Math Methods for Political Science

Lecture 9: Differentiability of univariate functions

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Differentiable functions

Definition 1 (Differentiable functions)

A function f is **differentiable** at $x_0 \in D_f$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists. In this case, $f'(x_0)$ is the **derivative of f at x_0**. For a set $S \subseteq D_f$:

- f is **differentiable on S** if f is differentiable at x_0 $\forall x_0 \in S$.
- f is **continuously differentiable on S** if f is differentiable on S and f' is continous on S.

Remark: writing $x = x_0 + h \implies f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

Example 1 (Differentiable functions)

- $f(x) = 1/x \implies f'(x) = -1/x^2$
- $f(x) = e^x \implies f'(x) = e^x$

Rate of change interpretation

- If $f(x)$ is the position of a particle at time $x \neq x_0$, then

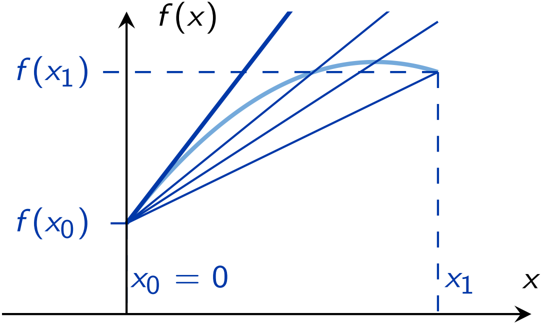
$$\frac{f(x) - f(x_0)}{x - x_0}$$

is the average velocity of the particle between times x_0 and x.

- As $x \to x_0$, one averages over shorter and shorter intervals $\implies$ the limit (if it exists) $\equiv$ the instantaneous velocity at x_0.
- When f is not the position of a particle $\implies$ the limit $\equiv$ **instantaneous rate of change of f at x_0**.

Geometric interpretation

Let $g_{x_0,x_1}(x) = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$.



Then $g_{x_0,x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$, i.e. $f'(x_0)$ is **the tangent of f** at x_0.

Higher-order derivatives

Definition 2 (Higher-order derivatives)

- If f' is differentiable at x_0, then $f''(x_0)$ is the **second derivative of f at x_0**.
- Notations: $f^{(0)} = f$, $f^{(1)} \equiv f'$, $f^{(2)} \equiv f''$.
- If $f^{(n-1)}$ is differentiable at x_0, the **n^{th} derivative of f at x_0**, denoted by $f^{(n)}(x_0)$ is the derivative of $f^{(n-1)}$ at x_0.

Example 2 (Higher-order derivatives)

- $f(x) = e^x \implies f^{(n)}(x) = e^x$
- $f(x) = x^m \implies f'(x) = mx^{m-1}$

$$\implies f^{(n)} = \begin{cases} 0 & \text{if } n > m \\ x^{m-n} \prod_{j=0}^{n-1}(n-j) & \text{otherwise} \end{cases}$$

Differentiability and continuity

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Theorem 1 (Differentiability implies continuity)

If f is differentiable at x_0, then f is continuous at x_0.

Example 3 (Continuity does not implies differentiability)

Let $f(x) = \mid x \mid$, then

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0$$
$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} -x = 0,$$

$\implies$ f is continuous at 0. However,

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x}{x} = 1$$
$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

$\implies$ f is not differentiable at 0.

Differentiation & arithmetic operations

Theorem 2 (Differentiation and arithmetic operations)

If f and g are differentiable at x_0, then so are $f+g$, $f-g$ and fg, with

$$\begin{aligned} (f+g)'(x_0) &= f'(x_0) + g'(x_0), \\ (f-g)'(x_0) &= f'(x_0) - g'(x_0), \\ \text{and } (fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0), \end{aligned}$$

and f/g is differentiable if $g(x_0) \neq 0$, with

$$(f/g)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Example 4 (Differentiation and arithmetic operations)

If $f(x) = \sin(x)$ and $g(x) = 1/x$ for $x \neq 0$, then $f'(x) = \cos(x)$ and $g'(x) = -1/x^2$ for $x \neq 0$, which implies

$$(f/g)'(x) = \frac{\cos(x)/x + \sin(x)/x^2}{(1/x)^2} = \cos(x)x + \sin(x)$$

Composite functions

Let f and g be functions s.t. $\exists T \subseteq D_g$ with $g(x) \in D_f \; \forall x \in D_h$.

