Smooth Functions

1. Prove that every continuous linear functional; is differentiable with $Df[\boldsymbol{x}] = \boldsymbol{\alpha}$.[1]

We need to find a $g(\boldsymbol{x})$ such $f(\boldsymbol{x}_0 + \boldsymbol{x}) - f(\boldsymbol{x}_0) \approx g(\boldsymbol{x})$. Since f is a linear function, we can write

$$f(\boldsymbol{x}) = \boldsymbol{\alpha}^T \boldsymbol{x}$$

for some $\boldsymbol{\alpha}$. Then,

$$\begin{aligned} f(\boldsymbol{x}_0 + \boldsymbol{x}) - f(\boldsymbol{x}_0) &= \boldsymbol{\alpha}^T(\boldsymbol{x}_0 + \boldsymbol{x}) - \boldsymbol{\alpha}^T \boldsymbol{x}_0 \\ &= \boldsymbol{\alpha}^T \boldsymbol{x} = g(\boldsymbol{x}) \end{aligned}$$

Then $\boldsymbol{x} \to \boldsymbol{0} \implies f(\boldsymbol{x}_0 + \boldsymbol{x}) - f(\boldsymbol{x}_0) - g(\boldsymbol{x}) \to \boldsymbol{0} \implies \eta(\boldsymbol{x}) \to \boldsymbol{0}$ as desired.

2. Prove that if a differentiable functional $f : \mathbb{R}^n \to \mathbb{R}$ is increasing, then $Df[\boldsymbol{x}_0](\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in X$, or $\frac{\partial f}{\partial x_i} \geq 0$ for all $i \in \{x_1, ..., x_n\}$.

We prove the contrapositive. Suppose

$$\frac{\partial f}{\partial x_i} < 0$$

for some i and take $\boldsymbol{dx} = \{0, ..., dx_i, ..., 0\}$. By the mean value theorem, there exists some $\bar{\boldsymbol{x}} = \{0, ..., \bar{x}_i, ..., 0\}$ with $0 < \bar{x}_i < dx_i$ such that

$$f(\boldsymbol{x}_0 + \boldsymbol{dx}) + f(\boldsymbol{x}_0) = Df[\bar{\boldsymbol{x}}](\boldsymbol{dx})$$

Our hypothesis implies

$$f(\boldsymbol{x}_0 + \boldsymbol{dx}) + f(\boldsymbol{x}_0) < 0$$

But then f is not increasing. We conclude $\frac{\partial f}{\partial x_i} \geq 0$. ■

3. Let f be a differentiable functional. Prove that the $\nabla f(\boldsymbol{x}_0)$ is orthogonal to the hyperplane tangent to the contour through $f(\boldsymbol{x}_0)$.

Let $c = f(\boldsymbol{x}_0)$. The contour through $f(\boldsymbol{x}_0)$ is

$$f^{-1}(c) = \{\boldsymbol{x} : f(\boldsymbol{x}) = c\}$$

[1] Carter 4.6

Define an implicit function $h(t)$ where

$$h(t) = f(\boldsymbol{x}(t)) = c$$

for all t. Since f is differentiable, we have by the chain rule that

$$Dh(t) = Df(\boldsymbol{x}(t))^T D\boldsymbol{x}(t) = 0$$

because c is constant. Since f is a functional this can be written

$$Dh(t) = \nabla f(\boldsymbol{x}(t))^T D\boldsymbol{x}(t) = 0$$

Since $\nabla f(\boldsymbol{x})$ is the gradient of f and $\nabla f(\boldsymbol{x})^T D\boldsymbol{x}(t) = 0$ we know $\nabla f(\boldsymbol{x}_0)$ and $D\boldsymbol{x}(t)$ are orthogonal. Note that $D\boldsymbol{x}(t)$ is a linear approximation of the function $\boldsymbol{x}(t)$ at t – a tangent hyperplane to the contour. ∎

4. Let the policy production function discussed above be written

$$f(x, y) = x^\alpha y^\beta$$

   Give a sufficient condition for this function to be concave on $\{\mathbb{R}_{++} \times \mathbb{R}_{++}\}$. **Hint:** A 2×2 symmetric matrix A is negative definite if $A_{11} < 0$ and $A_{11}A_{22} - A_{12}A_{21} > 0$.

We need to find conditions under which

$$\boldsymbol{z}^T H_f(x, y)\boldsymbol{z} \leq 0$$

for arbitrary $\boldsymbol{z}$. The Hessian is given by

$$H_f(x, y) = \begin{pmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial x \partial y} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix}$$

Computing second derivatives and cross partials,

$$H_f(x, y) = \begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2}y^\beta & \alpha x^{\alpha-1}\beta y^{\beta-1} \\ \alpha x^{\alpha-1}\beta y^{\beta-1} & x^\alpha \beta(\beta-1)y^{\beta-2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(\alpha-1)f(x,y)}{x^2} & \frac{\alpha\beta f(x,y)}{xy} \\ \frac{\alpha\beta f(x,y)}{xy} & \frac{\beta(\beta-1)f(x,y)}{y^2} \end{pmatrix} = f(x, y) \begin{pmatrix} \frac{\alpha(\alpha-1)}{x^2} & \frac{\alpha\beta}{xy} \\ \frac{\alpha\beta}{xy} & \frac{\beta(\beta-1)}{y^2} \end{pmatrix}$$

Applying the hint, we need

$$H_{11} < 0 \qquad H_{11}H_{22} > H_{12}^2$$

$H_{11} < 0$ and $H_{22} < 0$ require $\alpha, \beta < 1$. Rearranging the final condition,

$$\begin{aligned}
H_{11}H_{22} &> H_{12}^2 \\
\frac{\alpha(\alpha-1)}{x^2}\frac{\beta(\beta-1)}{y^2} &> \left(\frac{\alpha\beta}{xy}\right)^2 \\
\frac{\alpha(\alpha-1)\beta(\beta-1)}{y^2x^2} &> \frac{\alpha^2\beta^2}{x^2y^2} \\
\frac{\alpha(\alpha-1)\beta(\beta-1)-\alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{(\alpha^2-\alpha)(\beta^2-\beta)-\alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{\alpha^2\beta^2-\alpha\beta^2-\beta\alpha^2+\alpha\beta-\alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{\alpha\beta(1-\beta-\alpha)}{x^2y^2} &> 0
\end{aligned}$$

which requires $\alpha + \beta < 1$.