

# Two Sector Models

Prof. Lutz Hendricks

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## Two Sector Models

We relax the assumption that there is only one good at each date.

There are no major changes in methods.

Multi-sector models are used to study issues such as:

- ▶ technical change that is “embodied” in capital goods,
- ▶ human capital,
- ▶ international trade.

# The Environment

Demographics:

- ▶ There is a unit mass of **households** who live forever.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

- ▶  $v$  is work;  $1 - v$  is leisure.

## Technologies

Consumption goods are produced according to

$$Y_1 = F(K_1, L_1)$$

and capital goods according to

$$Y_2 = G(K_2, L_2)$$

The resource constraints are

$$L_{1t} + L_{2t} = v_t$$

$$K_{1t} + K_{2t} = K_t$$

$$Y_{1t} = c_t$$

$$Y_{2t} = K_{t+1} - (1 - \delta)K_t$$

# Capital mobility

There is only one type of capital.  
It can be moved freely between sectors.

# Planning Problem

The **planner** maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

subject to the resource constraints.

Since capital can be costlessly reallocated between sectors, the state variable is  $K_t$ .

The controls  $c_t, L_{1t}, L_{2t}$ , and  $\varphi_t$ .

$\varphi_t$  is the fraction of capital employed in sector 1

$$K_{1t} = \varphi_t K_t$$

$$K_{2t} = (1 - \varphi_t) K_t$$

## Exercise

Suppose that reallocating capital across sectors is not possible.

Solve the planner's problem.

We now need 2 states  $(K_{1t}, K_{2t})$ .

Resource constraint:

$$Y_{2t} = K_{1,t+1} + (1 - \delta_1) K_{1,t} + K_{2,t+1} + (1 - \delta_1) K_{2,t} \quad (1)$$

The difference:  $K_{j,t+1}$  has to be chosen in  $t$ .

# Planning Problem

The Bellman equation is

$$\begin{aligned} V(K) = \max u(F(\varphi K, L_1), 1 - L_1 - L_2) \\ + \beta V(K(1 - \delta) + G([1 - \varphi]K, L_2)) \end{aligned}$$

where the choice variables are  $L_1, L_2, \varphi$ .



# Planning Problem

FOCs:

$$\begin{aligned}u_l &= \beta V'(K')G_L = u_c F_L \\u_c F_K &= \beta V'(K')G_K\end{aligned}$$

Envelope:

$$\begin{aligned}V'(K) &= \varphi F_K u_c + \beta V'(K')\{1 - \delta + (1 - \varphi)G_K\} \\&= u_c F_K + (1 - \delta)\beta V'(K')\end{aligned}$$

Interpretation...

# Planning Problem

**Euler** equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.) \frac{F_K(.)}{G_K(.)} \{1 - \delta + G_K(.)\}$$

Static conditions

$$F_K/F_L = G_K/G_L$$

$$u_l = u_c F_L$$

Interpretation below...

## Solution: Planning Problem

Sequences  $\{c_t, v_t, K_{t+1}, \varphi_t, L_{1t}, L_{2t}\}$  that satisfy:

- ▶ 3 FOCs;
- ▶ 3 feasibility conditions;
- ▶ TVC:  $\lim_{t \rightarrow \infty} \beta^t u_c(t) K_t = 0$ .
- ▶  $K_0$  given.

## Intuition: Static condition

$$F_K/F_L = G_K/G_L$$

The static condition equates marginal rates of substitution in the two sectors.

This is necessary for maximizing output for given inputs.

## Intuition: Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.) \frac{F_K(.)}{G_K(.)} \{1 - \delta + G_K(.)\}$$

Consider first the case  $F_K = G_K$ .

Then we get the conventional Euler equation

$$u_c = \beta u_c(.) \{1 - \delta + G_K(.)\}$$

## Intuition: Euler equation - General case

- ▶ At any point in time, consumption can be converted into next period capital at a marginal rate of transformation  $G_K/F_K$ .
- ▶ Period  $t$ : Convert 1 unit of  $c$  into  $G_K/F_K$  units of  $K'$ .
- ▶ Period  $t+1$ : Produce an additional

$$\{(1 - \delta) + G_K(\cdot)\} G_K/F_K$$

units of date  $t+2$  capital.

- ▶ Convert the additional date  $t+2$  capital into date  $t+1$  consumption at the rate of transformation

$$F_K(\cdot)/G_K(\cdot)$$

- ▶ Eat this. This leaves all variables after  $t+1$  unchanged.

## Planner: Steady State

In steady state, the Euler equation simplifies to

$$\beta\{1 - \delta + G_K\} = 1$$

Because the MRT,  $G_K/F_K$ , is constant this is the same as in the one sector model.

## Competitive Equilibrium



# Competitive Equilibrium

Watch your units!

## Notation

- ▶  $P_i$  are the prices of the goods.
- ▶  $p_2 = P_2/P_1$ .
- ▶  $RP_1$  and  $wP_1$  are the rental prices of capital and labor.

All prices are in units of account.

