

# An Introduction to Mathematics for Political Science

## Problem Set 1

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions or scans of handwritten solutions as a PDF. Note that starting with problem set 3 all solutions must be written in LaTeX.

### Set Basics

1. Let  $A = (40, 60)$  and  $B = [50, 70]$ .
  - a) Is  $A \subset B$ ,  $B \subset A$ , both, or neither?
  - b) What is  $A \cup B$ ?
  - c) What is  $A \cap B$ ?
  - d) Write three elements of the Cartesian product  $A \times B$ .

**Solution:**

- a) Neither
  - b)  $(40, 70]$
  - c)  $[50, 60)$
  - d)  $(50, 50)$ ,  $(55, 60)$ ,  $(59, 70)$
2. Identify whether the following sets are (a) open, closed, or neither; (b) bounded; (c) compact; (d) convex:

- a)  $(0, 1)$
- b)  $[0, 1]$
- c)  $(0, 1]$
- d)  $[0, \infty)$
- e)  $(0, \infty)$
- f)  $[0, 1] \cup [2, 3]$
- g)  $[0, 1] \times [0, 1]$
- h)  $(0, 3] \cap [1, 4]$
- i)  $[0, 5] \setminus \{1, 2\}$

**Solution:**

- a) open, bounded, not compact, convex
- b) closed, bounded, compact, convex
- c) neither, bounded, not compact, convex
- d) closed, unbounded (above), not compact, convex
- e) open, unbounded (above), not compact, convex
- f) closed, bounded, compact, not convex
- g) closed, bounded, compact, convex
- h) closed, bounded, compact, convex
- i) neither, bounded, not compact, not convex

3. Express the following sentences in mathematical notation:

- a)  $A$  is the set of all real numbers less than or equal to seven, excluding zero and four.
- b)  $B$  is the intersection of the natural numbers and the real numbers between  $\pi$  and 30.5.
- c) For all epsilon greater than zero, there exists a delta greater than zero.
- d) The set of all even integers between 5 and 21.

**Solution:**

- a)  $A = \{x \in \mathbb{R} \setminus \{0, 4\} \mid x \leq 7\}$
- b)  $B = \mathbb{N} \cap (\pi, 30.5) = \{x \in \mathbb{N} \mid 4 \leq x \leq 30\}$

c)  $\forall \epsilon > 0, \exists \delta > 0$

d)  $\{x \in \mathbb{Z} | 5 \leq x \leq 21, x \text{ even}\}$

## Algebra

4. Simplify into one term or evaluate the following:

a)  $y \cdot y \cdot y \cdot y$

b)  $(-a)(-b)^3 - b^2 + a^3$

c)  $(4b + 2)(a - 5)$

d)  $\frac{5!}{2!}$

e)  $\sum_{i=1}^3 \left(\frac{1}{3}\right)^i$

f)  $\sum_{i=2}^5 2^i$

g)  $\prod_{i=1}^3 \left(\frac{1}{3}\right)^i$

h)  $\frac{48}{4} - 6 \cdot 9$

i)  $(3^3 + (-5)) \cdot 3 - (-7)$

j)  $\left[6 + \left(\frac{-66}{11}\right)\right] \cdot (-2)^3$

k)  $\frac{y-11}{5} + \frac{y+12}{3}$

**Solution:**

a)  $y^4$

b)  $ab - b^2 + a^3$

c)  $4ab + 2a - 20b - 10$

d) 60

e)  $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{9+3+1}{27} = \frac{13}{27}$

f)  $4 + 8 + 16 + 32 = 60$

g)  $\frac{1}{3 \cdot 9 \cdot 27} = \frac{1}{729}$

h) -42

i) 73

j) 0

k)  $\frac{8y+27}{15}$

5. Solve the following for  $x$ :

a)  $5(-3x - 2) - (2x - 3) = -4(4x + 5) + 13$

b)  $8x^2 = 15 - 14x$

c)  $x^2 - 3x + 4 = 2(x - 1)$

d)  $-6 > 5x + 5 + 4$

e)  $-2(x + 1) + 4 < 10$

f)  $2\ln(2x + 1) - 10 = 0$

g)  $3^x e^{4x} = e^7$

**Solutions**

a)  $x = 0$

b)  $x \in \{-2.5, .75\}$

c)  $x \in \{3, 2\}$

d)  $x < -3$

e)  $x > 4$

f)  $x = \frac{\exp(5)-1}{2}$

g)  $x = \frac{7}{\ln(3)+4}$

## Functions

6) Find the image/range of  $x^2$  on the domain  $[-3, 3]$ .

**Solution:**  $[0, 9]$

7) Let  $f(x) = x^2 - 4x + 2$  and  $g(x) = 3x - 7$ . Find (a)  $f(x) - g(x)$ ; (b)  $f(x) + g(x)$ ; (c)  $f(g(x))$ ; and (d)  $g(f(x))$ . Evaluate each for  $x = 2$ .

