

Math Methods for Political Science

Lecture 6: Orthogonality and Least Squares

Thibault Vatter <tv2233@columbia.edu>
Department of Statistics, Columbia University

September 27, 2017

1 Inner product, length, and orthogonality

2 Orthogonal sets

3 Orthogonal projections

4 Gram-Schmidt

5 Least-squares

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Definition 1 (Inner product)

The **inner product** (or **dot product**) is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^n u_i v_i$.

Example 1 (Inner product)

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \implies \mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

Theorem 1 (Inner product)

Let $\mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

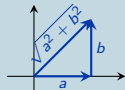
- | | |
|---|--|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | 3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ |
| 2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ | 4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \iff \mathbf{u} = \mathbf{0}$ |

Remark: 2. and 3. $\implies (\sum_{i=1}^p c_i \mathbf{u}_i) \cdot \mathbf{v}_j = \sum_{i=1}^p c_i (\mathbf{u}_i \cdot \mathbf{v}_j)$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Definition 2 (Length and unit vectors)

- $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ is the **length** (or **norm**) of \mathbf{v} .
- If $\|\mathbf{u}\| = 1$, then \mathbf{u} is a **unit vector**.

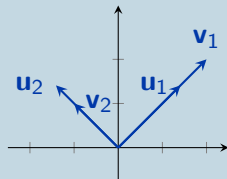


Theorem 2 (Length of scalar multiple and normalization)

- $\|c\mathbf{v}\| = |c| \|\mathbf{v}\|$
- $\mathbf{u} = \mathbf{v} / \|\mathbf{v}\|$ is a unit vector obtained by **normalizing** \mathbf{v} .

Example 2 (Length and unit vectors)

- $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- $\|\mathbf{v}_1\| = \sqrt{8}$ and $\|\mathbf{v}_2\| = \sqrt{2}$
- $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$



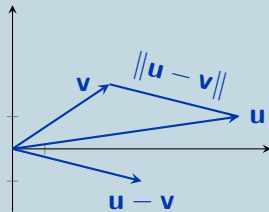
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Definition 3 (Distance)

The **distance** between \mathbf{u} and \mathbf{v} is $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

Example 3 (Distance)

- $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$
 $\implies \text{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{17}$



Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Definition 4 (Orthogonal vectors)

\mathbf{u} and \mathbf{v} are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

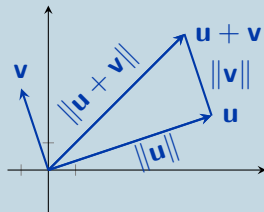
Remark: $\mathbf{0}$ is orthogonal to any \mathbf{u} .

Theorem 3 (The Pythagorean theorem)

\mathbf{u} and \mathbf{v} are orthogonal



$$\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$$



1 Inner product, length, and orthogonality

2 Orthogonal sets

3 Orthogonal projections

4 Gram-Schmidt

5 Least-squares

Let W be a subspace of \mathbb{R}^n .

Definition 5 (Orthogonal complement)

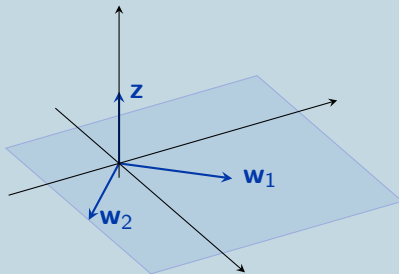
- $\mathbf{z} \in \mathbb{R}^n$ is **orthogonal to W** if $\mathbf{z} \cdot \mathbf{u} = 0 \ \forall \mathbf{u} \in W$.
- The **orthogonal complement** is
$$W^\perp = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} \text{ is orthogonal to } W\}.$$

Theorem 4 (Orthogonal complement)

- Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be s.t. $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = W$, then
$$\mathbf{z} \in W^\perp \iff \mathbf{z} \cdot \mathbf{v}_i = 0 \ \forall i.$$
- W^\perp is a subspace of \mathbb{R}^n .
- If A is an $m \times n$ matrix, then $(\text{Row } A)^\perp = \text{Nul } A$ and $(\text{Col } A)^\perp = \text{Nul } A^\top$.

Example 4 (Orthogonal complement)

Let $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Then if $L = \text{Span} \{ \mathbf{z} \}$ and $W = \text{Span} \{ \mathbf{w}_1, \mathbf{w}_2 \}$, we have $L = W^\perp$ and $W = L^\perp$.



