Columbia University

Math Methods for Political Science
Fall 2017

Exercise Set 1

Due: October 4, 2017

1. Determinants

Exercise 1. Compute the determinant of the following matrices.

a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

b) Same question for A^T, B^T, C^T, D^T.

Exercise 2. For which values c_1, c_2, c_3 is the following matrix invertible ?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ c_1^2 & c_2^2 & c_3^2 \end{bmatrix}$$

Hint: show $\det A = (c_2 - c_1)(c_3 - c_1)(c_3 - c_2)$.

Exercise 3. Let A be an $n \times n$ matrix. We say that A is *triangular* if either $A_{i,j} = 0$ for $j > i$ or $A_{i,j} = 0$ for $i > j$. If $A_{i,j} = 0$ for $j > i$, then the matrix is called *lower triangular.* If $A_{i,j} = 0$ for $i > j$, then the matrix is called *upper triangular.* If $A_{i,j} = 0$ for $i > j$ and $i < j$ (i.e., $A_{i,j} = 0$ for $i \neq j$), then the matrix is called *diagonal.*

$$\underbrace{\begin{bmatrix} A_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & A_{n-2,n-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & A_{n-1,n-2} & A_{n-1,n-1} & 0 \\ A_{n,1} & A_{n,2} & \cdots & \cdots & A_{n,n-3} & A_{n,n-1} & A_{n,n} \end{bmatrix}}_{\text{lower triangular}}$$

$$\underbrace{\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \vdots & \vdots & \vdots & A_{1,n} \\ 0 & A_{2,2} & A_{2,3} & \ddots & \vdots & \vdots & A_{2,n} \\ 0 & 0 & A_{3,3} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \cdots \\ \vdots & \cdots & \cdots & \ddots & A_{n-2,n-2} & A_{n-2,n-1} & A_{n-2,n} \\ 0 & 0 & 0 & \vdots & 0 & A_{n-1,n-1} & A_{n-1,n} \\ 0 & 0 & 0 & \vdots & 0 & 0 & A_{n,n} \end{bmatrix}}_{\text{upper triangular}}$$

$$\underbrace{\begin{bmatrix} A_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & A_{3,3} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \vdots & \ddots & \ddots & A_{n-2,n-2} & 0 & 0 \\ 0 & 0 & \vdots & \ddots & 0 & A_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_{n,n} \end{bmatrix}}_{\text{diagonal}}$$

Show that if A is triangular or diagonal, then its determinant is equal to the product of the diagonal elements, namely $\det A = \prod_{i=1}^{n} A_{i,i}$.

Exercise 4. Let A and B be $n \times n$ matrices. Show:

a) If A is invertible, then $\det \left[A^{-1}\right] = \frac{1}{\det A}$.
b) If A and B are invertible, then $\det \left[BAB^{-1}\right] = \det A$.
c) If B is such that $B^T B = I_n$, then $\det B = \pm 1$.
d) If A is such that $\det \left[A^3\right] = 0$, then A is not invertible.
e) If either A or B is not invertible, then AB is not invertible.

Exercise 5. Solve the following linear systems using Cramer's rule:

a)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

b)

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 2 \\ 3x_1 & + & 7x_2 & + & 2x_3 & = & 1 \end{array}$$

2. Vector spaces

Exercise 6. Show:

a) If V is a vector space, then $\mathbf{0} \in V$ (i.e., the zero vector) is unique.

b) If V be a vector space and $\mathbf{u} \in V$ a vector, then $-\mathbf{u} \in V$ (i.e., the inverse of $\mathbf{u}$) is unique.

c) The set of polynomials of degree at most n, namely

$$\{a_0 + a_1 t + ... + a_n t^n \mid a_0, ..., a_n \in \mathbb{R}\},$$

is a vector space.

d) The set of polynomials of degree exactly 2, namely

$$\left\{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R},\ a_2 \neq 0\right\},$$

is not a vector space.

e) The set of $m \times n$ matrices is a vector space.

f) If A is an $n \times n$ invertible matrix, then its columns are linearly independent.

Exercise 7. (a) What is the dimension of the subspace W of $\mathbb{R}^2$ defined as $W = \text{span}\{v_1, v_2, v_3\}$, where $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Find a subset B of $\{v_1, v_2, v_3\}$ such that B is a basis of W.

(c) Grow the subset $\{v_1 + v_2\} \subset W$ to obtain a basis of W.

Exercise 8. (a) Consider the vector $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ expressed in the standard basis for $\mathbb{R}^2$. Find the coordinates of v in the basis $\{b_1, b_2\}$ of $\mathbb{R}^2$, where $b_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Same question for $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ given in the standard basis for $\mathbb{R}^3$ to express in the basis $\{b_1, b_2, b_3\}$ where $b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 9. Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(a) Compute the rank of A and the dimension of its null space.

(b) Same question for A^T.

(c) Same question for A, a 7×7 matrix with a pivot in every row.

(d) Consider A, an $n \times m$ matrix, and a vector $b \in \mathbb{R}^n$. What relationship between the rank of $[\,A\ b\,]$ and the rank of A would guarantee the equation $Ax = b$ to be consistent?

Exercise 10. Let

$$w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 & -\frac{5}{2} \\ -3 & -2 & 4 \\ 2 & 4 & -4 \end{bmatrix}.$$

Which of the following proposals is correct? Justify briefly.

(a) w belongs to $\text{Col}\,A$, but not to $\text{Nul}\,A$.

(b) w belongs to $\text{Nul}\,A$, but not to $\text{Col}\,A$.

(c) w belongs to $\text{Nul}\,A$ and to $\text{Col}\,A$.

(d) w belongs neither to $\text{Nul}\,A$ nor to $\text{Col}\,A$.

