Ordered Sets

Relations

- Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$.

Relations

- Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$.
- We use the notation $(a, b) \in R$ or more often aRb to denote the relation R holding for an ordered pair (a, b).

Relations

- Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$.
- We use the notation $(a, b) \in R$ or more often aRb to denote the relation R holding for an ordered pair (a, b).
- This is read "a is in the relation R to b."

Relations

- Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$.
- We use the notation $(a, b) \in R$ or more often aRb to denote the relation R holding for an ordered pair (a, b).
- This is read "a is in the relation R to b."
- If $R \subset A \times A$, we say that R is a **relation on** A.

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.
- Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of."

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.
- Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of."
- Let $A = \{a, b, c\}$.

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.
- Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of."
- Let $A = \{a, b, c\}$.
- Let $R = \{(a, b), (a, c), (b, c)\}$

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.
- Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of."
- Let $A = \{a, b, c\}$.
- Let $R = \{(a, b), (a, c), (b, c)\}$
- What relation does R express?

Examples

- Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$.
- Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of."
- Let $A = \{a, b, c\}$.
- Let $R = \{(a, b), (a, c), (b, c)\}$
- What relation does R express?
- aRb: "a *occurs earlier in the alphabet than* b."

Properties of Binary Relations

A relation R on a nonempty set X is

- **reflexive** if xRx for each $x \in X$

Properties of Binary Relations

A relation R on a nonempty set X is

- **reflexive** if xRx for each $x \in X$
- **complete** if xRy or yRx for all $x, y \in X$

Properties of Binary Relations

A relation R on a nonempty set X is

- **reflexive** if xRx for each $x \in X$
- **complete** if xRy or yRx for all $x, y \in X$
- **symmetric** if for any $x, y \in X$, xRy implies yRx

Properties of Binary Relations

A relation R on a nonempty set X is

- **reflexive** if xRx for each $x \in X$
- **complete** if xRy or yRx for all $x, y \in X$
- **symmetric** if for any $x, y \in X$, xRy implies yRx
- **antisymmetric** if for any $x, y \in X$, xRy and yRx imply $x = y$

Properties of Binary Relations

A relation R on a nonempty set X is

- **reflexive** if xRx for each $x \in X$
- **complete** if xRy or yRx for all $x, y \in X$
- **symmetric** if for any $x, y \in X$, xRy implies yRx
- **antisymmetric** if for any $x, y \in X$, xRy and yRx imply $x = y$
- **transitive** if xRy and yRz imply xRz for any $x, y, z \in X$

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.
- A set on which a preorder is defined is called a **preordered set**.

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.
- A set on which a preorder is defined is called a **preordered set**.
- In our theory of choice, we refer to a preorder on X as a **preference relation** on X

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.
- A set on which a preorder is defined is called a **preordered set**.
- In our theory of choice, we refer to a preorder on X as a **preference relation** on X
- Preorders fall into two fundamental categories, depending on whether or not the relation is symmetric.

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.
- A set on which a preorder is defined is called a **preordered set**.
- In our theory of choice, we refer to a preorder on X as a **preference relation** on X
- Preorders fall into two fundamental categories, depending on whether or not the relation is symmetric.
- A *symmetric preorder* is called an **equivalence relation**.

Preorders

- Any relation which is reflexive and transitive is called a **preorder**.
- A set on which a preorder is defined is called a **preordered set**.
- In our theory of choice, we refer to a preorder on X as a **preference relation** on X
- Preorders fall into two fundamental categories, depending on whether or not the relation is symmetric.
- A *symmetric preorder* is called an **equivalence relation**.
- A preorder that is *not symmetric* is called an **order relation**.

Equivalence Relations

- A preorder $\sim$ on a nonempty set X is called an equivalence relation if it is reflexive, symmetric, and transitive.

