

Math Methods for Political Science
Fall 2017

Exercise Set 1

Due: September 27, 2017

1. DETERMINANTS

Exercise 1. Compute the determinant of the following matrices.

a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

b) Same question for A^T, B^T, C^T, D^T .

Exercise 2. For which values c_1, c_2, c_3 is the following matrix invertible ?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ c_1^2 & c_2^2 & c_3^2 \end{bmatrix}$$

Hint: show $\det A = (c_2 - c_1)(c_3 - c_1)(c_3 - c_2)$.

Exercise 3. Let A be an $n \times n$ matrix. We say that A is *triangular* if either $A_{i,j} = 0$ for $j > i$ or $A_{i,j} = 0$ for $i > j$. If $A_{i,j} = 0$ for $j > i$, then the matrix is called *lower triangular*. If $A_{i,j} = 0$ for $i > j$, then the matrix is called *upper triangular*. If $A_{i,j} = 0$ for $i > j$ and $i < j$ (i.e., $A_{i,j} = 0$ for $i \neq j$), then the matrix is called *diagonal*.

$$\underbrace{\begin{bmatrix} A_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & A_{n-2,n-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & A_{n-1,n-2} & A_{n-1,n-1} & 0 \\ A_{n,1} & A_{n,2} & \cdots & \cdots & A_{n,n-3} & A_{n,n-1} & A_{n,n} \end{bmatrix}}_{\text{lower triangular}}$$

upper triangular

diagonal

Exercise 4. Let A and B be $n \times n$ matrices. Show:

- a)

b)

2. VECTOR SPACES

a) If V is a vector space, then $\mathbf{0} \in V$ (i.e., the zero vector) is unique.

b) If V be a vector space and $\mathbf{u} \in V$ a vector, then $-\mathbf{u} \in V$ (i.e., the inverse of \mathbf{u}) is unique.

c) The set of polynomials of degree at most n , namely

$$\{a_0 + a_1t + \dots + a_nt^n \mid a_0, \dots, a_n \in \mathbb{R}\},$$

is a vector space.

d) The set of polynomials of degree exactly 2, namely

$$\{a_0 + a_1t + a_2t^2 \mid a_0, a_1, a_2 \in \mathbb{R}, a_2 \neq 0\},$$

is not a vector space.

e) The set of $m \times n$ matrices is a vector space.

f) If A is an $n \times n$ invertible matrix, then its columns are linearly independent.

Exercise 7. (a) What is the dimension of the subspace W of \mathbb{R}^2 defined as $W = \text{span}\{v_1, v_2, v_3\}$,

$$\text{where } v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(b) Find a subset B of $\{v_1, v_2, v_3\}$ such that B is a basis of W .

(c) Grow the subset $\{v_1 + v_2\} \subset W$ to obtain a basis of W .

Exercise 8. (a) Consider the vector $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ expressed in the standard basis for \mathbb{R}^2 .

Find the coordinates of v in the basis $\{b_1, b_2\}$ of \mathbb{R}^2 , where $b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Same question for $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ given in the standard basis for \mathbb{R}^3 to express in the basis $\{b_1, b_2, b_3\}$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 9. Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(a) Compute the rank of A and the dimension of its null space.

(b) Same question for A^T .

(c) Same question for A , a 7×7 matrix with a pivot in every row.

(d) Consider A , an $n \times m$ matrix, and a vector $b \in \mathbb{R}^n$. What relationship between the rank of $[A \ b]$ and the rank of A would guarantee the equation $Ax = b$ to be consistent?

Exercise 10. Let

$$w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 & -\frac{5}{2} \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{bmatrix}.$$

Which of the following proposals is correct? Justify briefly.

(a) w belongs to $\text{Col } A$, but not to $\text{Nul } A$.

(b) w belongs to $\text{Nul } A$, but not to $\text{Col } A$.

(c) w belongs to $\text{Nul } A$ and to $\text{Col } A$.

(d) w belongs neither to $\text{Nul } A$ nor to $\text{Col } A$.

Exercise 11. Determine whether each proposal is true or false and justify briefly your answer.

- (a) Let V be a vector space and H a subspace of V . Then V is a subspace of itself and H is a vector space.
- (b) If H is a subset of V , then $0 \in H$ implies that H is a subspace of V .
- (c) A square matrix A is invertible if and only if $\text{Nul } A = \{0\}$.
- (d) The null space of a matrix A is not always a vector space.

Exercise 12. (a) Let A be an 5×6 matrix. If $\dim \text{Nul } A = 3$, what is $\text{Rank } A$?

(b) Let A be an 7×3 matrix. What is the maximal rank for A ? What is the minimal dimension of its null space? Same question if A is a 3×7 matrix.

(c) Let A be an $n \times n$ matrix. Give a condition on $\text{Rank } A$ for A^T to be invertible ?

Exercise 13. (a) Show that the matrices $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 2 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$ et $B = \begin{bmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -\frac{7}{2} \end{bmatrix}$ are row equivalent.

(b) Compute $\text{Rank } A$, $\dim \text{Nul } A$, $\text{Rank } B$, $\dim \text{Nul } B$.

(c) Find a basis of $\text{Nul } A$ and $\text{Nul } B$.

(d) Find a basis of $\text{Col } A$ and $\text{Col } B$.

(e) Find a basis of $\text{Row } A$ and $\text{Row } B$.