Homework #6

Due Wednesday, November 29

1. Let $N(\mu, \sigma^2)$ denote the Normal distribution with mean μ and variance σ^2, with density for $x \in \mathbb{R}$ written as

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Note that we can also stack the two parameters in a two-dimensional vector $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$, in which case we write $f(x; \theta) = f(x; \theta_1, \theta_2) = f(x; \mu, \sigma^2)$. If $\{x_1, x_2, \cdots, x_n\}$ is an i.i.d. random sample with each observation drawn from a Normal distribution with mean μ and variance σ^2, then the joint density of the sample is

$$f(x_1, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta).$$

To estimate the parameters μ and σ^2 from the sample, one can write the log-likelihood of this model, namely

$$\ell(\theta) = \log f(x_1, \cdots, x_n; \theta) = \sum_{i=1}^{n} \log f(x_i; \theta).$$

The log-likelihood is to be taken as a function of the two-dimensional parameter vector θ, with the data being fixed.

(a) Compute $\nabla \ell(\theta)$ and $H(\theta)$.

(b) Use (a) to find the (unique) stationary point of $\ell(\theta)$.

(c) Use (a) and (b) to show that the (unique) stationary point of $\ell(\theta)$ is a global maximizer.

Hint: for (c), use the relationships between global optima, concave functions and their Hessian's definiteness.

2. Consider the following linear program:

$$\begin{array}{lll}
\underset{x}{\text{Maximize}} & -x_1 + 3x_2 & \\
\text{subject to} & -x_1 + x_2 & \leq 2 \\
& 2x_1 + x_2 & \leq 8 \\
& x_1 + x_2 & \leq 5 \\
& x_1, x_2 \geq 0 &
\end{array}$$

(a) Represent graphically the feasible domain.

(b) Give the corners of the feasible domain.

(c) Give the optimal solution to the linear program.

Hint: for (a), notice that a constraint like $\alpha x_1 + \beta x_2 \leq \gamma$ "separates" $\mathbb{R}^2$ into a feasible and an infeasible part on each "side" of the line $x_2 = \gamma/\beta - \alpha/\beta x_2$. As such, the feasible domain can be represented graphically by drawing the lines corresponding to the problem's constraints.
Hint: for (c), use the fact that the optimal solution of a linear program with bounded feasible domain is given by one of its corners.

3. A company with 10,000 m^2 of cardboard stores, manufactures and sells 2 types of cardboard boxes. The manufacture of a box of type 1 or 2 requires, respectively, 1 and 2 m^2 of cardboard as well as 2 and 3 minutes of assembly time. Only 200 hours of work are available during the coming week. The boxes are stapled and it takes four times more staples for a box of the second type than for one of the first. The stock of staples available can assemble up to 15,000 boxes of the first type. The boxes are sold, respectively, 3 and 5 CHF.

   (a) Formulate the problem of finding a production plan that maximizes the profit as a general linear program.

   (b) Transform the linear program into slack form.

   (c) Give an interpretation to each of the slack variables.

   (d) Solve for the optimum using the simplex algorithm.

   Hint: to make computations for (d) easier, you can multiply the decision variables by one thousand and solve in thousands of boxes of each type.

4. The following tableaus were obtained in the course of solving linear programs with 2 nonnegative variables x_1 and x_2, 2 inequality constraints, where the objective function was to be maximized, and slack variables x_3 and x_4 were added. In each case, indicate (with justification) whether the linear program is (1) unbounded, (2) has a unique optimum solution, (3) has an alternate optimum solution, (4) is degenerate. For degenerate linear programs, indicate additionally whether (1), (2) or (3) holds.

(a) $\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & 3 & 2 & 0 & 20 \\ \hline 0 & 1 & -2 & -1 & 0 & 4 \\ 0 & 0 & -1 & 0 & 1 & 2 \end{array}\right]$

(b) $\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & -1 & 0 & 2 & 20 \\ \hline 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 1 & -2 & 0 & 3 & 6 \end{array}\right]$

(c) $\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 2 & 0 & 0 & 1 & 8 \\ \hline 0 & 3 & 1 & 0 & -2 & 4 \\ 0 & -2 & 0 & 1 & 1 & 0 \end{array}\right]$

(d) $\left[\begin{array}{c|ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & 0 & 2 & 0 & 5 \\ \hline 0 & 0 & -1 & 1 & 1 & 4 \\ 0 & 1 & 1 & -1 & 0 & 4 \end{array}\right]$