

Math Methods for Political Science

Lecture 4: Vector Spaces

- 1 Vector spaces and subspaces
- 2 Null space, column space and linear transformations
- 3 Linearly independent sets and bases
- 4 Coordinate systems
- 5 Dimension of a vector space
- 6 Rank

Definition 1 (Vector space)

A **vector space** V is a nonempty set of elements, called **vectors**, on which are defined two operations called **addition** and **scalar multiplication** satisfying, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in \mathbb{R}$,

- | | |
|--|---|
| 1. $\mathbf{u} + \mathbf{v} \in V$ | 6. $c\mathbf{v} \in V$ |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u}$ |
| 3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| 4. $\exists \mathbf{0} \in V$ s.t. $\mathbf{v} + \mathbf{0} = \mathbf{v}$ | 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ |
| 5. $\exists -\mathbf{v} \in V$ s.t. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ | 10. $1\mathbf{u} = \mathbf{u}$. |

Remark: $\mathbf{0}$ and $-\mathbf{v}$ are unique, $0\mathbf{v} = \mathbf{0}$, $c\mathbf{0} = \mathbf{0}$, and $-\mathbf{v} = (-1)\mathbf{v}$.

Example 1 (Vector space)

- | | |
|-----------------------------|---|
| ■ \mathbb{R}^n | ■ Polynomials of degree $\leq n$ |
| ■ Doubly infinite sequences | ■ Real-valued functions on \mathbb{R} |

Definition 2 (Vector subspace)

A **subspace** of a vector space V is subset $H \subseteq V$ satisfying

1. $\mathbf{0} \in H$
2. $\mathbf{u} + \mathbf{v} \in H \forall \mathbf{u}, \mathbf{v} \in H$
3. $c\mathbf{v} \in H \forall \mathbf{u} \in H$ and $c \in \mathbb{R}$.

Remark: the definition implies that H is also a vector space.

Example 2 (Vector space)

- $H = \{\mathbf{0}\}$ (**zero subspace**)
- Polynomials form a subspace of the real-valued functions
- Polynomials of degree at most n form a subspace of the polynomials
- Since $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$, \mathbb{R}^2 is not a subspace of \mathbb{R}^3 .
- $H = \left\{ \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} \mid t, s \in \mathbb{R} \right\}$ is a subspace of \mathbb{R}^3 .

Let V be a vector space.

Theorem 1 (Subspace spanned by a set)

If $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$, then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

Definition 3 (Subspace spanned by a set and spanning set)

- $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the **subspace spanned** (or **generated by**) by $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.
- A **spanning** (or **generating**) set for a subspace H is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ s.t. $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = H$.

Example 3 (Subspace spanned by a set and spanning set)

Let $H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$. Notice that,

$$\mathbf{v} \in H \iff \exists a, b \in \mathbb{R} \text{ s.t. } \mathbf{v} = \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}.$$

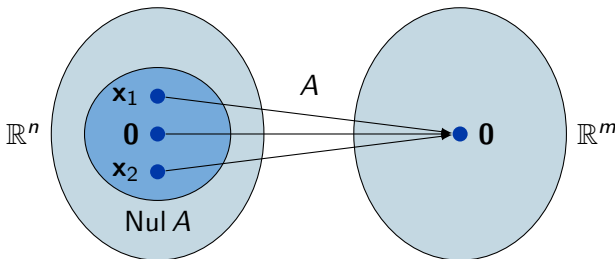
Hence $H = \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2 \} \implies H$ is a subspace of \mathbb{R}^4 .

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Definition 4 (Null space)

The **null space** of an $m \times n$ matrix A is

$$\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$



Example 4 (Null space)

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$, then $\mathbf{u} \in \text{Nul } A$.

Theorem 2 (Null space)

If A is a $m \times n$ matrix, then $\text{Nul } A$ is a subspace of \mathbb{R}^n .

Remark: \iff the solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace of \mathbb{R}^n .

