## 1 Deterministic Fruit Tree Prices

Consider the following version of a "fruit tree" economy a la Lucas. There is a unit measure of identical, infinitely lived households. Each receives an endowment of  $e_t$  in each period. In addition, each household owns one tree at the beginning of time (date 1) which yields  $d_t$  units of the consumption good in each period. Goods cannot be stored, but trees live forever. Trees can be bought or sold at a price of  $p_t$ , which is of course endogenous. The number of trees in the economy cannot be altered; it is always 1 per household.

Hence, the household solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t)$$

subject to

$$p_t k_{t+1} = k_t (d_t + p_t) + e_t - c_t$$

with  $k_1 = 1$ . In the budget constraint, the household receives as income the endowment  $e_t$  and capital income proportional to the number of trees owned  $(k_t)$ . This is spent on consumption ct and the purchase of new trees  $(k_{t+1})$ .

- (a) State the household problem as a Dynamic Program.
- (b) State the conditions that define a solution to the household problem.
- (c) Define a competitive equilibrium.
- (d) Assume that  $e_t$  and  $d_t$  are constant over time. Derive the steady state price of a tree (p) and the steady state rate of return of holding a tree.

## 1.1 Answer: Deterministic Fruit Tree Prices

(a) Bellman equation

$$V(k) = \max u \left( e + k(d+p) - pk' \right) + \beta V(k')$$

(b) A solution consists of sequences  $(c_t, k_t)$  that satisfy the budget constraint and the Euler equation

$$u'(c) = \beta u'(c')(d' + p')/p$$

- (c) A CE consists of sequences  $(c_t, k_t, p_t)$  that satisfy the 2 household optimality conditions and the market clearing conditions  $k_t = 1$  and  $c_t = e_t + d_t$ .
- (d) The EE implies that  $p = d\beta/(1-\beta)$ .

## 2 Deterministic Asset Prices

Consider the following deterministic Lucas fruit tree economy. There is a single representative agent who values consumption streams according to  $\sum_{t=0}^{\infty} \beta^t u(c_t)$ . There are (measure)  $K_i > 0$  fruit trees of type  $i \in \{1,2\}$  which yield exogenous dividend streams of  $d_{it} > 0$  units of the good. Trees are traded at (endogenous) prices  $p_{it}$ . The household also receives an endowment stream  $e_t$ .

(a) Solve the household problem using Dynamic Programming. Note that the household's budget constraint is

$$e_t + \sum_{i=1}^{2} (p_{it} + d_{it}) k_{it} = c_t + \sum_{i=1}^{2} p_{it} k_{i,t+1}$$

where  $k_{it}$  denotes the amount of type i trees held by the household in period t.

(b) Define a competitive equilibrium.

- (c) Solve in closed form for the prices of all trees in terms of (future) endowments and dividends. *Hint:* Note that the marginal rate of substitution terms  $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$  are essentially exogenous because agents eat essentially exogenous amounts in each period in equilibrium.
- (d) Suppose that endowments and dividends cycle. The endowments cycle between  $e_t = e^{odd}$  in odd periods and  $e_t = e^{even}$  in even periods. Assume  $e^{odd} < e^{even}$ . Tree 1 pays  $d_{1t} = d$  in odd periods and nothing in even periods. Tree 2 pays  $d_{2t} = d$  in even periods and nothing in odd periods. Calculate the asset price ratio,  $p_{1t}/p_{2t}$ , for odd and even t. Explain what you find. If you can't solve this, explain what you would expect to find.

## 2.1 Answer: Deterministic Asset Prices

(a) The Bellman equation is

$$V(k_1, k_2) = \max u \left( e + \sum (p_i + d_i) k_i - \sum p_i k_i' \right) + \beta V(k_1', k_2')$$

The Euler equation is standard:

$$u'(c) = \beta R_i' u'(c')$$

where  $R'_i = (p'_i + d'_i)/p_i$  is the rate of return of a type *i* tree. A solution consists of sequences  $c_t$  and  $a_t = \sum p_{it} k_{it}$  which satisfy the Euler equation and the flow budget constraint (and a TVC). Note that the portfolio composition is indeterminate.

