Introduction to Mathematics for Political Science

Problem Set 8: Solving Systems of Equations

Solutions

Instructions: You are encouraged to work in groups and actively participate on the Piazza page. Submitted solutions must be your individual work. You may use a calculator for basic arithmetic, but do not search the internet for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Solve the following systems of equations using substitution.

   - $$\begin{aligned} 2x + 3y &= 10 \\ 2y &= 4 \end{aligned}$$

     $$y = 2 \implies 2x + 6 = 10 \implies x = 2$$

     $$y = 2 \qquad x = 2$$

   - $$\begin{aligned} 3x + 5y - 2z &= 1 \\ 3y + z &= 8 \\ 5z &= 25 \end{aligned}$$

     $$z = 5 \implies 3y + 5 = 8 \implies y = 1 \implies 3x + 5 - 10 = 1 \implies x = 2$$

     $$x = 2 \qquad y = 1 \qquad z = 5$$

   - $$\begin{aligned} 10x - 3y - 2z &= 6 \\ 5y - 4z &= 9 \\ 3z &= 12 \end{aligned}$$

$$z = 4 \implies 5y - 16 = 9 \implies y = 5 \implies 10x - 15 - 8 = 6 \implies x = 2.9$$

$$x = 2.9 \qquad y = 5 \qquad z = 4$$

2. Solve the following systems of equations using elimination.

- $$\begin{aligned} 2x + 2y &= 8 \\ 6x + 2y &= 4 \end{aligned}$$

  $$-4x = 4 \implies x = -1 \implies -2 + 2y = 8 \implies y = 5$$

  $$x = -1 \qquad y = 5$$

- $$\begin{aligned} 4x + y + z &= 7 \\ x + 7y - z &= 23 \\ 10x + 4y + 3z &= 20 \end{aligned}$$

$$\begin{array}{r} 2(4x + y + z = 7) \\ +2(x + 7y - z = 23) \\ \hline \end{array}$$

$$\begin{array}{r} 10x + 16y + 0z = 60 \\ -(10x + 4y + 3z = 20) \\ \hline \end{array}$$

$$12y - 3z = 40$$

$$\begin{array}{r} 4x + y + z = 7 \\ -4(x + 7y - z = 23) \\ \hline \end{array}$$

$$-27y + 5z = -85$$

$$\begin{array}{r} -27y + 5z = -85 \\ +2.25(12y - 3z = 40) \\ \hline -\frac{7}{4}z = 5 \end{array}$$

This leaves the upper triangular system

$$\begin{aligned}
4x + y + z &= 7 \\
-27y + 5z &= -85 \\
-\frac{7}{4}z &= 5
\end{aligned}$$

which can be solved by back substitution leaving

$$x = \frac{38}{21} \qquad y = \frac{55}{21} \qquad z = -\frac{20}{7}$$

- $$\begin{aligned}
x - 3y - 2z &= 6 \\
2x - 4y - 3z &= 8 \\
-3x + 6y + 8z &= -5
\end{aligned}$$

$$\begin{array}{r}
x - 3y - 2z = 6 \\
\underline{+2x - 4y - 3z = 8}
\end{array}$$

$$\begin{array}{r}
3x - 7y - 5z = 14 \\
\underline{-3x + 6y + 8z = -5}
\end{array}$$

$$-y + 3z = -9$$

$$\begin{array}{r}
2(x - 3y - 2z = 6) \\
\underline{-2x - 4y - 3z = 8}
\end{array}$$

$$-2y - z = 4$$

$$\begin{array}{r}
-2y - z = 4 \\
\underline{-2(-y + 3z = -9)}
\end{array}$$

$$-7z = -14$$

This leaves the upper triangular system

$$\begin{aligned} x - 3y - 2z &= 6 \\ -2y - z &= 4 \\ -7z &= -14 \end{aligned}$$

which can be solved by back substitution leaving

$$x = 1 \qquad y = -3 \qquad z = 2$$

3. If possible, solve the following systems. If not, state whether each is over- or under-determined. If the system is over-determined, find a contradiction. If the system is underdetermined, solve it up to its free variables.