Orthogonal set and basis

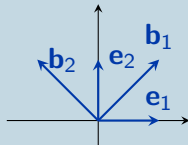
Let $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Definition 6 (Orthogonal set and basis)

- S is an **orthogonal set** if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.
- An **orthogonal basis** is a basis that is also an orthogonal set.

Example 5 (Orthogonal set and basis)

- The standard basis is orthogonal.
- $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is another orthogonal basis of \mathbb{R}^2



Theorem 5 (Orthogonal set)

If $\mathbf{v}_i \neq \mathbf{0} \forall i$, then S is linearly independent.

Corollary 1 (Orthogonal basis)

If $\mathbf{v}_i \neq \mathbf{0} \forall i$, then S is an orthogonal basis for $\text{Span } S$.

Theorem 6 (Coordinates in an orthogonal basis)

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be an orthogonal basis of $W \subseteq \mathbb{R}^n$, then

$$\forall \mathbf{y} \in W, \quad \mathbf{y} = \sum_{i=1}^p c_i \mathbf{v}_i \iff c_i = \frac{\mathbf{y} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \forall i.$$

Example 6 (Coordinates in an orthogonal basis)

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$.

- $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0 \implies \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis of \mathbb{R}^3 .

- $$\begin{array}{lll} \mathbf{y} \cdot \mathbf{v}_1 = 11 & \mathbf{y} \cdot \mathbf{v}_2 = -12 & \mathbf{y} \cdot \mathbf{v}_3 = -33 \\ \mathbf{v}_1 \cdot \mathbf{v}_1 = 11 & \mathbf{v}_2 \cdot \mathbf{v}_2 = 6 & \mathbf{v}_3 \cdot \mathbf{v}_3 = 33/2 \end{array}$$

$$\implies \mathbf{y} = \frac{11}{11} \mathbf{v}_1 + \frac{-12}{6} \mathbf{v}_2 + \frac{-33}{33/2} \mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3.$$

Let $\mathbf{u}_1, \dots, \mathbf{u}_p \in \mathbb{R}^n$ and $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$.

Definition 7 (Orthonormal set and basis)

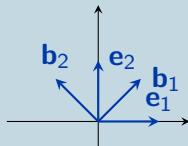
- S is an **orthonormal set** if it is orthogonal and $\|\mathbf{u}_i\| = 1 \forall i$.
- An **orthonormal basis** is a basis that is also orthonormal.

Example 7 (Orthonormal set and basis)

- The standard basis is orthonormal.

- $\mathbf{b}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

is another orthogonal basis of \mathbb{R}^2



Theorem 7 (Coordinates in an orthonormal basis)

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthonormal basis of $W \subseteq \mathbb{R}^n$, then
 $\forall \mathbf{y} \in W$,

$$\mathbf{y} = \sum_{i=1}^p c_i \mathbf{u}_i \iff c_i = \mathbf{y} \cdot \mathbf{u}_i \forall i.$$

Let U be an $m \times n$ matrix.

Theorem 8 (Orthonormal columns)

1. U has orthonormal columns $\iff U^\top U = I$.
2. If U has orthonormal columns and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then
 - a) $\|U\mathbf{x}\| = \|\mathbf{x}\|$,
 - b) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

Let U be an $n \times n$ matrix.

Definition 8 (Orthogonal matrix)

U is **orthogonal** if $U^{-1} = U^\top$.

Theorem 9 (Orthogonal matrix)

The following statements are equivalents:

1. U is an orthogonal matrix.
2. U has orthonormal columns.
3. U has orthonormal rows.

1 Inner product, length, and orthogonality

2 Orthogonal sets

3 Orthogonal projections

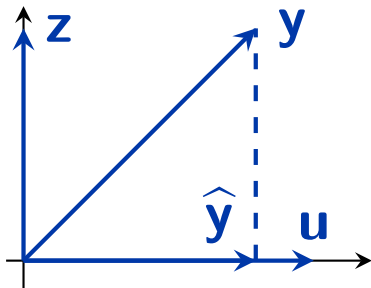
4 Gram-Schmidt

5 Least-squares

The general idea

Let $\mathbf{y}, \mathbf{u} \in \mathbb{R}^n$ and assume that we want

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}, \text{ with } \begin{cases} \hat{\mathbf{y}} = \alpha \mathbf{u} \text{ for } \alpha \in \mathbb{R}, \\ \mathbf{z} \cdot \mathbf{u} = 0 \end{cases}.$$



$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} \implies \mathbf{z} \cdot \mathbf{u} = 0 \iff \alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \text{ and } \hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

Let $\mathbf{y} \in \mathbb{R}^n$, $W \subseteq \mathbb{R}^n$ a subspace, $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ a basis for W .

