

# Asset Pricing

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# Pricing Assets: An Example

The infinite horizon model can be used to price long-lived assets.

- ▶ This is more interesting in stochastic economies.
- ▶ It then yields the famous  $\beta$  measure of risk and the CAPM.

# Environment

Demographics:

- ▶ a representative, infinitely lived household
- ▶ mass 1

Preferences:  $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Endowments:

- ▶ in each period:  $N$  units of labor time
- ▶ at  $t = 0$ :  $L$  units of land

# Environment

Technologies:  $F(N_t, L_t; A_t) = c_t$

- ▶ constant returns to  $(N, L)$
- ▶ the productivity sequence  $\{A_t\}$  is given.

Markets:

- ▶ goods (numeraire)
- ▶ land rental:  $r_t$
- ▶ labor:  $w_t$
- ▶ land purchase:  $p_t$

## Firm's problem

The firm's problem is standard:

$$\max F(N_t, L_t; A_t) - w_t N_t - r_t L_t \quad (1)$$

FOCs:

$$r = F_L \quad (2)$$

$$w = F_N \quad (3)$$

Solution:  $N_t, L_t$  that satisfy the 2 FOCs.

# Household

Budget constraint:

Bellman equation:

First-order conditions:

# Household

The Euler equation is standard

$$u'(c) = \beta u'(c') \frac{r' + p'}{p}$$

Solution:  $\{c_t, l_t\}$  that solve the Euler equation and budget constraint.

# Equilibrium

A competitive equilibrium is a set of sequences  $(c_t, l_t, p_t, r_t, w_t)$  that satisfy:

- ▶ household: Euler equation and budget constraint;
- ▶ firm: 2 FOCs:
- ▶ market clearing for land:
- ▶ market clearing for goods:



## The price of land

We find a difference equation for  $p_t$ .

Substitute the goods market clearing condition and the first-order condition for  $r$  into the Euler equation to obtain

$$u'(F(N, L)) = \beta u'(F(N', L')) \frac{F_L(N', L') + p'}{p} \quad (4)$$

Note that this difference equation only contains  $p$  and exogenous variables.

## Lucas asset pricing equation

We can derive the famous Lucas (1978) asset pricing equation:

Define:

$$MRS(t, t+j) = \frac{\beta^j u'(t+j)}{u'(t)} \quad (5)$$

$$R_{t+1} = \frac{F_L(N_{t+1}, L_{t+1}) + p_{t+1}}{p_t}$$

Then we have

$$1 = MRS(t, t+1) \times R_{t+1} \quad (6)$$

It turns out that this is a very general asset pricing equation (see section on stochastic models).

## The price of land I

We solve the difference equation for  $p_t$  by **forward iteration**.

Start from

$$p_t = MRS(t, t+1) [d_{t+1} + p_{t+1}] \quad (7)$$

where  $d_t = F_L(N_t, L_t, A_t)$  is the “dividend.”

Now iterate forward

$$\begin{aligned} p_t &= MRS(t, t+1) \{d_{t+1} + MRS(t+1, t+2) [d_{t+2} + p_{t+2}]\} \\ &= MRS(t, t+1) d_{t+1} + MRS(t, t+2) [d_{t+2} + p_{t+2}] \end{aligned} \quad (8)$$

$$= \sum_{j=1}^{\infty} MRS(t, t+j) d_{t+j} \quad (9)$$

## The price of land II

In deriving (9), I made use of the fact that

$$\begin{aligned} MRS(t, t+j) &= \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \cdots \frac{\beta u'(c_{t+1+j})}{u'(c_{t+j})} \\ &= \frac{\beta^j u'(c_{t+1+j})}{u'(c_t)} \end{aligned}$$

## The price of land

Key result: The asset price equals the discounted present value of "divdends."

$$p_t = \sum_{j=1}^{\infty} d_{t+1} MRS(t, t+j) \quad (10)$$

The discount factor is the Marginal Rate of Substitution

$$MRS(t, t+j) = \frac{\beta^j u'(t+j)}{u'(t)} \quad (11)$$

This is a fairly general result.