Equivalence Relations

- A preorder $\sim$ on a nonempty set X is called an equivalence relation if it is reflexive, symmetric, and transitive.
- For any $x \in X$, the **equivalence class** of x relative to $\sim$ is defined as the set $[x]_\sim \equiv \{y \in X : y \sim x\}$

Equivalence Relations

- A preorder $\sim$ on a nonempty set X is called an equivalence relation if it is reflexive, symmetric, and transitive.
- For any $x \in X$, the **equivalence class** of x relative to $\sim$ is defined as the set $[x]_\sim \equiv \{y \in X : y \sim x\}$
- We often exploit indifference in our theory of choice and equivalence classes help us formalize this.

Example

- Equality is an equivalence relation on $\mathbb{R}$

Example

- Equality is an equivalence relation on $\mathbb{R}$
- Reflexiveness: $x = x$

Example

- Equality is an equivalence relation on $\mathbb{R}$
- Reflexiveness: $x = x$
- Symmetry: if $x = y$, then $y = x$

Example

- Equality is an equivalence relation on $\mathbb{R}$
- Reflexiveness: $x = x$
- Symmetry: if $x = y$, then $y = x$
- Transitivity: if $x = y$ and $y = z$, then $x = y = z$ so $x = z$

Example

- Equality is an equivalence relation on $\mathbb{R}$
- Reflexiveness: $x = x$
- Symmetry: if $x = y$, then $y = x$
- Transitivity: if $x = y$ and $y = z$, then $x = y = z$ so $x = z$
- The equivalence class of $\sim$ is $[x]_\sim = \{y \in X : y = x\}$

Example

- Equality is an equivalence relation on $\mathbb{R}$
- Reflexiveness: $x = x$
- Symmetry: if $x = y$, then $y = x$
- Transitivity: if $x = y$ and $y = z$, then $x = y = z$ so $x = z$
- The equivalence class of $\sim$ is $[x]_\sim = \{y \in X : y = x\}$
- For any x, its equivalence class is a singleton: on $\mathbb{R}$, each element is equal only to itself.

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$
- Transitivity?

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$
- Transitivity?
- $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which yields $af = be$

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$
- Transitivity?
- $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which yields $af = be$
- Equivalence class?

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$
- Transitivity?
- $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which yields $af = be$
- Equivalence class?
- $[(a, b)]_{\sim} = \{(c, d) \in X : \frac{c}{d} = \frac{a}{b}\}$

Example

- Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad = bc$
- Reflexiveness?
- $(a, b) \sim (a, b)$ is true iff $ab = ab$
- Symmetry?
- Does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$?
- $ad = bc$ implies $cb = da$
- Transitivity?
- $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which yields $af = be$
- Equivalence class?
- $[(a, b)]_\sim = \{(c, d) \in X : \frac{c}{d} = \frac{a}{b}\}$
- With this relation, $\frac{4}{2}$ is equivalent to $\frac{8}{4}$. This is obvious to us precisely because the rational numbers are constructed as equivalence classes such that $(4, 2), (8, 4) \in [(2, 1)]_\sim$.

Partitions

- There is a close relationship between equivalence relations on a set and partitions of that set.

Partitions

- There is a close relationship between equivalence relations on a set and partitions of that set.
- A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X.

Partitions

- There is a close relationship between equivalence relations on a set and partitions of that set.
- A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X.
- Any equivalence relation on a set X partitions X: each element of X belongs to one and only one equivalence class.

Partitions

- There is a close relationship between equivalence relations on a set and partitions of that set.
- A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X.
- Any equivalence relation on a set X partitions X: each element of X belongs to one and only one equivalence class.
- Equality partitions $\mathbb{R}$ into singletons

Partitions

- There is a close relationship between equivalence relations on a set and partitions of that set.
- A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X.
- Any equivalence relation on a set X partitions X: each element of X belongs to one and only one equivalence class.
- Equality partitions $\mathbb{R}$ into singletons
- On the previous slide, $\sim$ partitions X into the rational numbers.

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.
- $x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.
- $x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$
- The relation $\succ$ is transitive but not reflexive (why?).

