Normed Linear Spaces

Motivation

Often when carrying out theoretical and empirical political analysis, we want to work in sets that are simultaneously linear spaces and metric spaces. In particular, we want to work in spaces where the algebraic structure of the space is consistent with its geometric structure. A normed linear space accomplishes this. In the quantitative sequence, you will be working almost exclusively in normed linear spaces.

Norms

For any linear space X, a **norm**, denoted $\|x\|$, is a measure of the size of the elements satisfying the following properties:

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

A norm on X induces a metric on X,

$$d(x, y) = \|x - y\|$$

Proof:

$d(x,y) = \|x-y\| \geq 0$. Norm axiom 1

$d(x,y) = 0 \iff \|x-y\| = 0 \iff x-y=0 \iff x=y$ Norm axiom 2

$d(x,y) = \|x-y\| = \|(-1)(y-x)\| = (-1)\|y-x\| = d(x,y)$ Norm axiom 3

$d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$ Norm axiom 4

A linear space with a norm, $(X, \|\cdot\|)$, is called a **normed linear space**. A normed linear space is a subclass of **metric linear spaces** which is a linear space equipped with a metric. It is a special type of metric linear space with rich interaction of the algebraic and linear structures.

Example: Say a government agency is devising a proposal to build a new highway. A production plan $y = (y_1, ..., y_n)$ is a list of the net outputs of various goods and services where y_i is net output of commodity i. How could we measure the overall "size" of the plan? What if we simply added up the net outputs of all the goods and services i.e. $\sum y_i$? Since some of the net outputs are negative, it would be more appropriate to take their absolute value first. Therefore one possible measure of size is

$$\|y\|_1 = \sum_{i=1}^{n} |y_i|$$

Another way to compensate for the negativity of some net inputs is to square the individual inputs as in

$$\|y\|_2 = \sqrt{\sum_{i=1}^{n} y_i^2}$$

Of course what we probably really care about is the "output" of the project i.e. how much road will the agency produce. Can we measure the size of the plan y by the output of road, y_r? We cannot because $|y_r|$ does not satisfy the requirements of a norm: $|y_r| = 0$ does not imply $\|y\| = 0$. As political scientists, we are quite aware that it is possible for a government program to consume inputs and produce no outputs. A related measure that does meet the requirements of a norm uses the largest component of y as the measure of the size or the plan:

$$\|y\|_\infty = \max_{i=1}^{n} |y_i| \quad \blacksquare$$

The **Euclidean norm**, $\|x\|_2$ generalizes the conventional Pythagorean notion of the length of a vector in two and three-dimensional space. In $\mathbb{R}^2$, the Euclidean norm expresses the Pythagorean theorem that in a right triangle, the squared length of the hypotenuse is equal to the sum of the squares of the two sides: $\|x\|^2 = |x_1|^2 + |x_2|^2$. The space $(\mathbb{R}^n, \|\cdot\|_2)$ is referred to as a **Euclidean space**.

As we saw in the metric spaces lecture, completeness is one of the most desirable properties of a metric space. A complete normed linear space is referred to as a **Banach space**. Recall that a metric space is called complete if every Cauchy sequence converges in the space. Euclidean space is an example of a Banach space. It turns out that any finite dimensional normed linear space is a Banach space.

Series

Given a sequence $x_1, x_2, x_3, ...$ of elements of a normed linear space, their sum is called a **series**. We typically deal with infinite series in game theory when we study repeated games. For an infinite series, we can define a sequence of **partial sums** whose nth term is

$$s_n = \sum_{i=1}^{n} x_i$$

If $s_1, s_2, ...$ converges to some s, we say that the series $x_1 + x_2 + ...$ converges and we call $s = x_1 + x_2 + ...$ the sum of the infinite series.

Example:

$$(\frac{1}{2^k})_{k=1}^{\infty} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_1, s_2, s_3, ... = \frac{1}{2}, \frac{3}{4}, \frac{7}{8} ... \to 1$$

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 1$$

If an infinite series does not have a sum, we say that the series is divergent.

Example: $(1)_{k=1}^{\infty} = 1 + 1 + 1 + ...$ Then $s_1, s_2, s_3, s_4, ... = 1, 2, 3, 4, ...$ which tends to infinity. ■

When we have a convergent series, the terms in the sequence tend towards zero. The reverse of this, however, is not true. A sequence that converges does not necessarily induce a sequence of partial sums that converges.

Example: The series $(\frac{1}{k})_{i=1}^{\infty}$ is called the *harmonic series*. The sequence $\frac{1}{n}$ converges to zero. The sequence of partial sums of the series however diverges.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + ...$

$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + ...$

$= 1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + ... + \frac{1}{16})$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = \infty$ ■

A special type of sequence in which each term in the sequence is a constant multiple of the previous term is called a **geometric series**: $x + \delta x + \delta^2 x + ...$ A geometric series converges if and only if $|\delta| < 1$ and converges to $\frac{x}{1-\delta}$.