# Competitive Equilibrium

Consumption sector **firms** maximize period profits:

$$\max P_1 [Y_1 - RK_1 - wL_1]$$

The FOCs are as usual:

$$R = F_K$$

$$w = F_L$$

# Competitive Equilibrium

Capital sector firms:

$$\max P_2 Y_2 - P_1 R K_2 - P_1 w L_2$$

Divide through by  $P_1$  to obtain

$$\max p_2 Y_2 - R K_2 - w L_2$$

The FOCs are

$$R/p_2 = G_K$$

$$w/p_2 = G_L$$

# Households

The budget constraint is

$$P_{2t}k_{t+1} = P_{2t}(1 - \delta)k_t + P_{1t}R_tk_t + P_{1t}(w_tv_t - c_t)$$

Divide through by  $P_1$  to obtain the budget constraint in real terms:

$$p_{2t}k_{t+1} = (1 - \delta)p_{2t}k_t + R_tk_t + w_tv_t - c_t$$

## Rate of return

At  $t$ : Give up  $p_{2t}$  units of consumption and buy  $dk_{t+1} = 1$

At  $t+1$ :

- ▶ Receive rental income  $R_{t+1}dk_{t+1}$ .
- ▶ Sell the undepreciated capital:  $(1 - \delta)p_{2,t+1}dk_{t+1}$ .

The rate of return is

$$\begin{aligned}1 + r_{t+1} &= \frac{R_{t+1} + (1 - \delta)p_{2,t+1}}{p_{2,t}} \\ &= R_{t+1}/p_{2,t} + (1 - \delta)\pi_{t+1}\end{aligned}$$

$\pi_{t+1} \equiv p_{2,t+1}/p_{2,t}$  is the price appreciation of  $k$ .

## Household problem I

$$V(k) = \max_{k', v} u(wv + Rk + (1 - \delta)p_2k - p_2k', 1 - v) \quad (2)$$

$$+ \beta V(k') \quad (3)$$

FOCs:

$$u_1 w = w_2 \quad (4)$$

$$u_1 p_2 = \beta V'(k') \quad (5)$$

Envelope:

$$V'(k) = u_1 \times (R + (1 - \delta)p_2) \quad (6)$$

## Household problem II

Euler:

$$u_1 = \beta u_1 (.' ) \frac{R' + [1 - \delta] p_2'}{p_2} \quad (7)$$

$$\beta u_1 (.' ) (1 + r') \quad (8)$$

Note that we could have written that down without even deriving it.

- All the household needs to know about an asset is the rate of return.

## Market clearing

Labor:  $L_{1t} + L_{2t} = v_t$ .

Capital:  $K_{1t} + K_{2t} = K_t$ .

Goods:

$$Y_{1t} = c_t$$

$$Y_{2t} = K_{t+1} - (1 - \delta)K_t$$

These are just the technological constraints.



# Equilibrium Definition

A CE is a sequence of prices  
and quantities  
which satisfy (11 equations in 10 unknowns):

## Equilibrium prices

The firms' FOCs imply that

$$R = p_2 G_K = F_K$$

and therefore

$$p_2 = F_K / G_K$$

In words: the relative price equals the marginal rate of transformation.

**Exercise:** Show that the solutions of the planning problem and the CE coincide by substituting prices for derivatives of  $F$  and  $G$  in the planner's FOCs.

## One-sector Reduced Form

## A One-sector Reduced Form

- ▶ We can construct a two sector model that looks very much like a one sector model.
- ▶ This requires the assumption

$$G(K,L) = AF(K,L)$$

for some constant  $A$

## A One-sector Reduced Form

Then static optimality (same MRS in both sectors)

$$F_K/F_L = G_K/G_L$$

implies

$$k_1 = k_2$$

where  $k = K/L$ .

The relative price of capital is constant

$$p_2 = 1/A$$

Intuition ...

## A One-sector Reduced Form

We can write a single **aggregate resource constraint**:

Define aggregate real output as

$$\begin{aligned} Y &= Y_1 + Y_2/A \\ &= F(K_1, L_1) + F(K_2, L_2) \\ &= (L_1 + L_2)f(k) \\ &= F(K, L) \\ &= c + (K_{t+1} - [1 - \delta]K_t)/A \end{aligned}$$

But isn't this adding apples and oranges?

Choose units of capital such that  $A = 1$ :  $\tilde{K} = K/A$ .

Then the resource constraint looks like a one sector model:

$$F(\tilde{K}_t, L_t) = c_t + \tilde{K}_{t+1} - (1 - \delta)\tilde{K}_t$$

## Why is this useful?

We can write down a model with cross-country (or cross-industry) productivity differentials without having to construct a full-blown multi-sector model with endogenous prices.

In the data, the relative **price of capital** varies greatly across countries. We can model that.

## Example: Investment specific technical change

Assume that  $A_t$  grows at some rate.

Then the relative price of capital  $p_{2,t}/1/A_t$  falls over time (as it does in the data).

The model generates an evolution of the industrial structure

- ▶ capital intensive sectors grow over time
- ▶ e.g. movement from ag to industry.

Greenwood, Hercowitz, and Krusell (1997) find that such technical change accounts for 60 percent of overall productivity growth.



# Summary

- ▶ Nothing fundamental changes when there are multiple sectors.
- ▶ The main additional complexity is in the household budget constraint because there may be capital gains terms.
- ▶ The dynamics of two sector models is much more complex than that of one sector models.