Columbia University

Math Methods for Political Science

Fall 2017

Exercise Set 1

Due: September 19, 2017

1. Systems of linear equations

Exercise 1. Among the following equations, determine which ones are linear.

a) $x_1^2 + x_2^2 = 1$
b) $2^2 x_1 + 2^2 x_2 = 1$
c) $\sqrt{3}x_1 + \left[1 - \sqrt{2}\right] x_2 + 3 = \pi x_1$
d) $3x_1 + 2x_2 + 4\, x_3\, x_4 = 5$
e) $\left[\frac{1}{\sqrt{2}} - 1\right] x_1 - 2 = 2x_1 + 4x_2 + \sqrt{3}\, x_3 + x_9$

Exercise 2. For the linear systems below:

(1) Write their augmented matrices.
(2) Solve them using elementary row operations on the augmented matrices.

a) $\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$

b) $\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & x_3 & = & 12 \\ x_3 & + & 2x_1 & - & 4x_2 & = & -1 \\ x_2 & + & 2x_3 & - & 4x_1 & = & -8 \end{array}$

c) $\begin{array}{rcrcrcr} 6x_1 & - & 3x_2 & + & 2x_3 & = & 11 \\ -3x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 5x_1 & - & 3x_2 & + & 2x_3 & = & 9 \end{array}$

Exercise 3. Determine whether each proposal is true or false and justify briefly your answer.

a) Elementary row operations are reversible.
b) A 5×6 matrix has 6 rows.
c) The solution set of a linear system in $x_1, x_2, \ldots, x_n$ is a set of the form $(s_1, s_2, \ldots, s_n)$ which, when substituted for $x_1, x_2, \ldots, x_n$ respectively, make each equation in the system true.
d) An inconsistent system has more than one solution.
e) If two augmented matrices are equal, then so are the solution set of their corresponding linear systems.

Exercise 4. Show that elementary row operations do not modify the solution set of a linear system.

Exercise 5. For each of the following systems:

1) Write its augmented form
2) Compute its RREF.
3) Identify the free and basic variables, and determine its general solution.

a) $\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 8 \\ 4x_1 & - & 3x_2 & = & 6 \end{array}$

b) $$\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -2x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & - & x_2 & + & 2x_3 & = & -1 \end{array}$$

c) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 3 \end{array}$$

d) $$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 3 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & = & 2 \end{array}$$

Exercise 6. Consider $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

i) Is it possible to write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$?
ii) Give a geometric interpretation of your result.

Exercise 7. Is $v = \begin{bmatrix} 5 \\ -3 \\ -6 \end{bmatrix}$ in the subset spanned by the columns of $A = \begin{bmatrix} 3 & 5 \\ 1 & 1 \\ -2 & -8 \end{bmatrix}$? Justify.

Exercise 8. Let A be a $m \times n$ matrix with columns $\mathbf{a}_1, \cdots, \mathbf{a}_n$.
Show that $\text{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \mathbb{R}^m \iff$ its REF has a pivot in each row.

2. Matrix algebra

Exercise 9. Consider the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 4 \end{bmatrix}.$$

Compute the following product if they exist. If they don't, explain why.

a) AB, BA, AC, CA, BC, CB, CD, EC, EA
b) AA^T, A^TA, BA^T, BC^T, C^TA, BD^T, D^TB

Exercise 10. (a) Compute the inverse of $A = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$,

(i) using the general formula for a 2×2 matrix,
(ii) by finding the RREF of $[A\, I_2]$.

(b) Compute the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ by finding the RREF of $[A\, I_3]$.

Exercise 11. Let A and B be matrices such that AB is well defined. Show $(AB)^T = B^TA^T$.

Exercise 12. Determine whether each proposal is true or false and justify briefly your answer.

(a) Let A and B be two 2×2 matrices whose columes are $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{b}_1, \mathbf{b}_2$, then $AB = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 \end{bmatrix}$.
(b) Let A, B and C be three 3×3 matrices, then $AB + AC = (B + C)A$.

(c) Let A and B two $n \times n$ matrices, then $A^T + B^T = (A+B)^T$.
(d) The transpose of a matrix product is equal to the product of their transpose in the same order.
(e) If A is invertible, then A^{-1} is also invertible.
(f) The product of invertible $n \times n$ matrices is not invertible.
(g) If A is an invertible $n \times n$ matrix, then $Ax = b$ has a solution for each $b \in \mathbb{R}^n$.
h) A $m \times n$ matrix can be multiplied from the left by a $p \times m$ matrix.
i) The matrix product is commutative.
j) If A and B are such that $AB = 0$, then $A = 0$ or $B = 0$.
k) $(ABC)^T = C^T B^T A^T$.

Exercise 13. Compute the determinant of the following matrices.

a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

b) Same question for A^T, B^T, C^T, D^T.

Exercise 14. For which values c_1, c_2, c_3 is the following matrix invertible ?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ c_1^2 & c_2^2 & c_3^2 \end{bmatrix}$$

Hint: show $\det A = (c_2 - c_1)(c_3 - c_1)(c_3 - c_2)$.

Exercise 15. Let A and B be $n \times n$ matrices. Show that if either A or B is not invertible, then AB is not invertible.

Exercise 16. Show:
a) If A is invertible, then $\det\left(A^{-1}\right) = \frac{1}{\det A}$.
b) If A and Q are $n \times n$ invertible matrices, then $\det\left(QAQ^{-1}\right) = \det A$.
c) If U is a $n \times n$ matrix such that $U^T U = I_n$, then $\det U = \pm 1$.
d) If A is a $n \times n$ matrix such that $\det\left(A^3\right) = 0$, then A is not invertible.

Exercise 17. Solve the following linear systems using Cramer's rule:

a)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

b)

$$\begin{array}{rcrcrcl} x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 2 \\ 3x_1 & + & 7x_2 & + & 2x_3 & = & 1 \end{array}$$