## The price of land: Intuition

The price should equal the value of the dividend stream.

What is the derivative of  $p$  w.r.to dividends?

- ▶ Start from the equilibrium price.
- ▶ Add  $\varepsilon$  to the date  $t+j$  payoff.
- ▶ The household gains  $\beta^j u'(t+j) \varepsilon$ .
- ▶ The household pays for this:  $u'(t) \varepsilon$ .

In equilibrium, the price change should equal the willingness to pay for additional dividends:

$$\partial p_t / \partial d_{t+j} = \frac{\beta^j u'(t+j)}{u'(t)} \quad (12)$$

## The price of land: stationary economy

In a stationary economy:  $A_t = A^S$  and  $d_t = F_L(N, L, A^S)$ .

Then:

$$\begin{aligned} p_t^S &= \sum_{j=1}^{\infty} \beta^j F_L(N, L, A^S) \\ &= \frac{\beta}{1 - \beta} F_L(N, L, A^S) \end{aligned}$$

## The price of land: fluctuating economy

We calculate  $p_t$  for an economy which is subject to deterministic fluctuations.

- ▶ In even periods  $A_t = A^H$
- ▶ In odd periods  $A_t = A^L \leq A^H$ .

Simplifying assumptions:

- ▶ the marginal product of land is independent of  $A_t$ .
- ▶  $2F^S = F^H + F^L$ , where  $F^j = F(N, L; A^j)$ .



## The price of land: fluctuating economy

The trick is to break the sum in (9) into two parts, one for even and one for odd periods:

$$p_t^R = F_L \{ MRS(t, t+1) + \beta MRS(t, t+2) + \beta^2 MRS(t, t+3) + \dots \}$$

$$\begin{aligned} p_t^R &= F_L \left\{ \sum_{j=0}^{\infty} \beta^{2j} MRS(t, t+1) + \beta \sum_{j=0}^{\infty} \beta^{2j} MRS(t, t+2) \right\} \\ &= F_L \left\{ MRS(t, t+1) \frac{1}{1-\beta^2} + MRS(t, t+2) \frac{\beta}{1-\beta^2} \right\} \end{aligned}$$

Denote the marginal rate of substitution between odd and even periods by

$$\alpha = u'(F^H)/u'(F^L) < 1$$

## The price of land: fluctuating economy

If  $t$  is even, then

$$p_t^{even} = (1/\alpha + \beta) F_L \frac{\beta}{1 - \beta^2}$$

If  $t$  is odd, then

$$p_t^{odd} = (\alpha + \beta) F_L \frac{\beta}{1 - \beta^2}$$

Intuition:

- ▶ we get alternating payoffs that are either  $(1/\alpha, 1)$  or  $(\alpha, 1)$  times  $F_L$
- ▶ discounting at  $\beta^2$  produces the present value factor  $1/(1 - \beta^2)$

## The price of land: fluctuating economy

Since  $1 - \beta^2 = (1 + \beta)(1 - \beta)$ ,

$$\frac{p_t^{even}}{p_t^S} = \frac{1/\alpha + \beta}{1 + \beta} > 1$$
$$\frac{p_t^{odd}}{p_t^S} = \frac{\alpha + \beta}{1 + \beta} < 1$$

The price of the asset fluctuates around its price in the stationary economy.

# The price of land: Intuition

Consider an even period.

- ▶ Times are good, so that saving is easy.
- ▶ And the return tomorrow is worth a lot because times will be bad.
- ▶ Hence, the demand for land is high and so is the price.

In odd periods, saving is painful and the return won't be worth much tomorrow. So the price is low.

# Summary

The standard growth model is also the standard framework for pricing assets.

The price of an asset equals the present value of "dividends."

The discount factors are the Marginal Rates of Substitution.

This survives in stochastic environments. Just add  $E[.]$ .