COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Math Methods for Political Science

Lecture 13: Midterm recap & Integral Calculus II

Thibault Vatter <tv2233@columbia.edu>
Department of Statistics, Columbia University

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Outline

1 About the midterm

2 Integral calculus II

Reminder

How to succeed:

- Attend lectures AND exercise sessions.
- **Work on your own and only seek help when stuck.**
- Stay on top of things.
- Practice.

Solutions set and cardinality

Definition 1 (Solutions to system of linear equations)

$(s_1, \cdots, s_n)$ is a **solution** of a linear system in $n \in \mathbb{N}_+$ variables if it makes every equation in the system true when $s_1, \cdots, s_n$ are substituted for $x_1, \cdots, x_n$ respectively.

Lemma 1 (Cardinality of the solution set)

Let S be the solution set of a linear system, then $\mid S \mid \in \{0, 1, \infty\}$.

Example 1 (Solutions set and cardinality)

$$\left.\begin{array}{rr} x_1 & -2x_2 = -1 \\ -x_1 & +3x_2 = 3 \end{array}\right\} \begin{array}{l} \implies (s_1, s_2) = (3, 2) \text{ is the unique solution.} \\ \implies \text{ the cardinality of the solution set is 1.} \end{array}$$

Let A be a $m \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$.

Definition 2 (Null space and column space)

- The **null space** is $\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$.
- The **column space** is $\operatorname{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \exists \mathbf{x} \in \mathbb{R}^n \text{ with } A\mathbf{x} = \mathbf{b}\} = \operatorname{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$.

Nul A:

- a subspace of $\mathbb{R}^n$,
- implicitly characterized by $A\mathbf{x} = \mathbf{0}$ & needs the RREF of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to find its vectors,
- $\mathbf{v} \in \operatorname{Nul} A \iff A\mathbf{v} = \mathbf{0}$,
- easy to know if $\mathbf{v} \in \mathbb{R}^n$ is in it by computing $A\mathbf{v}$,
- $\operatorname{Nul} A = \{\mathbf{0}\} \iff \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.

Col A:

- a subspace of $\mathbb{R}^m$,
- explicitly defined by $\operatorname{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ & does not need intermediate steps,
- $\mathbf{v} \in \operatorname{Col} A \Leftrightarrow \exists \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{v}$,
- to know if $\mathbf{v} \in \mathbb{R}^m$ is in it, needs the RREF of $\begin{bmatrix} A & \mathbf{v} \end{bmatrix}$,
- $\operatorname{Col} A = \mathbb{R}^m \iff A\mathbf{x} = \mathbf{b}$ has a solution $\forall \mathbf{b} \in \mathbb{R}^m$.

Necessary and sufficient conditions

Definition 3 (Necessary and sufficient conditions)

- x is **necessary** for y if $y \Rightarrow x$.
- x is **sufficient** for y if $x \Rightarrow y$.
- x is **necessary and sufficient** for y if $x \Leftrightarrow y$ (i.e., x and y are logically equivalent).

Example 2 (Necessary and sufficient conditions)

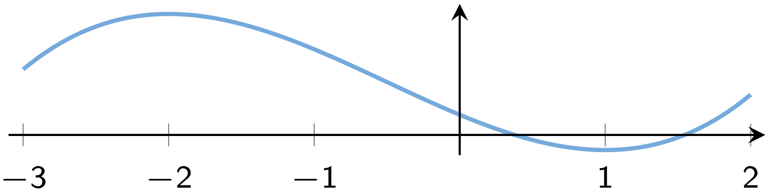
"An $n \times n$ matrix **needs** to have n distinct eigenvalues to be diagonalizable."

- Lecture 5, Thm 7-8-9
- Is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ diagonalizable?

