COLUMBIA UNIVERSITY
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Math Methods for Political Science

Lecture 15: Linear Programming I

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Linear programming: an example

Policy	Urban	Suburban	Rural
Building roads	-2	5	3
Gun control	8	2	-5
Farm subsidies	0	0	10
Gasoline tax	10	0	2
Population	100,000	200,000	50,000

Table: votes obtained per dollar spent advertising in support of an issue

What is the minimum amount of money we can spend to guarantee majority in all demographics?

Make the following hypothesis:

- linearity of the objective function and constraints,
- divisibility of the variables.

General form

$$
\begin{aligned}
\underset{x}{\text{Opt}} \quad & c^\top x \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i \in I \subseteq \{1, \ldots, m\} \\
& \sum_{j=1}^{n} a_{kj} x_j \geq b_k,\ k \in K \subseteq \{1, \ldots, m\} \\
& \sum_{j=1}^{n} a_{rj} x_j = b_r,\ r \in R \subseteq \{1, \ldots, m\} \\
& l_j \leq x_j \leq u_j
\end{aligned}
$$

Opt is maximize or minimize, I, K, R are disjoints and $I \cup K \cup R = \{1, \ldots, m\}$, $l_j = -\infty$ and $u_j = \infty$ are possible.

Terminology

- A **solution** is any $x = (x_1, \cdots, x_n)$.
- A solution is **feasible** if it satisfies the constraints.
- A solution's **value** is the value of the objective for the solution.
- The **feasible domain** is the set of feasible solutions.
- The **optimal solution** (if it exists) is the feasible solution solving the LP.

The admissible domain can be

- **empty**: no feasible solution $\implies$ no optimal solution.
- **bounded** (and non-empty): $\exists$ at least one optimal solution.
- **unbounded**, i.e. depending on the objective:
  - $\exists$ an optimal solution;
  - $\exists$ feasible solutions with arbitrarily large/small values $\implies$ the LP has no finite optimal solution and is said **unbounded**.

Canonical form

$$\begin{aligned}
\underset{x}{\text{Maximize}} \quad & c^\top x \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i \in \{1, \dots, m\} \\
& x_j \geq 0
\end{aligned}$$

- Maximization problem
- Only constraints of type "$\leq$"
- All variables are non-negative

Slack form

$$\begin{aligned} \underset{x}{\text{Maximize}} \quad & c^\top x \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i \in \{1, \dots, m\} \\ & x_j \geq 0 \end{aligned}$$

- Maximization problem
- Only equality constraints
- All variables are non-negative

Why particular forms?

- Simplex and variants use slack form.
- Interior point methods use canonical form.

Note that form definitions depend on the author!

Transformation rules

- **Minimize $\leftrightarrow$ maximize:**
  since $\min f(x) = -\max(-f(x))$, minimize cx by maximizing $-cx$ and conversely
- **Inequality "$\leq$" $\leftrightarrow$ inequality "$\geq$":**
  $ax \geq b \iff -ax \leq -b$
- **Equation $\rightarrow$ inequality "$\leq$":**
  $$ax = b \iff \begin{cases} ax \leq b \\ ax \geq b \end{cases} \iff \begin{cases} ax \leq b \\ -ax \leq -b \end{cases}$$
- **Inequality $\rightarrow$ equation:** add a "slack" (surplus) variable
  $$ax \leq b \iff ax + s = b,\ s \geq 0$$
  $$ax \geq b \iff ax - s = b,\ s \geq 0$$
- **Real variable $\rightarrow$ non-negative variable:**
  $x \in \mathbb{R} \iff x = x^+ - x^-,\ x^+, x^- \geq 0$

How to convert into canonical form?

- **Minimize an objective function:** negate the coefficients and maximize.
- **Variable x_j does not have a non-negativity constraint:** add the equality constraint $x_j = x_j^+ - x_j^-$ and the inequalities $x_j^+ \geq 0$ and $x_j^- \geq 0$.
- **Equality constraints:** replace $\sum_{j=1}^n a_{rj}x_j = b_r$ with two inequalities $\sum_{j=1}^n a_{rj}x_j \geq b_r$ and $\sum_{j=1}^n a_{rj}x_j \leq b_r$.
- **Greater than or equal to constraints:** replace $\sum_{j=1}^n a_{kj}x_j \geq b_k$ by $\sum_{j=1}^n -a_{kj}x_j \leq -b_k$.

Convert the following LP to canonical form:

$$\begin{aligned}
\underset{x}{\text{Minimize}} \quad & -3x_1 + 4x_2 \\
\text{subject to} \quad & x_1 + x_2 = 6 \\
& x_1 - x_2 \geq 4 \\
& x_1 \in \mathbb{R}, x_2 \geq 0
\end{aligned}$$

How to convert into slack form?

- **Minimize an objective function:** negate the coefficients and maximize.
- **Variable x_j does not have a non-negativity constraint:** add the equality constraint $x_j = x_j^+ - x_j^-$ and the inequalities $x_j^+ \geq 0$ and $x_j^- \geq 0$.
- **Inequality constraints:** replace $\sum_{j=1}^n a_{ij} x_j \leq b_i$ by adding a "slack" variable x_{n+i} with $\sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i$ and $x_{n+1} \geq 0$.

Recall the baking problem from the last lecture:

- Write it as an LP.
- Convert to slack form.

Another example

A company manufactures two products, A and B. The relevant production data is as follows

- Profit per unit: \$2 and \$5 respectively
- Labor time per unit: 2 hours and 1 hour respectively
- Machine time per unit: 1 hour and 2 hours respectively
- Available labor and machine time: 80 hours and 65 hours respectively

To do:

- Write the problem in standard form.
- Consider what happens when labor and machine overtime cost are \$15 and \$10 per hour, respectively