Definition 3 (Composite function)

The **composite function** $f \circ g$ is defined on T by $(f \circ g)(x) = f(g(x))$.

Theorem 3 (Continuity of the composite function)

If g is continuous at x_0 and f at $g(x_0)$, then so is $f \circ g$ at x_0.

Theorem 4 (The chain rule)

If g is differentiable at x_0 and f at $g(x_0)$, then so is $f \circ g$ at x_0, with $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$.

Example 5 (Composite functions and the chain rule)

If $f(x) = \sin(x)$ and $g(x) = 1/x$ for $x \neq 0$, then $h(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, which implies $(f \circ g)'(x) = \cos\left(\frac{1}{x}\right)\frac{-1}{x^2}$.

Local extrema

Definition 4 (Local extrema)

If $\exists \delta > 0$ s.t., $\forall x \in (x_0 - \delta, x_0 + \delta) \cap D_f$,

1. $f(x) \geq f(x_0)$, then x_0 is a **local minimum**,
2. $f(x) \leq f(x_0)$, then x_0 is a **local maximum**.

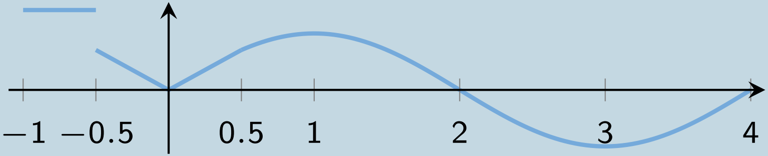
If 1. or 2. is true, then x_0 is a **local extremum**.

Example 6 (Local extrema)

Let

$$f(x) = \begin{cases} 1, & -1 < x \leq -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2} \sin \frac{\pi x}{2}}, & \frac{1}{2} < x \leq 4 \end{cases}$$

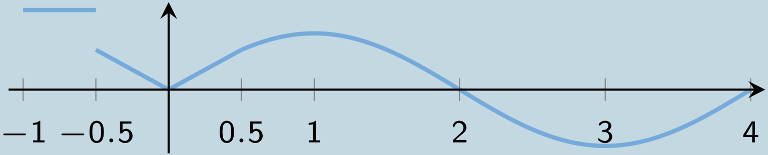
Example 6 (Local extrema cont'd)



$$f(x) = \begin{cases} 1, & -1 < x \leq -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}\sin\frac{\pi x}{2}}, & \frac{1}{2} < x \leq 4 \end{cases}$$

Local extrema cont'd

Example 6 (Local extrema cont'd)

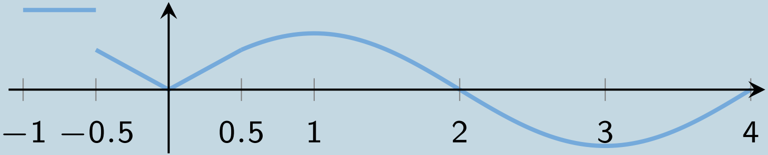


$$f(x) = \begin{cases} 1, & -1 < x \leq -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}\sin\frac{\pi x}{2}}, & \frac{1}{2} < x \leq 4 \end{cases}$$

- 0, 3, and every x in $(-1, -\frac{1}{2})$ are local minimum points.

