

Inner Product Spaces, Orthogonality, Projection

Exercises

1. Let X and Y be normed linear spaces. Let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a basis for X and $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ a basis for Y . Prove that if $x_i \perp y_j$ for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, then X and Y are orthogonal spaces.

Because $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis for X , all $\mathbf{x} \in X$ can be expressed as linear combinations of $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. We therefore need to show that all linear combinations of $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ are orthogonal to all linear combinations of $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$. More formally, we need

$$\alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j \perp \alpha_y \mathbf{y}_i + \beta_y \mathbf{y}_j$$

for arbitrary $\alpha_x, \beta_x, \alpha_y, \beta_y, \mathbf{x}_i, \mathbf{x}_j, \mathbf{y}_i, \mathbf{y}_j$. Equivalently,

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i + \beta_y \mathbf{y}_j \rangle$$

By the bilinearity (additivity) of the inner product, we can write

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i + \beta_y \mathbf{y}_j \rangle = \underbrace{\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i \rangle}_A + \underbrace{\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \beta_y \mathbf{y}_j \rangle}_B$$

Focusing on A, we also know by the symmetry of inner products that

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i \rangle = \langle \alpha_x \mathbf{x}_i, \alpha_y \mathbf{y}_i \rangle + \langle \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i \rangle$$

Again by the bilinearity (homogeneity) of the inner product, this can be written

$$\langle \alpha_x \mathbf{x}_i, \alpha_y \mathbf{y}_i \rangle + \langle \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i \rangle = \alpha_x \alpha_y \underbrace{\langle \mathbf{x}_i, \mathbf{y}_i \rangle}_{=0} + \beta_x \alpha_y \underbrace{\langle \mathbf{x}_j, \mathbf{y}_i \rangle}_{=0}$$

where $\langle \mathbf{x}_i, \mathbf{y}_i \rangle$ and $\langle \mathbf{x}_j, \mathbf{y}_i \rangle = 0$ by the orthogonality of the basis vectors. We conclude

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i \rangle = 0$$

Repeating the same argument for B gives

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \beta_y \mathbf{y}_j \rangle = 0$$

which gives

$$\langle \alpha_x \mathbf{x}_i + \beta_x \mathbf{x}_j, \alpha_y \mathbf{y}_i + \beta_y \mathbf{y}_j \rangle = 0$$

as desired. ■

2. Prove: If a vector α is in the null space of a set of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, then it is orthogonal to the space spanned by $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ where

$$\mathbf{y}_i = \{x_{1i}, \dots, x_{ni}\}$$

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

store the \mathbf{x} vectors. If $\alpha \in N(X)$, then

$$X\alpha = \mathbf{0}$$

or

$$\alpha^T X^T = \mathbf{0}$$

Now let

$$X^T = \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}$$

We can represent the space spanned by these vectors with $X^T\beta$ with β taking arbitrary linear combinations of the columns of X^T . We want to show

$$\alpha^T X^T \beta = \mathbf{0}$$

Because $\alpha^T X^T = \mathbf{0}$, this must be the case. ■.

3. Donald Trump tweeted 100 times in April, 150 times in May, and 110 times in June.¹ Let $\mathbf{b} = (100, 150, 110)$ represent the number of tweets in each month. Project \mathbf{b} onto the linear space spanned by $\mathbf{a} = (1, 1, 1)$. Interpret your result.

We have $\mathbf{p} = \hat{x}\mathbf{a}$ and $\mathbf{e} = \mathbf{b} - \mathbf{p}$. We need \mathbf{e} to be orthogonal to \mathbf{a} , or

$$\mathbf{a}^T (\mathbf{b} - \hat{x}\mathbf{a}) = 0$$

$$\mathbf{a}^T \mathbf{b} - \hat{x} \mathbf{a}^T \mathbf{a} = 0$$

$$\mathbf{a}^T \mathbf{b} = \hat{x} \mathbf{a}^T \mathbf{a}$$

$$\frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} = \hat{x}$$

Substituting our values, this becomes

$$\frac{\sum_i b_i}{\sum_i 1} = \frac{100 + 150 + 110}{3} = 120$$

Notice that for n months of tweeting, this is

$$\frac{1}{n} \sum_i b_i$$

or simply the mean number of tweets.

¹Disclaimer: these data are of suspect quality.