The Growth Model in Continuous Time (Ramsey Model)

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Econ720

October 31, 2018

The Growth Model in Continuous Time

We add optimizing households to the Solow model.

We first study the planner's problem, then the CE.

Planning Problem

Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{1}$$

subject to the resource constraint

$$\begin{aligned} \dot{k}_t &= f(k_t) - (n+\delta)k_t - c_t && (2) \\ & k_0 \text{ given} && (3) \\ k_t &\geq 0 && (4) \end{aligned}$$

Planning Problem

The current value Hamiltonian is

The state is k and the control is c.

The optimality conditions are

Planner: TVC

The TVC is:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t) = 0 \tag{5}$$

To check this:

- we need u and $g(k,c)$ to be monotone
- u is obvious.
- $g(k,c) = f(k) - c - \delta k$ is monotone in c but not k.
- However, we "know" that k never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.
- Then g is increasing in k.

Sufficiency

This is an example where the easiest (1st) set of sufficiency conditions applies:

- u is strictly concave in c (only).
- $g(k,c)$ is jointly concave in k and c.

First order conditions are sufficient.

Planner: Solution

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

1. The first-order conditions (2)
2. The resource constraint
3. The boundary conditions k_0 given and the TVC

$$\lim e^{-(\rho-n)t} \mu_t k_t = 0 \tag{6}$$

Planner: Euler Equation

We eliminate the multiplier.

Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \tag{7}$$

and therefore

$$\dot{\mu}/\mu = u''(c)\dot{c}/u'(c) \tag{8}$$

Substitute into the law of motion for μ:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \tag{9}$$

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \tag{10}$$

where

$$\sigma = -u''_c c/u' \tag{11}$$

$$= -\frac{du'(c)}{dc}\frac{c}{u'(c)} \tag{12}$$

is the intertemporal elasticity of substitution (and the coefficient of relative risk aversion).

Note: $u(c) = c^{1-\phi}/1-\phi$ implies $\sigma = \phi$.

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \tag{13}$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \tag{14}$$

The same idea:

- consumption growth rises with the interest rate
- declines with the discount rate.

Planner: Summary

- The planner's problem solves for functions of time $c(t)$ and $k(t)$.
- These satisfy two differential equations

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \tag{15}$$

$$\dot{k} = f(k) - (n+\delta)k - c \tag{16}$$

and two boundary conditions

$$k_0 \text{ given}$$

$$\lim_{t \to \infty} \beta^t u'(c(t)) \; k(t) = 0$$

- How can we analyze the dynamics of this system?

Phase Diagram

Phase Diagram

- Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.
- Consider the example

$$\begin{aligned}\dot{x} &= A - ax + by \\ \dot{y} &= B + cx - dy\end{aligned}$$

- Assume $a, b, c, d > 0$.

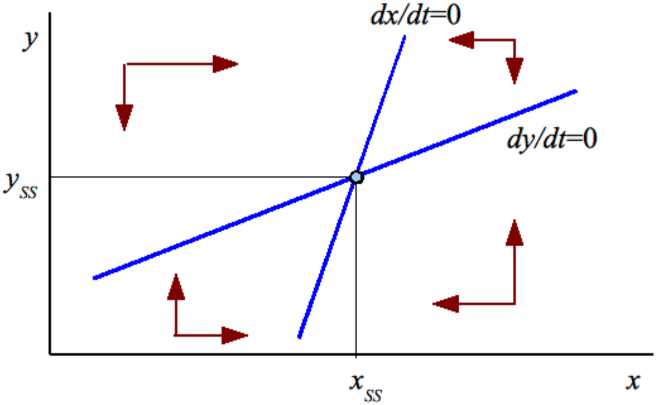
Phase Diagram: Steps

- Step 1: In an (x, y) plane, plot combinations of (x, y) that yield $\dot{x} = 0$ or $\dot{y} = 0$.

$$\begin{aligned} \dot{x} &= 0 \Rightarrow y = \frac{ax - A}{b} \\ \dot{y} &= 0 \Rightarrow y = \frac{B + cx}{d} \end{aligned}$$

- Step 2: Find out in which direction the system moves when off the $\dot{x} = 0$ or $\dot{y} = 0$ lines.
  - raise x: $\dot{x}$ falls - move left
  - raise y: $\dot{y}$ falls - move down
- Step 3: Divide phase diagram into 4 quadrants.
  - draw arrows of movement and think...

