

Perpetual Youth Model

Prof. Lutz Hendricks

Econ720

November 14, 2018

Perpetual youth

- ▶ The standard growth model is very tractable.
- ▶ But it has an important limitation: all households are identical.
- ▶ For some questions, it is important to have households of **different ages**:
 - ▶ fiscal policies that redistribute across ages
 - ▶ models with life-cycle features: job search, matching, ...
- ▶ An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

Poisson Process

The Poisson process is the continuous time analog of i.i.d.

It is a counting process: it describes the distribution of the **number of events** occurring during a particular time interval.

$$\Pr(N(t) = n) = \frac{(vt)^n}{n!} e^{-vt} \quad (1)$$

The parameter $v > 0$ is the arrival rate:

$$\mathbb{E}\{N(t)\} = vt \quad (2)$$

Mental image:

- ▶ randomly distribute points on a real line
- ▶ on average, there are v points per unit length
- ▶ as time passes, move along the line and count the points

Poisson Process

The probability of an event over a short period ε is $v\varepsilon$.

- ▶ to show this: evaluate $\partial \mathbb{E}\{N(t+\varepsilon) | N(t)\} / \partial \varepsilon = v$

The probability of **no event** over a period of length τ is $\exp(-v\tau)$.

- ▶ the continuous time analogue of $(1-p)^t$

Demographics

At $t = 0$, there are $L(0) = 1$ identical persons.

- ▶ They are all newborns.

At each instant, $nL(t)$ identical persons are born.

- ▶ n is the Poisson arrival rate of newborns

Each person dies at each instant with **Poisson** probability v .

The population growth rate is $n - v > 0$:

$$L(t) = \exp([n - v]t) \quad (3)$$

Demographics

- ▶ The mass of persons at t aged $t - \tau$ is

$$\begin{aligned} L(t|\tau) &= \exp(-v(t-\tau)) \times n \exp((n-v)\tau) \\ &= \Pr(\text{live beyond } t - \tau) \, nL(\tau) \end{aligned}$$

- ▶ Notation: $x(t|\tau)$ means x at t for those born at τ .

Preferences

Households are indexed by i .

Conditional on surviving, households utility at date t is $e^{-\rho t} \ln(c_i(t))$.

The probability of being alive after t "periods" is $\exp(-vt)$.

Expected utility for date t is $e^{-vt} e^{-\rho t} \ln(c_i(t))$.

Expected lifetime utility is

$$\int_0^{\infty} e^{-(\rho+v)t} \ln(c_i(t)) dt \quad (4)$$

Interesting: mortality simply increases the discount factor: $\rho + v$.

Endowments

Households work 1 unit of time.

Newborn households do not own any assets.

This is now age matters: older households are richer.

Technology

- ▶ The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- ▶ In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \quad (5)$$

- ▶ $k = K/L$ is capital per capita and capital per worker.

Markets

Competitive markets for

- ▶ goods (numeraire)
- ▶ labor rental: w
- ▶ capital rental: q
- ▶ annuities...

Annuities

The problem: what to do with the wealth of households who die?

- ▶ “accidental bequests”

Assumption: households buy fair **annuities**.

Each cohort τ household gives $a(t|\tau)$ to the insurance company.

He gets paid:

1. interest $r(t)a(t|\tau)$
2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)|t, \tau) = \nu a(t|\tau) \quad (6)$$

Effectively, the interest rate, conditional on survival, is $r(t) + \nu$.

Firms

- ▶ A representative firm solves the standard problem.
- ▶ Factor prices are

$$q = f'(k)$$

$$w = f(k) - f'(k)k$$

Equilibrium

Definition

A CE is an allocation $[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty}$ and a price system $[w(t), q(t), r(t)]$ such that:

1. $c(t|\tau)$ and $a(t|\tau)$ solve the household's problem for cohort $t - \tau$.
2. $w(t)$ and $q(t)$ solve the firm's problem.
3. markets clear (below).
4. identities: $L(t), C(t), r(t) = q(t) - \delta$

Important: we have to keep track of assets and consumption by age.

Equilibrium

Market clearing:

- ▶ labor: implicit
- ▶ capital: $K(t) = \int_0^t L(t|\tau) a(t|\tau) d\tau$.
- ▶ goods: same as resource constraint.

