# Logic and Proofs

## Motivation

Political scientists build and test theories. Our object of study, the political world, is sufficiently complex that we must use comparatively simple theories to identify causal relationships and the mechanisms that produce these relationships. We build our theories using assumptions. You will discover quickly that even ostensibly simple assumptions often have non-obvious implications that conflict with our intuition. It is therefore not sufficient to assert "H1" in a paper and jump straight into your data analysis. You must first prove that H1 is a plausible hypothesis given a set of assumptions. This lecture teaches you how to go about doing this.

## Statements and Their Truth Values

Let $A$ be a **statement**. A statement is sentence that can be true or false.

*Example* Let $X$ be a set and let $A$ be the statement "$x \in X$." $A$ can be true or false. If $x$ is an irrational number and $X$ is $\mathbb{Z}$, then $A$ is false. If $x$ is a banana and $X$ is the set of all types of fruit, then $A$ is true. ■

We are often interested in establishing the truth of **conditional statements**. One conditional statement we often care about takes the form "$A$ implies $B$," denoted $A \implies B$. This is equivalent to saying that "$A$ is **sufficient** for $B$" or "if $A$ then $B$."

*Example*: Let $A$ be "there was a rainstorm" and let $B$ be "the sidewalks are wet." Then $A \implies B$ is true: rainstorms are sufficient for wet sidewalks. ■

We also often care about statements of the form "$A$ if and only if $B$" or $A \iff B$. This is equivalent to saying "$A$ is **necessary and sufficient** for $B$."

*Example*: Again let $A$ and $B$ refer to a rainstorm and wet sidewalks. While $A$ is sufficient for $B$, it is clearly not necessary for $B$. $B$ can occur without $A$. For example, $B$ could occur when $A$ is false if somebody was washing their car or having a water balloon fight nearby. ■

*Example*: Now let $A$ be the event "the nominee was confirmed by the Senate" and $B$ be the event "the nominee became a Supreme Court justice." Procedural quibbles aside, the statement $A \iff B$ is true. One cannot become a Supreme court justice without formal Senate approval and Senate approval makes one a Supreme court justice. ■

Finally, we are also interested in conditional statements of the form $\neg A \implies \neg B$ where $\neg A$ is read "not $A$." This is equivalent to saying "$A$ is **necessary** for $B$" or "$B$ only if $A$."

*Example*: Let $A$ be the event "she was nominated by the President" and $B$ be the event "she became a Supreme court justice." $A$ is necessary for $B$: if $A$ is false, $B$ must be false. Note however that $A$ is not sufficient for $B$. She must subsequently be approved by the Senate for $B$ to be true. ■

New statements can be constructed from other statements by using the connectives "and" or "or." $A \vee B$ is read "$A$ or $B$" and corresponds to the concept of union in set theory. $A \wedge B$ is read "$A$ and $B$" and corresponds to the concept of intersection in set theory.

*Example*: Let $A$ be the statement "the child is a boy," $B$ be the statement "the child is a twin," and $C$ be the statement "the child is a brother." Then

$A \implies C$ is false

$B \implies C$ is false

$\neg A \implies \neg C$ is true

$\neg B \implies \neg C$ is false

$A \wedge B \implies C$ is true

$A \vee B \implies C$ is false

$C \implies A$ is true ■

Note in the previous example that $C \implies A$ and $\neg A \implies \neg C$ are both true. This is no accident but rather an example of the logical equivalence of a conditional statement with its **contrapositive**. The proposition "If something is a member of the UN security council, then it is a state" is logically equivalent to "if something is not a state, then it is not a member of the UN security council." A formal proof of this equivalence is beyond the scope of this course but you are encouraged to come up with your own examples to verify for yourself that this is true. In practice, proving the contrapositive of a conditional statement may be easier than proving the statement itself. The logical equivalence of the two is therefore quite useful.

The statement $(A \wedge \neg A)$ is always false. A statement must be either true or false. This is a logical principle that we will exploit when we prove statements by contradiction.

There are two types of statements that we typically encounter in mathematics. The first is a **universal statement**: "$A$ is always true within a given mathematical system."

*Example*: Let $A$ be the statement "$x$ is a real number" and let $B$ be the statement "for all $x$, $x \leq |x|$." To prove $A \implies B$, we need to prove the statement for a generic $x$ where we can only use the properties common to every value of $x$. To disprove it, we need to find only a single counterexample. ■

The other type of statement we encounter is an **existential statement**: "There are conditions under which $A$ is true."

*Example*: Let $A$ be the statement "$x$ is a real number" and let $B$ be the statement "$\exists x$ such that $x = |x|$." To prove this statement, we must only find one value of $x$ in the reals such that $B$ is true. ■

Which type of statement we are dealing with will inform our method for proving or disproving a statement.

Mathematics has well-defined procedures for verifying that a given statement is true. We

will explore four such procedures or "proof strategies."

