COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Math Methods for Political Science

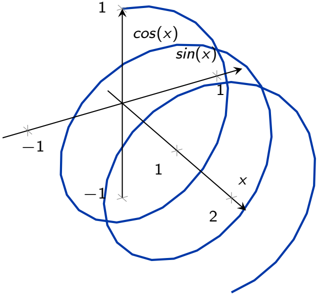
Lecture 10: Multivariate Calculus I

Thibault Vatter <tv2233@columbia.edu>
Department of Statistics, Columbia University

October 25, 2017

Objects of interests

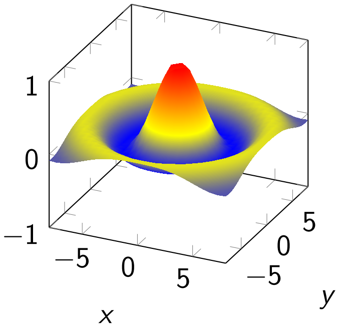
Curves	$f : \mathbb{R} \to \mathbb{R}^n$	• Curvature • Lengths of curves/line integrals



Objects of interests cont'd

Surfaces	$f : \mathbb{R}^2 \to \mathbb{R}^n$	• Curvature • Areas/flux of/through surfaces & surface integrals

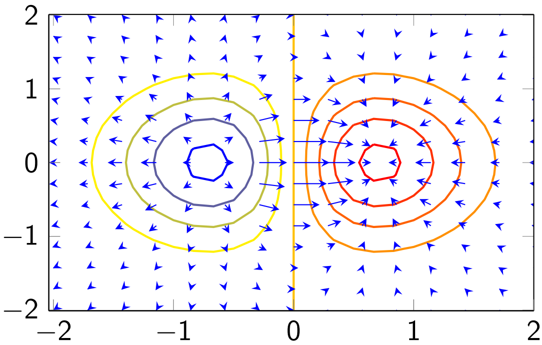
$$sin(\sqrt{x^2+y^2})/\sqrt{x^2+y^2}$$



Objects of interests cont'd

Scalar fields	$f : \mathbb{R}^m \to \mathbb{R}$	• Gradient & Hessian • Maxima and minima, Lagrange multipliers & level sets
Vector fields	$f : \mathbb{R}^m \to \mathbb{R}^n$	• All of the above • Jacobian/divergences/curl

$x \exp(-x^2 - y^2)$ and its gradient



Limits of functions of n variables

Let $f : \mathbb{R}^n \to \mathbb{R}$

Definition 1 (Limits)

L is the limit of f at $\mathbf{z}_0$, $\lim_{\mathbf{z}\to\mathbf{z}_0} f(\mathbf{z}) = L$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $\mathbf{z}$ in D_f,

$$0 < \|\mathbf{z} - \mathbf{z}_0\| < \delta \implies |f(\mathbf{z}) - L| < \epsilon$$

Example 1 (Limits)

Let $\mathbf{z} = (x, y) \in \mathbb{R}^2$, then

- $f(x, y) = x/y \implies \lim_{(x,y)\to(1,1)} f(x, y) = 1$
- $f(x, y) = 1 - x^2 - 2y^2 \implies \lim_{(x,y)\to(x_0,y_0)} f(x, y) = 1 - x_0^2 - 2y_0^2$ for every (x_0, y_0).

Limits of functions of n variables

Let $f : \mathbb{R}^n \to \mathbb{R}$

Theorem 1 (Uniqueness of the limit)

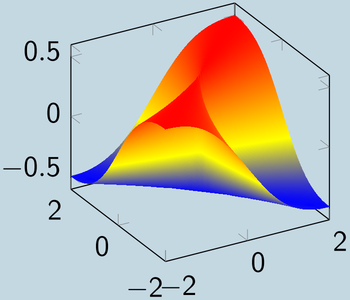
If $\lim_{\mathbf{z} \to \mathbf{z}_0} f(\mathbf{z}) = L$ exists, then it is unique.

Example 2 (Limits)

Let $\mathbf{z} = (x, y) \in \mathbb{R}^2$ and $f(x, y) = \frac{xy}{x^2+y^2}$, does the limit at $\mathbf{z}_0 = (0, 0)$ exists?

- Along the $y = x$ line, $f(x, x) = 1/2$.
- Along the $y = -x$ line, $f(x, -x) = -1/2$.

$\implies \lim_{\mathbf{z} \to \mathbf{z}_0} f(\mathbf{z})$ undefined!



Limits of functions of n variables

Theorem 2 (Limits and arithmetic operations)

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are such that $\lim_{\mathbf{z}\to\mathbf{z}_0} f(\mathbf{z}) = L_1$ and $\lim_{\mathbf{z}\to\mathbf{z}_0} g(\mathbf{z}) = L_2$, then

$$\lim_{\mathbf{z}\to\mathbf{z}_0} (f+g)(\mathbf{z}) = L_1 + L_2,$$

$$\lim_{\mathbf{z}\to\mathbf{z}_0} (f-g)(\mathbf{z}) = L_1 - L_2,$$

$$\lim_{\mathbf{z}\to\mathbf{z}_0} (fg)(\mathbf{z}) = L_1 L_2,$$

$$\text{and if } L_2 \neq 0,\ lim_{\mathbf{z}\to\mathbf{z}_0} \left(\frac{f}{g}\right)(\mathbf{z}) = \frac{L_1}{L_2}.$$

Example 3 (Limits and arithmetic operations)

- $f(x,y) = x + y \implies \lim_{(x,y)\to(1,1)} f(x,y) = 2$
- $f(x,y) = xy \implies \lim_{(x,y)\to(1,1)} f(x,y) = 1$

Continuity of functions of n variables

Let $f : \mathbb{R}^n \to \mathbb{R}$

Definition 2 (Continuity)

f **is continuous at $\mathbf{z}_0$** if $\lim_{\mathbf{z}\to\mathbf{z}_0} f(\mathbf{z}) = f(\mathbf{z}_0)$.

Example 4 (Continuity)

- Since $f(x, y) = 1 - x^2 - 2y^2 \implies \lim_{(x,y)\to(x_0,y_0)} f(x, y) = 1 - x_0^2 - 2y_0^2$ for every (x_0, y_0), f is continuous in $\mathbb{R}^2$.
- Since $f(x, y) = \frac{xy}{x^2+y^2} \implies \lim_{(x,y)\to(0,0)} f(x, y)$ does not exists, f is not continuous at 0.

Continuity of functions of n variables

Columbia University in the City of New York

Let $f : \mathbb{R}^n \to \mathbb{R}$

Theorem 3 (Continuity)

- *f and g continuous on $S \in \mathbb{R}^n \implies$ so are $f+g$, $f-g$, fg, and f/g at each $\mathbf{z}_0$ in S such that $g(\mathbf{z}_0) \neq 0$.*
- *f is continuous at $\mathbf{z}_0 \iff \forall \epsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{z} - \mathbf{z}_0\| < \delta$ and $\mathbf{z} \in D_f \implies |f(\mathbf{z}) - f(\mathbf{z}_0)| < \epsilon$..*

Example 5 (Continuity)

- $f(x, y) = xy$
- $f(x, y) = 5x^3 - x^2y^2$, $x_0 = y_0 = 0$, $D_f = \{(x, y) \mid |y| \leq |x|\}$.