# Asset Pricing

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# Topics

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

- ▶ We use the Lucas (1978) fruit tree model.
- ▶ The implications are far more general than the simple model.
- ▶ The model forms the basis for the CAPM and the $\beta$ risk measure.

# The Lucas (1978) Fruit Tree Model

- **Agents**:
  - A single representative household.
- **Preferences**:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- $E_0$ is the expectation as of time $t = 0$.

# Technology

- This is an endowment economy.
- There are $K$ identical fruit trees.
- Each tree yields $d_t$ units of consumption goods in period $t$.
- $d_t$ is random and the same for all trees.
- Trees cannot be produced.
- Fruits cannot be stored.

# Technology

▶ The aggregate resource constraint:

$$c_t = Kd_t \qquad (2)$$

▶ Assume that $d$ is a finite Markov chain with transition matrix $\pi(d', d)$.

▶ An important feature: All uncertainty is **aggregate**.

▶ There are no opportunities for households to insure each other.

▶ This is why we can work with a representative household.

# Markets

- ▶ There are markets for fruits and for trees.
- ▶ There is also a one period bond, issued by households (in zero net supply).
  - ▶ Its purpose is to determine a risk-free interest rate.

# Household problem

- The household starts out with bonds ($b_0$) and shares ($k_0$).
- At each date, he chooses $c_t, b_{t+1}, k_{t+1}$.
- The **budget constraint** is

$$p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t \qquad (3)$$

- Notation:
    - $p$ : the price of trees. Suppressing dependence on the state.
    - $R$ : the real interest rate on bonds.
    - the price of bonds is normalized to 1 (how?).

# Household problem

$$V(k,b,d) = \max u(c) + \beta E V(k',b',d') \qquad (4)$$

subject to

$$Rb + (p+d)k - c + pk' - b' = 0 \qquad (5)$$

# Household problem

First-order conditions:

$$
\begin{aligned}
c &: & u'(c) &= \lambda \\
k' &: & \lambda p &= EV_k(k',b',d') \\
b' &: & \lambda &= EV_b(k',b',d')
\end{aligned}
$$

Envelope:

$$
\begin{aligned}
V_k &= \lambda(p+d) \\
V_b &= \lambda R
\end{aligned}
$$

# Euler equations

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) R_{t+1} \right\}$$
$$= \beta E_t \{ u'(c_{t+1}) \underbrace{\frac{p_{t+1} + d_{t+1}}{p_t}}_{R_{t+1}^S} \}$$

This is very general - holds for any number of assets / for any type of asset.

# Solution

- A solution consists of state contingent plans $\{c(d^t), k(d^t), b(d^t)\}$ for all histories $d^t$.
- These satisfy:
    - 2 Euler equations
    - 1 budget constraint.
    - $b_0$ and $k_0$ given.
    - Transversality: $\lim_{t\to\infty} E_0 \beta^t u'(c_t) \, [b_t + p_t k_t] = 0$.

# Market clearing

For every history we need:
Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

# Competitive Equilibrium

▶ A CE consists of:
   1. an allocation: $\{c(d^t), b(d^t), k(d^t)\}$.
   2. a price system: $\{p(d^t), R(d^t)\}$

▶ These satisfy:
   1. household: 2 Euler equations and 1 budget constraint.
   2. 3 market clearing conditions.

# Recursive Competitive Equilibrium

Objects:

- ▶ Solution to the household problem: $V(k,b,d)$ and $c(k,b,d)$, $k' = \kappa(k,b,d)$, $b' = B(k,b,d)$.
- ▶ Price functions: $p(d), R(d)$.

Equilibrium conditions:

- ▶ Household: 4
- ▶ Market clearing: 2
- ▶ No need for consistency: law of motion of the aggregate state is exogenous.