## Proof by Deduction

The simplest form of proof uses the principle of deduction. We often refer to proofs of this sort as "direct proofs." Proof by deduction simply demonstrates how the truth of one statement implies the truth of another, often by demonstrating that the truth of one statement implies a series of statements that also must be true which then imply that the statement of interest is true.

Formally, let's say we want to prove $A \implies B$. Assume too that we know that there exists some $C$ such that $C \implies B$. If we can show that $A \implies C$, then we have proven that $A \implies B$.

*Example* Claim: $x^2 - 4x + 9$ is always positive. Proof: We know that for any $x \in \mathbb{R}$, $x^2 \geq 0$. Therefore $x^2 + y$ for any $y > 0$ is always positive. Now note that $x^2 - 4x + 9$ can be rewritten as $(x-2)^2 + 5$ by completing the square. Therefore $x^2 - 4x + 9$ is always positive. ■

*Example* Claim: if $f(x)$ is even, then it is not one-to-one. Proof: Recall that $f(x)$ is even if $f(-x) = f(x)$ for all $x$ and is one-to-one if for all $x$, $f(x)$ is unique. If $f(x)$ is even, then $f(x) = f(-x)$. Thus there exists an $a$ and a $b$ such that $f(a) = f(b)$. Therefore $f(x)$ is not one-to-one. ■

## Proof by Contradiction

A second proof strategy exploits the fact that $(\neg A \wedge A)$ is logically invalid. Formally, we prove $A$ by showing that $\neg A \implies (\neg B \wedge B)$. Note that $B$ can be any statement, not necessarily one that we are trying to prove or disprove.

*Example*: Claim: $\sqrt{2}$ is irrational. Proof: Suppose that $\sqrt{2}$ is rational. Then there exist two integers, $a$ and $b$ such that $\frac{a}{b} = \sqrt{2}$. Let the fraction be fully reduced. This implies that

$a$ and $b$ are not both even (why?). Our assumption implies that $a^2 = 2b^2$. We know that $a^2$ must be even which implies that $a$ is even. Therefore $b$ must be odd. Since $a$ is even, there must be some integer $c$ such that $a = 2c$. This yields $(2c)^2 = 2b^2$ so $4c^2 = 2b^2$ and hence $b^2 = 2c^2$. This implies that $b^2$ is even which implies that $b$ is also even. But we just deduced that $b$ is odd. Therefore we have a contradiction: $b$ is both even and odd. Therefore our presumption that $\sqrt{2}$ is rational must be false. ■

Contradiction is often a good strategy for proving statements of the form "for all $x$, $A$ is true of $x$." The setup for contradiction involves assuming that "there exists an $x$ such that $A$ is not true of $x$." This gives us a specific $x$ for which $A$ is false which is often enough to produce a contradiction.

*Example*: Claim: For every $x \in [0, \pi/2]$, $\sin x + \cos x \geq 1$. Proof: suppose there exists an $x \in [0, \pi/2]$ for which $\sin x + \cos x < 1$. Since $x \in [0, \pi/2]$, neither $\sin x$ nor $\cos x$ is negative so $0 \leq \sin x + \cos x < 1$. Thus $0^2 \leq (\sin x + \cos x)^2 < 1^2$ which gives $0^2 \leq \sin^2 x + 2 \sin x \cos x + \cos^2 x^2 < 1^2$. Since $\sin^2 x + \cos^2 x = 1$, we have $0 \leq 1 + 2 \sin x \cos x < 1$ so $1 + 2 \sin x \cos x < 1$. Subtracting 1 from both sides gives $2 \sin x \cos x < 0$. But this contradicts the fact that neither $\sin x$ nor $\cos x$ is negative. ■

*Example*: Claim: There exists an integer $n > 0$ such that $n^2 + n + 17$ is not a prime number. Proof: Assume that for all integers $n > 0$, $n^2 + n + 17$ is a prime number. This implies that $n + 1 + 17/n$ is not an integer for all $n$ (the sum is greater than 1 for all integers). This therefore implies that $17/n$ is not an integer for all $n$ which implies that 1 is not an integer which is false. ■.

Of course, we typically are interested in proving conditional statements in political science. To prove $A \implies B$, we assume $\neg(A \implies B)$. That is, we assume $A$ is true while $B$ is false (why?). Our setup now is to assume $(A \wedge \neg B)$ and show that $(C \wedge \neg C)$ for some statement $C$.

*Example*: Claim: Assume $a \in \mathbb{Z}$. If $a^2$ is even, then $a$ is even. Proof: Assume $a$ is odd and $a^2$ is even. Since $a$ is odd, there exists an integer $c$ for which $a = 2c + 1$. Then

$a^2 = 2(2c^2 + 2c) + 1$ which implies that $a^2$ is odd, a contradiction. Therefore $a$ must be even.

■

## Proof by Induction

Some statements describe a property of an index number $n$ and may be written as $A(n)$. One way to prove that $A(n)$ is true for all natural numbers $n$ is to demonstrate that $A(1)$ is true and that if $A(n)$ is true then $A(n+1)$ must be true.