Local extrema cont'd

Example 6 (Local extrema cont'd)



$$f(x) = \begin{cases} 1, & -1 < x \leq -\frac{1}{2} \\ |x|, & -\frac{1}{2} < x \leq \frac{1}{2}, \\ \frac{1}{\sqrt{2}\sin\frac{\pi x}{2}}, & \frac{1}{2} < x \leq 4 \end{cases}$$

- 0, 3, and every x in $(-1, -\frac{1}{2})$ are local minimum points.
- 1, 4, and every x in $(-1, -\frac{1}{2}]$ are local maximum points.

Differentiability at local extrema

Definition 5 (Stationary point)

f is differentiable at x_0 and $f'(x_0) = 0 \Rightarrow x_0$ is a **stationary point**.

Theorem 5 (Differentiability at local extrema)

1. *If f is differentiable at a local extremum x_0, then $f'(x_0) = 0$.*
2. *If f is twice differentiable at a local extremum x_0, then*
   - $f''(x_0) > 0 \implies x_0$ *is a local minimum.*
   - $f''(x_0) < 0 \implies x_0$ *is a local maximum.*

Example 7 (Differentiability at local extrema)

- $f(x) = x^2 \implies f'(x) = 2x$ & $f''(x) = 2 \implies f'(0) = 0$ & $f''(0) > 0 \implies 0$ is a local minimum.
- $f(x) = x^3 \implies f'(x) = 3x^2$ & $f''(x) = 6x \implies f'(0) = 0$ & $f''(0) = 0 \implies 0$ is only a stationary point.

Boundary and interior

Definition 6 (Boundary and interior)

Let S be a set of real numbers.

- x_0 is
  - a **boundary point** of $\forall \epsilon > 0$ $(x_0 - \epsilon, x_0 + \epsilon)$ contains at least one point in S and one not in S,
  - an **interior point** is $\exists \epsilon > 0$ s.t. $(x_0 - \epsilon, x_0 + \epsilon) \subset S$.
- The **boundary** of S, ∂S, is the set of boundary points.
- The **interior** of S, $S^0 = S \setminus \partial S$ is the set of interior points.
- The **closure** of S is $\bar{S} = S \cup \partial S$.

Example 8 (Boundary point and boundary)

Let $S_1 = [a, b]$, $S_2 = (a, b)$, and $S_3 = \mathbb{R}$, then:

- $\partial S_1 = \{a, b\}$, $\partial S_2 = \{a, b\}$ and $\partial S_3 = \emptyset$.
- $S_1^0 = (a, b)$, $S_2^0 = (a, b)$, and $S_3^0 = \mathbb{R}$.
- $\bar{S}_1 = [a, b]$, $\bar{S}_2 = [a, b]$, and $\bar{S}_3 = \mathbb{R}$.

Finding local extrema

Assume that f is twice continuously differentiable on D_f^0.

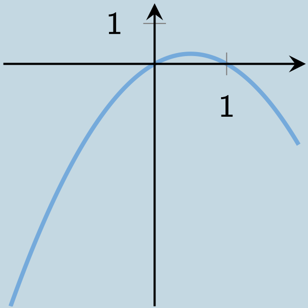
1. $\forall x \in \partial D_f$, check whether f decreases/increases if you move slightly to the interior to determine whether it is a local maximum/minimum.
2. Let $S_f = \{x \mid x \in D_f \text{ and } f'(x) = 0\}$, and compute $f''(x)$ $\forall x \in S_f$.
   2.1 If $f''(x) > 0$, then x is a local minimum.
   2.2 If $f''(x) < 0$, then x is a local maximum.
   2.3 If $f''(x) = 0$, then x is a stationary point.