Definition 9 (Orthogonal projection)

The **orthogonal projection** $\text{proj}_W \mathbf{y}$ is s.t. $\mathbf{y} = \text{proj}_W \mathbf{y} + \mathbf{z}$ with

1. $\text{proj}_W \mathbf{y} \in W$
2. $\mathbf{z} = \mathbf{y} - \text{proj}_W \mathbf{y} \in W^\perp$

Remark: we often use $\hat{\mathbf{y}}$ to denote $\text{proj}_W \mathbf{y}$.

Theorem 10 (Orthogonal projection)

1. $\hat{\mathbf{y}}$ is unique.
2. $\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad \forall \mathbf{v} \in W^\perp, \mathbf{v} \neq \hat{\mathbf{y}}$ (**best approximation**),
3. \mathcal{B} orthogonal $\implies \hat{\mathbf{y}} = \sum_{i=1}^p \frac{(\mathbf{y} \cdot \mathbf{v}_i)}{(\mathbf{v}_i \cdot \mathbf{v}_i)} \mathbf{v}_i$,
4. \mathcal{B} orthonormal $\implies \hat{\mathbf{y}} = \sum_{i=1}^p (\mathbf{y} \cdot \mathbf{v}_i) \mathbf{v}_i$.

Remark: for 3. and 4., we sometimes write $\hat{\mathbf{y}} = \sum_{i=1}^p \hat{\mathbf{y}}_i$.

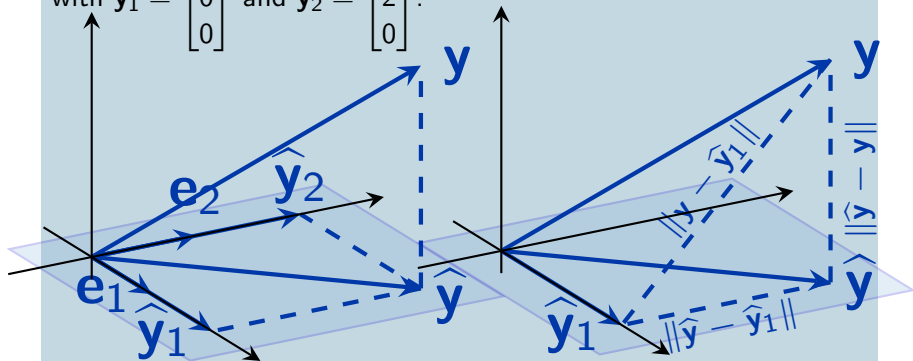
Remark: for 4., we can rewrite $\hat{\mathbf{y}} = UU^\top \mathbf{y}$ for $U = [\mathbf{u}_1 \quad \dots \quad \mathbf{u}_p]$.

Orthogonal projection cont'd

Example 8 (Orthogonal projection)

Let $W = \text{Span} \{ \mathbf{e}_1, \mathbf{e}_2 \}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, then $\hat{\mathbf{y}} = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

with $\hat{\mathbf{y}}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\hat{\mathbf{y}}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.



1 Inner product, length, and orthogonality

2 Orthogonal sets

3 Orthogonal projections

4 Gram-Schmidt

5 Least-squares

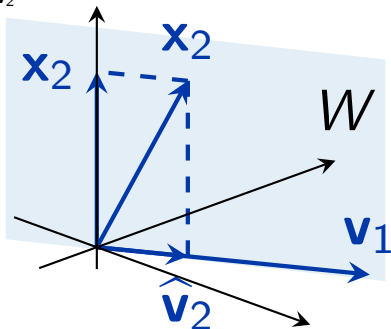
The general idea

Let $W = \text{Span} \{ \mathbf{x}_1, \mathbf{x}_2 \}$ where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

1. $\mathbf{v}_1 = \mathbf{x}_1$

2. $\mathbf{v}_2 = \mathbf{x}_2 - \underbrace{\text{proj}_{\mathbf{v}_1} \mathbf{x}_2}_{\hat{\mathbf{v}}_2} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

3. $\{ \mathbf{v}_1, \mathbf{v}_2 \}$ is lin.
ind. and
 $\dim W = 2 \implies$
orthogonal basis



Theorem 11 (Gram-Schmidt)

Given $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ a basis of a nonzero subspace $W \subseteq \mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} - \dots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}.$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis of W and $W_k = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \forall k$.

