Problem Set 3: Growth Model in Discrete Time

Econ720. Fall 2017. Prof. Lutz Hendricks. September 29, 2017

1 Bonds Of Different Maturities

This question examines Ricardian Equivalence when the government has bonds of different maturities to finance spending. Consider a standard growth model in discrete time where the government issues two types of bonds:

- b_{t+1} one-period bonds are issued at date t; each has a price of 1 and pay R_{t+1} units of consumption at $t+1$.
- $B_{t+1} - B_t$ infinitely lived bonds are issued at date t; each costs p_t and pays one unit of consumption at dates $s \geq t+1$.

The government also imposes a lump-sum tax τ_t and spends g_t units of the good on a useless purpose.

Firms are standard with first-order conditions $r = f'(k)$ and $w = f(k) - f'(k)\,k$.

(a) The household maximizes

$$\sum_{t=0}^{\infty} \beta^t \, u(c_t)$$

subject to the budget constraint

$$k_{t+1} + c_t + \tau_t + b_{t+1} + p_t\,(B_{t+1} - B_t) = (r_t + 1 - \delta)\,k_t + w_t + R_t\,b_t + B_t$$

with initial endowments (k_0, b_0, B_0) given. Solve the household problem using Dynamic Programming.

(b) The government budget constraint is

$$g_t + R_t\,b_t + B_t = \tau_t + b_{t+1} + p_t(B_{t+1} - B_t)$$

Show that the present value budget constraint of the government can be written as

$$b_0 + (1 + p_0)\,B_0/R_0 = \sum_{t=0}^{\infty} \frac{\tau_t - g_t}{D_t}$$

where $D_t = R_0 \cdot ... \cdot R_t$ is a cumulative discount factor.

(c) Show that the household's present value budget constraint is given by

$$b_0 + \frac{(1 + p_0)\,B_0}{R_0} + k_0 = \sum_{t=0}^{\infty} \frac{c_t + \tau_t - w_t}{D_t}$$

(d) Show that Ricardian Equivalence holds in this economy. That is, a change in the timing of taxation does not affect the equilibrium allocation (for a given sequence g_t). The best way of answering this part is to define a competitive equilibrium in such a way that a set of equations that does not depend on $\tau's$ determines the allocation.

2 Wealth in the utility function

Consider the following modification of the standard growth model where the households derives utility from holding wealth.

Demographics: There is a representative household of unit mass who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$ where c_t is consumption and k_{t-1} is last period's capital (wealth). The utility function is strictly concave and increasing in both arguments.

Endowments: At $t = 0$ the household is endowed with capital K_0. In each period the household works 1 unit of time.

Technologies:

$$K_{t+1} = A\,F(K_t, L_t) + (1 - \delta)\,K_t - c_t \tag{1}$$

The production function has constant returns to scale.

Markets: Production takes place in a representative firm which rents capital and labor from households. There are competitive markets for goods (price 1), capital rental (r_t), and labor rental (w_t).

1. State the household's dynamic program.
2. Derive and explain the conditions that characterize a solution to the household problem (in sequence language).
3. Define a competitive equilibrium.
4. Derive a single equation that determines the steady state capital stock.
5. Is the steady state unique? Explain the intuition why the steady state is or is not unique.

3 Answers

3.1 Answer: Bonds Of Different Maturities

(a) The Bellman equation is

$$V(k,b,B) = \max\, u\left([r+1-\delta]\,k + w - \tau + Rb + B - k' - b' - p(B'-B)\right) + \beta\, V(k',b',B')$$

The first-order conditions may be written as

$$\begin{aligned} u'(c) &= \beta\, R'\, u'(c') \\ R &= r+1-\delta \\ R' &= (1+p')/p \end{aligned}$$

A solution consists of sequences (c,k,b,B) that satisfy the 3 FOCs and the budget constraints. Boundary conditions are k_0, b_0, B_0 given and the transversality condition

$$\lim_{t\to\infty} \beta^t\, u'(c_t)\,[k_t + b_t + p_t B_t] = 0$$

(b) This is a standard forward replacement argument. Start from

$$b_t = R_t^{-1}\,\left[\tau_t - g_t + p_t\, B_{t+1} - (1+p_t)\, B_t + b_{t+1}\right]$$

For $t = 0$:

$$b_0 = \frac{\tau_0 - g_0}{R_0} + \frac{p_0\, B_1 - (1+p_0)\, B_0}{R_0} + \frac{\tau_1 - g_1 + p_1\, B_2 + (1+p_1)\, B_1 + b_2}{R_0\, R_1}$$

Iterating over this implies

$$\begin{aligned} b_0 &= \sum_{t=0}^{\infty} \frac{\tau_t - g_t}{D_t} + \sum_{t=0}^{\infty} \frac{p_t\, B_{t+1} - (1+p_t)\, B_t}{D_t} \\ &= \sum_{t=0}^{\infty} \frac{\tau_t - g_t}{D_t} + \sum_{t=1}^{\infty} \left[\frac{p_{t-1}\, B_t}{D_{t-1}} - \frac{(1+p_t)\, B_t}{D_t}\right] - (1+p_0)\, B_0/R_0 \end{aligned}$$

Note that the lower bound of the sum has been changed to 1 and the only B_0 term has been pulled out of the sum. Next, I show that the term in the square brackets equals zero for each t. To see this, note that

$$\frac{p_{t-1}\, B_t}{D_{t-1}} = \frac{p_{t-1} R_t\, B_t}{D_t} = \frac{(1+p_t)\, B_t}{D_t}$$

where the last equality holds because all assets pay the same rate of return. The intuition why the term in square brackets is zero is that future bond issues change the timing of government surpluses, but it does not add to the present value of resources the government can spend.