Local extrema

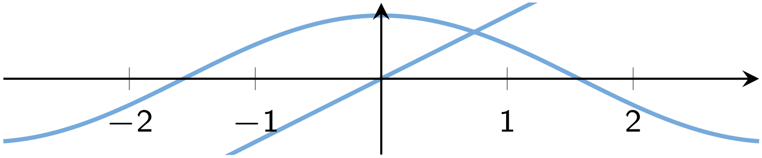
1. $\forall x \in \partial D_f$, check whether f decreases/increases if you move slightly to the interior to determine whether it is a local maximum/minimum.
2. Let $S_f = \{x \mid x \in D_f \text{ and } f'(x) = 0\}$, and compute $f''(x)$ $\forall x \in S_f$.
   2.1 If $f''(x) > 0$, then x is a local minimum.
   2.2 If $f''(x) < 0$, then x is a local maximum.
   2.3 If $f''(x) = 0$, then x is a stationary point.

How about $f(x) = 2x^3 + 3x^2 - 12x + 4$ and $D_f = [-3, 2]$?



Be creative!

Prove that $\cos x = x$ has at least one solution.



$\implies$ The two curves intersect, so $\cos x = x$ must have a solution.

Outline

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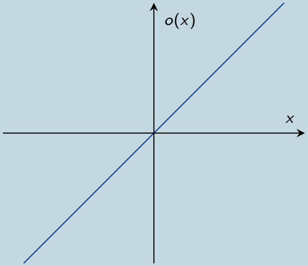
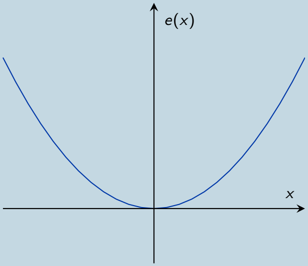
Even and odd functions

Definition 4 (Even and odd functions)

- An **odd function** $o(x)$ satisfies
  $o(x) = -o(-x) \implies \int_{-t}^{t} o(x)dx = 0.$
- An **even function** $e(x)$ satisfies
  $e(x) = e(-x) \implies \int_{-t}^{t} e(x)dx = 2\int_{0}^{t} e(x)dx.$

Example 3 (Even and odd functions)





Change of variables

Theorem 1 (Change of variables)

Let g be a differentiable function and f be continuous, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy.$$

The idea: let $y = g(x)$, then $dy/dx = g'(x)$ and $dy = g'(x)dx$.

Example 4 (Change of variables)

- $\int_0^2 x\cos(x^2+1)dx$
- $\int_2^3 \sqrt{7x+9}dx$

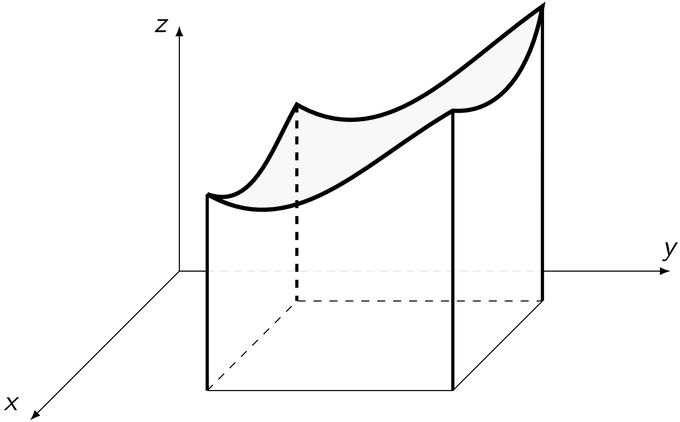
Note: this works for antiderivatives too!

Multiple integral

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $a_1 < b_1$, $a_2 < b_2$ and $T = [a_1, b_1) \times [a_2, b_2)$.

Definition 5 (Multiple integral)

The **multiple integral** over S is written $\int \int_T f(x, y) dx dy$.



Multiple integral cont'd

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $a_1 < b_1$, $a_2 < b_2$ and $S = [a_1, b_1) \times [a_2, b_2)$.

Theorem 2 (Fubini's theorem)

$\int \int_T f(x, y) dxdy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y) dx \right) dy =$
$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y) dy \right) dx$

Example 5 (Multiple integral and Fubini's theorem)

Let $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$.

- $f(x, y) = x + y$
- $f(x, y) = xy^2$