Columbia University in the City of New York

Math Methods for Political Science

Lecture 16: Linear Programming II

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Reminder: LP in slack form

$$
\begin{aligned}
\underset{x}{\text{Maximize}} \quad & c^\top x \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i \in \{1, \dots, m\} \\
& x_j \geq 0
\end{aligned}
$$

- Maximization problem
- Only equality constraints
- All variables are non-negative

LP in slack form cont'd

$$
\begin{aligned}
\underset{z}{\text{Maximize}} \quad & z \\
\text{subject to} \quad & z - c^\top x = 0 \\
& \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \; i \in \{1, \dots, m\} \\
& x_j \geq 0, \; j \in \{1, \dots, n+m\}
\end{aligned}
$$

- Maximization problem
- Only equality constraints
- All variables are non-negative
- **Add a variable for the maximization**
- **Differentiate between "standard" and "slack" variables**

Tableau form, z-row & basic variables

Transform:

$$\begin{cases} z - c^\top x & = 0 \\ \sum_{j=1}^n a_{ij} x_j + x_{n+i} & = b_i, \; i \in \{1, \ldots, m\} \end{cases}$$

Into:

$$T = \left[\begin{array}{c|cccccccc|c}
z & x_1 & x_2 & \cdots & x_n & x_{n+1} & x_{n+2} & \cdots & x_{n+m} & b \\
\hline
1 & -c_1 & c_2 & \cdots & -c_n & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 & b_1 \\
0 & a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 & b_m
\end{array}\right]$$

Definition 1 (Tableau form, z-row and basic variables)

T is the (initial) **tableau form**, the first row is the **z-row** and variables appearing in only one row are called **basic variables**.

The tableau form: an example

$$\begin{aligned}
\underset{x}{\text{Maximize}} \quad & x_1 + x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 4 \\
& x_1 + 2x_2 \leq 3 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

A basic (feasible) solution

If $b_i \geq 0$ for $i \in \{1, \ldots, m\}$, then $x_j = 0$ for $j \in \{1, \ldots, n\}$ is feasible with:

$$\begin{cases} z - c^\top x & = 0 \\ \sum_{j=1}^n a_{ij} x_j + x_{n+i} & = b_i \end{cases} \implies \begin{cases} z & = 0 \\ x_{n+i} & = b_i \end{cases}$$

Definition 2 (Initial basic solution)

If $b_i \geq 0$ for $i \in \{1, \ldots, m\}$, then $x_j = 0$ for $j \in \{1, \ldots, n\}$, $z = 0$ and $x_{n+i} = b_i$ for $i \in \{1, \ldots, m\}$ is the initial **basic solution**.

Theorem 1 (Optimality of a basic solution)

If the z-row contain only nonnegative numbers, then the current basic solution is optimal.

First rule of the simplex

First rule the simplex:

- If the z-row contain only nonnegative numbers, then the current basic solution is optimal.
- Otherwise, pick a variable x_j with a negative coefficient in the z-row.

Then, "pivot" to "exchange" x_j and a basic variable.

Definition 3 (Entering and leaving variable)

The picked variable x_j is the **entering variable** and the basic one selected for the exchange is the **leaving variable**.

Second rule of the simplex

Second rule the simplex:

- For each Row i, $i \geq 2$, where there is a strictly positive "entering variable coefficient", compute the ratio of the RHS to the "entering variable coefficient".
- Choose the pivot row as being the one with MINIMUM ratio.

Then, pivot and iterate until "convergence".

Alternate optimal solutions

Consider a slightly different LP:

$$\begin{aligned} \underset{x}{\text{Maximize}} \quad & x_1 + \frac{x_2}{2} \\ \text{subject to} \quad & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Degeneracy

Definition 4 (Degenerate basic solution)

A basic solution is **degenerate** if it has (at least) one basic variable equal to zero.

$$
\begin{aligned}
\underset{x}{\text{Maximize}} \quad & 2x_1 + x_2 \\
\text{subject to} \quad & 3x_1 + x_2 \leq 6 \\
& x_1 - x_2 \leq 2 \\
& x_2 \leq 3 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

Cycling

Definition 5 (Cycling)

Cycling is a sequence of pivots that goes through the same tableaus and repeats itself indefinitely.

Degeneracy may lead to cycling, but it may be avoided by:

- Choosing the entering variable with smallest index in Rule 1, among all those with a negative coefficient in the z-row
- Breaking ties in Rule 2 test by choosing the leaving variable with smallest index (**Bland's rule**).

Unboundded LPs

$$
\begin{aligned}
\underset{x}{\text{Maximize}} \quad & 2x_1 + x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leq 1 \\
& x_1 - 2x_2 \leq 2 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$