# Bewley Models

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# Bewley Models

For many applications we need models with **heterogeneous agents**.

In Bewley models, agents are ex ante identical.

They are ex post heterogeneous because they are hit by idiosyncratic shocks.

Incomplete markets prevents sharing these risks.

In this section, we study a simple endowment economy.

The goal is to get the mechanics down in a simple setting.

# Endowment Economy

# An Endowment Economy

- Demographics:
  - There is a unit measure of households.
  - Each lives forever.
- Preferences:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- Technology:
  - Households receive random endowments $y_t \in Y$ (finite).
  - Transition matrix: $\pi(y'|y)$.

# No aggregate uncertainty

- Assume a "law of large numbers."
- Let $\Pi(y)$ be the stationary distribution of $y$.
- Assume that the fraction of households with endowment $y$ is $\Pi(y)$.
- The aggregate endowment $\bar{y}$ is constant over time.
- With complete markets, households would not face any uncertainty.

# Household problem

- Flow budget constraint:

$$a' = y + (1+r)a - c \tag{2}$$

- Borrowing constraint:

$$a' \geq -b \tag{3}$$

# Household problem

- Focus on a **stationary equilibrium**.
  - Meaning: Aggregates & prices are constant over time.
- State vector: $(a, y)$.
- Given: $r$.
- Bellman equation:

$$v(a,y) = \max u(c) + \beta \sum_{y'} \pi(y'|y) \; v(a', y') \tag{4}$$

s.t. budget constraint and borrowing constraint.

# Household problem I

The borrowing constraint may bind. We need Kuhn-Tucker.

Bellman equation

$$
\begin{aligned}
v(a,y) &= \max u\left(y+(1+r)a-a'\right) && (5)\\
&\quad +\beta \sum_{y'} \pi\left(y'|y\right)\ v\left(a',y'\right)+\mu(a'+b) && (6)
\end{aligned}
$$

First-order conditions:

$$
\begin{aligned}
u'(c) &= \beta \sum \pi\left(y'|y\right) \frac{\partial v(a',y')}{\partial a'} + \mu && (7)\\
\partial v/\partial a &= u'(c)\ (1+r) && (8)\\
\mu(a'+b) &= 0 && (9)
\end{aligned}
$$

# Household problem

- Euler equation
$$u'(c) \geq E\beta(1+r)u'(c') \tag{10}$$
with equality if $a' > -b$.
- Solution: $v(a,y), c(a,y), a'(a,y)$ that satisfy the usual conditions.

# How does the household behave?

Assume (for the moment) that $y \sim iid$

Then (obviously?), consumption and saving depend on "cash on hand"

$$x = y + (1+r)a \tag{11}$$

Also (obviously?), $a'$ is increasing in $x$

If $x$ is sufficiently high: Choose $a' > -b$ and satisfy standard Euler equation.

If $x$ is below a cutoff, set $a' = -b$ and "violate" the Euler equation.

The borrowing constraint depresses current consumption.

# Stationary Recursive Competitive Equilibrium

- Aggregate state: The joint distribution of assets and endowments: $\Phi(a, y)$.
    - This is needed to compute aggregates.
- Objects:
    - Household: $v(a, y), c(a, y), a'(a, y)$.
    - $\Phi(a, y)$.
    - Price function: $r(\Phi)$.
- Equilibrium conditions:
    - Household: see above.
    - Market clearing.
    - Time invariance of $\Phi$.

# Stationary Recursive Competitive Equilibrium

## Market clearing

- Goods:

$$C = \int \int c(a,y)\,\Phi(da,dy) = \int y\Pi(dy) = \bar{y} \tag{12}$$

- Bonds:

$$\int \int a'(a,y)\,\Phi(da,dy) = 0 \tag{13}$$

# Time invariance of the distribution

- Informally, household choices determine tomorrow's distribution $\Phi'$.
- The policy function $a'(a,y)$ implies a law of motion for $\Phi$.
- In stationary equilibrium, the law of motion must imply $\Phi' = \Phi$.

# Law of motion for the distribution

- Define a transition function $Q((a,y),(A,Y))$.
- Its value is the probability (mass) of households in state $(a,y)$ today that end up in $(a',y') \in (A,Y)$ tomorrow.

$$Q((a,y),(A,Y)) = \begin{cases} \sum_{y' \in Y} \pi(y'|y) & \textit{if } a'(a,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

- This is because $a'$ is deterministic.

# Law of motion for the distribution

- Law of motion

$$\Phi'(A,Y) = \int \int Q((a,y),(A,Y)) \ \Phi(da,dy) \qquad (14)$$

- In words:
  - $\Phi'(A,Y)$: What is the mass of households in the set of states $(A,Y)$ tomorrow?
  - For any $(a,y)$, this is given by $Q((a,y),(A,Y))$.
  - Sum over all $(a,y)$ to get the total mass.
- Stationarity then means: $\Phi'(A,Y) = \Phi(A,Y)$ for all $(A,Y)$.

# Non-stationary Equilibrium

- The equilibrium concept generalizes easily to economies where $\Phi$ evolves over time.
- Household:
  - Add the aggregate state $\Phi$ to the household's state: $v(a, y, \Phi)$ and $a'(a, y, \Phi)$.
  - The household takes prices as functions of the aggregate state: $r(\Phi)$.
  - The household knows the law of motion for $\Phi$: $\Phi' = H(\Phi)$.
- Equilibrium:
  - Drop stationarity of $\Phi$.

# Where is this useful?

- Models of the wealth distribution:
  - Krusell and Smith (1998)
- Models of business cycles with heterogeneous agents:
  - Castaneda et al. (1998)

# Reading

- Acemoglu (2009), ch. 17.4.
- Krueger, "Macroeconomic Theory," ch. 10.

# References I


Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Castaneda, A., J. Dıaz-Giménez, and J.-V. Rıos-Rull (1998): "Exploring the income distribution business cycle dynamics," *Journal of Monetary economics*, 42, 93–130.

Krusell, P. and J. Smith, Anthony A. (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *The Journal of Political Economy*, 106, 867–896.