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Math Methods for Political Science

Lecture 7: Functions and limits

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Outline

Correspondences and functions

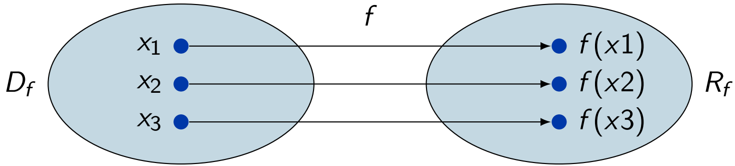
Let D_f, R_f be two sets.

Definition 1 (Corresp., domain, range, function, image)

- A **correspondence** is a mapping $f : D_f \rightrightarrows R_f$ with D_f the **domain** and R_f the **range**.
- A **function** is a correspondence $f : D_f \to R_f$ s.t. $\exists! f(x) \in R_f \, \forall x \in D_f$, with $f(x)$ the **image** of x.

Example 1 (Corresp., domain, range, function, image)

- A mapping between students and majors is a correspondence. The domain/range are the sets of students /majors.
- $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = 1 - 2x$ is a fct with $\mathbb{R}$ as domain/range.



Arithmetic operations on functions

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Definition 2 (Arithmetic operations on functions)

If $D_f \bigcap D_g \neq \emptyset$, then $f+g$, $f-g$ and fg are defined by

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x),$$
$$\text{and } (fg)(x) = f(x)g(x),$$

$\forall x \in D_f \bigcap D_g$, and f/g is defined by $(f/g)(x) = f(x)/g(x)$ $\forall x \in D_f \bigcap D_g$ s.t. $g(x) \neq 0$.

Example 2 (Arithmetic operations on functions)

Let $f(x) = \sqrt{4-x^2}$ and $g(x) = \sqrt{x-1}$ with $D_f = [-2,2]$ and $D_g = [1,\infty)$. Then $D_f \bigcap D_g = [1,2]$ and

$$(f \pm g)(x) = \sqrt{4-x^2} \pm \sqrt{x-1},$$
$$(f-g)(x) = \sqrt{(4-x^2)(x-1)},$$

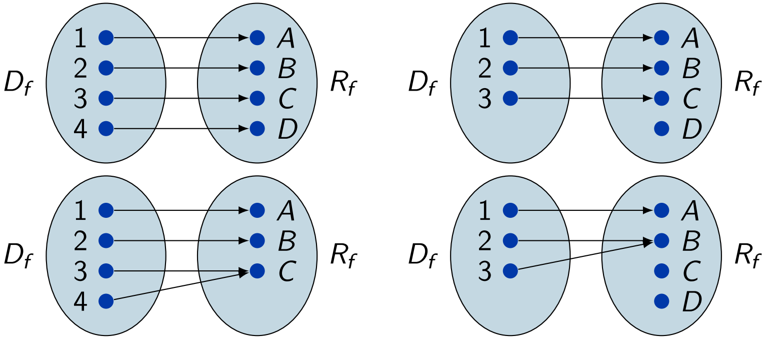
for $x \in [1,2]$, and $(f/g)(x) = \sqrt{(4-x^2)/(x-1)}$ for $x \in (1,2]$.

Injective/surjective/bijective functions

Let D_f, R_f be two sets and $f : D_f \to R_f$ be a function.

Definition 3 (Injective, surjective and bijective functions)

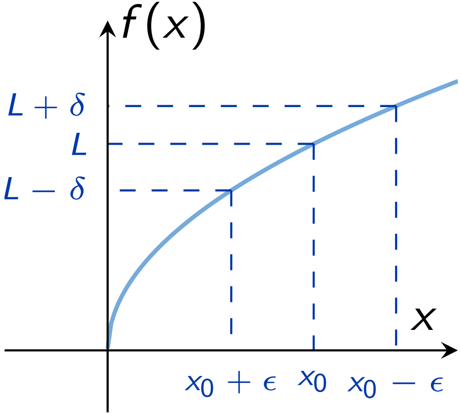
- If $x_2 \neq x_1 \implies f(x_2) \neq f(x_1)$, f is **injective** (or **one-to-one**).
- If $\forall y \in R_f$, $\exists x \in D_f$ s.t. $y = f(x)$, f is **surjective** (or **onto**).
- If f is injective and surjective, f is **bijective**.



Outline

Definition 4 (Limits)

$f(x)$ approaches the **limit** L as x approaches x_0, $\lim_{x \to x_0} f(x) = L$, if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $0 \mid x - x_0 \mid < \delta \implies \mid f(x) - L \mid < \epsilon$.



Example 3 (Limits)

- $f(x) = cx \implies \lim_{x \to x_0} f(x) = cx_0$
- $f(x) = xsin(1/x) \implies \lim_{x \to 0} f(x) = 0$

Theorem 1 (Properties of the limit)

- *If* $\lim_{x \to x_0} f(x)$ *exists, then it is e.*
- *If* $\lim_{x \to x_0} f(x) = L_1$ *and* $\lim_{x \to x_0} g(x) = L_2$*, then*
  1. $\lim_{x \to x_0} (f \pm g)(x) = L_1 \pm L_2$,
  2. $\lim_{x \to x_0} (fg)(x) = L_1 L_2$,
  3. *and* $\lim_{x \to x_0} (f/g)(x) = L_1/L_2$ *if* $L_2 \neq 0$.

