Columbia University

Math Methods for Political Science

Fall 2017

Mock Final Exam

Date: December 6, 2017

Exercise 1 (5 points)

Let $f(x) = x\sqrt{x+1}$

(1) Compute $\int f(x)dx$ using integration by parts.
(2) Without using your result from (a), compute $\int_1^2 f(x)dx$ using a change of variables.

Exercise 2 (10 points)

Let $f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2$.

(1) Compute the gradient vector ∇f.
(2) Compute the Hessian matrix H.
(3) Find all stationary points of function f.
(4) Determine whether any of the stationary points are also local extrema.

Exercise 3 (7 points)

A farmer has 10 acres to plant in wheat and rye. He has to plant at least 7 acres. However, he has only \$1200 to spend and each acre of wheat costs \$200 to plant and each acre of rye costs \$100 to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye. If the profit is \$500 per acre of wheat and \$300 per acre of rye how many acres of each should be planted to maximize profits?

(1) Formulate the farmer's problem as a linear program, namely define
    (a) the decision variables,
    (b) the objective function,
    (c) the constraints.
(2) Represent graphically the feasible domain.
(3) Give the corners of the feasible domain.
(4) Give the optimal solution to the linear program.

Exercise 4 (12 points)

In the context of consumer theory, consider the problem of maximization of a utility function with a fixed amount of wealth to spend on various commodities. Assume that

- there are two commodities with amounts x_1, x_2, and x_3, and prices p_1, p_2, and p_3,
- the total wealth is fixed with $p_1x_1 + p_2x_2 + p_3x_3 = w$, where $w > 0$ is a positive constant,
- the utility is given by $f(x_1, x_2, x_3) = x_1x_2x_3$.

(1) Formulate the constrained utility maximization as an optimization problem.
(2) Write the Lagragian corresponding to the problem.
(3) Write the first-order conditions and solve for the stationary point.
(4) Compute the bordered Hessian corresponding to the problem.

(5) Verify that the stationary point is a maximum using the second-order conditions.

Hint: using $A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & d & 0 & f \\ c & e & f & 0 \end{bmatrix} \implies \begin{cases} \det A_4 = 2((af)^2 + (be)^2 + (cd)^2) - (af + be + cd)^2 \\ \det A_3 = 2abd \end{cases}$,

express $\det H_4$ and $\det H_3$ as a function of of the wealth w and the prices p_1, p_2, and p_3.

(6) Express the maximum utility as a function of the wealth w and the prices p_1, p_2, and p_3.

(7) Interpret the shadow price in relation to your answer to (6).

Exercise 5 (10 points)

Consider the following problem:

$$\begin{aligned} \underset{x,y}{\text{maximize}} \quad & xy \\ \text{subject to} \quad & x + y^2 \leq 2 \\ & x, y \geq 0 \end{aligned}$$

Note: since the feasible region is bounded and a continuous function on a closed and bounded set has a maximum, a solution to this problem must exist.

(1) Write the problem with inequality constraints of the type $\leq$ and zeros on the right-hand-side, that is find $g_1(x,y)$, $g_2(x,y)$, and $g_3(x,y)$ such that

$$\begin{aligned} \underset{x,y}{\text{maximize}} \quad & xy \\ \text{subject to} \quad & g_1(x,y) \leq 0 \\ & g_2(x,y) \leq 0 \\ & g_3(x,y) \leq 0 \end{aligned}$$

is equivalent to the considered problem, with $g_1(x,y) \leq 0$ replacing $x + y^2 \leq 2$, $g_2(x,y) \leq 0$ for $x \geq 0$ and $g_3(x,y) \leq 0$ for $y \geq 0$.

(2) Write the Karush-Kuhn-Tucker conditions.

(3) Find the solution resulting from considering $\lambda_1 = 0$.
Hint: use the stationarity conditions along with the fact that $x, y, \lambda_2, \lambda_3 \geq 0$.