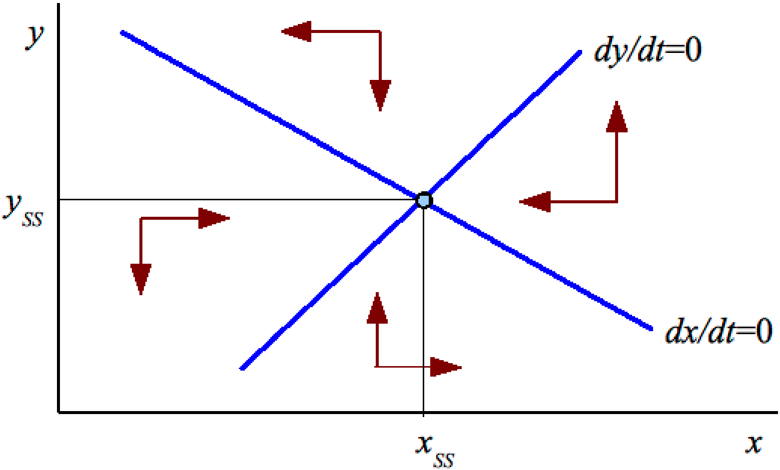
Phase Diagram



Recall: $\dot{x} = A - ax + by$. $\dot{y} = B + cx - dy$.

The steady state is stable.

Phase Diagram



With other coefficients: there are oscillations.

Applications

Galor (2000)

- studies transition from Malthusian stagnation to industrialization using a sequence of phase diagrams

Models of human capital accumulation over the life-cycle:

- Heckman (1976)

Phase Diagram: Growth Model

The $\dot{c} = 0$ locus is characterized by

$$f'(k^*) = \rho + \delta \tag{17}$$

The $\dot{k} = 0$ locus is hump-shaped:

$$c = f(k) - (n + \delta)k \tag{18}$$

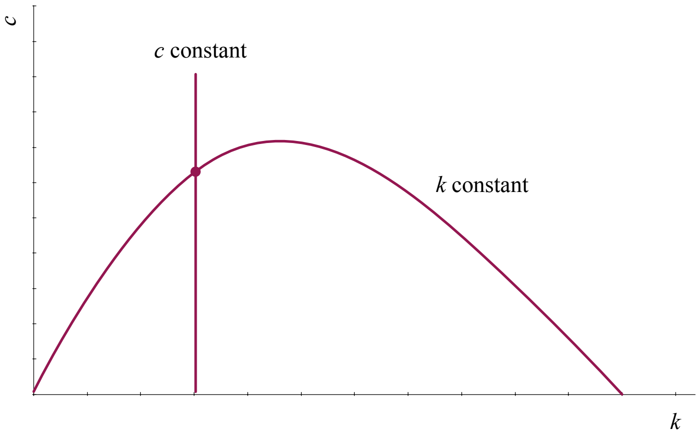
with a maximum at

$$f'(k^*) = n + \delta \tag{19}$$

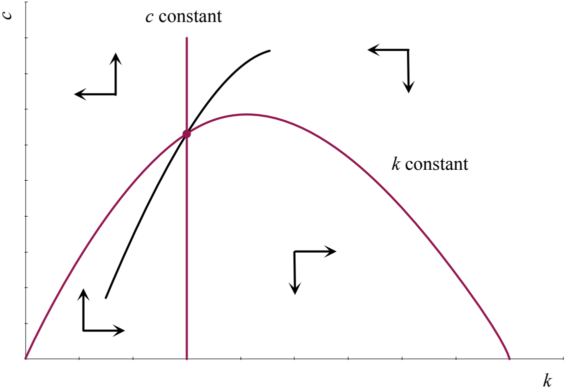
Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

The steady state is located at the intersection of the two curves.

Phase Diagram



Dynamics



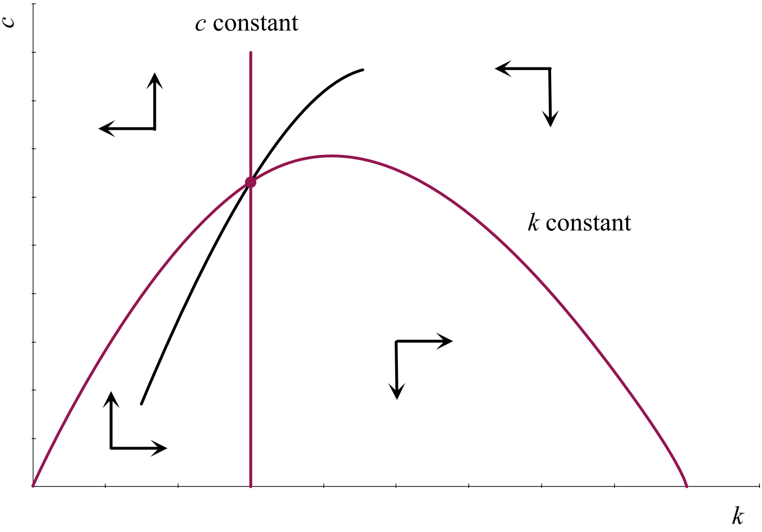
$\dot{k} = f(k) - (n+\delta)k - c$

- $c \uparrow \implies \dot{k} \downarrow$
- $k \uparrow \implies \dot{k} \downarrow$