Remark: $\mathbf{v}_k = \mathbf{x}_k - \text{proj}_{W_k} \mathbf{x}_k = \mathbf{x}_k - \sum_{i=1}^{k-1} \frac{\mathbf{x}_k \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \mathbf{v}_i$.

Remark: an orthonormal basis is obtained with $\mathbf{u}_k = \mathbf{v}_k / \|\mathbf{v}_k\|$.

Example 9 (Gram-Schmidt)

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ lin. ind.

$\Rightarrow W = \text{Span} \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a subspace of \mathbb{R}^4 .

1. $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{Span} \{\mathbf{v}_1\}$.
- 2a. $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$.
- 2b. (optional scaling) $\mathbf{v}'_2 = 4\mathbf{v}_2$.
3. $W_2 = \text{Span} \{\mathbf{v}_1, \mathbf{v}'_2\}$.
4. $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}'_2}{\mathbf{v}'_2 \cdot \mathbf{v}'_2} \mathbf{v}'_2 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.
5. $\{\mathbf{v}_1, \mathbf{v}'_2, \mathbf{v}_3\}$ is an orthogonal basis of W .

1 Inner product, length, and orthogonality

2 Orthogonal sets

3 Orthogonal projections

4 Gram-Schmidt

5 Least-squares

Let A be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$.

Definition 10 (Least-squares solution)

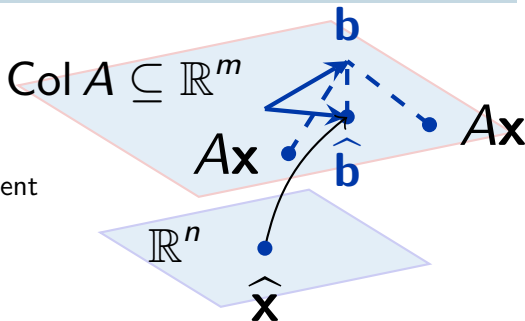
A **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} \in \mathbb{R}^n$ s.t.

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\| \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Let $\hat{\mathbf{b}} = A\hat{\mathbf{x}} \in \text{Col } A$, then

Thm 10 $\Rightarrow \hat{\mathbf{b}} = \text{proj}_{\text{Col } A} \mathbf{b}$

$\Rightarrow A\mathbf{x} = \hat{\mathbf{b}}$ consistent
with $\hat{\mathbf{x}}$ the solution



Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ & $\mathbf{b} \in \mathbb{R}^m$, then

$$\begin{aligned}\hat{\mathbf{b}} = A\hat{\mathbf{x}} = \text{proj}_{\text{Col } A} \mathbf{b} &\iff \mathbf{b} - \hat{\mathbf{b}} \in (\text{Col } A)^\perp \iff \mathbf{a}_j \cdot (\mathbf{b} - A\hat{\mathbf{x}}) \forall j \\ &\iff A^\top (\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \iff A^\top A\hat{\mathbf{x}} = A^\top \mathbf{b}\end{aligned}$$

Definition 11 (Normal equations)

The **normal equations** for $A\mathbf{x} = \mathbf{b}$ are $A^\top A\mathbf{x} = A^\top \mathbf{b}$.

Theorem 12 (Normal equations)

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is equal to the (nonempty) set of solutions of $A^\top A\mathbf{x} = A^\top \mathbf{b}$.

Furthermore, the following statements are equivalent:

1. $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution $\forall \mathbf{b} \in \mathbb{R}^m$.
2. The columns of A are linearly independent.
3. $A^\top A$ is invertible.

Remark: 1.-2.-3. $\implies \hat{\mathbf{x}} = (A^\top A)^{-1} A^\top \mathbf{b}$.

Example 10 (Normal equations I)

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \implies A^\top A = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \text{ and } A^\top \mathbf{b} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Then $A^\top A\mathbf{x} = A^\top \mathbf{b}$ becomes $\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$. Since $A^\top A$ is invertible, we have

$$\begin{aligned} \hat{\mathbf{x}} &= (A^\top A)^{-1} A^\top \mathbf{b} \\ &= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Example 11 (Normal equations II)

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix} \implies A^T A = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \text{ and}$$

$$A^T \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}. \text{ Then } \begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \xrightarrow{\text{RREF}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$