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# Math Methods for Political Science

*Lecture 12: Integral Calculus I*

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# An introducing example

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Suppose:

- A country's population growth at time $t$ is given by $f(t) = 3t$.
- The population is initially 2 at time $t = 0$.

Questions:

- What is the country's population at time $t = 10$?
- How much does the country's population change between time $t = 5$ and time $t = 10$?

# Introducing example cont'd

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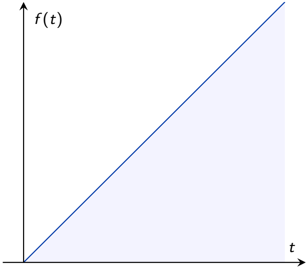

# Area under the curve

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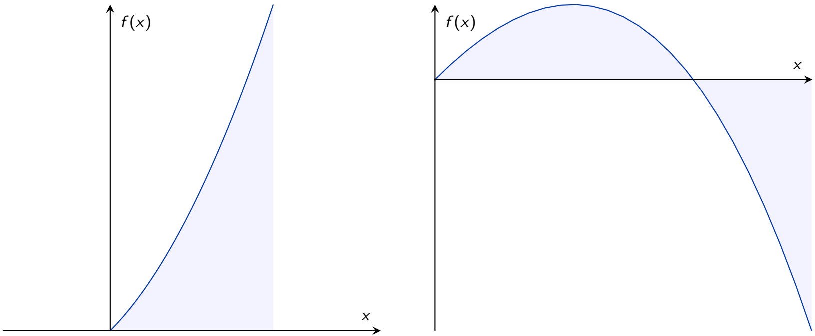



Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $[a, b]$ be a closed interval in $\mathbb{R}$.

**Definition 1 (Definite integral)**

The **definite integral** of $f$ over $[a, b]$ is $\int_a^b f(x)dx$ is the area under the curve $f$ between $a$ and $b$.

# Indefinite integral

Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

**Can we find $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$?**

**Definition 2 (Indefinite integral)**

The **indefinite integral (or antiderivative)** of $f$ is $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) + C = \int f(x)dx$ with $C \in \mathbb{R}$.

**Example 1 (Indefinite integral)**

- $f(x) = x^r$ for $r \in \mathbb{R}$
- $f(x) = 1/x$
- $f(x) = e^{ax}$
- $f(x) = x^{k-1}e^{x^k}$
- $f(x) = ah(x) + bg(x)$

# The fundamental theorem of calculus

## Theorem 1 (First fundamental theorem of calculus)

*Let $f$ be continuous on $[a, b]$ and $F$ be defined, for all $x \in [a, b]$, by*

$$F(x) = \int_a^x f(t)dt.$$

*Then $F$ is uniformly continuous on $[a, b]$, differentiable on $(a, b)$ and $F'(x) = f(x)$ for all $x \in (a, b)$.*

## Corollary 1

*If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and $F : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $f$ on $[a, b]$, then $\int_a^b f(t)\, dt = F(b) - F(a)$.*

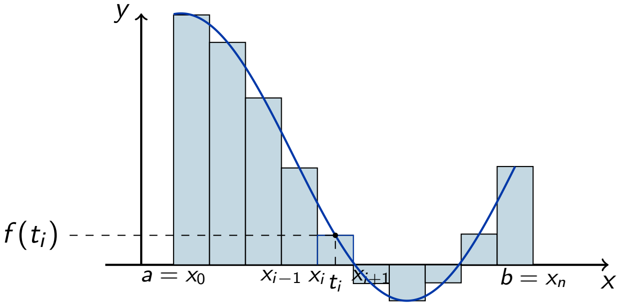
## Theorem 2 (Second fundamental theorem of calculus)

*If $f$ is the derivative of $F$ "almost everywhere" on $[a, b]$ and $f$ is "Riemann integrable", then $\int_a^b f(t)\, dt = F(b) - F(a)$.*

## Definition 3 (Riemann integrability)

$f$ has **Riemann integral** $\int_a^b f(t)\,dt$ over $[a, b]$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. for any $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with $x_i - x_{i+1} < \delta$ and $t_0 < t_1 < \cdots < t_{n-1}$ with $t_i \in [x_i, x_{i+1}]$,

$$\left| \sum_{i=1}^{n-1} f(t_i)(x_{i+1} - x_i) - \int_a^b f(t)\,dt \right| < \epsilon.$$



# Integration by parts



**Theorem 3 (Integration by parts)**

*Let $f$ and $g$ be two differentiable functions, then*

- $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx,$
- $\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx.$

**Example 2 (Integration by parts)**

- $\int log(x)dx$
- $\int xe^x dx$
- $\int x^2 e^x dx$
- $\int \sin(x)e^x dx$