\mathbf{x} = \lambda \mathbf{x}$.

Example 1 (Eigenvectors and eigenvalues)

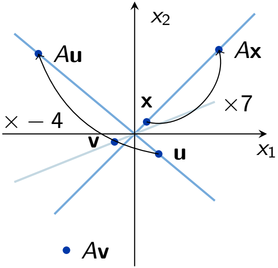
If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, then $\begin{cases} \mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} & \implies A\mathbf{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\mathbf{u} \\ \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} & \implies A\mathbf{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \mathbf{v} \end{cases}$.

Example 2 (Eigenvectors and eigenvalues cont'd)

If $\lambda = 7$, then $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - 7I)\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \mathbf{x} = \mathbf{0}$.

To find $\mathbf{x}$: $\begin{bmatrix} A - 7I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, $\operatorname{Nul}(A - 7I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \implies \mathbf{x} \in \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is an eigenvector corresponding to $\lambda = 7$.



Sets of eigenvectors

Let A be an $n \times n$ matrix.

Theorem 1 (Sets of eigenvectors)

If $\mathbf{v}_1, \cdots, \mathbf{v}_r$ *are eigenvectors corresponding to distinct eigenvalues* $\lambda_1, \cdots, \lambda_r$*, then the set* $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ *is linearly independent.*

Example 3 (Eigenvectors and eigenvalues)

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ correspond to eigenvalues -4 and 7. Hence, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \iff c_1 = 0,\ c_2 = 0$.

Triangular matrices

Let A be an $n \times n$ matrix.

Theorem 2 (Eigenvalues of triangular matrices)

If A is triangular, its eigenvalues are the entries on its diagonal.

Example 4 (Eigenvalues of triangular matrices)

- The only eigenvalue of I is 1.
- The eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $A^\top = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ are 1 and 3.

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The characteristic equation

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$.

Definition 2 (Eigenspace)

The **eigenspace** of A corresponding to λ is $\mathrm{Nul}\,(A - \lambda I)$.

Remark: the eigenspace is a subspace of $\mathbb{R}^n$.

Theorem 3 (The characteristic equation)

The following statements are equivalent:

1. *λ is an eigenvalue of A.*
2. $\exists \mathbf{x} \neq \mathbf{0}$ *s.t.* $(A - \lambda I)\mathbf{x} = \mathbf{0}$
3. $\dim Nul\,(A - \lambda I) > 0$.
4. $\det(A - \lambda I) = 0$ **(the ch.eq.)**

Example 5 (The characteristic equation)

If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, then $A - \lambda I = \begin{bmatrix} 1 - \lambda & 6 \\ 5 & 2 - \lambda \end{bmatrix}$.
Hence, $\det(A - \lambda I) = (2 - \lambda)(1 - \lambda) - 30 = \lambda^2 - 3\lambda - 28$.
Finally, $\det(A - \lambda I) = 0 \iff \lambda \in \{7, -4\}$

Invertible matrices cont'd

Theorem 4 (Characterization of the matrix inverse cont'd)

Let A be a $n \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, then the following statements are equivalent:

1. *A is invertible.*
2. *A is r.e. to I_n.*
3. *A has n pivot positions.*
4. *The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
5. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.*
6. *$Span\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \mathbb{R}^n$.*
7. *$\exists C$ $n \times n$ s.t. $CA = I_n$.*
8. *$\exists B$ $n \times n$ s.t. $AB = I_n$.*
9. *$A^\top$ is invertible.*
10. *$\mathbf{a}_1, \cdots, \mathbf{a}_n$ form a basis of $\mathbb{R}^n$.*
11. *$Col\,A = \mathbb{R}^n$.*
12. $\dim Col\,A = n$.
13. *$Rank\,A = n$.*
14. *$Nul\,A = \{\mathbf{0}\}$.*
15. $\dim Nul\,A = 0$.

Furthermore:

16. $\det A \neq 0$.
17. *0 is not an eigenvalue of A.*

The characteristic polynomial

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Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$.

Definition 3 (The characteristic polynomial)

- $\det(A - \lambda I)$ is the **characteristic polynomial** of A.
- The (algebraic) **multiplicity** of an eigenvalue λ, $\text{Mult}\,(A, \lambda)$, is its multiplicity as a root of the characteristic equation.

