Review Questions: AK Models

Econ720. Fall 2019. Prof. Lutz Hendricks

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1 Growth model with two capital goods

Consider the following endogenous growth model with two capital goods.

Households: There is a single, representative household who lives forever. Preferences over consumption streams are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Households own two capital goods, K_1 and K_2. The income obtained from renting these capital goods to firms is their only source of income.

Firms: Production takes place in two sectors ($i = 1, 2$). The resource constraints for sector 1 is

$$A_1 F_1(K_{11t}, K_{12t}) + (1 - \delta) K_{1t} = K_{1t+1} + c_t$$

where K_{ist} is the amount of capital of type s used in sector i and $K_{st} = K_{1st} + K_{2st}$ is the total amount of capital good s used in both sectors. The resource constraint for sector 2 is similar, except that good 2 is not consumed:

$$A_2 F_2(K_{21t}, K_{22t}) + (1 - \delta) K_{2t} = K_{2t+1}$$

In each sector, a representative firm maximizes period profits. Assume that both production functions exhibit constant returns to scale.

(a) Define a solution to the firm's problem in each sector. Be careful to define the purchase and rental prices of the various goods consistently. Good 1 is the numeraire.

(b) State the household problem and define a solution.

(c) Define a competitive equilibrium. Make sure that the number of objects equals the number of equations.

(d) Consider the balanced growth path. Derive the balanced growth rates of c, k_s, r_s, p_s for $s = 1, 2$, where $k_s = K_{s2}/K_{s1}$ is the input ratio in sector s, r_s is the rental price of capital good s, and p_s is its purchase price. Assume log utility: $u(c) = \ln(c)$.

(e) Derive 7 equations that solve for 7 (constant) objects and thus define the balanced growth path.

(f) Using the 7 equations from (e), determine qualitatively how the balanced growth rate and prices change when A_1 rises.

1.1 Answer: Growth model with two capital goods

To begin, we define prices. r_{st} is the rental price of capital good s in terms of good 1.

(a) The firm in sector i solves

$$\max A_i F_i(K_{i1t}, K_{i2t}) p_i - \sum_s r_{st} K_{ist}$$

The first-order conditions are $r_s = A_i F_{is}(K_{i1}, K_{i2}) p_i$ for $s = 1, 2$. A solution is a pair (K_{i1t}, K_{i2t}) which satisfies the 2 first order conditions.

(a) We anticipate that both capital goods must pay the same rate of return in equilibrium; call it R. Denote household wealth by $a_t = p_{1t} K_{1t} + p_{2t} K_{2t}$. Then the budget constraint is $a_{t+1} = R_t a_t - c_t$. The household problem is entirely standard with Euler equation $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. A solution is a sequence (c_t, a_t) which satisfies Euler equation and budget constraint (and a transversality condition).

(c) A competitive equilibrium is a set of sequences $(c_t, a_t, K_{it}, K_{ist}, r_{it}, p_{it}, R_t)$ (13 objects) which satisfy:

- 2 household conditions (see b).
- 4 firm conditions (see a).

- Definition of the rate of return: $R_{t+1} = [(1-\delta)\,p_{st+1} + r_{st+1}]\,/p_{st};\ \ s = 1, 2$. Giving up $1/p_{st}$ units of good s today and investing the good as capital pays $[(1-\delta)\,p_{st+1} + r_{st+1}]$ units of the same good tomorrow. This rate of return must be the same for both goods (2 equations)
- Goods market clearing in both sectors (given in the question).
- Capital market clearing (also given): $K_{st} = \sum_i K_{ist}$.
- Definition of a_t.
- The normalization $p_{1t} = 1$.

There are $2+4+2+2+2+1 = 14$ equations. One is redundant by Walras' law.

(d) We know that R must be constant, otherwise the consumption growth rate would not be. By the definition of R, this requires constant prices and rental prices. The quantities grow, all at rate γ.

(e) The Euler equation implies

$$1 + \gamma = \beta R.$$

The definition of R yields 2 additional equations

$$R = 1 - \delta + r_s/p_s.$$

The firms' first-order conditions are

$$r_s = A_i p_i \ F_{is}\,(K_{i1}, K_{i2})$$

Note that the marginal products (b/c of constant returns to scale) only depend on the inputs ratios $k_i = K_{i2}/K_{i1}$:

$$r_s = A_i p_i \ F_{is}\,(1, k_i)$$

(slightly abusing notation) (4 equations). With better notation: define $f_i(k_i) = F_i(1, K_{i2}/K_{i1})$. Then the firms' FOCs become

$$r_1 = A_i\,p_i\ [f_i(k_i) - f_i'(k_i)\,k_i] \tag{1}$$

$$r_2 = A_i\,p_i\,f_i'(k_i) \tag{2}$$

for $i = 1, 2$. This is entirely analogous to a model with capital and labor. Note that a higher k_i reduces $f_i'(k_i)$ but increases $f_i(k_i) - f_i'(k_i)\,k_i$.