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.
- $x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$
- The relation $\succ$ is transitive but not reflexive (why?).
- $x \sim y \iff x \succsim y \ \wedge \ y \succsim x$ for all $x, y \in X$

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.
- $x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$
- The relation $\succ$ is transitive but not reflexive (why?).
- $x \sim y \iff x \succsim y \ \wedge \ y \succsim x$ for all $x, y \in X$
- $\sim$ is the symmetric part of $\succsim$

Order Relations

- An order relation is a preorder, $\succsim$, that is reflexive and transitive but not symmetric.
- An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.
- Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.
- $x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$
- The relation $\succ$ is transitive but not reflexive (why?).
- $x \sim y \iff x \succsim y \ \wedge \ y \succsim x$ for all $x, y \in X$
- $\sim$ is the symmetric part of $\succsim$
- *Example* The natural order $\geq$ on $\mathbb{R}$ induces the strict order $>$ and the equivalence relation $=$ on $\mathbb{R}$

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$
- $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$.

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$
- $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$.
- For any ordered set $(X, \succsim)$, every maximum of a nonempty subset of X is maximal in that set (why?).

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$
- $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$.
- For any ordered set $(X, \succsim)$, every maximum of a nonempty subset of X is maximal in that set (why?).
- Maximal is a weaker concept than maximum.

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$
- $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$.
- For any ordered set $(X, \succsim)$, every maximum of a nonempty subset of X is maximal in that set (why?).
- Maximal is a weaker concept than maximum.
- Let A be a nonempty ordered subset of X. An element $x \in X$ is called an **upper bound** for A if $x \succsim a$ for all $a \in A$.

Maximal and Maximum Elements, Least Upper Bound

- Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$
- $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$.
- For any ordered set $(X, \succsim)$, every maximum of a nonempty subset of X is maximal in that set (why?).
- Maximal is a weaker concept than maximum.
- Let A be a nonempty ordered subset of X. An element $x \in X$ is called an **upper bound** for A if $x \succsim a$ for all $a \in A$.
- An upper bound is called a **least upper bound** or **supremum** for A if it precedes every other upper bound for A.

Example

- Consider the ordered set $([a, b], \geq)$

Example

- Consider the ordered set $([a, b], \geq)$
- b is the maximal element, the maximum element, and the least upper bound

Example

- Consider the ordered set $([a, b], \geq)$
- b is the maximal element, the maximum element, and the least upper bound
- Now consider $((a, b), \geq)$.

Example

- Consider the ordered set $([a, b], \geq)$
- b is the maximal element, the maximum element, and the least upper bound
- Now consider $((a, b), \geq)$.
- The set has no maximal or maximum element.

Example

- Consider the ordered set $([a, b], \geq)$
- b is the maximal element, the maximum element, and the least upper bound
- Now consider $((a, b), \geq)$.
- The set has no maximal or maximum element.
- b is its least upper bound in $\mathbb{R}$

Simplexes

- A simplex is the generalization of the notion of triangle.

Simplexes

- A simplex is the generalization of the notion of triangle.
- A standard k-simplex is defined a

$$\Delta^k \equiv \{x \in \mathbb{R}^{k+1} : x_0 + ...x_k = 1, x_i \geq 0, i = 0, ..., k\}$$

Simplexes

- A simplex is the generalization of the notion of triangle.
- A standard k-simplex is defined a

$$\Delta^k \equiv \{x \in \mathbb{R}^{k+1} : x_0 + ...x_k = 1, x_i \geq 0, i = 0, ..., k\}$$

- Δ^1 is a line segment connecting (0,1) and (1,0).

Simplexes

- A simplex is the generalization of the notion of triangle.
- A standard k-simplex is defined a

$$\Delta^k \equiv \{x \in \mathbb{R}^{k+1} : x_0 + ...x_k = 1, x_i \geq 0, i = 0, ..., k\}$$

- Δ^1 is a line segment connecting (0,1) and (1,0).
- Δ^2 is a triangular plane connecting (1,0,0), (0,1,0), and (0,0,1).

