Continuous Functions

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- Now we reintroduce functions as relationships between elements in two or more spaces.
- "Sets and spaces provide the basic characters of mathematical analysis, functions provide the plot" (Carter 2001).
- In this lecture we study functions that preserve the geometric structure of the sets they associate, continuous functions.

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- We typically refer to x as an **independent variable** and $y = f(x)$ as a **dependent variable**.
- The **range** or "image" of $f : X \to Y$ is the set of all elements in Y that are images of elements in X, denoted

$$f(X) = \{y \in Y : y = f(x) \text{ for some } y \in Y\}$$

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- A **graph** of a function

$$\mathrm{graph}(f) = \{(x, y) \in X \times Y : y = f(x), x \in X\}$$

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- It is also not one-to-one: $-x^2 = x^2$.

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- Utility functions such as $-(x-z)^2$ are functionals.

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- The composition of these two functions is

$$g(f(x)) = \alpha \ln(x) + \beta \ln(y)$$

  with $g(f(x)) : \mathbb{R}^2_+ \to \mathbb{R}$

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- This is a discontinuous function.

Continuity

- Now consider the function

$$f(x, y, z) = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z + \epsilon$$

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- A continuous function preserves the geometric structure of the sets it associates i.e. the metric spaces it associates.
- For the rest of this lecture we will assume that X and Y are metric spaces.

Continuity: Epsilon-Delta Characterization

- A function $f : X \to Y$ is continuous at x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in X$,

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- If a function is continuous for all x_0 in its domain, we call the function continuous.

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- Therefore $|x - 0| < \delta$ implies $|f(x) - f(0)| < \epsilon$ so f is continuous at $x = 0$.

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- Note that $d(x, x_0) = |x - x_0|$ and observe that

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- For $2|x - x_0|(2|x_0| + k) < \epsilon$ to hold, it is sufficient that $|x - x_0| < \frac{\epsilon}{2(2|x_0|+k)}$ and $|x - x_0| < k$

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- This alternative definition implies that values of $f(x)$ are close to $f(x_0)$ when values of x are close to x_0.

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- Therefore f is discontinuous at $x = 0$.

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- A function $f : X \to Y$ is continuous if and only if the inverse image of any open (closed) subset of Y is an open (closed) subset of X.

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- Note that this is an open subset of $X = \mathbb{R}$.

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- The inverse image under $(0, 2)$ is $\mathbb{R}$.

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- Because $\lim f(x_n) = f(x_0)$, there exists an N such that $n > N$ implies $||f(x_n)| - |f(x_0)|| < \epsilon$ and thus $\lim |f(x_n)| = |f(x_0)|$.

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- Therefore $f + g$ is continuous at x_0.
- A similar proof establishes that fg and f/g are continuous at x_0.

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- Because f is continuous at x_0, $\lim f(x_n) = f(x_0)$.
- Because $f(x_n)$ converges to $f(x_0)$ and g is continuous at $f(x_0)$, we have

$$\lim g(f(x_n)) = g(f(x_0))$$

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- The same proof can be applied to the infimum of the set.