**Solutions:**

a)  $x^2 - 7x + 9$ ,  $f(2) - g(2) = -1$

b)  $x^2 - x - 5, f(2) + g(2) = -3$

c)  $9x^2 - 54x + 77, f(g(2)) = 5$

d)  $3x^2 - 12x - 1, g(f(2)) = -13$

8) Identify whether each of the following functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  is (a) surjective/onto; (b) injective/one-to-one; (c) bijective.

a)  $f(x) = x^2$

b)  $f(x) = x^3 - x$

c)  $f(x) = e^x$

d)  $f(x) = x^3$

**Solutions:**

a) Neither; negative reals are not mapped into and positives are mapped into twice

b) Onto, not one-to-one; all reals are mapped into, reals near  $f(0)$  mapped into twice

c) Not onto, one-to-one; negative reals not mapped into, each positive real mapped into only once

d) Both.

9) Find an equation for the inverse for each of the following functions:

a)  $f(x) = (5x - 1)^3$

b)  $f(x) = \frac{x+4}{3x-5}$

c)  $f(x) = e^{5x-1}$

**Solutions:**

a)  $f^{-1}(x) = \frac{1}{5}(x^{\frac{1}{3}} + 1)$

b)  $f^{-1}(x) = \frac{5x+4}{3x-1}$  [Multiply by  $3x - 5$ , distribute, add  $5y$ , subtract  $x$ , factor  $x$ , divide by  $3y - 1$ , swap]

c)  $f^{-1}(x) = \frac{\ln(x)+1}{5}$

10) Evaluate each of the following limits or show that they do not exist:

a)  $\lim_{x \rightarrow 5} \frac{x^2-25}{x^2+x-30}$

b)  $\lim_{x \rightarrow -1} \frac{x^3}{(x+1)^2}$

- c)  $\lim_{x \rightarrow 2} \frac{x^2+4x-12}{|x-2|}$   
d)  $\lim_{x \rightarrow \infty} \frac{x^2-1}{2x^2+1}$   
e)  $\lim_{x \rightarrow \infty} (\frac{x^3}{x^2+2} - x)$   
f)  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x}$

**Solutions:**

a)  $\frac{x^2-25}{x^2+x-30} = \frac{(x-5)^2}{(x-5)(x+6)} = \frac{x+5}{x+6} \cdot \lim_{x \rightarrow 5} \frac{x+5}{x+6} = 10/11$

b) Limit does not exist; function approaches  $-\infty$  as  $x$  approaches  $-1$  from each side.

c) To evaluate the RHS limit, find  $\lim_{x \rightarrow 2^+} \frac{x^2+4x-12}{x-2} = \lim_{x \rightarrow 2^+} \frac{(x-2)(x+6)}{(x-2)} = \lim_{x \rightarrow 2^+} x + 6 = 8$ . To evaluate the LHS limit, find  $\lim_{x \rightarrow 2^-} \frac{x^2+4x-12}{-(x-2)} = \lim_{x \rightarrow 2^-} \frac{(x-2)(x+6)}{-(x-2)} = \lim_{x \rightarrow 2^-} -(x+6) = -8$ . Because the left and right limits are not equivalent, the limit does not exist.

d)  $\lim_{x \rightarrow \infty} \frac{x^2-1}{2x^2+1} = \lim_{x \rightarrow \infty} \frac{x^2}{2x^2} = 1/2$

e)  $(\frac{x^3}{x^2+2} - x) = \frac{-2x}{x^2+2} \cdot \lim_{x \rightarrow \infty} \frac{-2x}{x^2+2} = -2 \lim_{x \rightarrow \infty} \frac{x}{x^2+2} = -2 \lim_{x \rightarrow \infty} \frac{1/x}{1+2/x^2} = -2 \frac{\lim_{x \rightarrow \infty} 1/x}{\lim_{x \rightarrow \infty} (1+2/x^2)} = -2 \cdot \frac{0}{1} = 0$ .

11) Determine whether the following functions are continuous at the specified value. Use limits where appropriate. You may find sketching the graph of each function to be helpful.

a)  $f(x) = \frac{x^2+1}{x^3+1}$  at  $x = -1$

b)  $f(x) = \begin{cases} 3x-5 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \quad \text{at } x = 1$

c)  $f(x) = \begin{cases} \frac{x-6}{x-3} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{4+x^2} & \text{if } x > 0 \end{cases} \quad \text{at } x = 0$

**Solutions:**

a) Function is undefined at  $x = -1$  and is therefore discontinuous at  $-1$ .

b) The function is defined at 1 since  $f(1) = 2$ . To determine whether it is continuous, consider  $\lim_{x \rightarrow 1} 3x-5 = 3-5 = -2$ . Since  $\lim_{x \rightarrow 1} f(x) \neq f(1)$ , the function is not continuous at  $x = 1$ .

c) The function is clearly defined at  $x = 0$ . Now check limits on either side of 0.  
 $\lim_{x \rightarrow 0^-} \frac{x-6}{x-3} = -6/-3 = 2$ .  $\lim_{x \rightarrow 0^+} \sqrt{4+x^2} = \sqrt{4} = 2$ . Conclude that the function is continuous at  $x = 0$ .