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?
- Every element.

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?
- Every element.
- There exists no $(a, b) \in \Delta^1$ such that $(a, b) > (1, 0)$

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?
- Every element.
- There exists no $(a, b) \in \Delta^1$ such that $(a, b) > (1, 0)$
- This follows from the fact that $1 > a$ for all $(a, b) \neq (1, 0)$

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?
- Every element.
- There exists no $(a, b) \in \Delta^1$ such that $(a, b) > (1, 0)$
- This follows from the fact that $1 > a$ for all $(a, b) \neq (1, 0)$
- Now consider $(1 - \epsilon, \epsilon)$

Example

- Let $X = \Delta^1$ and $\geq$ be the natural vector order.
- For $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$.
- What is the set of maximal elements?
- Every element.
- There exists no $(a, b) \in \Delta^1$ such that $(a, b) > (1, 0)$
- This follows from the fact that $1 > a$ for all $(a, b) \neq (1, 0)$
- Now consider $(1 - \epsilon, \epsilon)$
- Since $\epsilon > b$ for all $(a, b) \neq (1, 0)$, there is no $(a, b) > (1 - \epsilon, \epsilon)$.

Example (cont.)

- What is the set of maximum elements?

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$
- Assume (a, b) is a maximum.

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$
- Assume (a, b) is a maximum.
- This implies that $(a, b) \succsim (c, d)$ for all (c, d)

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$
- Assume (a, b) is a maximum.
- This implies that $(a, b) \succsim (c, d)$ for all (c, d)
- But for all (a, b), $(c, d) = (a - \epsilon, b + \epsilon)$ which implies that (a, b) and (c, d) cannot be compared

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$
- Assume (a, b) is a maximum.
- This implies that $(a, b) \succsim (c, d)$ for all (c, d)
- But for all (a, b), $(c, d) = (a - \epsilon, b + \epsilon)$ which implies that (a, b) and (c, d) cannot be compared
- Therefore it cannot be true that $(a, b) \succsim (c, d)$ which implies that (a, b) is not a maximum.

Example (cont.)

- What is the set of maximum elements?
- $\emptyset$
- Assume (a, b) is a maximum.
- This implies that $(a, b) \succsim (c, d)$ for all (c, d)
- But for all (a, b), $(c, d) = (a - \epsilon, b + \epsilon)$ which implies that (a, b) and (c, d) cannot be compared
- Therefore it cannot be true that $(a, b) \succsim (c, d)$ which implies that (a, b) is not a maximum.
- What is the least upper bound in $\mathbb{R}^2$?

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.
- *Example* Let $X = \Delta^1$ and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.
- *Example* Let $X = \Delta^1$ and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.
- The points (1,0) and (0,1) are both maximal and maximum elements. ■

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.
- *Example* Let $X = \Delta^1$ and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.
- The points (1,0) and (0,1) are both maximal and maximum elements. $\blacksquare$
- Uniqueness may be achieved by imposing the additional requirement of antisymmetry.

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.
- *Example* Let $X = \Delta^1$ and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.
- The points (1,0) and (0,1) are both maximal and maximum elements. ■
- Uniqueness may be achieved by imposing the additional requirement of antisymmetry.
- A **partial order** is a relation that is reflexive, transitive, and antisymmetric.

Partially Ordered Sets

- In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.
- *Example* Let $X = \Delta^1$ and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.
- The points (1,0) and (0,1) are both maximal and maximum elements. ■
- Uniqueness may be achieved by imposing the additional requirement of antisymmetry.
- A **partial order** is a relation that is reflexive, transitive, and antisymmetric.
- The result is a **partially ordered set**.

Example

- The natural order on $\mathbb{R}^n$ is only partial.

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?
- For example, $(1,1) \geq (1,0)$, but $(0,1)$ and $(1,0)$ are not comparable.

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?
- For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable.
- It is antisymmetric.