Exercise 11. Determine whether each proposal is true or false and justify briefly your answer.

(a) Let V be a vector space and H a subspace of V. Then V is a subspace of itself and H is a vector space.

(b) If H is a subset of V, then $0 \in H$ implies that H is a subspace of V.

(c) A square matrix A is invertible if and only if $\operatorname{Nul} A = \{0\}$.

(d) The null space of a matrix A is not always a vector space.

Exercise 12. (a) Let A be an 5×6 matrix. If $\dim \operatorname{Nul} A = 3$, what is $\operatorname{Rank} A$?

(b) Let A be an 7×3 matrix. What is the maximal rank for A? What is the minimal dimension of its null space? Same question if A is a 3×7 matrix.

(c) Let A be an $n \times n$ matrix. Give a condition on $\operatorname{Rank} A$ for A^T to be invertible ?

Exercise 13. (a) Show that the matrices $A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 2 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -\frac{7}{2} \end{bmatrix}$ are row equivalent.

(b) Compute $\operatorname{Rank} A$, $\dim \operatorname{Nul} A$, $\operatorname{Rank} B$, $\dim \operatorname{Nul} B$.

(c) Find a basis of $\operatorname{Nul} A$ and $\operatorname{Nul} B$.

(d) Find a basis of $\operatorname{Col} A$ and $\operatorname{Col} B$.

(e) Find a basis of $\operatorname{Row} A$ and $\operatorname{Row} B$.

3. Eigenvalues and eigenvectors

Exercise 14. Let

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix},$$

and $E = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 4 & 17 & 1 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Compute the characteristic polynomial, the eigenvalues and eigenvectors of matrices A, B, C, D, E.

Exercise 15. Using a minimal number of steps, determine whether the following matrices are diagonalizable:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \ D = \begin{bmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

Exercise 16. Determine whether each proposal is true or false and justify briefly your answer.

a) A matrix A is not invertible if and only if 0 is an eigenvalue of A.

b) A square matrix is invertible if and only if it is diagonalizable.

c) The eigenvalues of a square matrix are its diagonal entries.

d) We can find the eigenvalues of a matrix by computing its RREF.

e) If A and B are similar, they have the same eigenvalues.

f) An $n \times n$ matrix needs to have n distinct eigenvalues to be diagonalizable.

g) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are two eigenvectors linearly independent, then their associated eigenvalues are distinct.

h) Let A, B and C be matrices. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Exercise 17. Diagonalize the following matrices

$$B = \begin{bmatrix} 2 & 0 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}, C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Exercise 18. Show or find a counter-example:
Let A be an $n \times n$ matrix, $n \geq 2$ and $k \geq 2$.

(a) If A is diagonalizable, then A^k is diagonalizable.
(b) If A^k is diagonalizable, then A is diagonalizable.
(c) The eigenvectors of A and $A^\top$ are the same.

4. Orthogonality and least-squares

Exercise 19. a) Find a nonzero vector that is orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

b) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}$. Compute

$$\mathbf{u} \cdot \mathbf{v}, \quad \mathbf{v} \cdot \mathbf{w}, \quad \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{v}\|}, \quad \frac{1}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}, \quad \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{v}\|}\mathbf{v}.$$

c) Compute the distance between $\mathbf{u}$ and $\mathbf{v}$, and the distance between $\mathbf{u}$ and $\mathbf{w}$.
d) Compute the unit vectors corresponding to $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Exercise 20. Let $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and let $V = \text{Span}\,\{\mathbf{v}\}$. Find a basis of $W = V^\perp$.

Exercise 21. Determine whether each proposal is true or false and justify briefly your answer.

a) For any vector $\mathbf{v}$ and scalar c, $\|c\mathbf{v}\| = c\,\|\mathbf{v}\|$.
b) Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2$.
c) If vector $\mathbf{v}$ is orthogonal to every vector of a basis of subspace W except one, then $\mathbf{v} \in W^\perp$.
d) Let W be a subspace of a vector space V. If $\dim W^\perp = 1$, then it is possible to form a basis for V using vectors in W.

Exercise 22. Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$.

(i) Verify that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.
(ii) Compute $\text{proj}_W\mathbf{v}$ with $W = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$.
(iii) Give the decomposition $\mathbf{v} = \mathbf{z} + \text{proj}_W\mathbf{v}$ where $\mathbf{z} \in W^\perp$.

Exercise 23. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ et $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ two orthonormal bases for $\mathbb{R}^n$, $U = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ and $V = [\mathbf{v}_1 \cdots \mathbf{v}_n]$. Show that $U^TU = I$, $V^TV = I$ and that UV is invertible.

Exercise 24. Use the Gram-Schmidt process to orthogonalize the bases of the following subspaces of $\mathbb{R}^n$

(i) Span $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix} \right\}$

(ii) Span $\left\{ \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \right\}$

(iii) Give the orthonormal basis corresponding to (i) and (ii).

Exercise 25. Give the least-squares solution(s) to $A\mathbf{x} = \mathbf{b}$,

(i) $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$,

(ii) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$,

(iii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$;

Exercise 26. a) Show that if Q is an orthogonal matrix, then so is Q^T.

b) Show that if U, V are orthogonal matrices, then UV is also orthogonal.

c) Let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. Show that $Q = I - 2\mathbf{u}\mathbf{u}^T$ is orthogonal.

d) Show that any real eigenvalue λ of an orthogonal matrix Q verifies $\lambda = \pm 1$.

e) Let Q be an $n \times n$ orthogonal matrix and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ an orthogonal basis of $\mathbb{R}^n$. Show that $\{Q\mathbf{u}_1, \cdots, Q\mathbf{u}_n\}$ is also an orthogonal basis of $\mathbb{R}^n$.