Identities:

- ▶ $C(t) = \int_0^t L(t|\tau) c(t|\tau) d\tau$ etc

Math Digression: Leibniz's Rule

We want to differentiate an integral

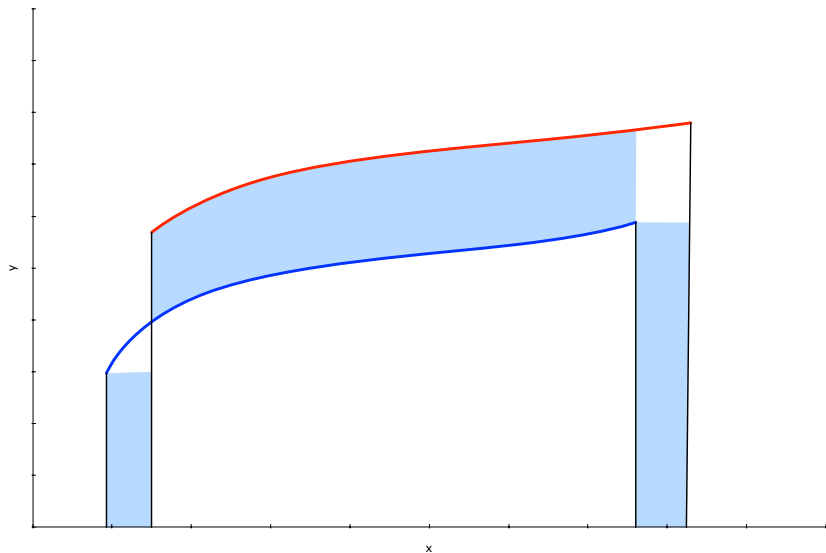
Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx \quad (7)$$

We have

$$\frac{\partial F}{\partial \theta} = f(b(\theta), \theta) b'(\theta) - f(a(\theta), \theta) a'(\theta) + \int_{a(\theta)}^{b(\theta)} f_{\theta}(x, \theta) dx \quad (8)$$

Leibniz's Rule



Households

The representative member of cohort τ solves

$$\max \int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau)) dt$$

subject to

$$\dot{a}(t|\tau) = [r(t) + \nu]a(t|\tau) - c(t|\tau) + w(t) \quad (9)$$

Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t) + v] - [\rho + v] = r(t) - \rho \quad (10)$$

budget constraint and TVC

$$\lim_{t \rightarrow \infty} D_{t,\tau} a(t|\tau) = 0 \quad (11)$$

where

$$D_{t,\tau} = \exp \left(- \int_{\tau}^t [r(z) + v] dz \right) \quad (12)$$

Notation

- ▶ $D_{t,\tau}$ discounts a date t payment to τ .
- ▶ $D_{\tau,t} = 1/D_{t,\tau}$ discounts a date τ payment to t .
- ▶ $PV(x, t) = \int_{s=t}^{\infty} D_{s,t} x(s) ds$ is the present value of x .

Household: PIH

Claim: the household consumes a constant fraction of wealth:

$$c(t|\tau) = (\rho + \nu)[a(t|\tau) + \omega(t)] \quad (13)$$

Human wealth is the present value of lifetime earnings

$$\omega(t) = PV(w, t) = \int_t^\infty D_{s,t} w(s) ds \quad (14)$$

Note: all persons alive at t have the same ω .

Intuition...

Proof: PIH

Claim: We have a standard present value budget constraint:

$$PV(c(.|\tau), \tau) = a(\tau|\tau) + \omega(\tau) \quad (15)$$

In words: present value of c = present value of earnings + initial assets.

Claim:

$$PV(c(.|\tau), \tau) = \frac{c(\tau|\tau)}{\rho + v} \quad (16)$$

Together, these imply $c(\tau|\tau) = (\rho + v)[a(\tau|\tau) + \omega(\tau)]$.

From the derivation, we see that this holds for any age, not just for $t = \tau$.

Present value of consumption I

Integrate the Euler equation to get consumption:

$$c(t|\tau) = c(\tau|\tau) \exp \left(\int_{\tau}^t [r(z) - \rho] dz \right) \quad (17)$$

Verify by differentiating and comparing with Euler.

Multiply both sides by $D_{t,\tau}$:

$$D_{t,\tau} c(t|\tau) = c(\tau|\tau) \exp \left(\int_{\tau}^t [r(z) - \rho - r(z) - v] dz \right) \quad (18)$$

$$= c(\tau|\tau) \exp(-[\rho + v][t - \tau]) \quad (19)$$

In words: The present value of $c(t|\tau)$ grows at a rate that equals the difference between the consumption growth rate and the interest rate.

Present value of consumption II

Present value of consumption

$$\int_{\tau}^{\infty} D_{t,\tau} c(t|\tau) dt = c(\tau|\tau) \int_{\tau}^{\infty} e^{-(\rho+v)t} dt = \frac{c(\tau|\tau)}{\rho+v} \quad (20)$$

Lifetime Budget Constraint

Claim:

$$D_{t,\tau}a(t|\tau) = a(\tau, \tau) + \int_{\tau}^t D_{z,\tau} [w(z) - c(z|\tau)] dz \quad (21)$$

In words: The present value of “terminal” assets $a(t|\tau)$ equals initial assets + the present value of savings.