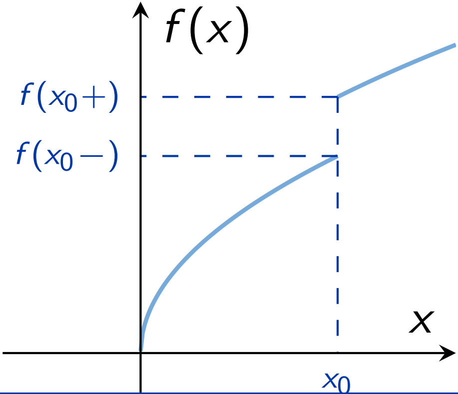
Example 4 (Properties of the limit)

Since $\lim_{x \to 2} 9 - x^2 = 5$ and $\lim_{x \to 2} x + 1 = 3$, $\lim_{x \to 2} \frac{9-x^2}{x+1} = 5/3$ and $\lim_{x \to 2} (9 - x^2)(x + 1) = 15$.

One-sided limits

Definition 5 (One-sided limits)

- **left-hand limit:** $\lim_{x \to x_0-} f(x) = f(x_0-)$, if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $x_0 - \delta < x < x_0 \implies \mid f(x) - L \mid < \epsilon$.
- **right-hand limit:** $\lim_{x \to x_0+} f(x) = f(x_0+)$, if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $x_0 < x < x_0 + \delta \implies \mid f(x) - L \mid < \epsilon$.



One-sided limits cont'd

Remark: theorem 1 is still valid for one-sided limits.

Example 5 (One-sided limits)

- $f(x) = x/\mid x \mid, x \neq 0 \implies \begin{cases} f(x_0-) & = -1 \\ f(x_0+) & = 1 \end{cases}$
- $f(x) = \frac{x+|x|(1+x)}{x} \sin(1/x), x \neq 0 \implies \begin{cases} f(x_0-) = 0 \\ f(x_0+) \text{ undefined} \end{cases}$

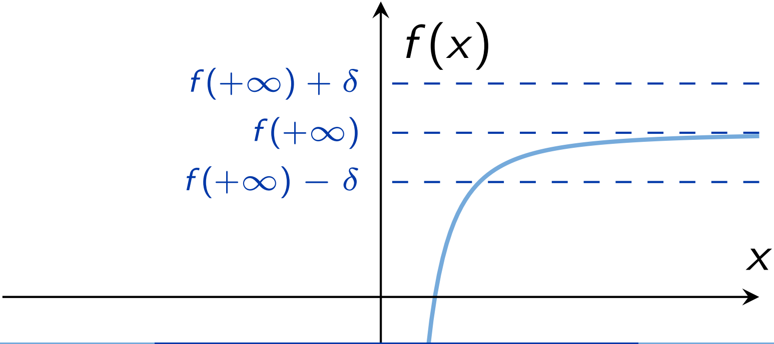
Theorem 2 (Existence of a limit)

$\lim_{x \to x_0} f(x) = L \iff f(x_0-) = f(x_0+) = L$

Definition 6 (Limits at $\pm\infty$)

- **Limits at $+\infty$:** $\lim_{x\to\infty} f(x) = f(+\infty)$, if $\forall\epsilon > 0$, $\exists\beta > 0$ s.t. $x > \beta \implies | f(x) - L | < \epsilon$.
- **Limits at $-\infty$:** $\lim_{x\to-\infty} f(x) = f(-\infty)$, if $\forall\epsilon > 0$, $\exists\beta < 0$ s.t. $x < \beta \implies | f(x) - L | < \epsilon$.

Remark: theorem 1 is still valid for limits at $\pm\infty$.



Outline

The idea

Let

$$\begin{cases} f(x) = 1/x \\ g(x) = 1/x^2 \\ p(x) = \sin(1/x) \\ q(x) = (1/x^2)\sin(1/x) \end{cases}$$

What are their (one-sided) limits at $x_0 = 0$?

Definition 7 (Infinite limits)

- **left-hand infinite limit:** $f(x_0-) = \pm$, if $\forall M \gtrless 0$, $\exists \delta > 0$ s.t. $x_0 - \delta < x < x_0 \implies f(x) \gtrless M$.
- **right-hand infinite limit:** $f(x_0+) = \pm$, if $\forall M \gtrless 0$, $\exists \delta > 0$ s.t. $x_0 < x < x_0 + \delta \implies f(x) \gtrless M$.

Remark: we'll say that a (one-sided) limit exists if it is finite.