(f) Take the ratio of (2), (1) for both sectors and write this as $r_1/r_2 = g_i(k_i)$. Note that $g_i'(k_i) > 0$. From $g_1(k_1) = g_2(k_2)$ it follows that k_1 and k_2 are positively related. Define this relationship as $k_2 = h(k_1) = g_2^{-1}(g_1(k_1))$. The positive relationship is not surprising. When r_1/r_2 increases, firms in both sectors substitute towards the cheaper capital good. Now consider the condition

$$r_1 = A_1\,[f_1(k_1) - f_1'(k_1)\,k_1] = A_2\,f_2'(k_2) = r_2/p_2 \tag{3}$$

This can be written as

$$\frac{f_1(k_1) - f_1'(k_1)\,k_1}{f_2'(h(k_1))} = \frac{A_2}{A_1} \tag{4}$$

The LHS of (4) is increasing in k_1. It follows that a higher A_1 reduces k_1 and k_2. The intuition is that K_1 is cheaper to produce and used relative more intensively. A lower k_2 implies a higher $r_1 = r_2/p_2$ by (3). Therefore R and the balanced growth rate γ must both increase.

2 Endogenous growth with two capital goods[1]

The economy is populated by a single representative consumer. Production of a single good takes place according to the production function

$$Y(t) = K_1(t)^\alpha\,K_2(t)^{1-\alpha}$$

[1] Based on a question due to Dirk Krueger.

A representative firm rents the two capital stocks at rental prices $r_i(t)$ from households. Capital is accumulated according to

$$\dot{K}_i(t) = x_i(t) - \delta\, K_i(t)$$

Assume that capital stocks can be adjusted instantaneously and that $\dot{K}_i(t)$ may be negative (you will see the significance of this information when working out the first-order conditions). Household preferences are

$$\int_0^\infty e^{-\rho\, t}\, \frac{c(t)^{1-\sigma}}{1-\sigma}\, dt$$

The aggregate resource constraint takes the form

$$Y(t) = c(t) + x_1(t) + x_2(t)$$

(a) Write down the social planner's Hamiltonian. What are the planner's state and control variables?

(b) Solve for the growth rate of consumption in closed form. Under what conditions is there endogenous growth?

(c) Now consider the competitive equilibrium. Investment into capital good 2 is taxed at rate τ, so that the household's budget constraint is given by

$$c(t) + x_1(t) + (1+\tau)\, x_2(t) = r_1(t)\, K_1(t) + r_2(t)\, K_2(t) + T(t)$$

where $T(t)$ is the lump-sum rebate from the tax. The government budget constraint is $\tau\, x_2(t) = T(t)$. Write down the household's Hamiltonian and solve for the optimal consumption growth rate as a function of prices and tax rates.

(d) Derive the equilibrium interest rate $r_1(t)$ from the firm's optimality conditions. Hint: Use the fact that in equilibrium $r_2(t)/r_1(t) = 1+\tau$ must be true.

(e) Derive the growth rate of consumption in competitive equilibrium. What is the tax rate $\tau \geq 0$ that maximizes the growth rate? Which tax rate maximizes welfare?

2.1 Endogenous growth with two capital goods

(a) The planner's Hamiltonian is

$$H = \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\left\{K_1(t)^\alpha\, K_2(t)^{1-\alpha} - \delta\, K_1(t) - x_2(t) - c(t)\right\} + \mu(t)\left\{x_2(t) - \delta\, K_2(t)\right\}$$

The states are $K_1(t)$ and $K_2(t)$. Note that one could also set up the problem with a single state variable, $K_1(t)+K_2(t)$ as long as the capital stocks can be converted into each other instantaneously. The controls are $c(t)$ and $x_2(t)$.