Example 9 (Finding local extrema)

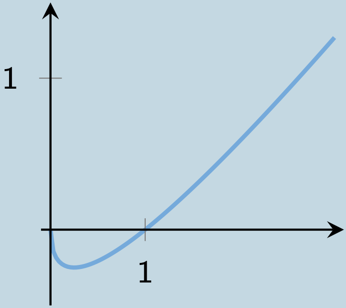
- $f(x) = x - x^2$
- $f(x) = x - \sqrt{x}$

Example 9 (Finding local extrema cont'd)

$x - x^2$



$x - \sqrt{x}$



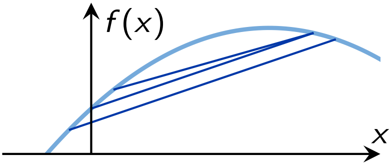
Concavity and convexity

Definition 7 (Concavity and convexity)

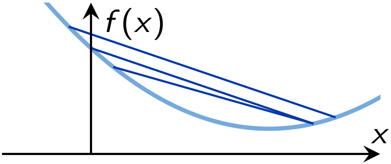
Let $S \subseteq D_f$. If $\forall x, y \in S$ and $\lambda \in [0,1]$,

- $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$, then f is **concave**,
- $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, then f is **convex**.

Concave function



Convex function



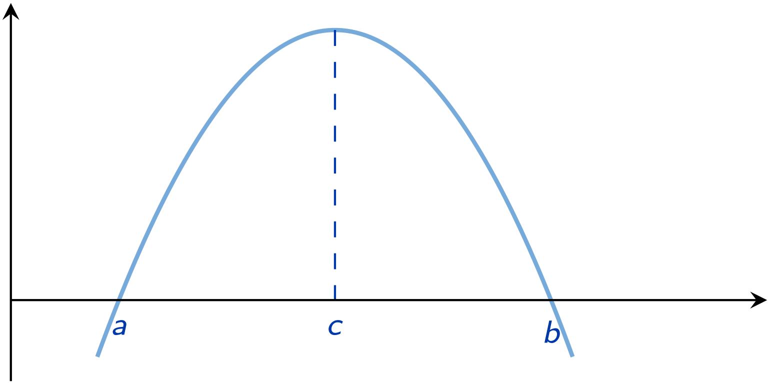
Theorem 6 (Concavity and convexity)

If f is twice continuously differentiable on S, then $f''(x) \lesseqgtr 0$ $\forall x \in S \implies f$ is concave/convex on S.

Rolle's theorem

Theorem 7 (Rolle's theorem)

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.



Intermediate values of theorems

Theorem 8 (Intermediate values of functions)

If f is continuous on $[a, b]$, $f(a) \neq f(b)$, and μ is between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = \mu$.

Theorem 9 (Intermediate values of derivatives)

If f is differentiable on $[a, b]$, $f'(a) \neq f'(b)$, and μ is between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = \mu$.

Mean value theorem

Theorem 10 (Mean value theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary 1

If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

Corollary 2

If f' exists and does not change sign on (a, b), then f is monotonic on (a, b) : increasing, nondecreasing, decreasing, or nonincreasing as $f'(x) > 0$, $f'(x) \geq 0$, $f'(x) < 0$, or $f'(x) \leq 0$, respectively $\forall x \in (a, b)$.

L'Hospital's rule

Theorem 11 (L'Hospital's rule)

Suppose

1. $\exists \epsilon > 0$ *s.t.* f *and* g *are differentiable on* $(b-\epsilon, b)$,
2. $\exists \epsilon > 0$ *s.t.* g' *has no zero on* $(b-\epsilon, b)$,
3. $\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0$ *(0/0* **form***) or*
   $\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = \pm\infty$ *(*∞/∞ **form***).*

If $\lim_{x \to b-} f'(x)/g'(x) = L$ *(with* L *finite or* $\pm\infty$*), then* $\lim_{x \to b-} f(x)/g(x) = L$*.*

Remark: similar rule for $\lim_{x \to b+}$ & $\lim_{x \to \pm\infty}$.

Example 10 (L'Hospital's rule)

- $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$ (0/0 form).
- $\lim_{x \to -\infty} \frac{e^{-x}}{x} = \lim_{x \to -\infty} \frac{-e^{-x}}{1} = -\infty$ (∞/∞ form).