(4) Find the solution resulting from considering $x + y^2 = 2$.
Hints:
- Use the fact that $x + y^2 = 2$ implies that at least $x = 2 - y^2$ or y must be positive.
- Show that $x = 0$ generates a contradiction but $x > 0$ does not.

(5) Compute the value of the objective for each candidate and find the maximum.

EXERCISE 2

(1) $\nabla f = \begin{pmatrix} 3x_1^2 - 3x_2 \\ 3x_2^2 - 3x_1 \end{pmatrix}$

(2) $H = \begin{pmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{pmatrix}$

(3) $(0,0)$ & $(1,1)$

(4) $H(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\Rightarrow$ neither positive nor negative definite

$H(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\det H_1 = 6 > 0$ $\Rightarrow$ pos def

$\det H_2 = 25 > 0$ $\Rightarrow$ local ~~max~~ min

EXERCISE 1

(a) $u = x$ $\quad dv = \sqrt{x+1}\,dx$

$du = dx$ $\quad v = \frac{2}{3}(x+1)^{3/2}$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}dx$$

$$= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + c$$

(b) $y = x+1$ $\quad x = y-1$ $\quad dx = dy$

$$\int_1^2 x\sqrt{x+1}\,dx = \int_2^3 (y-1)\sqrt{y}\,dy$$

$$= \int_2^3 \left(y^{3/2} - y^{1/2}\right)dy = \left[\frac{2}{5}y^{5/2} - \frac{2}{3}y^{3/2}\right]_2^3 =$$

EXERCISE 3

(1) (a) x acres of wheat
y acres of rye

(b) $500 \cdot x + 300 \cdot y$

(c) $x + y \leq 10$

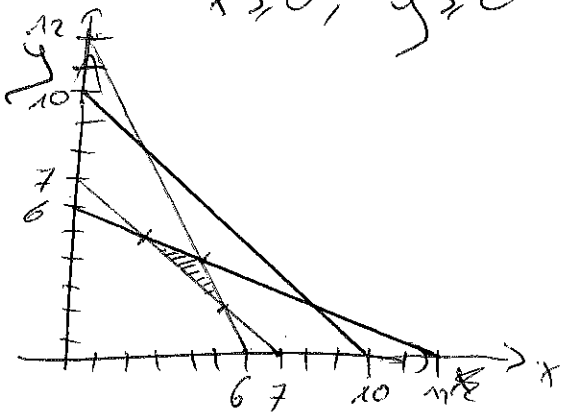
$x + y \geq 7$

$200x + 100y \leq 1200$

$x + 2y \leq 12$

$x \geq 0, \; y \geq 0$

(2)



(3)

Ⓐ $y = 7 - x$

Ⓑ $y = 12 - 2x$

Ⓒ $y = 6 - \frac{x}{2}$

<u>A + B</u> $7 - x = 12 - 2x$

$\Rightarrow x = 5 \;\&\; y = 2$

<u>A + C</u> $7 - x = 6 - x/2$

$\Rightarrow x = 2 \;\&\; y = 5$

<u>B + C</u> $12 - 2x = 6 - \frac{x}{2}$ $\qquad \frac{3x}{2} = 6$

$\Rightarrow x = 4 \;\&\; y = 8$

(4) <u>A + B</u> $500 \cdot 5 + 300 \cdot 2 = 3100$

<u>A + C</u> $500 \cdot 2 + 300 \cdot 5 = 2500$

<u>B + C</u> $500 \cdot 4 + 300 \cdot 4 = 320 \Rightarrow$ <u>OK</u>

EXERCISE 4

(1) $\max\limits_{x} \; x_1 x_2 x_3$

s.t. $p_1 x_1 + x_2 p_2 + x_3 p_3 = w$

(2) $L(x_1, x_2, x_3, \lambda) = x_1 x_2 x_3 - \lambda(x_1 p_1 + x_2 p_2 + x_3 p_3 - w)$