   - $$\begin{aligned} 3x + 2y + z &= 10 \\ 2x + y + 3z &= 12 \\ 7x + 4y + 7z &= 34 \end{aligned}$$

$$\begin{aligned} 3x + 2y + z &= 10 \\ 2(2x + y + 3z &= 12) \\ \\ 7x + 4y + 7z &= 34 \end{aligned}$$

We see equation 3 is eliminated entirely, implying that is is linearly dependent on 1 and 2. Eliminating y gives

$$\begin{aligned} 3x + 2y + z &= 10 \\ -2(2x + y + 3z &= 12) \\ \\ -x - 5z &= -14 \end{aligned}$$

or $x = 14 - 5z$. Eliminating z gives

$$\begin{aligned} -3(3x + 2y + z &= 10) \\ 2x + y + 3z &= 12 \\ \\ -7x + 5y &= -18 \end{aligned}$$

or $5y = 7x - 18$. Thus, the system is underdetermined with $x = 14 - 5z$. We can solve the system up to z by substituting this into $3x + 2y + z = 10$ giving $y = 7z - 16$.

- $$\begin{aligned} 3x + y + 4z &= 8 \\ 9x - 3y - 2z &= 4 \\ x + y + z &= 3 \end{aligned}$$

$$\begin{array}{r} 3x + y + 4z = 8 \\ -3(x + y + z = 3) \\ \hline \\ -2y + z = -1 \end{array}$$

$$\begin{array}{r} -3(3x + y + 4z = 8) \\ 9x - 3y - 2z = 4 \\ \hline \\ -6y - 14z = -20 \end{array}$$

$$\begin{array}{r} -3(-2y + z = -1) \\ -6y - 14z = -20 \\ \hline \\ -17z = -17z \end{array}$$

This yields the upper triangular matrix

$$\begin{aligned} 3x + y + 4z &= 8 \\ -6y - 14z &= -20 \\ -17z &= -17 \end{aligned}$$

that can be solved by back substitution giving

$$x = 1 \qquad y = 1 \qquad z = 1$$

- $$
\begin{aligned}
3x + y + 4z &= 8 \\
9x - 3y - 2z &= 4 \\
x + y + z &= 4
\end{aligned}
$$

We can first rearrange the rows giving

$$
\begin{aligned}
x + y + z &= 4 \\
9x - 3y - 2z &= 4 \\
3x + y + 4z &= 8
\end{aligned}
$$

$$
\begin{array}{r}
9x - 3y - 2z = 4 \\
\underline{-3(3x + y + 4z = 8)} \\
\\
-6y - 14z = -20
\end{array}
$$

$$
\begin{array}{r}
-9(x + y + z = 4) \\
\underline{+9x - 3y - 2z = 4} \\
\\
-12y - 11z = -32
\end{array}
$$

$$
\begin{array}{r}
(-2(-6x - 14z = -20) \\
\underline{-12y - 11z = -32} \\
\\
17z = 8
\end{array}
$$

This gives the upper triangular system

$$
\begin{aligned}
x + y + z &= 4 \\
-12y - 11z &= -32 \\
17z &= 8
\end{aligned}
$$

which can be solved by back substitution leaving

$$x = \frac{22}{17} \qquad y = \frac{38}{17} \qquad z = \frac{8}{17}$$

4. For what values of the parameter k does the following system have a) no solution, b) one solution, and c) more than one solution?[1]

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

Consider $k \neq -1$. Subtracting the equations gives

$$(1+k)x_2 = 0$$

which requires $x_2 = 0$ for all $k \neq -1$. Then, $x_1 = 1$. So when $k \neq -1$, the system has exactly one solution. Now consider $k = -1$. The equations are now redundant with $x_1 + x_2 = 1$. Infinitely many solutions lie along this line.

[1] **Source:** Moore and Siegel 13.4 #3