$g(c) = \frac{f'(k) - \delta - \rho}{\sigma}$

- $k \uparrow \implies \dot{c} \downarrow$

Dynamics: Possible Paths



Ruling out the "north-west" path

$g(c)$ rises over time as $k \to 0$.

Eventually, this violates feasibility.

Ruling out the "south-east" path

Properties of that path:

- $c \to 0 \implies k \to k_{max} > k_{GR}$
- Euler: $g(u') = \rho + \delta - f'(k)$

Note:

- Even though $g(c)$ is strictly negative, $\dot{c} \to 0$. Therefore c does not turn negative.
- Any such path asymptotes towards $c = 0$ and $k = k_{max}$.

Transversality

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t) k_t = 0 \tag{20}$$

This is violated because

$$g(u') - \rho - n = -\left[f'(k) - \delta - n\right] > 0 \tag{21}$$

when $k > k_{GR}$

Dynamics: Saddle-path Stability

Only one value of c avoids moving into “forbidden” regions for given k.

For this c, the economy converges to the steady state.

Such a system is called "saddle-path stable."

Technical notes: Unique saddle path

Theorem

Take as given
$\dot{x}(t) = G[x(t)]$ *with initial value* $x(0)$ *given, where* G *is continuously differentiable.*
The steady state is $G(x^*) = 0$. *Define* $A = DG(x^*)$.
Suppose that m *eigenvalues of* A *have negative real parts while* $n - m$ *have positive real parts.*
Then there exists an m *dimensional manifold in the neighborhood around the steady state such that starting from any* $x(0)$ *in that manifold a unique* $x(t) \rightarrow x^*$.

See Acemoglu (2009), Theorem 7.15.

What this says in words

Suppose we have a system of $n = 2$ differential equations (in c and k).

The local dynamics around the steady state can be approximated by a linear differential equation with matrix A.

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = A \begin{bmatrix} k \\ c \end{bmatrix} \tag{22}$$

If that matrix has $m = 1$ negative eigenvalues, then **locally** around the steady state there is a line (dimension $m = 1$) of points (c, k) that converge to the steady state.

- This is the saddle path.

Other points could, in principle, converge as well, but we can rule that out as above.

Application to the growth model

First, establish that the saddle path is locally unique.

Start from a linear approximation to the two differential equations:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*) \sigma & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix} \tag{23}$$

The eigenvalues solve

$$\begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*) \sigma & 0 \end{bmatrix} x = \lambda x \tag{24}$$

or

$$\det \begin{bmatrix} f'(k^*) - n - \delta - \lambda & -1 \\ c^* f''(k^*) \sigma & 0 - \lambda \end{bmatrix} = 0 \tag{25}$$

Application to the growth model

Therefore

$$\lambda = 0.5\left\{f'(k^*) - n - \delta + 0 \pm \sqrt{-4c^* f'' \sigma + (f' - n - \delta)^2}\right\} \quad (26)$$

Since the square root term is positive and greater in absolute value than $f' - n - \delta$, there is exactly one negative eigenvalue.

Therefore: in a neighborhood of the steady state, the saddle path is unique.

Application to the growth model

How do we know that the saddle is globally unique?

Define one saddle path that converges.

Take a point not on it. We know:

1. The path cannot reach or cross the saddle path in finite time.
2. The path cannot asymptote to the saddle path because that would get into a neighborhood of the steady state where the saddle is unique.
3. Therefore, the path cannot converge to the steady state.

Reading

- Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro and Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- Blanchard and Fischer (1989), ch. 2
- Romer (2011), ch. 2A
- Phase diagram: Barro and Martin (1995), ch. 2.6

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Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Galor, O. (2000): "Ability Biased Technological Transition, Wage Inequality, and Economic Growth," *Quarterly Journal of Economics*, 115, 469–498.

Heckman, J. J. (1976): "A Life-Cycle Model of Earnings, Learning, and Consumption," *Journal of Political Economy*, 84, pp. S11–S44.

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