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Math Methods for Political Science

Lecture 4: Vector Spaces

Thibault Vatter <tv2233@columbia.edu>
Department of Statistics, Columbia University

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Outline

Vector space

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Definition 1 (Vector space)

A **vector space** V is a nonempty set of elements, called **vectors**, on which are defined two operations called **addition** and **scalar multiplication** satisfying, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in \mathbb{R}$,

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\exists \mathbf{0} \in V$ s.t. $\mathbf{v} + \mathbf{0} = \mathbf{v}$
5. $\exists -\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
6. $c\mathbf{v} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$.

Remark: $\mathbf{0}$ and $-\mathbf{v}$ are unique, $0\mathbf{v} = \mathbf{0}$, $c\mathbf{0} = \mathbf{0}$, and $-\mathbf{v} = (-1)\mathbf{v}$.

Example 1 (Vector space)

- $\mathbb{R}^n$
- Doubly infinite sequences
- Polynomials of degree $\leq n$
- Real-valued functions on $\mathbb{R}$

Vector subspace

Definition 2 (Vector subspace)

A **subspace** of a vector space V is subset $H \subseteq V$ satisfying

1. $\mathbf{0} \in H$
2. $\mathbf{u} + \mathbf{v} \in H \; \forall \mathbf{u}, \mathbf{v} \in H$
3. $c\mathbf{v} \in H \; \forall \; \mathbf{u} \in H$ and $c \in \mathbb{R}$.

Remark: the definition implies that H is also a vector space.

Example 2 (Vector space)

- $H = \{\mathbf{0}\}$ (**zero subspace**)
- Polynomials form a subspace of the real-valued functions
- Polynomials of degree at most n form a subspace of the polynomials
- Since $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$.
- $H = \left\{ \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} \mid t, s \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

Subspace spanned by a set

Let V be a vector space.

Theorem 1 (Subspace spanned by a set)

If $\mathbf{v}_1, \cdots, \mathbf{v}_p \in V$, then Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is a subspace of V.

Definition 3 (Subspace spanned by a set and spanning set)

- Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the **subspace spanned** (or **generated by**) by $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.
- A **spanning** (or **generating**) set for a subspace H is a set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ s.t. Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} = H$.

Example 3 (Subspace spanned by a set and spanning set)

Let $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Notice that,

$$\mathbf{v} \in H \iff \exists a, b \in \mathbb{R} \text{ s.t. } \mathbf{v} = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}.$$

Hence $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \implies H$ is a subspace of $\mathbb{R}^4$.

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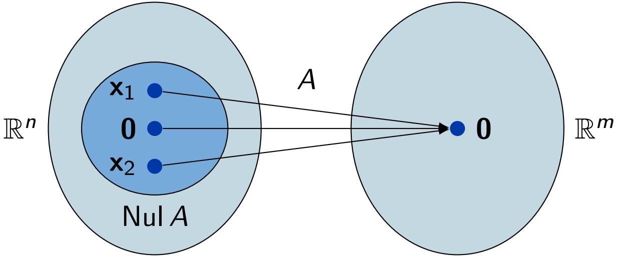
Null space

Definition 4 (Null space)

The **null space** of an $m \times n$ matrix A is
$\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$.



Example 4 (Null space)

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$, then $\mathbf{u} \in \operatorname{Nul} A$.

Null space cont'd

Theorem 2 (Null space)

If A is a $m \times n$ matrix, then $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

Remark: $\iff$ the solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Example 5 (Row reduction to characterize the null space)

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, then the RREF of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} \implies \begin{aligned} x_1 &= x_3 + 2x_3 \\ x_2 &= -2x_3 - 3x_4 \end{aligned}$$

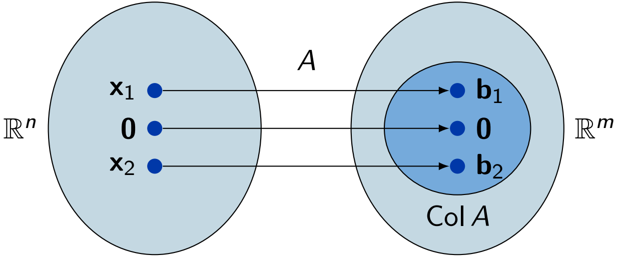
$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

$$\implies \mathrm{Nul}\, A = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$$

Column space

Definition 5 (column space)

The **column space** of an $m \times n$ matrix A with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$ is $\text{Col}\, A = \{\mathbf{b} \in \mathbb{R}^m \mid \exists \mathbf{x} \in \mathbb{R}^n \text{ with } A\mathbf{x} = \mathbf{b}\} = \text{Span}\, \{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$.



Theorem 3 (Column space)

- *If A is a $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$.*
- *Col $A = \mathbb{R}^m \iff A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$.*

Null space and column space

Let A be a $m \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$.

Nul A:

1. a subspace of $\mathbb{R}^n$,
2. implicitly characterized by $A\mathbf{x} = \mathbf{0}$ & needs the RREF of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to find its vectors,
3. $\mathbf{v} \in \operatorname{Nul} A \iff A\mathbf{v} = \mathbf{0}$,
4. easy to know if $\mathbf{v} \in \mathbb{R}^n$ is in it by computing $A\mathbf{v}$,
5. $\operatorname{Nul} A = \{\mathbf{0}\} \iff \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.