Example 5 (Row reduction to characterize the null space)

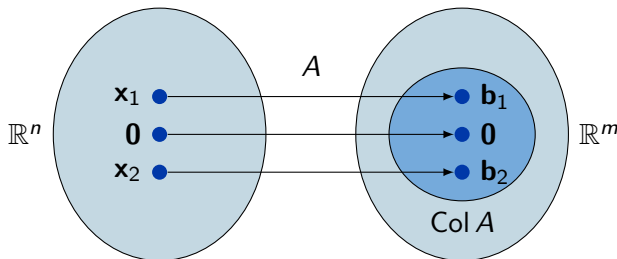
Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, then the RREF of $[A \ \mathbf{0}]$ is

$$\begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} \implies \begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - 3x_4 \end{aligned}$$

$$\begin{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}} + x_4 \underbrace{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}} \\ \implies \text{Nul } A &= \text{Span} \{ \mathbf{u}, \mathbf{v} \} \end{aligned}$$

Definition 5 (column space)

The **column space** of an $m \times n$ matrix A with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ is $\text{Col } A = \{\mathbf{b} \in \mathbb{R}^m \mid \exists \mathbf{x} \in \mathbb{R}^n \text{ with } A\mathbf{x} = \mathbf{b}\} = \text{Span } \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$.



Theorem 3 (Column space)

- If A is a $m \times n$ matrix, then $\text{Col } A$ is a subspace of \mathbb{R}^m .
- $\text{Col } A = \mathbb{R}^m \iff A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$.

Let A be a $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$.

Nul A :

1. a subspace of \mathbb{R}^n ,
2. implicitly characterized by $A\mathbf{x} = \mathbf{0}$ & needs the RREF of $[A \ \mathbf{0}]$ to find its vectors,
3. $\mathbf{v} \in \text{Nul } A \iff A\mathbf{v} = \mathbf{0}$,
4. easy to know if $\mathbf{v} \in \mathbb{R}^n$ is in it by computing $A\mathbf{v}$,
5. $\text{Nul } A = \{\mathbf{0}\} \iff \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.

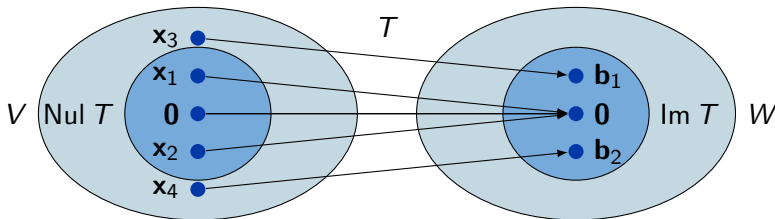
Col A :

1. a subspace of \mathbb{R}^m ,
2. explicitly defined by $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ & does not need intermediate steps,
3. $\mathbf{v} \in \text{Col } A \iff \exists \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{v}$,
4. to know if $\mathbf{v} \in \mathbb{R}^m$ is in it, needs the RREF of $[A \ \mathbf{v}]$,
5. $\text{Col } A = \mathbb{R}^m \iff A\mathbf{x} = \mathbf{b}$ has a solution $\forall \mathbf{b} \in \mathbb{R}^m$.

Definition 6 (Linear transformation, kernel, image)

Let V and W be two vector spaces.

- For each $\mathbf{x} \in V$, a **linear transformation** $T : V \rightarrow W$ assigns a unique $T(\mathbf{x}) \in W$ and is s.t.
 1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$,
 2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and $c \in \mathbb{R}$.
- The **null space** (or **kernel**) is $\text{Nul } T = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{0}\}$.
- The **image** is $\text{Im } T = \{\mathbf{b} \in W \mid \exists \mathbf{x} \in V \text{ with } T(\mathbf{x}) = \mathbf{b}\}$.



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Let V be a vector space and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ a set of vectors.

Definition 7 (Linear independence)

S is **linearly independent** $\sum_{i=1}^p c_i \mathbf{v}_i = \mathbf{0} \iff c_i = 0 \forall i$.

Theorem 4

For $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, S is linearly dependent $\iff \exists j > 1$ s.t. $\mathbf{v}_j \in \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_{j-1}\}$.

