Normed Linear Spaces Exercises

1) Prove the following: for any x, y in a normed linear space,

$$||x|| - ||y|| \le ||x - y||$$

Solution: $||x|| = ||x - y + y|| \le ||x - y|| + ||y||$ by property (4). Rearranging yields $||x|| - ||y|| \le ||x - y||$

2) Prove that if $x_n \to x$ is a convergent sequence in a normed linear space, then $||x_n|| \to ||x||$

Solution: We are given that for all $\epsilon > 0$, there exists an N such that for all n > N, $||x_n - x|| < \epsilon$. The sequence $||x_n||$ lies in the metric space (\mathbb{R}, d) . Therefore to show that $||x_n|| \to ||x||$, we need to show that for all $\epsilon > 0$, there exists an N such that for all n > N, $d(||x_n||, ||x||) = |||x_n|| - ||x||| < \epsilon$. Note that $|||x_n|| - ||x||| \le ||x_n - x||$: for $a, b \in \mathbb{R}$, $||y - x|| = \max\{(y - x), (x - y)\}$. By the reverse triangle inequality, $||x_n|| - ||x|| \le ||x_n - x||$ and $||x|| - ||x_n|| \le ||x - x_n||$. Since $||x_n - x|| = ||x - x_n||$, we have that $|||x_n|| - ||x||| \le ||x_n - x|| < \epsilon$ for N sufficiently high. \blacksquare

3) Prove that $\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1-\delta}$ for $\delta \in (0,1)$.

Solution: The partial sums are given by

$$s_n = \sum_{k=0}^{n} a\delta^k = a \frac{1 - \delta^{n+1}}{1 - \delta}$$
:

$$(1-\delta)\sum_{n=0}^{\infty}a\delta^n = \frac{a}{1-\delta} = \sum_{k=0}^{n}a\delta^k - \sum_{k=0}^{n}a\delta^{k+1}$$

$$= a + a\delta + a\delta^{2} + \dots + a\delta^{n} - \delta(a + a\delta + a\delta^{2} + \dots + a\delta^{n})$$

$$=a-a\delta^{n+1}$$

For $\delta \in (0,1)$, $\lim_{n\to\infty} a\delta^{n+1} = 0$. Therefore $\lim_{n\to\infty} s_n = \frac{a}{1-\delta}$.