Homework #4

Due Monday, November 6

1 Limits and continuity of functions of n variables

1. Let $(x_0, y_0) \in \mathbb{R}^2$ and use the $\delta - \epsilon$ method to show that:

   (a) $f(x,y) = x - y \implies \lim_{(x,y)\to(x_0,y_0)} f(x,y) = x_0 - y_0$ **without** using the theorem about limits and arithmetic operations.

   (b) $f(x,y) = yx^2 - y^3 \implies \lim_{(x,y)\to(x_0,y_0)} f(x,y) = y_0x_0^2 - y_0^3$ using the theorem about limits and arithmetic operations.

2. Let $f(x,y) = \frac{x^2y}{x^4+y^2}$ and show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
   Hint: consider what happens when the origin is approached along a line (e.g., $y = x$) and along a parabola (e.g., $y = x^2$).

3. Let $f(x,y) = \begin{cases} x + y & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y,z) = (0,0,0) \end{cases}$

   (a) Show that f is **not** continuous at $(0,0)$.

   (b) Show that f is continuous on $S = \mathbb{R}^2 \setminus \{0\}$.

2 Differentiability of functions of n variables

1. Consider the function $f(x_1,x_2) = x_1^3 + x_2^3 - 3x_1x_2$.

   (a) Find the gradient vector of this function.

   (b) Find the Hessian matrix of this function.

   (c) Find all stationary points of this function.

   (d) For each stationary point, investigate whether it is a local extremum.

   (e) For each local extremum, investigate whether it is a local maximum or a local minimum.

2. Consider the function $f(x_1,x_2,x_3) = x_1^2 + (x_1+x_2)^2 + (x_1+x_3)^2$.

   (a) Find the gradient vector of this function.

   (b) Find the Hessian matrix of this function.

   (c) Find all stationary points of this function.

   (d) For each stationary point, investigate whether it is a local extremum.

   (e) For each local extremum, investigate whether it is a local maximum or a local minimum.