# Consumption smoothing

▶ The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \qquad (6)$$

▶ Define: Marginal rate of substitution:

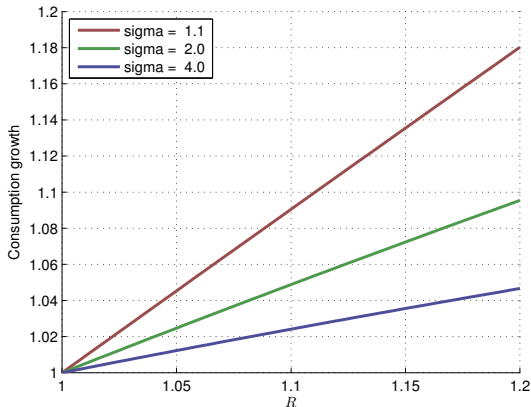$$MRS_{t+1} = \beta u'(c_{t+1})/u'(c_t) \qquad (7)$$

▶ $MRS_{t+1}$ is inversely related to consumption growth.

▶ With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$u'(c) = c^{-\sigma} \qquad (8)$$
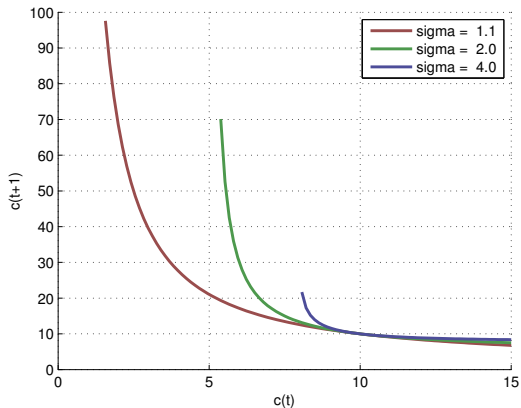$$MRS_{t+1} = \beta \ (c_{t+1}/c_t)^{-\sigma} \qquad (9)$$

# Consumption smoothing

▶ The coefficient of relative risk aversion ($\sigma$) determines how much *MRS* fluctuates with $c$.

▶ High $\sigma$ implies that the household chooses smooth consumption.

▶ Illustration for the deterministic case:

# Consumption smoothing

▶ With high $\sigma$, marginal utility changes a lot when $c$ changes.

▶ The household then keeps $c$ smooth.

# Asset Prices

# Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: $R_{t+1}^S = (p_{t+1} + d_{t+1})/p_t$.
- Directly from the 2 Euler equations:

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^S \right\} = 1$$

- Or

$$E\{MRS_{t+1} R_{t+1}\} = E\{MRS_{t+1} R_{t+1}^S\} = 1 \qquad (10)$$

# When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x,y) = E(xy) - E(x)E(y)$$

as

$$
\begin{aligned}
1 &= E\{MRS\}\,E\{R\} + Cov(MRS,R) & (11) \\
E(R) &= \frac{1 - Cov(MRS,R)}{E(MRS)} & (12)
\end{aligned}
$$

# When do assets pay high returns?

$$\mathbb{E}(R) = \frac{1 - Cov(MRS, R)}{\mathbb{E}(MRS)} \qquad (13)$$

▶ Take a "safe" asset with fixed $R$.
  ▶ $Cov(MRS, R) = 0$
  ▶ $\mathbb{E}(R) = 1/\mathbb{E}(MRS)$.
▶ If $Cov(MRS, R) < 0$: the asset pays higher return than the safe asset
  ▶ a **risk premium**
▶ If $Cov(MRS, R) > 0$: the asset pays **lower** return than the safe asset
  ▶ important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  ▶ examples?

# When do assets pay high returns?

- High returns require low / negative $Cov(MRS, R)$.
- Example: log utility
  - $u'(c) = 1/c$
  - $MRS = \beta u'(c_{t+1})/u'(c_t) = \beta c_t/c_{t+1}$.
- High $MRS$ means low consumption growth.
- Therefore: Assets pay high returns if their returns are **positively correlated with consumption growth**.