Col A:

1. a subspace of $\mathbb{R}^m$,
2. explicitly defined by Span $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ & does not need intermediate steps,
3. $\mathbf{v} \in \operatorname{Col} A \Leftrightarrow \exists \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{v}$,
4. to know if $\mathbf{v} \in \mathbb{R}^m$ is in it, needs the RREF of $\begin{bmatrix} A & \mathbf{v} \end{bmatrix}$,
5. $\operatorname{Col} A = \mathbb{R}^m \iff A\mathbf{x} = \mathbf{b}$ has a solution $\forall \mathbf{b} \in \mathbb{R}^m$.

Outline

Linear independence

Let V be a vector space and $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ a set of vectors.

Definition 6 (Linear independence)

S is **linearly independent** $\sum_{i=1}^{p} c_i \mathbf{v}_i = \mathbf{0} \iff c_i = 0 \, \forall i$.

Theorem 4

For $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, S is linearly dependent $\iff \exists j > 1$ s.t. $\mathbf{v}_j \in \mathit{Span}\,\{\mathbf{v}_1, \cdots, \mathbf{v}_{j-1}\}$.

Example 6

- $V = \mathbb{R}^n$ and $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_p \end{bmatrix} \implies S$ linearly independent $\iff \operatorname{Nul} A = \{\mathbf{0}\} \iff \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.
- Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$, $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent because $\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$.
- $S = \{\cos(t), \sin(t)\}$ is linearly independent in $V = C[0,1]$ because there exists no $c \in \mathbb{R}$ s.t. $\cos(t) = c\sin(t) \; \forall t \in [0,1]$.

Basis

Definition 7 (Basis)

Let H be a subspace of a vector space V. A set of vectors $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ is a **basis** for H if

1. $\mathcal{B}$ is linearly independent,
2. $H = \text{Span}\,\{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$.

Example 7

- The columns of an $n \times n$ invertible matrix form a basis of $\mathbb{R}^n$.
- The basis $\mathcal{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ formed with the columns of I_n is called the **standard basis** for $\mathbb{R}^n$.
- $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ is a basis of $\mathbb{R}^3$.

The spanning set theorem

Theorem 5 (The spanning set theorem)

Let $S = \{\mathbf{v}_1, \cdots, \mathbf{b}_p\}$ be a set in V and $H = \mathit{Span}\, \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

1. *If $\exists k$ s.t. $\mathbf{v}_k = \sum_{j \neq k} c_j \mathbf{v}_j$, then $H = \mathit{Span}\, \{\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \cdots, \mathbf{v}_p\}$.*
2. *If $H \neq \{\mathbf{0}\}$, then $\exists R \subseteq S$ s.t. R is a basis for H.*

Corollary 1 (A basis for Col A)

The pivot columns of a matrix A form a basis for Col A.

The spanning set theorem cont'd

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Example 8 (The spanning set theorem)

If $B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$, then

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

form respectively bases for $\operatorname{Col} B$ and $\operatorname{Col} A$ (the later because B is the RREF of A).

Outline

Coordinates

Let V be a vector space, $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a basis and $\mathbf{x} \in V$.

Definition 8 (Coordinates)

- The **coordinates of x relative to $\mathcal{B}$** (or **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \cdots, c_n$ s.t. $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{b}_i$.
- If $V = \mathbb{R}^n$ and $c_1, \cdots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then

  $$\left[\mathbf{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$ is the **coordinate vector of x relative to $\mathcal{B}$**

  (or the **$\mathcal{B}$-coordinate vector of x**), and $\mathbf{x} = \left[\mathbf{x}\right]_{\mathcal{E}}$.

Theorem 6 (The unique representation theorem)

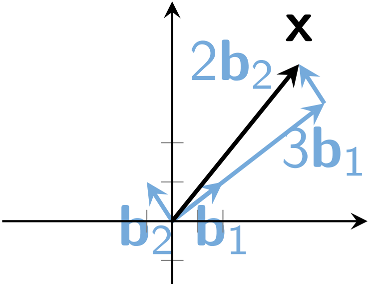
The $\mathcal{B}$-coordinates of $\mathbf{x}$ are unique.

Coordinates cont'd

Example 9 (Coordinates in $\mathbb{R}^2$)

$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ lin. ind. $\implies \mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$.

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

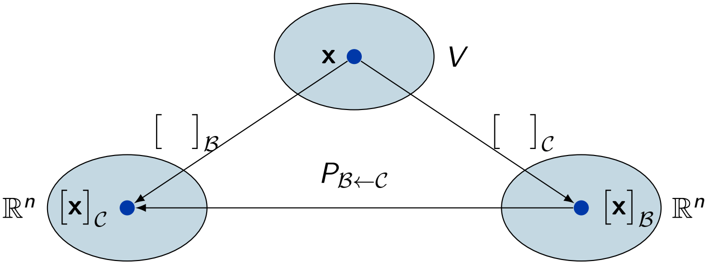


Theorem 7 (Change of basis)

Let $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \cdots, \mathbf{c}_n\}$ be two bases of a vector space V, then there exists a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t.

