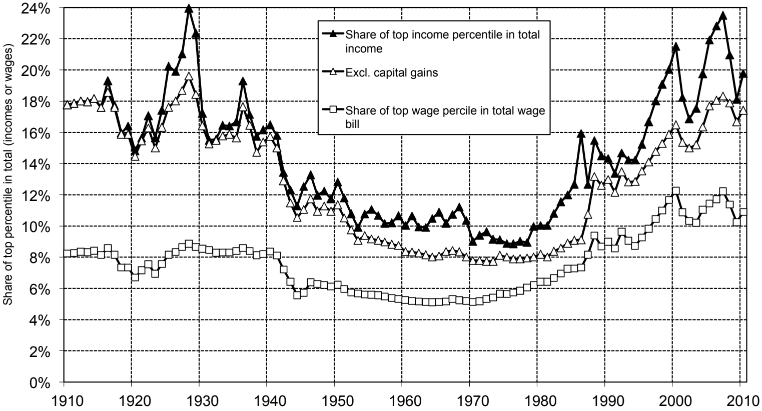
Labor Economics, Week 3
Estimating top income shares, and Distributional decompositions

Maximilian Kasy

Department of Economics, Oxford University

Top 1% income share in the US

Figure 8.8. The transformation of the top 1% in the United States



The rise in the top 1% highest incomes since the 1970s is largely due to the rise in the top 1% highest wages. Sources and series: see piketty.pse.ens.fr/capital21c.

How are these estimated?

- Using income tax data (for numerator)
  and national accounts (for denominator)
- Available for top incomes since the introduction of income taxes
- For lower incomes: only since the expansion of income taxes
- These slides: Econometric issues
- Student presentation: Data issues, interpretation, etc.

The Pareto distribution

- Top incomes are very well described by the Pareto distribution
- Defined by:

$$P(Y > y | Y \geq \underline{y}) = \left(\underline{y}/y\right)^{\alpha_0}$$

for $y \geq \underline{y}$, where $\alpha_0 > 1$.

- Corresponding density:

$$\begin{aligned} f(Y;\alpha_0) &= -\frac{\partial}{\partial y} P(Y > y | Y \geq \underline{y}) \\ &= -\frac{\partial}{\partial y}\left(\underline{y}/y\right)^{\alpha_0} \end{aligned}$$

Questions for you

Calculate $f(Y;\alpha_0)$

Answer:

$$f(Y;\alpha_0) = \alpha_0 \cdot \underline{y}^{\alpha_0} \cdot y^{-\alpha_0 - 1}.$$

Key property

- Pareto distribution satisfies:

$$E[Y|Y \geq y] = \frac{\alpha_0}{\alpha_0 - 1} \cdot y.$$

- This is true for all y!!

Questions for you

Describe this equation in words.

- We can therefore calculate average incomes of the 1% as:

$$\bar{y}^{1\%} = \frac{\alpha_0}{\alpha_0 - 1} \cdot q^{99},$$

where

$$P(Y \leq q^{99}) = .99$$

- To get top income shares, we need estimates of
  1. α_0
  2. q^{99}
  3. National income for the denominator
- We will discuss α_0.
- Smaller $\alpha_0 \Rightarrow$ fatter tails $\Rightarrow$ more inequality, larger top income shares.

Key problem

- Standard technique to construct estimators: maximum likelihood.
- Find the number α_0 which makes the observed incomes $y_1, \ldots, y_n$ "most likely"

$$\widehat{\alpha}^{MLE} = \underset{\alpha}{\operatorname{argmax}} \prod_{i=1}^{n} f(y_i; \alpha)$$
$$= \underset{\alpha}{\operatorname{argmax}} \sum_{i=1}^{n} \log(f(y_i; \alpha)).$$

- First order condition

$$\frac{\partial}{\partial \alpha} \sum_{i=1}^{n} \log(f(y_i; \alpha)) = 0.$$

Questions for you

Solve this first order condition for the Pareto density.