# Intuition

- Imagine there are good times (high $c$) and bad times (low $c$).
- There are 2 assets: A pays dividends in good times, B pays in bad times.
- The value of the dividend is $u'(c)$.
- Assets that pay in good times are not valuable: $u'(c)$ is low.
- Assets that pay in bad times provide insurance - they are valuable (have low expected returns).

# Risk (premia)

▶ The "risk free" assets has expected return

$$E(R_f) = \frac{1}{E(MRS)} \tag{14}$$

▶ A "risky" asset has expected return

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \tag{15}$$

▶ The **risk premium** is

$$E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \tag{16}$$

▶ This defines what **risk** means: **covariance with consumption growth**.

▶ Note that risk can be **negative** (insurance).

# The Equity Premium Puzzle

- ▶ Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- ▶ The equity premium is "high" (6-7% p.a.)
- ▶ The cov of $c$ growth and $R_s$ is low.
  - ▶ The reason: Consumption is very smooth.

# The Equity Premium Puzzle

| | | | |
|---|---|---|---|
| | | **TABLE 1** | |
| | | SUMMARY STATISTICS | |
| | UNITED STATES ANNUAL DATA, 1889–1978 | | |

| | Sample Means | | |
|---|---|---|---|
| $R_t^s$ | 0.070 | | |
| $R_t^b$ | 0.010 | | |
| $C_t/C_{t-1}$ | 0.018 | | |

| | Sample Variance-Covariance | | |
|---|---|---|---|
| | $R_t^s$ | $R_t^b$ | $C_t/C_{t-1}$ |
| $R_t^s$ | 0.0274 | 0.00104 | 0.00219 |
| $R_t^b$ | 0.00104 | 0.00308 | −0.000193 |
| $C_t/C_{t-1}$ | 0.00219 | −0.000193 | 0.00127 |

# The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = -\frac{Cov\left(\beta\left[c_{t+1}/c_t\right]^{-\sigma}, R_s\right)}{E\left\{\beta\left[c_{t+1}/c_t\right]^{-\sigma}\right\}} \qquad (17)$$

Take log utility: $\sigma = 1$.

- ▶ $Cov(MRS, R_s) \simeq -0.0022$.
- ▶ $E(MRS) \simeq 1$.
- ▶ $EP \simeq 0.2\%$.
- ▶ Replicating the observed equity premium requires **very high risk aversion** ($\sigma = 40$).

# How severe is the puzzle?

Investors forego very large returns.

| Investment Period | Stocks | | T-bills | |
|---|---|---|---|---|
| | Real | Nominal | Real | Nominal |
| 1802-1997 | $558,945 | $7,470,000 | $276 | $3,679 |
| 1926-2000 | $266.47 | $2,586.52 | $1.71 | $16.56 |

**Table 3**

**Terminal value of $1 invested in Stocks and Bonds**

Source: Mehra and Prescott (2003)

# Long holding periods

Over 20 year holding periods: stocks dominate bonds.



Equity Risk Premium Over 20-Year Periods
1926–2000
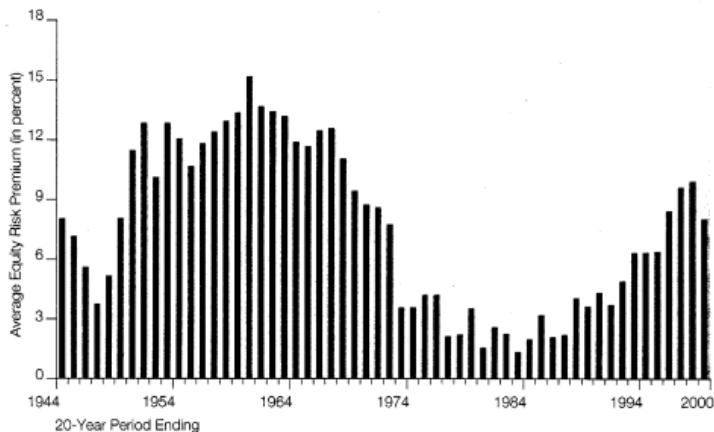
Source: Mehra and Prescott (2003)

# Why do we care?

▶ The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption

▶ This has implications for:
  1. The welfare costs of business cycles
     ▶ They are very low in standard models.
  2. Stock price volatility.
     ▶ Standard models fail to explain it (see below).