1. $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$,
2. $P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$,
3. *the RREF of* $\begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$ *is* $\begin{bmatrix} I_n & P_{\mathcal{C} \leftarrow \mathcal{B}} \end{bmatrix}$.

Remark: $P_{\mathcal{B} \leftarrow \mathcal{C}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} \implies [\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{x}]_{\mathcal{C}} = P^{-1}_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{C}}$.



Change of basis cont'd

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Example 10 (Change of basis)

Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, then

$$\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

$$\implies \left[\mathbf{b}_1\right]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \left[\mathbf{b}_2\right]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$$\implies P_{\mathcal{B}\leftarrow\mathcal{C}} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}.$$

Outline

Dimension of a vector space

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Let V be a vector space and $\mathcal{B}$ a basis for V.

Theorem 8

If $\mid \mathcal{B} \mid = n$, then any set S with $\mid S \mid > n$ is linearly dependent.

Corollary 2

If $\mathcal{B}_2$ is another basis for V, then $\mid \mathcal{B}_2 \mid = \mid \mathcal{B} \mid$.

Definition 9 (Dimension of a vector space)

The **dimension** of V is $\dim V = \mid \mathcal{B} \mid$. If $\dim V < \infty$, then V is **finite-dimensional**. Otherwise, it is **infinite-dimensional**.

Remark: by convention, if $V = \{\mathbf{0}\}$, then $\dim V = 0$.

Example 11 (Dimension of a vector space)

- If $V = \mathbb{R}^n$, then $\mid \mathcal{E} \mid = n \implies \dim V = n$.
- If $H = \text{Span}\,\{\mathbf{e}_1, \mathbf{e}_2\}$ (subspace of $V = \mathbb{R}^3$), then $\dim H = 2$.

The basis theorem

Let V be a vector space with $\dim V = n < \infty$.

Theorem 9 (Subspaces of finite-dimensional spaces)

Let H be a subspace of V, then

1. *in H, linearly independent sets can be extended into bases,*
2. *and* $\dim H \leq \dim V$.

Theorem 10 (The basis theorem)

Let S be a set of n vectors and $n \geq 1$. If S is linearly independent (or equivalently if $\operatorname{Span} S = V$*), then S is a basis for V.*

Example 12 (The basis theorem)

Let A be an $m \times n$ matrix.

- Pivot columns = basis of $\operatorname{Col} A \implies \dim \operatorname{Col} A = \#$ of pivots.
- $\dim \operatorname{Nul} A = \#$ of free variables in $A\mathbf{x} = \mathbf{0}$.

Outline

The row space

Let A be an $m \times n$ matrix.

Definition 10 (The row space)

Let $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{R}^n$ be the rows of A, namely

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix},$$

then **the row space** of A is $\operatorname{Row} A = \operatorname{Span} \{\mathbf{a}_1, \cdots, \mathbf{a}_m\}$.

Theorem 11 (Row equivalence and the row space)

Let B is another $m \times n$ matrix, then

1. *A is r.e. to B $\iff$ $\operatorname{Row} A = \operatorname{Row} B$,*
2. *if B is in REF, its nonzero rows form a basis for $\operatorname{Row} A / \operatorname{Row} B$.*

The row space cont'd

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Example 13 (The row space)

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\implies \text{Row } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}, \text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

The rank

Let A be an $m \times n$ matrix.

Definition 11 (The rank)

The **rank** of a matrix A is $\mathrm{Rank}\, A = \dim \mathrm{Col}\, A$.

Theorem 12 (The rank theorem)

$\dim \mathit{Row}\, A = \mathit{Rank}\, A$ *and* $\mathit{Rank}\, A + \dim \mathit{Nul}\, A = n$.

Example 14 (The rank theorem)

- If A is a 7×9 matrix with $\dim \mathrm{Nul}\, A = 2$, then $\mathrm{Rank}\, A = 7$.
- If A is a 6×9 matrix, then $\dim \mathrm{Nul}\, A \neq 2$.

Rank of invertible matrices

Theorem 13 (Characterization of the matrix inverse cont'd)

Let A be a $n \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, then the following statements are equivalent:

1. *A is invertible.*
2. *A is r.e. to I_n.*
3. *A has n pivot positions.*
4. *The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.*
5. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.*
6. *$Span\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \mathbb{R}^n$.*
7. *There exists C $n \times n$ s.t. $CA = I_n$.*
8. *There exists B $n \times n$ s.t. $AB = I_n$.*
9. *$A^\top$ is invertible.*

Furthermore:

10. *$\mathbf{a}_1, \cdots, \mathbf{a}_n$ form a basis of $\mathbb{R}^n$.*
11. *$Col\, A = \mathbb{R}^n$.*
12. $\dim Col\, A = n$.
13. *$Rank\, A = n$.*
14. *$Nul\, A = \{\mathbf{0}\}$.*
15. $\dim Nul\, A = 0$.