Example: $1 + \frac{1}{2} + \frac{1}{4} + ... = \sum_{i=1}^{\infty} (\frac{1}{2})^{i-1} = \frac{1}{1-1/2} = 2$ ■

Geometric series will play an important role in models where players discount the future.

Compactness in Euclidean Space

Recall that every compact set in a metric space is closed and bounded. The reverse, however, is not guaranteed to hold. We have seen already via the Heine-Borel theorem that the two are equivalent for a subset S of $(\mathbb{R}^n, d_2)$. Therefore the two are also equivalent in $(\mathbb{R}^n, \|\cdot\|_2)$ i.e. in Euclidean space. We originally took this powerful result for granted. Now we prove it. An additional definition and lemma will be useful in our first proof of this result.

A **limit point** (or "accumulation" point or "cluster" point) of a set A is a point a such that for all $r > 0$, there exists at least one point $c \in A\backslash\{a\}$ such that $c \in B_r(a)$. Intuitively, a limit point is a set where no matter how little you move off of it in all, you intersect points of the set. Cluster points need not be in the set.

Example: Every point in $[0, 1]$ is a limit point of $(0, 1)$ ■

Lemma 1 *A closed subset of a compact set is compact.*

Proof: Let K be a closed subset of a compact set $T \subset X$ and let C_K be an open cover of K. Then $U = X\backslash K$ is an open set and $C_T = C_K \cup U$ is an open cover of T. Since T is compact, then C_T has a finite subcover C'_T that also covers K. Since U does not contain any point of K, the set K is already covered by $C'_K = C'_T\backslash U$ i.e. a finite subcollection of the original collection C_K. It is therefore possible to extract from any open cover C_K of K a finite subcover. ■

Theorem 1 (Heine-Borel) *For a subset S of Euclidean space $\mathbb{R}^n$, S is compact if and only if it is closed and bounded.*

Proof: Recall the definition of compact: every open cover of S has a finite subcover. We start by showing that compact implies closed. Let a be a limit point of S and let C be an arbitrary collection of open sets such that every open set $U \in C$ is disjoint from some neighborhood, V_U of a. Then C fails to cover S. Indeed $\cap V_U$ is a neighborhood W of a in

$\mathbb{R}^n$. Since a is a limit point of S, W must contain a point $x \in S$. This point is not covered by C since every U in C is disjoint from V_U and hence disjoint from W which contains x.

If S is compact but not closed, then it has a limit point $a \notin S$. Consider a collection C' consisting of an open neighborhood $N(x)$ for every $x \in S$ chosen small enough not to intersect some neighborhood V_x of a. Then C' is an open cover of S but any finite subcollection of C' has the form of C discussed previously and thus cannot be a finite subcover of S. This contradicts the compactness of S. Therefore every limit point of S is in S so S is closed.

Now consider the open balls centered on a common point. This can cover any set, because all points in the set are some distance away from that point. Any finite subcover of this cover must be bounded because all balls in the subcover are contained in the largest open ball within that subcover. Therefore any set covered by this subcover must also be bounded.

If a set $S \in \mathbb{R}^n$ is bounded, then it can be enclosed in an n-box $T_0 = [-a, a]^n$ where $a > 0$. By (1), it is enough to show that T_0 is compact. Assume fsoc that T_0 is not compact. Then there exists an infinite open cover C of T_0 that does not admit any finite subcover. By bisecting each of the sides of the T_0 box, T_0 can be broken into 2^n sub n-boxes, each of which has a diameter equal to half the diameter of T_0. Then at least one of the 2^n sections of T_0 must require an infinite subcover of C, otherwise C itself would have a subcover formed by taking the union of the finite subcovers of the sections. Call this T_1. The sides of T_1 can be bisected as well which yields 2^n smaller sections of T_1, at least one of which must require an infinite subcover of C.

Continuing this process gives us a decreasing sequence of nested n-boxes: $T_0 \supset T_1 \supset T_2 \supset ... \supset T_k \supset ...$ The side length of T_k is $\frac{2a}{2^k} \to 0$. Define a sequence (x_k) such that each x_k is in T_k. This sequence is Cauchy: for any two elements of the set x_j, x_l with $l > j$. Since $x_l \in T_j$, $d(x_k, x_l) \leq \frac{2a}{2^j}$, the diameter of T_j. Since $\frac{2a}{2^j} < \frac{3a}{2^j}$ which is strictly decreasing in j, for any ϵ, we can choose a N sufficiently large such that $\frac{3a}{2^N} \leq \epsilon$. Because the sequence is Cauchy, it converges to a limit, L. Since each T_k is closed, $L \in T_k$ for all k.