# How to resolve the puzzle

Proposed explanations include:

1. Habit formation: $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$.
   - Implies high risk aversion when $c_t$ is close to $c_{t-1}$.

2. Heterogeneous agents
   - Implicit in the standard model: all idiosyncratic risk is perfectly insured.

3. Borrowing constraints
   - The young should hold stocks (long horizon), but cannot.
   - The old receive mostly capital income and find stocks risky.

4. Taxes / regulations (McGrattan and Prescott, 2000)
   - The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.

# Beta

Now we derive the famous "beta" measure of risk.

Suppose asset $m$ (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma\, R_{m,t+1} \tag{18}$$

The market's expected return is

$$E\, R_m - R = -\frac{Cov(MRS, R_m)}{E(MRS)} \tag{19}$$

# Beta

Now we relate the covariance term to marginal utility:

$$Cov(MRS, R_m) = Cov\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}, R_{m,t+1}\right) = \beta \frac{Cov(u'(c_{t+1}), R_{m,t+1})}{u'(c_t)}$$

$$E(MRS) = \beta \frac{E(u'(c_{t+1}))}{u'(c_t)}$$

Therefore:

$$E(R_m) - R = -\frac{Cov(u'(c_{t+1}), R_{m,t+1})}{E\, u'(c_{t+1})} = \frac{\gamma\, Var(R_{m,t+1})}{E\, u'(c_{t+1})}$$

# Beta

For any asset $i$:

$$E\, R_i - R = -\frac{Cov(u'(c_{t+1}), R_i)}{E\, u'(c_{t+1})} = \frac{\gamma\, Cov(R_m, R_i)}{E\, u'(c_{t+1})}$$

Take the ratio for assets $i$ and $m$:

$$\beta_i = \frac{\mathbb{E}R_i - R}{\mathbb{E}R_m - R} = \frac{Cov(R_m, R_i)}{Var(R_m)} \tag{20}$$

Note: $\beta_i$ is the coefficient of regressing $R_i$ on $R_m$ using OLS.

This is the famous **CAPM** asset pricing equation.

# Beta

- The risk premium for asset $i$ depends on:
  - it's **beta** (essentially the correlation with the market)
  - the market price of risk: $E\,R_m - R$.
- A stock's beta can be estimated from data on past returns of the stock $(R_i)$ and the market (using a broad stock index).
- Betas are used to
  - Measure the risk of an asset.
  - Calculate the required rate of return for investment projects.
  - Evaluation of mutual fund managers.
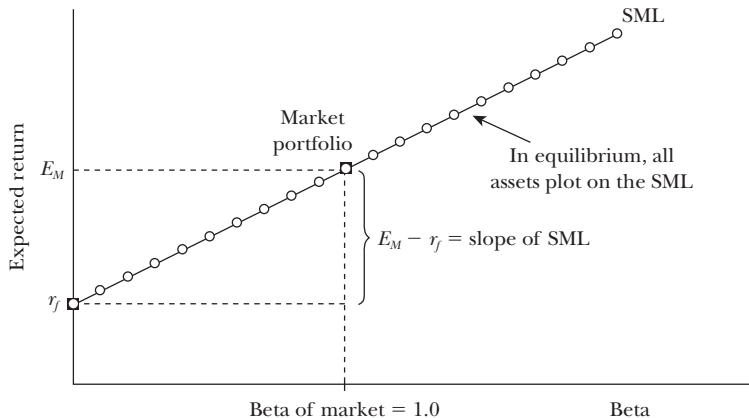
# Securities market line

CAPM prediction:

$$\mathbb{E}R_i = (1 - \beta_i)R + \beta_i\mathbb{E}R_m \qquad (21)$$
$$= R + \beta_i\mathbb{E}\{R_m - R\} \qquad (22)$$

If we plot expected returns against $\beta$s, we should get a straight line.