Example 6

- $V = \mathbb{R}^n$ and $A = [\mathbf{v}_1 \ \dots \ \mathbf{v}_p] \implies S$ linearly independent $\iff \text{Nul } A = \{\mathbf{0}\} \iff \mathbf{0}$ is the unique solution of $A\mathbf{x} = \mathbf{0}$.
- Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$, $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent because $\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$.
- $S = \{\cos(t), \sin(t)\}$ is linearly independent in $V = C[0, 1]$ because there exists no $c \in \mathbb{R}$ s.t. $\cos(t) = c \sin(t) \forall t \in [0, 1]$.

Definition 8 (Basis)

Let H be a subspace of a vector space V . A set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ is a **basis** for H if

1. \mathcal{B} is linearly independent,
2. $H = \text{Span} \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$.

Example 7

- The columns of an $n \times n$ invertible matrix form a basis of \mathbb{R}^n .
- The basis $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ formed with the columns of I_n is called the **standard basis** for \mathbb{R}^n .

- $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ is a basis of \mathbb{R}^3 .

Theorem 5 (The spanning set theorem)

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V and $H = \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

1. If $\exists k$ s.t. $\mathbf{v}_k = \sum_{j \neq k} c_j \mathbf{v}_j$, then
 $H = \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\}$.
2. If $H \neq \{\mathbf{0}\}$, then $\exists R \subseteq S$ s.t. R is a basis for H .

Corollary 1 (A basis for Col A)

The pivot columns of a matrix A form a basis for Col A.

Example 8 (The spanning set theorem)

$$\text{If } B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}, \text{ then}$$
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

form respectively bases for Col B and Col A (the later because B is the RREF of A).

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Let V be a vector space, $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis and $\mathbf{x} \in V$.

Definition 9 (Coordinates)

- The **coordinates of \mathbf{x} relative to \mathcal{B}** (or **\mathcal{B} -coordinates of \mathbf{x}**) are the weights c_1, \dots, c_n s.t. $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{b}_i$.

- If c_1, \dots, c_n are the \mathcal{B} -coordinates of \mathbf{x} , then $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is

the **coordinate vector of \mathbf{x} relative to \mathcal{B}**
(or the **\mathcal{B} -coordinate vector of \mathbf{x}**).

Remark: if $V = \mathbb{R}^n$, then $\mathbf{x} = [\mathbf{x}]_{\mathcal{E}}$.

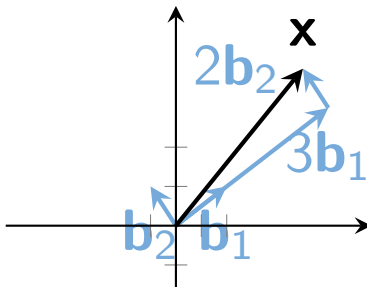
Theorem 6 (The unique representation theorem)

The \mathcal{B} -coordinates of \mathbf{x} are unique.

Example 9 (Coordinates in \mathbb{R}^2)

$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ lin. ind. $\implies \mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for \mathbb{R}^2 .

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

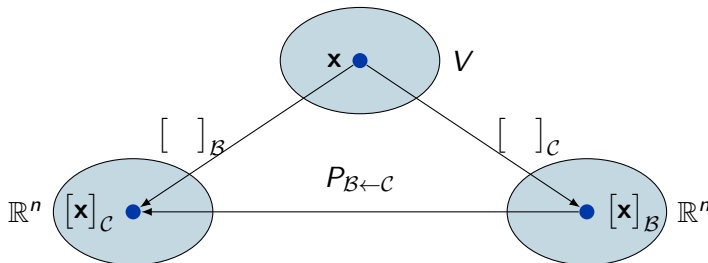


Theorem 7 (Change of basis)

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be two bases of a vector space V , then there exists a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ s.t.

1. $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$
2. $P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}}]$
3. the RREF of $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$ is $[I_n \ P_{\mathcal{C} \leftarrow \mathcal{B}}]$.

Remark: $P_{\mathcal{B} \leftarrow \mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} \implies [\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \mathcal{C}} [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} [\mathbf{x}]_{\mathcal{C}}$.