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?
- For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable.
- It is antisymmetric.
- $(a, b) \geq (c, d)$ implies that $a \geq c$ and $b \geq d$

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?
- For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable.
- It is antisymmetric.
- $(a, b) \geq (c, d)$ implies that $a \geq c$ and $b \geq d$
- $(c, d) \geq (a, b)$ implies $c \geq a$ and $d \geq b$

Example

- The natural order on $\mathbb{R}^n$ is only partial.
- It is not complete. Why?
- For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable.
- It is antisymmetric.
- $(a, b) \geq (c, d)$ implies that $a \geq c$ and $b \geq d$
- $(c, d) \geq (a, b)$ implies $c \geq a$ and $d \geq b$
- Therefore $a = c$ and $b = d$

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.
- Because x is a least upper bound, $y \succsim x$

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.
- Because x is a least upper bound, $y \succsim x$
- But antisymmetry implies $x = y$

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.
- Because x is a least upper bound, $y \succsim x$
- But antisymmetry implies $x = y$
- Therefore x is unique.

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.
- Because x is a least upper bound, $y \succsim x$
- But antisymmetry implies $x = y$
- Therefore x is unique.
- If a maximum exists, the maximum is a least upper bound and is therefore unique.

Partially Ordered Sets

- The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique.
- Consider a least upper bound of X, x
- By the definition of least upper bound, $x \succsim y$ for all other upper bounds y
- Let y be a second least upper bound.
- Because x is a least upper bound, $y \succsim x$
- But antisymmetry implies $x = y$
- Therefore x is unique.
- If a maximum exists, the maximum is a least upper bound and is therefore unique.
- Antisymmetry, however, is not sufficient to guarantee existence, only uniqueness.

Chains

- A partial order is partial in the sense that not all elements are comparable.

Chains

- A partial order is partial in the sense that not all elements are comparable.
- If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order**.

Chains

- A partial order is partial in the sense that not all elements are comparable.
- If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order**.
- A totally ordered set is referred to as a **chain**.

Chains

- A partial order is partial in the sense that not all elements are comparable.
- If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order**.
- A totally ordered set is referred to as a **chain**.
- *Example* $(\mathbb{R}, \geq)$ is a chain.

Chains

- A partial order is partial in the sense that not all elements are comparable.
- If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order**.
- A totally ordered set is referred to as a **chain**.
- *Example* $(\mathbb{R}, \geq)$ is a chain.
- *Example* $(\Delta^1, \geq)$ is not a chain since neither $(1, 0) \geq (0, 1)$ nor $(0, 1) \geq (1, 0)$ is true.

Lattices

- A **lattice** is a partially ordered set in which every pair of elements has a least upper bound.

Lattices

- A **lattice** is a partially ordered set in which every pair of elements has a least upper bound.
- If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**.

Lattices

- A **lattice** is a partially ordered set in which every pair of elements has a least upper bound.
- If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**.
- Their greatest lower bound $x \wedge y$, called their **meet**, is also an element of L

Lattices

- A **lattice** is a partially ordered set in which every pair of elements has a least upper bound.
- If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**.
- Their greatest lower bound $x \wedge y$, called their **meet**, is also an element of L
- We will see later on that lattices admit a powerful tool for theoretical analysis called "monotone comparative statics."

Lattices

- A **lattice** is a partially ordered set in which every pair of elements has a least upper bound.
- If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**.
- Their greatest lower bound $x \wedge y$, called their **meet**, is also an element of L
- We will see later on that lattices admit a powerful tool for theoretical analysis called "monotone comparative statics."
- They also allow us to identify when a subset of $\mathbb{R}^n$ will have a maximum element (or least upper bound).

Example

- We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element.

Example

- We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element.
- Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice

Example

- We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element.
- Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice
- In particular, let X be the unit square.