(b) The planner's first order conditions are

$$\begin{aligned}
c(t)^{-\sigma} &= \lambda(t) \\
\mu(t) &= \lambda(t) \\
\dot{\lambda}(t) &= \lambda(t)\, \left\{\rho - \alpha\, [K_1(t)/K_2(t)]^{\alpha-1} + \delta\right\} \\
\dot{\mu}(t) &= (\rho+\delta)\, \mu(t) - \lambda(t)\, (1-\alpha)\, [K_1(t)/K_2(t)]^{\alpha}
\end{aligned}$$

After a little bit of manipulation the standard Euler equation emerges in two versions:

$$\begin{aligned}
g(c(t)) &= \{(1-\alpha)\, [K_1(t)/K_2(t)]^{\alpha} - \delta - \rho\}/\sigma \\
&= \left\{\alpha [K_1(t)/K_2(t)]^{1-\alpha} - \delta - \rho\right\}/\sigma
\end{aligned}$$

Setting both equal implies

$$K_1(t)/K_2(t) = \alpha/(1-\alpha) \tag{5}$$

This is an important equation because it implies that this two-sector model is *equivalent to an AK model.* It has no transitional dynamics. At time 0, the planner immediately adjusts the ratio of capital stocks to match the target ratio (5). Thereafter, the growth rate is constant (as in an AK model) and given by the Euler equation:

$$g(c(t)) = \left\{\alpha\, [(1-\alpha)/\alpha]^{1-\alpha} - \delta - \rho\right\}/\sigma \tag{6}$$

Endogenous growth occurs, if the term in the brackets is positive.

(c) The household's Hamiltonian is given by

$$H = \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\,\{[r_1(t)-\delta]K_1(t) + r_2(t)\,K_2(t) + T(t) - (1+\tau)\,x_2(t) - c(t)\} + \mu(t)\,\{x_2(t) - \delta\,K_2(t)\}$$

The first-order conditions again imply two versions of the standard Euler equation:

$$\begin{aligned} g(c(t)) &= \{r_1(t) - \delta - \rho\}/\sigma \\ &= \{r_2(t)/(1+\tau) - \delta - \rho\}/\sigma \end{aligned}$$

(d) The firm's first-order conditions are

$$\begin{aligned} r_1(t) &= \alpha\,[K_1(t)/K_2(t)]^{1-\alpha} \\ r_2(t) &= (1-\alpha)\,[K_1(t)/K_2(t)]^{\alpha} \end{aligned}$$

Taking the ratio and using the fact that $r_2(t)/r_1(t) = 1+\tau$ yields $K_1(t)/K_2(t) = (1-\tau)\,\alpha/(1-\alpha)$ and thus

$$r_1(t) = \left[\frac{1-\alpha}{\alpha\,(1+\tau)}\right]^{1-\alpha}$$

(e) Substitute $r_1(t)$ into the household Euler equation to obtain the equilibrium growth rate

$$g(c(t)) = \left\{\alpha\,\left[\frac{1-\alpha}{\alpha\,(1+\tau)}\right]^{1-\alpha} - \delta - \rho\right\}/\sigma$$

It is obvious that the growth maximizing tax rate is zero. This is also the welfare maximizing growth rate by the Welfare Theorem.

3 Capital externality

The economy is populated by a representative household and a large number of firms, indexed by i, whose number is normalized to 1. The production function for the representative firm is given by

$$y_{it} = Ak_{it}^{\alpha}(K_t l_{it})^{1-\alpha}$$

where $0 < \alpha < 1$, $A > 0$, and y_{it}, k_{it} and l_{it} are output, capital and labor input of firm i at date t. The aggregate, economy-wide capital stock is given by K_t, that is, the sum of all capital stocks in the economy. When making their choices, firms act competitively, that is, take as given the aggregate capital stock K_t and the rental price of capital r_t and the wage rate w_t. The presence of K_t in the production function of firm i indicates a production externality.

The representative consumer has a unit time endowment to work and faces the standard problem of choosing a stream of consumption and asset holdings $(c_t, a_t)_{t=[0,\infty)}$ to maximize

$$\int_0^{\infty} e^{-\rho t}\frac{c_t^{1-\sigma}-1}{1-\sigma}dt$$

subject to

$$c_t + \dot{a}_t = i_t a_t + w_t$$

where i_t is the real interest rate and w_t is the wage rate. Let $a_0 = K_0$ be the initial endowment of capital of the representative consumer. Assume that the population is not growing over time, so that the aggregate resource constraint is given by

$$c_t + \dot{K}_t + \delta K_t = Y_t$$

where Y_t is the aggregate output of all firms and $\delta > 0$ is the depreciation rate of capital.

a. Define a competitive equilibrium for this economy

b. Along a balanced growth path c_t, Y_t and K_t are growing at a constant rate γ_c. Derive the formula for γ^{CE} as a function of the model parameters and interpret your results.

c. Compare the growth rate γ^{CE} to the growth rate γ^{SP} that a social planner would choose (along a BGP) who chooses consumption c_t and the aggregate capital stock K_t to maximize utility of the representative agent, subject to the aggregate resource constraint. Interpret your results.