Example 10 (Change of basis)

Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, then

$$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{bmatrix}$$

$$\implies [\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$$\implies P_{\mathcal{B} \leftarrow \mathcal{C}} = \frac{1}{2} \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}.$$

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Let V be a vector space and \mathcal{B} a basis for V .

Theorem 8

If $|\mathcal{B}| = n$, then any set S with $|S| > n$ is linearly dependent.

Corollary 2

If \mathcal{B}_2 is another basis for V , then $|\mathcal{B}_2| = |\mathcal{B}|$.

Definition 10 (Dimension of a vector space)

The **dimension** of V is $\dim V = |\mathcal{B}|$. If $\dim V < \infty$, then V is **finite-dimensional**. Otherwise, it is **infinite-dimensional**.

Remark: by convention, if $V = \{\mathbf{0}\}$, then $\dim V = 0$.

Example 11 (Dimension of a vector space)

- If $V = \mathbb{R}^n$, then $|\mathcal{E}| = n \implies \dim V = n$.
- If $H = \text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$ (subspace of $V = \mathbb{R}^3$), then $\dim H = 2$.

Let V be a vector space with $\dim V = n < \infty$.

Theorem 9 (Subspaces of finite-dimensional spaces)

Let H be a subspace of V , then

- 1. in H , linearly independent sets can be extended into bases,*
- 2. and $\dim H \leq \dim V$.*

Theorem 10 (The basis theorem)

Let S be a set of n vectors and $n \geq 1$. If S is linearly independent (or equivalently if $\text{Span } S = V$), then S is a basis for V .

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Dimension of Nul A and Col A

Let A be an $m \times n$ matrix.

- Pivot columns = basis of Col $A \implies \dim \text{Col } A = \#$ of pivots.
- $\dim \text{Nul } A = \#$ of free variables in $A\mathbf{x} = \mathbf{0}$.

Example 12 (Dimension of Nul A and Col A)

If $A = \begin{bmatrix} -3 & -6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$, then the RREF of $[A \ \mathbf{0}]$ is

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- 2 pivot columns $\implies \dim \text{Col } A = 2$
- 3 free variables $\implies \dim \text{Nul } A = 3$

Let A be an $m \times n$ matrix.

Definition 11 (The row space)

Let $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$ be the rows of A , namely

$$A = \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix},$$

then **the row space** of A is $\text{Row } A = \text{Span } \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$.

Theorem 11 (Row equivalence and the row space)

Let B is another $m \times n$ matrix, then

1. A is r.e. to $B \iff \text{Row } A = \text{Row } B$,
2. if B is in REF, its nonzero rows form a basis for $\text{Row } A / \text{Row } B$.

Example 13 (The row space)

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Rightarrow \text{Row } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}, \text{Nul } A = \text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$
$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

Let A be an $m \times n$ matrix.

Definition 12 (The rank)

The **rank** of a matrix A is $\text{Rank } A = \dim \text{Col } A$.

Theorem 12 (The rank theorem)

$\dim \text{Row } A = \text{Rank } A$ and $\text{Rank } A + \dim \text{Nul } A = n$.

Example 14 (The rank theorem)

- If A is a 7×9 matrix with $\dim \text{Nul } A = 2$, then $\text{Rank } A = 7$.
- If A is a 6×9 matrix, then $\dim \text{Nul } A \neq 2$.

Theorem 13 (Characterization of the matrix inverse cont'd)

Let A be a $n \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, then the following statements are equivalent:

1. A is invertible.
2. A is r.e. to I_n .
3. A has n pivot positions.
4. The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.
6. $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \mathbb{R}^n$.
7. There exists C $n \times n$ s.t. $CA = I_n$.
8. There exists B $n \times n$ s.t. $AB = I_n$.
9. A^\top is invertible.

Furthermore:

10. $\mathbf{a}_1, \dots, \mathbf{a}_n$ form a basis of \mathbb{R}^n .
11. $\text{Col } A = \mathbb{R}^n$.
12. $\dim \text{Col } A = n$.
13. $\text{Rank } A = n$.
14. $\text{Nul } A = \{\mathbf{0}\}$.
15. $\dim \text{Nul } A = 0$.