Since C covers T_0, then it has some member $U \in C$ such that $L \in U$. Since U is open, there is an open ball $B_n(L) \subseteq U$. For a large enough k, we have $T_k \subseteq B_n(L) \subseteq U$. But this contradicts our claim that we need an infinite number of members of C in order to cover T_k. Therefore T_0 is compact. Since S is a closed subset of T_0, by (1), S is also compact. ■

Closely related to the Heine-Borel theorem is the Bolzano-Weierstrass theorem. In metric spaces, a set is compact if and only if it is **sequentially compact**, i.e. if every infinite sequence has a convergent subsequence. The Bolzano Weierstrass theorem will allow us to use this equivalence to write a simpler proof of the Heine-Borel theorem.

Theorem 2 (Bolzano-Weierstrass) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof: Let (x_n) be a bounded sequence. Without loss of generality assume that every term in the sequence lies in $[0,1]$. Divide $[0,1]$ into two intervals, $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. At least one of the halves contains infinitely many terms of (x_n). Denote that interval I_1 which has length $\frac{1}{2}$. Let (x_{n_k}) be the subsequence of (x_n) consisting of every term that lies in I_1. Now divide I_1 into two halves each of length $\frac{1}{4}$. At least one of these contains infinitely many terms of the (x_{n_k}) and denote that half by I_2. Continue this way to construct a sequence of nested intervals $I_1 \supseteq I_2 \supseteq ...$ where the length of I_n is $(\frac{1}{2})^n$ and each interval contains an infinite number of terms of the original sequence (x_n). Now construct a subsequence (z_n) of (x_n) made up of one term for each interval I_n. This is a Cauchy sequence: for all $m,n > N$, $|z_m - z_n| < (\frac{1}{2})^N$. Since $\mathbb{R}^n$ is a complete metric space, (z_n) converges. ■

Now back to the last part of Heine-Borel. It is now much easier to prove that closed and bounded implies compact. Consider a closed bounded subset S of $\mathbb{R}^n$. Let x_n be a sequence in S. Because S is bounded, so is x_n. By BW, x_n has a convergent subsequence in $\mathbb{R}^n$. Since S is closed, the limit belongs to S. Therefore S is sequentially compact and therefore compact.

Exercises

1) Prove the following: for any x, y in a normed linear space,

$$\|x\| - \|y\| \leq \|x - y\|$$

Solution: $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ by property (4). Rearranging yields $\|x\| - \|y\| \leq \|x - y\|$ ■

2) Prove that if $x_n \to x$ is a convergent sequence in a normed linear space, then $\|x_n\| \to \|x\|$

Solution: We are given that for all $\epsilon > 0$, there exists an N such that for all $n > N$, $\|x_n - x\| < \epsilon$. The sequence $\|x_n\|$ lies in the metric space $(\mathbb{R}, d)$. Therefore to show that $\|x_n\| \to \|x\|$, we need to show that for all $\epsilon > 0$, there exists an N such that for all $n > N$, $d(\|x_n\|, \|x\|) = |\|x_n\| - \|x\|| < \epsilon$. Note that $|\|x_n\| - \|x\|| \leq \|x_n - x\|$: for $a, b \in \mathbb{R}$, $\|y - x\| = \max\{(y - x), (x - y)\}$. By the reverse triangle inequality, $\|x_n\| - \|x\| \leq \|x_n - x\|$ and $\|x\| - \|x_n\| \leq \|x - x_n\|$. Since $\|x_n - x\| = \|x - x_n\|$, we have that $|\|x_n\| - \|x\|| \leq \|x_n - x\| < \epsilon$ for N sufficiently high. ■

3) Prove that $\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1-\delta}$ for $\delta \in (0, 1)$.

Solution: The partial sums are given by

$$s_n = \sum_{k=0}^{n} a\delta^k = a\frac{1 - \delta^{n+1}}{1 - \delta} :$$

$$(1 - \delta)\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1 - \delta} = \sum_{k=0}^{n} a\delta^k - \sum_{k=0}^{n} a\delta^{k+1}$$

$$= a + a\delta + a\delta^2 + ... + a\delta^n - \delta(a + a\delta + a\delta^2 + ... + a\delta^n)$$

$$= a - a\delta^{n+1}$$

For $\delta \in (0,1)$, $\lim_{n\to\infty} a\delta^{n+1} = 0$. Therefore $\lim_{n\to\infty} s_n = \frac{a}{1-\delta}$. ■

Source Material

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