Answer

- Log density of y_i

$$\log(f(y_i;\alpha)) = \log(\alpha \left(\underline{y}/y_i\right)^{\alpha} \cdot y_i^{-1}) = \log(\alpha) + \alpha \log\left(\underline{y}/y_i\right) - \log(y_i).$$

- First order condition

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha} \sum_{i=1}^{n} \log(\alpha \left(\underline{y}/y_i\right)^{\alpha} \cdot y^{-1}) \\ &= \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \log\left(\underline{y}/y_i\right) \right). \end{aligned}$$

- Solving for α

$$\widehat{\alpha}^{MLE} = \frac{n}{\sum_i \log\left(y_i/\underline{y}\right)}. \tag{1}$$

Additional problem

- Available data do not list actual incomes,
- just the number of people in different tax brackets $[y_l, y_u]$.
- Technical term: The data are "censored."
- For the Pareto distribution:

$$\begin{aligned} P(Y \in [y_l, y_u] | Y \geq \underline{y}) &= P(Y > y_l | Y \geq \underline{y}) - P(Y > y_u | Y \geq \underline{y}) \\ &= \left(\underline{y}/y_l\right)^{\alpha_0} - \left(\underline{y}/y_u\right)^{\alpha_0}. \end{aligned} \tag{2}$$

Likelihood for two tax brackets

- Data on N people with incomes above $\underline{y}$
- N_2 people in the bracket $[y_l, \infty)$
- Probability of any given individual in the top bracket:

$$p(\alpha_0) = P(Y > y_l | Y > \underline{y}) = \left(\underline{y}/y_l\right)^{\alpha_0}.$$

- Probability of exactly N_2 individuals in the top bracket:

$$P(N_2 = n_2 | N = n; \alpha) = \binom{n}{n_2} \cdot p(\alpha_0)^{n_2} (1 - p(\alpha_0))^{n - n_2}.$$

- Remember the binomial distribution?

Questions for you

Calculate the maximum likelihood estimator for censored data

$$\widehat{\alpha}^{MLE} = \underset{\alpha}{\operatorname{argmax}}\ P(N_2 = n_2 | N = n; \alpha).$$

(Homework)

References

Atkinson, A. B., Piketty, T., and Saez, E. (2011). Top incomes in the long run of history. Journal of Economic Literature, *49(1):3–71.*

Piketty, T. (2014). Capital in the 21st Century. *Harvard University Press, Cambridge.*

Atkinson, A. B. and Morelli, S. (2015). Chartbook of economic inequality.

`http://www.chartbookofeconomicinequality.com/`

Decreasing unionization since the 1980s

- Union wages: higher and less unequal
- Thus: declining unionization
  $\Rightarrow$ increase in inequality?
- Just compare wages of union / non-union members?
- Problem: two groups might be different, in terms of
  - age,
  - education,
  - gender,
  - ethnicity,
  - sector of the economy,
  - state of residence,
  - ...
- Want to compare people who look similar along all these dimensions!

Distributional decompositions

Hypothetical questions of the form:

- What if
  1. distribution of demographic covariates
     had stayed the same,
  2. distribution of wages
     given demographics and union membership status
     had stayed the same, but
  3. we consider actual historical changes of union membership
     given demographics.
- How would the distribution of wages have changed?
- i.e., to what extent is de-unionization responsible for the rise in inequality?

Setup

- Observe repeated cross-sections
  of draws from the time t distributions P^t.
- Variables (Y, D, X)
  - Y: outcome, e.g., real earnings
  - X: demographic covariates, e.g., age, gender, ...
  - D: binary "treatment," e.g., union membership
- Effect of historical changes in D
  on the distribution $P(Y)$?
- In particular, on statistics $\nu(P(Y))$?
- Examples for ν: mean, variance, share below the poverty line, quantiles, Gini coefficient, top income shares, ...