This is called the **securities market line** (SML)
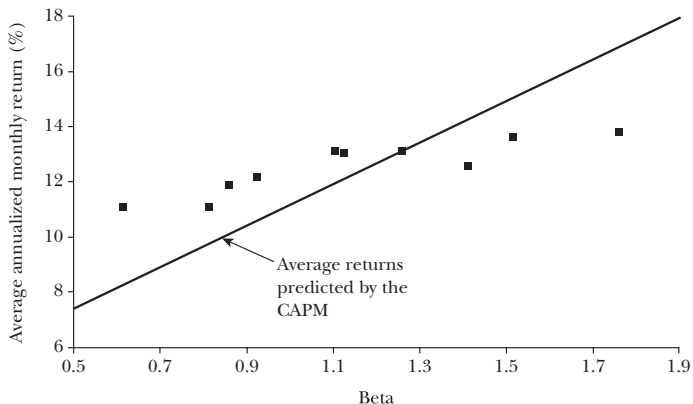
# Securities market line

**The Securities Market Line (SML)**



Source: Perold (2004)

# Securities market line: Evidence

**Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003**



Source: Fama (2004)

# Implications

Stocks with higher $\beta$s have higher expected returns, but the relationship is flatter than predicted.

Again: we don't understand how investors value / measure risk.

▶ a fundamental problem.

Oddly, $\beta$ remains popular, even though it does not work in the data.

# Solving for the asset price

We show that the asset price equals the present discounted value of dividends

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} d_{t+j} MRS(t, t+j) \qquad (23)$$

The discount factor is the MRS, called the **stochastic discount factor**.

# Solving for the asset price

Start from the Euler equation:

$$u'(c_t) = \beta \, E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \qquad (24)$$

Solve for the price:

$$p_t = E_t \left\{ \frac{\beta \, u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \qquad (25)$$

Replace $p_{t+1}$ with (25) shifted to $t+1$:

$$\begin{aligned}
p_t \;=\;& E_t \left\{ \frac{\beta \, u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\
& E_t \left\{ \frac{\beta \, u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[ \frac{\beta \, u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\} \quad (26)
\end{aligned}$$

# Solving for the asset price

The law of iterated expectations:

$$E_t\{E_{t+1}(x)\} = E_t(x) \qquad (27)$$

Eliminate the $E_{t+1}$:

$$p_t = E_t\left\{\frac{\beta\, u'(c_{t+1})}{u'(c_t)}\, d_{t+1}\right\} + E_t\left\{\frac{\beta^2\, u'(c_{t+2})}{u'(c_t)}\,(p_{t+2}+d_{t+2})\right\} \qquad (28)$$

Iterate forward for $T$ periods:

$$p_t = E_t\left\{\sum_{j=1}^{T}\frac{\beta^j\, u'(c_{t+j})}{u'(c_t)}\, d_{t+j}\right\} \qquad (29)$$

$$+ E_t\left\{\frac{\beta^{T+1}\, u'(c_{t+T+1})}{u'(c_{t+T})}\,(p_{t+T+1}+d_{t+T+1})\right\} \qquad (30)$$

# Solving for the asset price

Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j \, u'(c_{t+j})}{u'(c_t)} \, d_{t+j} \right\} \qquad (31)$$

- ▶ There is no good reason for this assumption!
- ▶ We will see later: other prices solve the asset pricing equation (bubbles)

The asset price equals the **discounted present value of dividends**.

The stochastic **discount factor** is the marginal rate of substitution.

## Example: Log Utility

In the Lucas model, assume: $u(c) = \ln(c)$. $K = 1$.

In equilibrium: $c_t = d_t$.

