Columbia University
In the City of New York

Math Methods for Political Science

Lecture 16: Lagrange's method

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Method of Substitution

Consider:

$$\begin{aligned} \underset{x,y}{\text{maximize}} \quad & x^2 + y^2 \\ \text{subject to} \quad & x + y = 1 \end{aligned}$$

Idea: plug $y = 1 - x$ in the objective and maximize for x!

How about the following problem?

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \frac{y}{x} + \frac{x^2}{2} \\ \text{subject to} \quad & 2x + y = 27 \end{aligned}$$

Lagrange's method

Consider:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x,y) \\ \text{subject to} \quad & g(x,y) = c. \end{aligned}$$

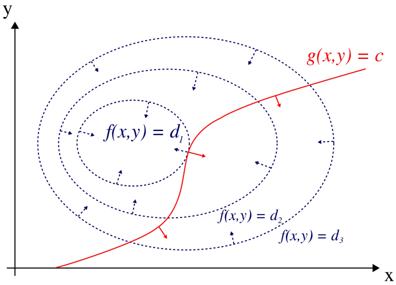

Figure: from wikipedia

At the optimum, $\nabla f(x,y) = \lambda \nabla g(x,y)$!

Lagrange's method cont'd

Define $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$, which implies

$$\nabla L(x, y, \lambda) = 0 \iff \begin{cases} \nabla f(x, y) & = \lambda \nabla g(x, y) \\ g(x, y) & = c \end{cases}$$

Solve the following problem again using Lagrange's method:

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \frac{y}{x} + \frac{x^2}{2} \\ \text{subject to} \quad & 2x + y = 27 \end{aligned}$$

Multiple constraints

Consider:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) = b_i, \; i = 1, \ldots, m. \end{array}$$

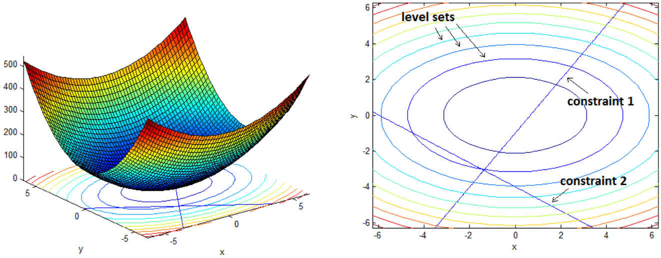

Figure: from wikipedia

Consider:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) = b_i, \; i = 1, \ldots, m. \end{aligned}$$

Example: find the point on the line of intersection of planes $3x_1 - 2x_2 + 4x_3 = 9$ and $x_1 + 2x_2 = 3$ which is closest to $(3, -1, 2)$.

Definition 1 (Lagrangian and Lagrange's multipliers)

$L(x, \lambda_1, \cdots, \lambda_m) = f_0(x) - \sum_{i=1}^m \lambda_i (f_i(x) - b_i)$ is the **Lagrangian** and $\lambda_1, \cdots, \lambda_m$ are the **Lagrange's multipliers**.

To solve the minimization problem above, notice that

$$\nabla L(x, \lambda_1, \cdots, \lambda_m) = 0 \iff \begin{cases} \nabla f_0(x) & = \sum_{i=1}^m \lambda_i \nabla f_i(x) \\ f_i(x) & = b_i \; i = 1, \ldots, m \end{cases}$$

Local maximum or minimum?

Definition 2 (Bordered Hessian)

The **bordered Hessian** is

$$H = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \frac{\partial f_m}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} & \frac{\partial^2 f_0}{\partial x_1^2} & \frac{\partial^2 f_0}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_0}{\partial x_1 \partial x_n} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_2} & \frac{\partial^2 f_0}{\partial x_1 \partial x_2} & \frac{\partial^2 f_0}{\partial x_2^2} & \cdots & \frac{\partial^2 f_0}{\partial x_2 \partial x_n} \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_1} & \frac{\partial^2 f_0}{\partial x_1 \partial x_n} & \frac{\partial^2 f_0}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f_0}{\partial \partial x_n^2} \end{bmatrix}$$

The $n - m$ largest principal minors

- alternate in sign with the smallest one having the sign of $(-1)^{m+1} \implies$ local maximum,
- have the sign of $(-1)^m \implies$ local minimum.

Interpretation of the multipliers

When solving $\nabla L(x, \lambda_1, \cdots, \lambda_m) = 0$, one obtains $(x_1^*, \cdots, x_n^*)$ and $(\lambda_1^*, \cdots, \lambda_m^*)$, and it is possible to show that

$$\lambda_k^* = \frac{\partial f_0(x_1^*(b_1, \cdots, b_m), \cdots, x_n^*(b_1, \cdots, b_m))}{\partial b_k}$$

- λ_k^* is the effect of the constraint on the value of the objective function at the optimum.
- In economics, when maximizing profit under constraints, λ_k^* is referred to as the **shadow price** corresponding to constraint k.

Finding a Maximum Production Level

A manufacturer's production is modeled by the Cobb-Douglas function:

$$p(x, y) = 100x^{3/4}y^{1/4}$$

where x represents the units of labor and y represents the units of capital. Each labor unit costs 200\$ and each capital unit costs 250\$. The total expenses for labor and capital cannot exceed 50,000\$.

- Find the maximum production level.
- Interpret the shadow price.