(c) The household budget constraint is given by

$$b_t = R_t^{-1}\,\left[c_t + \tau_t - w_t + k_{t+1} - R_t k_t + p_t\, B_{t+1} - (1+p_t)\, B_t + b_{t+1}\right]$$

Iterating over this expression yields

$$\begin{aligned}
b_0 &= \sum_{t=0}^{\infty} D_t^{-1}\left[c_t + \tau_t - w_t + k_{t+1} - R_t k_t + p_t\, B_{t+1} - (1+p_t)\, B_t\right] \\
&= \sum_{t=0}^{\infty} D_t^{-1}\left[c_t + \tau_t - w_t + k_{t+1} - k_{t+1} + p_t\, B_{t+1} - R_{t+1}\,(1+p_{t+1})\, B_{t+1}\right] - \frac{(1+p_0)\, B_0}{R_0} + k_0
\end{aligned}$$

The second equation is obtained by pulling the date 0 terms out of the sum. By the same argument as for the government we find that $p_t B_{t+1} - R_{t+1}\,(1+p_{t+1})\, B_{t+1} = 0$ and the asserted budget constraint follows. The logic is again that the present value of future dissaving must equal current wealth.

(d) Substitute the government budget constraint into the household budget constraint. Define an equilibrium as sequences $\{c_t, k_t, \tau_t, b_t, B_t, r_t, w_t, p_t\}$ that satisfy:

- Household: $\{c_t, k_t\}$ solve the Euler equation $u'(c_t) = (r_{t+1} + 1 - \delta)\,\beta\, u'(c_{t+1})$ and the present value budget constraint.
- Firms: $\{r_t, w_t\}$ solve the 2 first-order conditions
- Government: $\{b_t, B_t, \tau_t\}$ solve the government present value budget constraint and the flow budget constraint (some indeterminacy remains)
- The goods market clears: $f(k_t) + (1-\delta)\, k_t = k_{t+1} + c_t + g_t$.

Now the system of equilibrium conditions is block-recursive. The household, firm and market clearing conditions determine $\{c_t, k_t, r_t, w_t, p_t\}$; they do not depend on tax rates. The remaining variables, $\{\tau_t, b_t, B_t\}$, are determined by the government budget constraint. The timing of taxes is not determined.

3.2 Answer: Wealth in the utility function

(a) Bellman equation

$$V(k, z) = \max\, u(c, z) + \beta\, V(k', k) + \lambda\,\left[R\, k + w - c - k'\right]$$

where $z' = k$. Controls are c, k'.

Alternative: Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, k_{t-1}\right) + \lambda_t\left[R_t k_t + w_t - c_t - k_{t+1}\right]$$

(b) First-order conditions from DP:

$$\begin{aligned}
u_c(c, z) &= \lambda \\
\beta\, V_k(k', k) &= \lambda
\end{aligned}$$

Envelope conditions:

$$
\begin{aligned}
V_k(k,z) &= \beta\, V_z(k',z') + \lambda\, R \\
V_z(k,z) &= u_z(c,z)
\end{aligned}
$$

Euler equation:

$$u_c(c,z) = \beta\, R'\, u_c(c',z') + \beta^2\, u_z(c'',z'')$$

Solution: Sequences (c_t, k_t) that solve the Euler equation, budget constraint, and transversality condition. Intuition: The additional u_z term in the Euler equation reflects the effect of raising k' in the usual perturbation.

Alternative: Lagrangian FOCs:

$$
\begin{aligned}
c_t &: \beta^t u_c\,(t) = \lambda_t \\
k_{t+1} &: \beta^{t+2} u_k\,(t+2) = \lambda_t + \lambda_{t+1} R_{t+1}
\end{aligned}
$$

(c) CE: Totally standard. $\{c_t, k_{t+1}, w_t, R_t\}$ that satisfy: Household Euler equation and budget constraint. 2 firm first-order conditions. Market clearing: $K_t = k_t$, $L_t = 1$, $F\,(K_t, L_t) + (1-\delta)\,K_t = c_t + K_{t+1}$.

(d) The Euler equation implies in steady state:

$$1 = \beta\, \left[f'(k) + 1 - \delta\right] + \beta^2\, \frac{u_z\,(f(k) - \delta\, k, k)}{u_c\,(f(k) - \delta\, k, k)}$$

(e) Steady state is generally not unique. Household may choose low c and high k or vice versa.