Probability reminder

Let $p(y, x)$ denote a joint probability density.

1. Conditional distribution:
$$p(Y|X) = \frac{p(Y, X)}{p(X)}$$
2. Marginal distribution:
$$p(Y) = \int p(Y, X) dX$$
3. Thus:
$$p(Y) = \int p(Y|X) p(X) dX$$
4. Similarly (law of iterated expectations):
$$E[Y] = E[E[Y|X]]$$

Counterfactual distribution

- Two distributions $P^0(Y, D, X)$, $P^1(Y, D, X)$ (beginning and end of historical period)
- What would the wage distribution $P^*(Y)$ be, assuming
  1. dist of demographics stayed the same,
  2. dist of wages given demographics, union membership stayed the same
  3. actual historical change of union membership

$$\begin{aligned} P^*(X) &= & P^0(X) \\ P^*(Y \leq y | X, D) &= & P^0(Y \leq y | X, D) \\ P^*(D|X) &= & P^1(D|X). \end{aligned}$$

- Get the counterfactual distribution $P^*(Y)$:

$$P^*(Y \leq y) := \int_{X,D} P^0(Y \leq y | X, D) dP^1(D|X) dP^0(X).$$

Rewriting the counterfactual distribution

1. Multiply and divide the integrand by $P^0(D|X)$.
2. Rewrite the probability $P^0(Y \leq y|X, D)$ as an expectation $E^0[\mathbf{1}(Y \leq y)|X, D]$.
3. Give the fraction $P^1(D|X)/P^0(D|X)$ a new name: $\theta(D, X)$.
4. Pull θ into the conditional expectation.
5. Use the “law of iterated expectations” to get an unconditional expectation.

Questions for you

Execute these steps, and see what you get!

Solution

$$
\begin{aligned}
P^*(Y \leq y) &= \int_{X,D} P^0(Y \leq y|X, D)\frac{P^1(D|X)}{P^0(D|X)}P^0(D|X)P^0(X)dDdX \\
&= \int_{X,D} E^0[\mathbf{1}(Y \leq y)|X, D]\theta(D,X)P^0(D|X)P^0(X)dDdX \\
&= E^0[E^0[\mathbf{1}(Y \leq y) \cdot \theta(D,X)|X, D]] \\
&= E^0\left[\mathbf{1}(Y \leq y) \cdot \theta(D,X)\right],
\end{aligned}
$$

where

$$
\theta(D,X) := \frac{P^1(D|X)}{P^0(D|X)}.
$$

Questions for you

Interpret this representation of the counterfactual distribution.

Estimation

- Suppose X is discrete.
- Let $N^t(d,x)$ be the number of observations in period t with $D = d$, $X = x$,
- similar for $N^t(x)$.
- Then we can estimate $\theta(d,x)$ as

$$\widehat{\theta}(d,x) = \frac{N^1(d,x)}{N^1(x)} \Big/ \frac{N^0(d,x)}{N^0(x)}.$$

- Estimate $P^*(Y \leq y)$ as

$$\sum_i \mathbf{1}(Y_i \leq y) \cdot \widehat{\theta}(D_i, X_i) \Big/ \sum_i \widehat{\theta}(D_i, X_i),$$

where the sums are over all observations in period 0.

Questions for you

Implement this in Stata!
(Section)

References

Fortin, N. M. and Lemieux, T. (1997). Institutional changes and rising wage inequality: Is there a linkage? The Journal of Economic Perspectives*, 11(2):pp. 75–96.*

Firpo, S., Fortin, N., and Lemieux, T. (2011). Decomposition methods in economics. Handbook of Labor Economics*, 4:1–102.*

DiNardo, J., Fortin, N., and Lemieux, T. (1996). Labor market institutions and the distribution of wages, 1973-1992: A semiparametric approach. Econometrica*, 64:1001–1044.*