The logic is that if you want to show that somebody can climb the stairs to the nth floor of a building, you only need to show that you *can* climb to the first floor and then show that you know how to climb the stairs from any floor to the next floor.

*Example*: Claim: $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. Proof: First show that the equality holds for $n = 1$. $1 = 1(1+1)/2 = 1$. Now assume that the claim is true for $n = k$. This is called the *inductive hypothesis*. That is, we assume $1 + 2 + ... + k = \frac{k(k+1)}{2}$. Now we just need show that the claim holds for $n = k + 1$:

$$1 + 2 + ... + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Start with the left side of the equation. By the inductive hypothesis,

$$1 + 2 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2} \quad ■$$

The algorithm for proof by induction is simple. First prove the statement for a base case. Then assume the statement is true for some $n$. Then show that given the inductive hypothesis (step 2), show that the statement holds for $n+1$. While the algorithm is simple, intuition for why inductive proofs are valid may take a while to understand.

## Proof by Contraposition

We saw earlier that $A \implies B$ is equivalent to $\neg B \implies \neg A$. Proof by contraposition exploits this fact to prove $A \implies B$. Often $A \implies B$ is too hard to prove by deduction, contradiction, or induction while $\neg B \implies \neg A$ is relatively simple to prove by one of these techniques.

*Example*: Claim: if $7m$ is an odd integer, then $m$ is an odd integer $m > 1$. Proof: We will prove that $m$ is even implies $7m$ is even. If $m$ is even, then $m = 2k$ for some integer $k \implies 7m = 7(2k) \implies 7m = 2(7k) \implies 7m = 2n$ for some integer $n \implies 7m$ is even. ■

## If and Only If

So far we have been focusing on proving simple statements and conditional statements of the form $A \implies B$. To prove a conditional statement of the form $A \iff B$, we have to prove $A \implies B$ and $B \implies A$. We can use different proof techniques to prove both sides of the statement.

*Example*: The function $\lceil x \rceil$ is referred to as the "ceiling function." For any $x \in \mathbb{R}$, the function returns the smallest integer greater than $x$. The "floor function," $\lfloor x \rfloor$, returns the largest integer smaller than $x$. Claim: For any $x \in \mathbb{R}$, $\lfloor x \rfloor = \lceil x \rceil$ if and only if $x \in \mathbb{Z}$. Proof: First we prove that $\lfloor x \rfloor = \lceil x \rceil$ implies that $x$ is an integer. Assume $\lfloor x \rfloor = \lceil x \rceil$. Note that $\lfloor x \rfloor \leq \lceil x \rceil$. Since by assumption $\lfloor x \rfloor = \lceil x \rceil$, it follows that $x = \lfloor x \rfloor$. Since $\lfloor x \rfloor$ is an integer, $x$ must be an integer as well. Now we prove that if $x$ is an integer, then $\lfloor x \rfloor = \lceil x \rceil$. If $x$ is an integer, then $x = \lfloor x \rfloor$ and $x = \lceil x \rceil$. Therefore $\lfloor x \rfloor = \lceil x \rceil$. ■

## Exercises

1) Prove that an integer is even if and only if its square is even.

2) Let $A$, $B$, and $C$ be any sets. Prove that

$$A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$$

(Hint: show equality by proving that both sides of the equality are subsets of each other)

3) Every March in the United States, college basketball teams compete in a 64-team single-elimination tournament to determine the national champion. Assume for simplicity that rather than 64 teams, there are only 8 teams in the tournament. Let $X$ denote the set of all teams in the tournament. Matchups in the first round are determined as follows. Each team is assigned a number 1 though 8. Refer to such a permutation as a *seeding* and denote an individual seeding as $s$. Let $S$ represent the set of all seedings, i.e. the set of all ways that the 8 teams can be assigned a number 1 through 8. In the first round the team assigned number 1 plays team number 8, team 2 plays number 7, team 3 plays team number 6, and team 4 plays team number 5. In the second round, matchups are determined similarly. The lowest seeded team plays the highest seeded team and the second highest seed plays the second lowest seed. The two remaining teams after round two play in a championship game to decide the tournament. A *tournament winner* is a team that wins three games in a row and thus wins the championship. Note that for a given seeding, the tournament winner is unique.

Assume that the result of any individual matchup is deterministic. That is, for any $x$ and $x'$ in $X$, either $x \succ x'$ or $x' \succ x$ where $\succ$ connotes "defeats." Assume that $\succ$ is exogenous i.e. it is predetermined which team beats another for all matchups (there is no randomness).

Define a *Condorcet winner* as a team $x \in X$ such that for all $x' \neq x$, $x \succ x'$.

Prove or disprove the following three statements:

i) If $x$ is a Condorcet winner, then for all seedings $s \in S$, $x$ is the tournament winner.

ii) If $x$ is a tournament winner for some $s \in S$, then $x$ is a Condorcet winner.

iii) If $x$ is a tournament winner for all $s \in S$, then $x$ is a Condorcet winner.

## Source Material

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