COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Math Methods for Political Science

Lecture 1

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September 6, 2017

Outline

The basics

- Course website:

  https://tvatter.github.io/gu4700_2017/

- Lectures:
  - focus on introducing the theory and deriving the main results.
  - 4:10-5:25pm on Mondays & Wednesdays
  - Room 825, Seeley W. Mudd Building
- Exercise sessions:
  - focus on clarifying the theory, improving your math skills, help with the problem sets and preparation for the midterm/final.
  - To be determined
  - Please fill the form on the website!
- TA: Thomas Leavitt, `tl2624@columbia.edu`

Before the midterm

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- **Linear Algebra:**
  - System of linear equations
  - Matrix algebra
  - Vector spaces
  - Eigenvalues and eigenvectors
  - Orthogonality and least squares
  - Symmetric matrices and quadratic forms

- **Differential Calculus:**
  - Functions and limits
  - Continuity
  - Derivatives
  - Analysis of functions
  - Multivariate calculus

After the midterm

- **Integral Calculus:**
  - Concept of integral
  - Integration techniques
- **Optimization:**
  - Unconstrained optimization
  - Linear programming
  - Convex optimization
- **Probability and statistics:**
  - Combinatorics and probabilities
  - Random variables and vectors
  - Stochastic convergence
  - Statistical inference
  - Hypothesis testing

Grading and important dates

- 8 HW assignments (40%):
  - 2 for Linear Algebra (due 9/18 and 9/27)
  - 2 for Differential Calculus (due 10/9 and 10/18)
  - 1 for Integral Calculus (due 11/1)
  - 1 for Optimization (due 11/15)
  - 2 for Proba & Stats (due 11/27 and 12/6)
- Midterm (30% on 10/23):
  - Linear Algebra
  - Differential Calculus
- Final (30% on 12/11):
  - Integral Calculus
  - Optimization
  - Probability and Statistics

Maths are hard!

- No late HWs.
- Grades based on academic performance only.
- How to succeed:
  - Attend lectures AND exercise sessions.
  - Work on your own and only seek help when stuck.
  - Stay on top of things.
  - Practice.

Outline

Sets

Definition 1 (Set)

A **set** S is a collection of distinct objects (its **elements**).

Definition 2 (Cardinality)

$\mid S \mid$, the **cardinality** of set S, is the number of elements in S.

Example 1

$$
\begin{aligned}
S_1 &= \{1, 2, 3, 4, 5, 6\}, & \mid S_1 \mid &= 6 \\
S_2 &= \{Chris, Michael, Sara\}, & \mid S_2 \mid &= 3 \\
S_3 &= \emptyset, & \mid S_3 \mid &= 0
\end{aligned}
$$

Relations between sets

Let S and T be sets.

Definition 3 (Set equality, subset and proper subset)

- Sets S, T are **equal**, or $S = T$, when $x \in S \Leftrightarrow x \in T$.
- S is a **subset** of T, or $S \subseteq T$, if $x \in S \Rightarrow x \in T$.
- S is a **proper subset** is of T, or $S \subset T$, if $x \in S \Rightarrow x \in T$ and $\exists y \in T$ s.t. $y \notin S$ (i.e., $S \subseteq T$ but $S \neq T$).

$S \subseteq T$ and $T \subseteq S$ imply that $S = T$.

Example 2 (Set equality, subset and proper subset)

- Sets $S = \{A, B, C\}$ and $T = \{C, B, A\}$ are equal.
- $S = \{A, B, C\}$ is a proper subset of $T = \{A, B, C, D\}$.
- $\emptyset \subseteq S \; \forall S$

Relations between sets cont'd

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Definition 4 (Union, intersection, disjoint and complement)

- The **union** is $S \cup T = \{x \mid x \in S \text{ } or \text{ } x \in T\}$.
- The **intersection** is $S \cap T = \{x \mid x \in S \text{ } and \text{ } x \in T\}$.
- S, T are **disjoint** if $S \cap T = \emptyset$.
- The **complement** of S in T is $T \setminus S = \{x \mid x \in T, x \notin S\}$.

For $S_1, \cdots, S_n$, the union is $\bigcup_{i=1}^n S_i$ and the intersection is $\bigcap_{i=1}^n S_i$.

Example 3 (Union, intersection, disjoint and complement)

- $S \cup T = \{1, 2, 3, A, B\}$ when $S = \{1, 2, 3\}$ and $T = \{A, B\}$.
- $S = \{A, B\}$ and $T = \{snake, bumblebee\}$ are disjoint.
- For $S = \{A, B\}$ in $T = \{A, B, C, D, E\}$, $T \setminus S = \{C, D, E\}$.

Definition 5 (Laws of set operations)

- **Commutative** laws:

$$A \cup B = B \cup A;$$
$$A \cap B = B \cap A.$$

- **Associative** laws:

$$(A \cup B) \cup C = A \cup (B \cup C);$$
$$(A \cap B) \cap C = A \cap (B \cap C).$$

- **Distributive** laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Lists of numbers

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Definition 6 ($\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{R}$)

- **Natural** numbers (also called counting numbers):
  - $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ (positive natural numbers)
- **Integers** (positive and negative counting numbers):
  - $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
- **Rational** numbers (ratios of integers to non-zero integers):
  - $q \in \mathbb{Q}$ if $\exists x, y \in \mathbb{Z}, y > 0$ s.t. $q = x/y$.
- **Real** numbers:
  - $r \in \mathbb{R}$ if r has a "decimal representation" that has a finite or infinite sequence of digits to the right of the decimal point.