Example 6 (Infinite limits)

- $f(x) = 1/x \implies \begin{cases} f(0-) = -\infty \\ f(0+) = \infty \end{cases}$
- $f(x) = 1/x^2 \implies f(0-) = f(0+) = \infty$
- $f(x) = x^2 \implies f(-\infty) = f(\infty) = \infty$
- $f(x) = x^3 \implies \begin{cases} f(-\infty) = -\infty \\ f(\infty) = \infty \end{cases}$
- $f(x) = e^{2x} - e^x \implies f(\infty) = \infty$

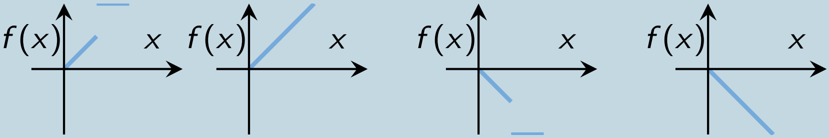
Outline

Monotonic functions

Definition 8 ((Non)decreasing and (non)increasing functions)

- f is **nondecreasing/nonincreasing** on an interval I if$\forall x_1, x_2 \in I \; x_1 > x_2 \implies f(x_1) \gtreqqless f(x_2)$.
- f is **decreasing/increasing** on an interval I if $\forall x_1, x_2 \in I$ $x_1 > x_2 \implies f(x_1) \gtrless f(x_2)$.
- f is **monotonic** on I if it is nondecreasing or nonincreasing on I.
- f is **strictly monotonic** on I if it is decreasing or increasing on I.

Example 7 (Monotone functions)



Monotonic functions cont'd

Theorem 3 (Monotonic functions)

Assume f monotonic on $I = (a, b)$ and define

$$\alpha = \inf_{x \in I} f(x), \text{ and } \beta = \sup_{x \in I} f(x).$$

1. f *nondecreasing* $\implies f(a+) = \alpha$ *and* $f(b-) = \beta$.
2. f *nonincreasing* $\implies f(a+) = \beta$ *and* $f(b-) = \alpha$.
3. $a < x_0 < b \implies f(x_0+)$ *and* $f(x_0-)$ *exist and are finite, with*

$$f(x_0-) \leq f(x_0) \leq f(x_0+) \text{ if } f \text{ is nondecreasing}$$
$$f(x_0-) \geq f(x_0) \geq f(x_0+) \text{ if } f \text{ is nonincreasing.}$$

Outline

Left limits inferior and superior

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Definition 9 (Bounded function)

f is **bounded** on a set S if $\exists M < \infty$ s.t. $\mid f(x) \mid < M \; \forall x \in S$.

Example 8 (Bounded function)

- $f(x) = x$ is bounded on $[-1, 1]$ but not in $\mathbb{R}$.
- $f(x) = 1/x$ is bounded on $[-1, 1] \setminus \{0\}$ but not on $[-1, 1]$.

Definition 10 (Left limits inferior and superior)

Let $S = [a, x_0)$ and assume f bounded on S.

- **Left limit superior:**
  $\limsup_{x \to x_0 -} f(x) = \lim_{x \to x_0 -} \sup_{x \leq t < x_0} f(t)$
- **Left limit inferior:**
  $\liminf_{x \to x_0 -} f(x) = \lim_{x \to x_0 -} \inf_{x \leq t < x_0} f(t)$

Example 9 (Left limits inferior and superior)

$f(x) = sin(1/x) \Rightarrow \limsup_{x \to 0-} f(x) = 1, \; \liminf_{x \to 0-} f(x) = -1$

Right limits inferior and superior

Definition 11 (Right limits inferior and superior)

Let $S = (x_0, a]$ and assume f bounded on S.

- **Right limit superior:**
  $\limsup_{x \to x_0+} f(x) = \lim_{x \to x_0+} \sup_{x_0 < t \leq x} f(t)$
- **Right limit inferior:**
  $\liminf_{x \to x_0+} f(x) = \lim_{x \to x_0+} \inf_{x_0 < t \leq x} f(t)$

Example 10 (Right limits inferior and superior)

$f(x) = sin(1/x) \Rightarrow \limsup_{x \to 0+} f(x) = 1, \liminf_{x \to 0+} f(x) = -1$

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Theorem 4 (Existence of the left limits inferior and superior)

Let $S = [a, x_0)$ and assume f bounded on S, then

$$\begin{cases} \alpha = \liminf_{x \to x_0 -} f(x) \\ \beta = \limsup_{x \to x_0 -} f(x) \end{cases}$$

exist and $\forall \epsilon > 0$,

1. $\exists \alpha_1, \beta_1 \in S$ *s.t.* $\begin{cases} \alpha_1 \leq x < x_0 \implies f(x) > \alpha - \epsilon \\ \beta_1 \leq x < x_0 \implies f(x) < \beta + \epsilon \end{cases}$,
2. $\exists \alpha_1, \beta_1 \in S$ *s.t.* $\begin{cases} f(\bar{x}_1) < \alpha + \epsilon \text{ for some } \bar{x}_1 \in [\alpha_1, a_0) \\ f(\bar{x}_2) > \beta - \epsilon \text{ for some } \bar{x}_2 \in [\beta_1, a_0) \end{cases}$.