

# Introduction to Mathematics for Political Science

## Problem Set 9: Matrix Inversion and Determinants

**Instructions:** You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$\begin{aligned}3x_1 - x_2 &= 10 \\ -x_1 + 4x_2 &= 4\end{aligned}$$

Write this system in  $A\mathbf{x} = \mathbf{b}$  form and solve via matrix inversion.

2. Let  $C = AB$  where  $C$  is invertible and  $A$  and  $B$  are square matrices. Solve for  $A^{-1}$ .<sup>1</sup>
3. Let  $M = ABC$  where  $M$  is invertible and  $A$ ,  $B$ , and  $C$  are square matrices. Solve for  $B^{-1}$ .<sup>2</sup>
4. If  $B$  is the inverse of  $A^2$ , show that  $AB$  is the inverse of  $A$ .<sup>3</sup>
5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

What are  $X^{-1}$  and  $Y^{-1}$ , assuming  $ad \neq bc$ .<sup>4</sup> **Hint:** First, multiply the two matrices  $XY$ . Then, attempt to solve for  $X^{-1}$  and  $Y^{-1}$ .

6.  $A$  is an *idempotent* matrix if and only if  $AA = A$ . Show that if  $A$  is symmetric ( $A^\top = A$ ) and idempotent then  $(I - A) = (I - A)(I - A)^\top$ , where  $I$  is the identity matrix.
7. Let  $R_{m \times n}$  be a rectangular matrix ( $m \neq n$ ) and  $A_{m \times m}$  be a symmetric matrix. Show  $R^T A R$  is also symmetric. What are the dimensions of this matrix?<sup>5</sup>

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<sup>1</sup>Strang p. 90 #12

<sup>2</sup>Strang p. 90 #13

<sup>3</sup>Strang p. 90 #18

<sup>4</sup>Strang p. 90 #16

<sup>5</sup>Strang 117 #19

8. Show every orthogonal matrix  $A$  has determinant 1 or -1. Hint: Apply the product rule ( $|AB| = |A||B|$ ) and the transpose rule ( $|A| = |A^T|$ ) for determinants.<sup>6</sup>
9. Let  $\mathbf{x} = \{x_1, \dots, x_{50}\}$  denote the number of electoral votes for each state. Let  $\mathbf{y}_i = \{y_{i1}, \dots, y_{ij}, \dots, y_{i50}\}$  denote whether or not candidate  $i \in \{R, D\}$  won the votes of a given state, where  $y_{ij} \in \{0, 1\}$ . Write an expression for the total number of votes for each candidate.

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<sup>6</sup>Strang p. 252 #8