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Definition 7 (Completeness and transitivity of $\mathbb{R}$)

Let $x, y \in \mathbb{R}$.

- **Completeness**: for all x, y, we have $x \geq y$ or $y \geq x$.
- **Transitivity**: if $x \geq y$ and $y \geq z$, then $x \geq z$.

Definition 8 (Intervals)

Let $a, b \in \mathbb{R}, a < b$.

- **Open interval**: $(a, b) = \{x \mid a < x < b\}$.
- **Closed interval**: $[a, b] = \{x \mid a \leq x \leq b\}$.
- **Half-closed intervals**: $\begin{matrix}[a, b) = \{x \mid a \leq x < b\} \\ (a, b] = \{x \mid a < x \leq b\}\end{matrix}$.

Necessary and sufficient conditions

Definition 9 (Necessary and sufficient conditions)

- x is **necessary** for y if $y \Rightarrow x$.
- x is **sufficient** for y if $x \Rightarrow y$.
- x is **necessary and sufficient** for y if $x \Leftrightarrow y$ (i.e., x and y are logically equivalent).

Example 4 (Necessary and sufficient conditions)

- "being a woman" is necessary for "being a Finnish woman".
- "being a Finnish woman" is sufficient for "being a woman".
- "Germany lost WW2" is necessary and sufficient for "Other countries prevailed over Germany in WW2".

Methods of proofs

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- Proof by **deduction**
  - If P is true and we establish that $P \Rightarrow Q$, then Q is also true.
  - Usually through series of steps, such as $P \Rightarrow A \Rightarrow B \Rightarrow Q$.
- Proof by **contraposition**
  - If we can prove that $\neg Q \Rightarrow \neg P$, then $P \Rightarrow Q$.
- Proof by **contradiction**
  - If we can prove that $\neg P$ is false, then P must be true.
  - In practice, use the following logic:
    1. Suppose $\neg P$ is true;
    2. Prove that it violates another proposition Q known to be true;
    3. Conclude that $\neg P$ cannot be true and that P must be true.

Proof by deduction

- If P is true and we establish that $P \Rightarrow Q$, then Q is also true.
- Usually through series of steps, such as $P \Rightarrow A \Rightarrow B \Rightarrow Q$.

Example 5 (Proof by deduction)

Suppose $x, y \in \mathbb{R}$ and $x, y > 0$. Then $x^2 < y^2 \Rightarrow x < y$.

Proof:

$$\begin{aligned} x^2 < y^2 &\Rightarrow 0 < y^2 - x^2 && \mid && -x^2; \\ &\Rightarrow 0 < (y+x)(y-x) && \mid && \text{factor}; \\ &\Rightarrow 0 < y - x && \mid && x, y > 0; \\ &\Rightarrow x < y && \mid && +x. \end{aligned}$$

$\square$

Proof by contraposition

- If we can prove that $\neg Q \Rightarrow \neg P$, then $P \Rightarrow Q$.

Example 6 (Proof by contraposition)

Suppose $x, y \in \mathbb{R}$ and $x, y > 0$. Then $x^2 < y^2 \Rightarrow x < y$.

Proof:

$$\begin{aligned} x > y &\Rightarrow x - y > 0 && \mid && -y;\\ &\Rightarrow (y + x)(x - y) > 0 && \mid && x, y > 0;\\ &\Rightarrow x^2 - y^2 > 0 && \mid && \text{distribute};\\ &\Rightarrow x^2 > y^2 && \mid && +y^2. \end{aligned}$$

Since $x > y \Rightarrow x^2 > y^2$, we conclude that $x^2 < y^2 \Rightarrow x < y$. $\square$

Proof by contradiction

- If we can prove that $\neg P$ is false, then P must be true.
- In practice, use the following logic:
  1. Suppose $\neg P$ is true;
  2. Prove that it violates another proposition Q known to be true;
  3. Conclude that $\neg P$ cannot be true and that P must be true.

Example 7 (Proof by contradiction)

If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Proof:

1. Suppose $\exists a, b \in \mathbb{Z}$ s.t. $a^2 - 4b = 2$;
2. $$\begin{aligned} a^2 - 4b = 2 &\Rightarrow a^2 = 2 + 4b && \mid && +4b; \\ &\Rightarrow a^2 = 2(1 + 2b) && \mid && \text{factor;} \\ &\Rightarrow a^2 \text{ is even} && \mid && \text{definition.} \end{aligned}$$

Proof by contradiction cont'd

Example 7 (Proof by contradiction cont'd)

If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Proof cont'd:

2. $$\begin{aligned} a^2 - 4b = 2 &\Rightarrow a^2 \text{ is even} & \mid & \quad \text{previous;} \\ &\Rightarrow a \text{ is even} & \mid & \quad \text{property;} \\ &\Rightarrow \exists c \in \mathbb{Z} \text{ s.t. } a = 2c & \mid & \quad \text{definition;} \\ &\Rightarrow (2c)^2 - 4b = 2 & \mid & \quad \text{initial assumption;} \\ &\Rightarrow 2c^2 - 2b = 1 & \mid & \quad \text{divide by 2;} \\ &\Rightarrow 2(c^2 - b) = 1 & \mid & \quad \text{factor;} \\ &\Rightarrow 1 \text{ is even} & \mid & \quad \text{definition.} \end{aligned}$$

3. Because 1 is not even, we conclude that $a^2 - 4b = 2$ is false and $a^2 - 4b \neq 2$ must be true. $\square$