Example 6 (The characteristic polynomial)

Let $A = \begin{bmatrix} 5 & -2 & 6 \\ 0 & 3 & -8 \\ 0 & 0 & 5 \end{bmatrix}$, then $A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 \\ 0 & 3-\lambda & -8 \\ 0 & 0 & 5-\lambda \end{bmatrix}$.
Hence, $\det(A - \lambda I) = (5-\lambda)^2(3-\lambda)$ and $\det(A - \lambda I) = 0$ implies $\lambda \in \{5, 3\}$ with $\text{Mult}\,(A, 5) = 2$ and $\text{Mult}\,(A, 3) = 1$.

Theorem 5 (Eigenspace dimension and multiplicity)

If λ is an eigenvalue, then $\dim Nul(A - \lambda I) \leq Mult(A, \lambda)$.

Outline

Similarity

Let A and B be $n \times n$ matrices.

Definition 4 (Similarity)

If $\exists P$ invertible s.t. $P^{-1}AP = B$, then A is **similar** to B, $A \sim B$.

Example 7 (Similarity)

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Then $P^{-1} = P$, $PAP = B \implies A \sim B$, $PBP = A \implies B \sim A$.

Theorem 6 (Similarity)

1. *$A \sim A$,*
2. *$A \sim B \implies B \sim A$*
3. *If C is an $n \times n$ matrix, then $A \sim B$ and $B \sim C \implies A \sim C$*

If $A \sim B$, then

4. $\det A = \det B$,
5. *$\mathrm{Rank}\, A = \mathrm{Rank}\, B$,*
6. *A invertible $\iff$ B invertible,*
7. $\det(A - \lambda I) = \det(B - \lambda I)$.

Diagonalization

Let A be an $n \times n$ matrix.

Definition 5 (Diagonalizable matrix)

A is **diagonalizable** if it is similar to a diagonal matrix D.

Example 8 (Diagonalizable matrix)

If A is diagonalizable, then $\exists P$ s.t. $A = P^{-1}DP$, which implies that $A^2 = (P^{-1}DP)(P^{-1}DP) = P^{-1}D^2P$ and $A^k = P^{-1}D^kP$.

Theorem 7 (The diagonalization theorem)

A is diagonalizable $\iff$ A has n linearly independent eigenvectors.

To diagonalize A:

1. Find its eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.
2. Find the corresponding eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$.
3. Let $P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$
4. Let the diagonal entries of D be $\{\lambda_1, \cdots, \lambda_n\}$.

Diagonalization cont'd

Example 9 (Diagonalization)

Let $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

1. $\det A - \lambda I = -(\lambda - 1)(\lambda + 2)^2 \implies$ the eigenvalues are 1 and -2 .
2. $\text{Nul}\,(A - I) = \text{Span}\ \{\mathbf{v}_1\}$, $\text{Nul}\,(A + 2I) = \text{Span}\ \{\mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, A is diagonalizable.
3. $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
4. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Conditions for diagonalization

Let A be an $n \times n$ matrix.

Theorem 8 (Sufficient condition)

If the n eigenvalues are distinct, then A is diagonalizable.

Example 10 (Sufficient condition)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \text{eigenvalues are } \{1, 0, 5\} \Rightarrow \text{diagonalizable.}$$

Conditions for diagonalization cont'd

Let A be an $n \times n$ matrix, $\lambda_1, \cdots, \lambda_p$ be the distinct eigenvalues with $\mathcal{B}_1, \cdots, \mathcal{B}_p$ the bases of their corresponding eigenspaces.

Theorem 9 (Necessary and sufficient conditions)

The following statements are equivalent:

1. *A diagonalizable*
2. $\mid U_{k=1}^{p} \mathcal{B}_k \mid = n$
3. $\lambda_k \in \mathbb{R}$ *and* $\mid \mathcal{B}_k \mid = Mult(A, \lambda_k) \; \forall k$

Example 11 (Necessary and sufficient condition)

$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \implies \det(A - \lambda I) = (1 - \lambda)^2 \implies$ 1 is the only eigenvalue and $\text{Mult}\,(A, 1) = 2.$ Since $\text{Nul}\,(A - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, A is not diagonalizable.