- (b) A competitive equilibrium is a set of sequences  $(c_t, k_{it}, p_{it})$  that satisfy: 2 household conditions;  $k_{it} = K_{it}$ ;  $c_t = e_t + \sum d_{it} k_{it}$ . Finally, for asset markets to clear, all trees must yield the same rate of return:  $R_{it} = R_t$ .
- (c) This is standard iterating over a difference equation stuff:

$$p_{it} = \beta \alpha_{t+1} (d_{it+1} + p_{it+1})$$

$$= \sum_{j=1}^{\infty} \beta^{j} \alpha_{t+j} d_{it+j}$$
(1)

where  $\alpha_{t+j} = u'(c_{t+j})/u'(c_t)$  is the marginal rate of substitution (which is essentially exogenous).

(d) Note that  $\alpha_{t+j}$  now takes on only 2 values. If t is even, then  $c_t = e^{even} = e^{even} + dK_2$ .

If t is odd, then  $c_t = c^{odd} = e^{odd} + dK_1$ . Therefore, if t is odd, then  $\alpha_{t-1+2j} = \alpha^{odd} = u'(c^{even})/u'(c^{odd})$  and  $\alpha_{t+2j} = 1$ . But if t is even, then  $\alpha_{t-1+2j} = \alpha^{even}$  and  $\alpha_{t+2j} = 1$ . In words: The MRS oscillates between two values. This is helpful for pricing the assets because each asset pays either in periods with  $MRS = \alpha^{even}$  or with  $MRS = \alpha^{odd}$ .

Now consider an **even period** t. Asset 1 pays in odd periods (t+1, t+3, ...). Hence, from (1), its price is given by

$$p_1^{even} = \alpha^{even} d \left[ \beta + \beta^3 + \ldots \right] = \beta \sum_{i=0}^{\infty} \beta^{2i} \alpha^{even} d = \frac{\beta \alpha^{even} d}{1 - \beta^2}$$

Here I used the fact that

$$\sum_{j=0}^{\infty} \beta^{2j} = \sum_{j=0}^{\infty} (\beta^2)^j = \frac{1}{1 - \beta^2}$$

Similarly, tree 2 pays in even periods (t+2,t+4,...) and its price in an even period t must be

$$p_2^{even} = d\,\beta^2\,\left[1 + \beta^2 + \ldots\right] = \beta^2 \sum_{j=0}^{\infty} \beta^{2j}\,d = \frac{\beta^2\,d}{1 - \beta^2}$$

Hence, the price ratio in even periods is given by

$$PR^{even} = \frac{p_1^{even}}{p_2^{even}} = \frac{\alpha^{even}}{\beta}$$

If t is odd, the calculation is similar and yields

$$p_1^{odd} = \frac{\beta^2 d}{1 - \beta^2}$$

$$p_2^{odd} = \frac{\beta \alpha^{odd} d}{1 - \beta^2}$$

$$PR^{odd} = \frac{\beta}{\alpha^{odd}}$$

Assuming that  $c^{odd} < c^{even}$ , it follows that  $\alpha^{odd} < 1$  and  $\alpha^{even} > 1$ . Therefore,  $PR^{even} > 1$  and  $PR^{odd} < 1$ . The intuition is simple. Tree 1 yields fruit when consumption is low and marginal utility is high. Hence, it tends to be more valuable than tree 2. In even periods, tree 1 has the additional advantage of paying dividends one period earlier that tree 2 (in t+1 instead of t+2). Hence,  $PR^{even} > 1$ . However, in odd periods, tree 1 pays dividends one period later than tree 2 and  $PR^{odd}$  may be less than 1.