Take $\lim_{t \rightarrow \infty}$ and the LHS goes to 0 due to TVC.

That gives the lifetime budget constraint b/c the RHS is $\omega - PV(c)$.

Lifetime budget constraint

To show that the claim implies the flow budget constraint:

Multiply by $D_{\tau,t}$:

$$a(t|\tau) = a(\tau|\tau) D_{\tau,t} + \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz \quad (22)$$

Differentiate with respect to t and check that the flow budget constraint

$$\dot{a}(t|\tau) = [r(t) + v]a(t|\tau) - c(t|\tau) + w(t) \quad (23)$$

emerges.

Lifetime Budget Constraint

$$\dot{a}(t|\tau) = a(\tau|\tau) \frac{\partial D_{\tau,t}}{\partial t} + D_{t,t} [w(t) - c(t|\tau)] + \int_{\tau}^t \frac{\partial D_{z,t} [w(z) - c(z|\tau)]}{\partial t} dz$$

and note that

1. $\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t} [r(t) - v]$, so that the first term becomes $(r(t) + v) a(\tau|\tau) D_{\tau,t}$
2. $D_{t,t} = \exp(0) = 1$, so that the second term becomes $w(t) - c(t|\tau)$
3. the 3rd term is

$$[r(t) + v] \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz = [r(t) + v] [a(t, \tau) - a(\tau|\tau) D_{\tau,t}]$$

Add all that up and the flow budget constraint emerges.

Summary

We now have a solution for the individual consumption function:

$$c(t|\tau) = (\rho + \nu) [a(t|\tau) + \omega(t)] \quad (24)$$

To characterize equilibrium, we need the aggregate consumption function:

$$c(t) = \int_{-\infty}^t L(t, \tau) c(t|\tau) d\tau / L(t) \quad (25)$$

A nice feature of this model: we can aggregate with paper and pencil.

Aggregation

$$c(t) = \int_{-\infty}^t L(t, \tau) c(t|\tau) d\tau / L(t) \quad (26)$$

$$= (\rho + \nu) [a(t) + \omega(t)] \quad (27)$$

where $a(t)$ and $\omega(t)$ are defined analogously.

This is a form of **aggregation**: Aggregate consumption behaves like individual consumption.

- ▶ As if a single individual made the choice.

The budget constraint aggregates in the same way.

How general is this?

Equilibrium Dynamics

It would be tempting to say:

- ▶ Euler is unchanged relative to growth model
- ▶ Resource constraint is unchanged
- ▶ Everything behaves like the growth model

But this would be wrong:

- ▶ each person has an Euler equation that looks “standard”
- ▶ that does not mean that aggregate consumption also behaves that way

Equilibrium Dynamics

- ▶ We have a system in c, a, ω .
- ▶ Equations: consumption function, budget constraint, def of lifetime wealth:

$$c(t) = (\rho + v)[a(t) + \omega(t)]$$

$$\dot{a}(t) = (r(t) - (n - v))a(t) + w(t) - c(t)$$

$$\omega(t) = \int_t^{\infty} \exp\left(-\int_t^s [r(l) + v] dl\right) w(s) ds$$

- ▶ The strategy: Derive an Euler equation for aggregate consumption by differentiating the $c(t)$ equation.

Equilibrium Dynamics

- Differentiate the consumption function:

$$\dot{c} = (\rho + v) [\dot{a} + \dot{\omega}] \quad (28)$$

- Sub in budget constraint for \dot{a} .
- Differentiate def of ω (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + v) \omega(t) - w(t) \quad (29)$$

- Sub that into \dot{c} and collect terms:

$$\dot{c}(t) = [r(t) - \rho] c(t) - (\rho + v) n a(t) \quad (30)$$

- Sub in $k(t) = a(t)$ and the firm foc for $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + v) n \frac{k(t)}{c(t)} \quad (31)$$

Intuition for $\omega(t)$

Think of human wealth as an asset with price $\omega(t)$.

Its instantaneous payoff consists of:

1. "dividend" $w(t)$
2. capital gain $\dot{\omega}(t)$

The asset price equals [required rate of return] \times [dividend + capital gain]

Required rate of return is $r(t) + v$.

$$[r(t) + v] \omega(t) = w(t) + \dot{\omega}(t) \quad (32)$$

Note: Differentiating $\omega(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + v] d\iota\right) w(s) ds \quad (33)$$

$\dot{\omega}(t)$ has 2 pieces:

1. Effect of changing lower bound of integral is integrand evaluated at $s = t$: $w(t)$.
2. Derivative of integrand w.r.to t :
 $-[r(t) + v] \omega(t) = \int_t^\infty w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + v] d\iota\right) ds.$

Now note that

$$\frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + v] d\iota\right) = \exp\left(-\int_t^s [r(\iota) + v] d\iota\right) \times [-(r(t) + v)].$$

Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + v)n \frac{k(t)}{c(t)} \quad (34)$$

$$\dot{k} = f(k) - c - (n - \delta - v)k \quad (35)$$

with boundary conditions $k(0)$ given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...

