Notes on slides 11-14 of lecture 3

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September 14, 2017

1 Slide 11

We discussed the definition of the inverse in class, and the proof of property 2 of the theorem in the subject of an exercise in HW1.

- *Proof of 1 (by contradiction):* Let B and C be 2 inverses such that $B \neq C$, then

$$\begin{aligned} B &= BI & &\mid \quad \text{identity for matrix multiplication;} \\ &= B(AC) & &\mid \quad \text{because } C \text{ is an inverse of } A; \\ &= (BA)C & &\mid \quad \text{associative law;} \\ &= IC & &\mid \quad \text{because } B \text{ is an inverse of } A; \\ &= C & &\mid \quad \text{identity for matrix multiplication.} \end{aligned}$$

  Since we assumed $B \neq C$, we have a contradiction. As such, $B \neq C$ must be false and $B = C$ must be true. Therefore, the inverse is unique. $\square$

- *Proof of 3 (by deduction):* Here, we do not prove that AB is invertible (as this is trivial using the properties of the determinant that we will discuss on Monday), we only show that the inverse of AB is $B^{-1}A^{-1}$.

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} & &\mid \quad \text{associative law;} \\ &= AIA^{-1} & &\mid \quad \text{because } B^{-1} \text{ is the inverse of } B; \\ &= AA^{-1} & &\mid \quad \text{identity for matrix multiplication;} \\ &= I & &\mid \quad \text{because } A^{-1} \text{ is the inverse of } A. \end{aligned}$$

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B & &\mid \quad \text{associative law;} \\ &= B^{-1}IB & &\mid \quad \text{because } A^{-1} \text{ is the inverse of } A; \\ &= B^{-1}B & &\mid \quad \text{identity for matrix multiplication;} \\ &= I & &\mid \quad \text{because } B^{-1} \text{ is the inverse of } B. \end{aligned}$$

  Therefore $(AB)^{-1} = B^{-1}A^{-1}$. $\square$

2 Slide 12

With theorem 5 of lecture 2 for $m \times n$ (i.e., non-square) matrices in mind, theorem 5 of lecture 3 gives a similar condition for the existence and uniqueness of the solution of linear systems corresponding to $n \times n$ (i.e., square) matrices. Recall that, applying theorem 4 of lecture 2 for A an $n \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$, $A\mathbf{x} = \mathbf{b}$ has the same solution set as the system whose augmented matrix is $\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b}\end{bmatrix}$ for $\mathbf{b} \in \mathbb{R}^n$. As such, theorem 5 of lecture 3 gives a way to compute directly the solution of $A\mathbf{x} = \mathbf{b}$ by computing A^{-1}. Its proof is quite instructive.

- *Existence of a solution (by deduction):*

$$\begin{aligned}
\text{Let } \mathbf{x} = A^{-1}\mathbf{b} &\implies A\mathbf{x} = AA^{-1}\mathbf{b} && \mid \quad \text{definition of } \mathbf{x}; \\
&\implies A\mathbf{x} = I_n\mathbf{b} && \mid \quad \text{because } A^{-1} \text{ is the inverse of } A; \\
&\implies A\mathbf{x} = \mathbf{b} && \mid \quad \text{identity for matrix multiplication.}
\end{aligned}$$

  As such $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. $\square$

- *Uniqueness of the solution (by contradiction):* let $\mathbf{u}$ be a solution such that $\mathbf{u} \neq \mathbf{x}$, then

$$\begin{aligned}
& A\mathbf{u} = \mathbf{b} && \mid \quad \text{definition of a solution;} \\
\implies & A^{-1}A\mathbf{u} = A^{-1}\mathbf{b} && \mid \quad \text{multiply both sides by } A^{-1}; \\
\implies & I\mathbf{u} = A^{-1}\mathbf{b} && \mid \quad \text{because } A^{-1} \text{ is the inverse of } A; \\
\implies & \mathbf{u} = A^{-1}\mathbf{b} && \mid \quad \text{identity for matrix multiplication.}
\end{aligned}$$

  Since we assumed $\mathbf{u} \neq \mathbf{x}$, we have a contradiction. As such, $\mathbf{u} \neq \mathbf{x}$ must be false and $\mathbf{u} = \mathbf{x}$ must be true. Therefore, the solution is unique. $\square$

3 Slide 13

Recall the definition of row equivalence from lecture 2. This theorem comes without proof but it is important because it gives a recipe to compute the inverse of an $n \times n$ matrix A by applying the row reduction algorithm to the matrix $\begin{bmatrix}A & I_n\end{bmatrix}$.

Let's have a detailed look at the example:

$$
\begin{aligned}
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies \begin{bmatrix} A & I_n \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} & \quad | \quad \text{definition of the identity matrix;} \\
\implies \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} & \quad | \quad r_2 \longmapsto r_2 - 3r_1; \\
\implies \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{bmatrix} & \quad | \quad r_1 \longmapsto r_1 - r_2; \\
\implies \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & 1/2 \end{bmatrix} & \quad | \quad r_2 \longmapsto r_2/2; \\
\implies A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & 1/2 \end{bmatrix} & \quad | \quad \text{application of the theorem.}
\end{aligned}
$$

4 Slide 14

Theorem 7 is essentially a summary of all the results from theorems 4-5 of lecture 2 (applied to square matrices) and theorems 4-5-6 of lecture 3. For a given matrix A, "the statements are equivalent" means that they are either all true or all false.

To prove this theorem (strongly encouraged), use previous results to establish first the main "circle of implications" (first on the left in Figure 1). Then, link the remaining statements in the theorem to statements in this circle by proving the three chain of implications on the right in Figure 1.

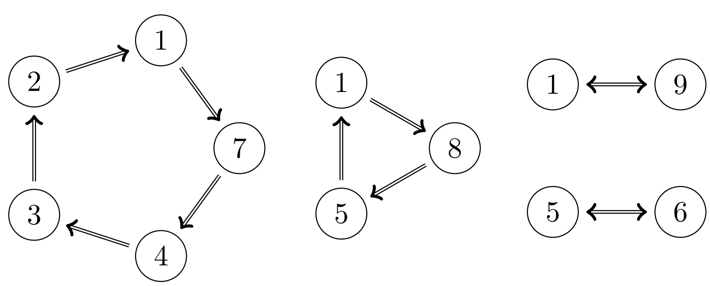

Figure 1: Chain of implications for the invertible matrix theorem