$MRS_{t+1} = \frac{\beta \ u'(c_{t+1})}{u'(c_t)} = \frac{\beta \ d_t}{d_{t+1}}$.

The asset pricing equation becomes

$$
\begin{aligned}
p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j d_t}{d_{t+j}} d_{t+j} \right\} \\
&= d_t \frac{\beta}{1 - \beta}
\end{aligned}
$$

# Example: Periodic dividends

In the Lucas model, assume:

- Utility is $u(c) = c^{1-\sigma}/(1-\sigma)$.
- $d_t$ alternates between $d^H$ and $d^L$.

Asset pricing equation:

$$
\begin{aligned}
p_t &= \sum \beta^j (d_t/d_{t+j})^\sigma d_{t+j} \\
&= d_t^\sigma \sum \beta^j d_{t+j}^{1-\sigma}
\end{aligned}
\qquad (32)
$$

On good days, $p_t$ is pulled up by low $u'(c')$, but is pushed down by low $d_{t+1}$.

# The Excess Volatility Puzzle

Consider a stock with dividend process $d_t$.

Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \qquad (33)$$

In the data:

▶ Dividends are very smooth (a goal of company policy).

▶ Stock prices are much more volatile than dividends.

But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.

This is the flip-side of the Equity Premium Puzzle. See Shiller (1981)

# Excess Volatility



Real price (log scale)

Real S&P 500 index
Present value of subsequent dividends
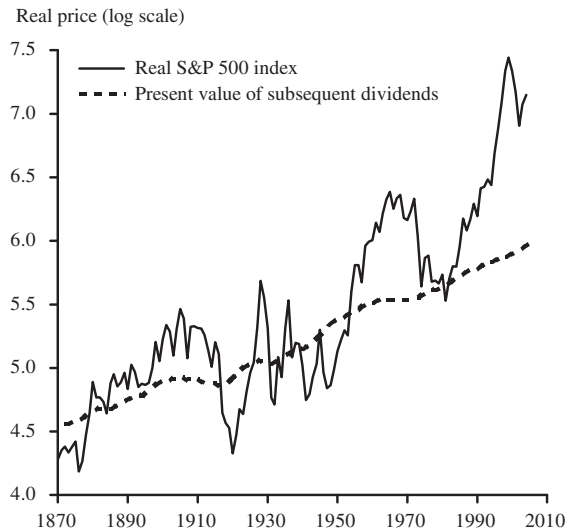
Source: FRSBSF Economic Letter Nov 2007

# Bubbles

- ▶ Recall how the asset pricing formula is derived:
- ▶ We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta \, u'(c_{t+1})}{u'(c_t)} \left( p_{t+1} + d_{t+1} \right) \right\} \qquad (34)$$

- ▶ We assume that the $p_{t+1}$ term vanishes in the limit.
- ▶ What if it does not vanish?
- ▶ Then **any** (current) **asset price** can satisfy the asset pricing equation.
- ▶ The deviation between $p_t$ and the **fundamental price** from (34) is called a **bubble**.
- ▶ It is purely a self-fulfilling expectation.

# Bubbles: Example

- ▶ Consider an asset that pays no dividends.
- ▶ Its **fundamental price** is 0.
- ▶ Assume that the MRS is constant at $\frac{\beta\ u'(c_{t+1})}{u'(c_1)} = 1$.
- ▶ The the asset pricing equation is

$$p_t = E_t p_{t+1} \tag{35}$$

- ▶ One price process that satisfies this: $p$ doubles with probability 1/2 and drops to 0 otherwise.
- ▶ This satsifies (35) for **any** $p_t$.
- ▶ Bubbles are a possible explanation for asset price volatility.
- ▶ Note that the bubble does not offer any excess return opportunities.

# Reading

- ▶ Romer (2011), ch. 7.5
- ▶ Ljungqvist and Sargent (2004), ch. 7.
- ▶ On the equity premium puzzle: Mehra and Prescott (1985, 2003)

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