(3) $\nabla L = \begin{bmatrix} x_2 x_3 - \lambda p_1 \\ x_1 x_3 - \lambda p_2 \\ x_1 x_2 - \lambda p_3 \\ w - \sum x_i p_i \end{bmatrix} = 0 \iff x_1 x_2 x_3 = \lambda x_1 p_1 = \lambda x_2 p_2 = \lambda x_3 p_3$

$\iff x_1 p_1 = x_2 p_2 = x_3 p_3$

$\iff w = 3 x_1 p_1 = 3 x_2 p_2 = 3 x_3 p_3$

$\iff x_i = \dfrac{w}{3 p_i}$ $\qquad \lambda = \dfrac{w^2}{9 p_1 p_2 p_3}$

(4) $H = \begin{bmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & x_3 & x_2 \\ p_2 & x_3 & 0 & x_1 \\ p_3 & x_2 & x_1 & 0 \end{bmatrix}$

(5) $\det H = \dfrac{-w^2}{3} < 0$ $\qquad \Rightarrow$ maximum

$\det H_3 = \dfrac{2 p_1 p_2 w}{3 p_3} > 0$

(6) $f(x_1, x_2, x_3) = \dfrac{w^3}{27 p_1 p_2 p_3}$

(7) $\lambda = \dfrac{\partial f(w)}{\partial w}$

A Karush-Kuhn-Tucker Example

It's only for very simple problems that we can use the Karush-Kuhn-Tucker conditions to solve a nonlinear programming problem. Consider the following problem:

$$\begin{array}{ll} \text{maximize} & f(x,y) = xy \\ \text{subject to} & x + y^2 \ \leq \ 2 \\ & x, y \quad \geq \quad 0 \end{array}$$

Note that the feasible region is bounded, so a global maximum must exist: a continuous function on a closed and bounded set has a maximum there.

We write the constraints as $g_1(x,y) = x + y^2 \leq 2$, $g_2(x,y) = -x \leq 0$, $g_3(x,y) = -y \leq 0$. Thus the KKT conditions can be written as

$$\begin{aligned} y - \lambda_1 + \lambda_2 &= 0 \\ x - 2y\lambda_1 + \lambda_3 &= 0 \\ \lambda_1(2 - x - y^2) &= 0 \\ \lambda_2 x &= 0 \\ \lambda_3 y &= 0 \\ x + y^2 &\leq 2 \\ x, y, \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

In each of the "complementary slackness" equations $\lambda_i(b_i - g_i(x_1, \ldots, x_n)) = 0$, at least one of the two factors must be 0. With n such conditions, there would potentially be 2^n possible cases to consider. However, with some thought we might be able to reduce that considerably.

Case 1: Suppose $\lambda_1 = 0$. Then the first KKT condition says $y + \lambda_2 = 0$ and the second says $x + \lambda_3 = 0$. Since each term is nonnegative, the only way that can happen is if $x = y = \lambda_2 = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$ (although clearly this is not a local maximum since $f(0,0) = 0$ while $f(x,y) > 0$ at points in the interior of the feasible region).

Case 2: Suppose $x + y^2 = 2$. Now at least one of $x = 2 - y^2$ and y must be positive.

Case 2a: Suppose $x > 0$. Then $\lambda_2 = 0$. The first KKT condition says $\lambda_1 = y$. The second KKT condition then says $x - 2y\lambda_1 + \lambda_3 = 2 - 3y^2 + \lambda_3 = 0$, so $3y^2 = 2 + \lambda_3 > 0$, and $\lambda_3 = 0$. Thus $y = \sqrt{2/3}$, and $x = 2 - 2/3 = 4/3$. Again all the KKT conditions are satisfied.

Case 2b: Suppose $x = 0$, i.e. $y = \sqrt{2}$. Since $y > 0$ we have $\lambda_3 = 0$. From the second KKT condition we must have $\lambda_1 = 0$. But that takes us back to Case 1.

We conclude there are only two candidates for a local max: $(0,0)$ and $(4/3, \sqrt{2/3})$.

The global maximum is at $(4/3, \sqrt{2/3})$.