Steady state

$$\dot{c} = 0 \implies$$

$$c = \frac{(\rho + v)n}{f'(k) - \delta - \rho} k \quad (36)$$

Properties:

1. $k \longrightarrow 0 \implies c \longrightarrow 0$ [as $f' \longrightarrow \infty$]
2. $k \longrightarrow k^{MGR}$ where $f'(k^{MGR}) = \delta + \rho \implies c \longrightarrow \infty$
3. $c''(k) > 0$ [verify]

Steady state

$$\dot{k} = 0 \implies c = f(k) - (n + \delta - v)k \quad (37)$$

Properties: as the standard growth model.

Steady state

Solution for steady state k^*

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \quad (38)$$

Unique steady state k^* : $f(k)/k \searrow$ in k . $-1/f'(k) \searrow$ in k .

Dynamic efficiency

- ▶ **Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - v)k^* \quad (39)$$

$$f'(k_{GR}) - \delta = n - v \quad (40)$$

- ▶ Steady state:

$$f'(k^*) - \delta > \rho \quad (41)$$

[otherwise $c/k < 0$]

- ▶ There can be overaccumulation relative to the Golden Rule.
- ▶ This happens when households are sufficiently impatient (high ρ).
- ▶ Similar to the finite lifetime OLG model.

Dynamic efficiency

- ▶ **Modified Golden Rule** for planner with discount factor ρ [effects of mortality and "annuities" cancel]:

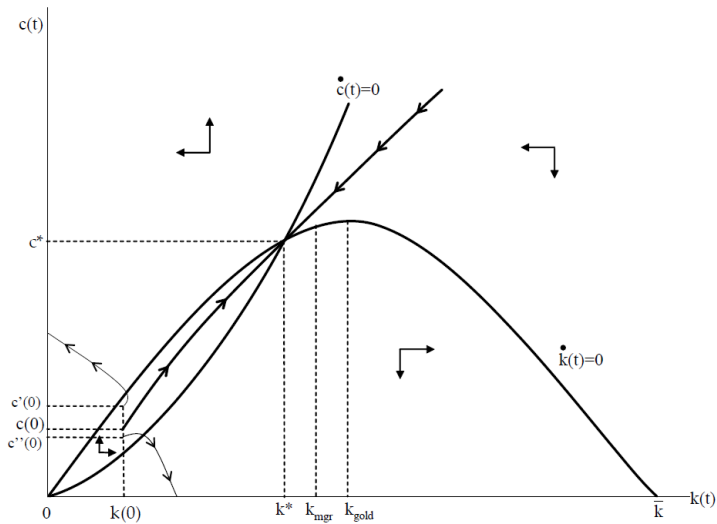
$$f'(k_{MGR}) - \delta = \rho \quad (42)$$

- ▶ Equilibrium avoids overaccumulation relative to MGR.
- ▶ This is not a robust feature of the model.
- ▶ Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- ▶ Example: labor efficiency declines with age.

Dynamic efficiency

- ▶ Finite lifetimes are not necessary to generate overaccumulation.
- ▶ In this model, it is the presence of overlapping generations that destroys the welfare theorems.

Phase diagram



Phase diagram

- ▶ The dynamics closely resemble the growth model.
- ▶ A unique, globally saddle path stable steady state exists.
- ▶ Convergence is monotone.
- ▶ An analytically tractable model with OLG.

Where Is This Used?

Models of human capital

- ▶ combine the convenience of an infinitely lived decision maker
- ▶ capture that only young invest in education
- ▶ Akyol and Athreya (2005)

Models of income / wealth distribution

- ▶ a version of perpetual youth: agents age stochastically
- ▶ Castaneda et al. (2003)

Reading

- ▶ Acemoglu (2009), ch. 9.7-9.8.
- ▶ Blanchard and Fischer (1989), ch. 3.3

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Akyol, A. and K. Athreya (2005): “Risky higher education and subsidies,” *Journal of Economic Dynamics and Control*, 29, 979–1023.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Castaneda, A., J. Diaz-Gimenez, and J. V. Rios-Rull (2003): “Accounting for the US earnings and wealth inequality,” *Journal of political economy*, 111, 818–857.