Example

- We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element.
- Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice
- In particular, let X be the unit square.
- X is clearly a lattice: $(1,0) \vee (0,1) = (1,1)$ and $(1,0) \wedge (0,1) = (0,0)$

Example

- We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element.
- Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice
- In particular, let X be the unit square.
- X is clearly a lattice: $(1,0) \vee (0,1) = (1,1)$ and $(1,0) \wedge (0,1) = (0,0)$
- This ordered set also has a well-defined maximum element: (1,1).

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.
- Is the set a lattice?

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.
- Is the set a lattice?
- No: $(1,0) \vee (0,1) = (1,1) \notin X$

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.
- Is the set a lattice?
- No: $(1, 0) \vee (0, 1) = (1, 1) \notin X$
- The set of maximal elements of X is Δ^1

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.
- Is the set a lattice?
- No: $(1,0) \vee (0,1) = (1,1) \notin X$
- The set of maximal elements of X is Δ^1
- As we saw above, Δ^1 has no maximum element when ordered by $\geq$

Example

- Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$.
- Is the set a lattice?
- No: $(1,0) \vee (0,1) = (1,1) \notin X$
- The set of maximal elements of X is Δ^1
- As we saw above, Δ^1 has no maximum element when ordered by $\geq$
- When working in $(\mathbb{R}^n, \geq)$ is useful to remember that "squares" are lattices, "triangles" are not.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.
- We can also impose completeness rather than antisymmetry.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.
- We can also impose completeness rather than antisymmetry.
- A **weak order** is an order relation that is complete, reflexive, and transitive.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.
- We can also impose completeness rather than antisymmetry.
- A **weak order** is an order relation that is complete, reflexive, and transitive.
- A weak order is often referred to as a **rational order** or a **rational preference relation**.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.
- We can also impose completeness rather than antisymmetry.
- A **weak order** is an order relation that is complete, reflexive, and transitive.
- A weak order is often referred to as a **rational order** or a **rational preference relation**.
- In a weakly ordered set, every element is related to every other element.

Weakly Ordered Sets

- As we have seen, we can impose antisymmetry on a preorder to assist our theory of choice.
- We can also impose completeness rather than antisymmetry.
- A **weak order** is an order relation that is complete, reflexive, and transitive.
- A weak order is often referred to as a **rational order** or a **rational preference relation**.
- In a weakly ordered set, every element is related to every other element.
- This is a desirable property for political scientists: we want the actors in our models to be able to compare any action or outcome in the model in terms of their preferences.

Example

- $(\mathbb{R}^2, \geq)$ is not a weakly ordered set (why?).

Example

- $(\mathbb{R}^2, \geq)$ is not a weakly ordered set (why?).
- Now let $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.

Example

- $(\mathbb{R}^2, \geq)$ is not a weakly ordered set (why?).
- Now let $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$.
- $(\mathbb{R}^2, \succsim)$ is a weakly ordered set (why?)

Weakly Ordered Sets

- Note the difference between a weakly ordered set and a chain.

Weakly Ordered Sets

- Note the difference between a weakly ordered set and a chain.
- A total order imposes antisymmetry on a set while a weak order does not.

Weakly Ordered Sets

- Note the difference between a weakly ordered set and a chain.
- A total order imposes antisymmetry on a set while a weak order does not.
- A weakly ordered set may have multiple maximum elements while a totally ordered set can only have one maximum element.

Weakly Ordered Sets

- Note the difference between a weakly ordered set and a chain.
- A total order imposes antisymmetry on a set while a weak order does not.
- A weakly ordered set may have multiple maximum elements while a totally ordered set can only have one maximum element.
- In political science a weak or rational order makes more sense than a total order.

Weakly Ordered Sets

- Note the difference between a weakly ordered set and a chain.
- A total order imposes antisymmetry on a set while a weak order does not.
- A weakly ordered set may have multiple maximum elements while a totally ordered set can only have one maximum element.
- In political science a weak or rational order makes more sense than a total order.
- Why?

Rationality

- In theoretical political science, "rational" here refers to something very specific and mathematical.

Rationality

- In theoretical political science, “rational” here refers to something very specific and mathematical.
- When do we have “irrational actors?”