

# Midterm Examination

POL 500 - Introduction to Mathematics for Political Science

August 28, 2019

This is a closed book examination. Calculators are not permitted. Attempt to answer all questions. Each question is worth ten points and should take about ten minutes. The exam ends sharply at 11:50.

## Question 1

Evaluate the following limits:

a.

$$\lim_{x \rightarrow 5} \frac{\sqrt{x^2 - 9} - 4}{x - 5}$$

**Solution:** Reduces to  $\frac{x+5}{\sqrt{x^2-9}+4}$  where we can plug in  $x = 5$  to get  $10/8$ .

b.

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^6 + 9}}{2x^3 + 6x + 1}$$

**Solution:** Note that  $\sqrt{4x^6}$  simplifies to  $-2x^3$  on this side of zero. The standard method will then show that the limit is  $-1$ .

c.

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

**Solution:** Limit DNE.

## Question 2

Let  $g(x) = \ln(x)$ . Use the limit definition of the derivative to find  $g'(x)$ . **Hint:**  $\lim_{k \rightarrow 0} -(1 + \frac{k}{x})^{1/k} = -e^{1/x}$

**Solution:**

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) = \lim_{h \rightarrow 0} \ln\left(\left(1 + \frac{h}{x}\right)^{1/h}\right)$$

Note that  $\lim_{h \rightarrow 0} (1 + \frac{h}{x})^{1/h} = e^{1/x}$  so

$$\lim_{h \rightarrow 0} \ln\left(\left(1 + \frac{h}{x}\right)^{1/h}\right) = \ln(e^{1/x}) = \frac{1}{x} \ln(e) = \frac{1}{x}$$

## Question 3

Compute the following derivatives with respect to  $x$ :

a)  $f(x) = \sqrt{x^3 - 7x}$

**Solution:**  $\frac{3x^2 - 7}{2\sqrt{x^3 - 7x}}$ .

b)  $f(x) = x^2 e^{-x}$

**Solution:**  $x(2 - x)e^{-x}$

c)  $f(x) = x^2 \sqrt{1 - x^2}$

**Solution:**  $2x\sqrt{1 - x^2} - \frac{x^3}{\sqrt{1 - x^2}}$

## Question 4

Let  $f(x) = x^2$  and  $g(x) = x$ . Let  $A = \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$  and  $B = \{(x, y) \in \mathbb{R}^2 : y \leq g(x)\}$ .

a)  $A \cap B$  is a region in  $\mathbb{R}^2$ . Find the area of this region.

**Solution:**  $f(x)$  and  $g(x)$  intersect at 0 and 1. The area is therefore given by

$$\int_0^1 (x - x^2) dx = \int_0^1 x dx - \int_0^1 x^2 dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

square units.

b) Now assume that a point is drawn completely at random from  $A \cap B$ . Find the joint probability density function that represents this process.

**Solution:**  $f(x, y) = 6$  if  $(x, y) \in A \cap B$ , 0 otherwise.

## Question 5

Evaluate the following integrals

a)  $\int_0^2 (x^2 - xb) dx$

**Solution:**  $-\frac{6b-8}{3}$

b)  $\int x \ln(x) dx$

**Solution:**  $\int x \ln(x) dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln(x) - (1/2) \int x dx = \frac{x^2}{2} \ln(x) - (1/2) \frac{x^2}{2} +$

$C$

## Question 6

A family has two children. Given that one of the children is a boy and that he was born on a Tuesday, what is the probability that both children are boys?

**Solution:** Let  $B$  be the event that the family has one boy born on Tuesday and  $A$  be the event that both children are boys. Note that there are 49 permutations for days of the week the boys were born on and 13 of these have a boy born on a Tuesday so  $Pr(B|A) = \frac{13}{49}$ .  $Pr(A) = \frac{1}{4}$ . There are  $14^2 = 196$  ways to select the gender and day of the week the child was born on. Of these,  $13^2 = 169$  ways do not have a boy born on Tuesday so  $P(B) = \frac{27}{196}$ . By Bayes' rule then  $Pr(A|B) = \frac{13}{27}$ .

## Question 7

Prove the following statement: If  $A$  is orthogonal, then the rows of  $A$  are orthogonal to each other and each row has a norm of 1. **Hint: A matrix is orthogonal if  $A^T A = I$**

**Solution:** Let  $n$  denote the number of rows and  $\mathbf{a}_i$  denote the  $i$ th row of  $A$ . Note that  $\mathbf{a}_i$  is a row vector, not a column vector. So its dot product is written as  $\mathbf{a}_i \mathbf{a}_i^\top$ .

$$\begin{aligned}
 AA^\top &= \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_n \end{bmatrix} [\mathbf{a}_1^\top \ \dots \ \mathbf{a}_n^\top] \\
 &= \begin{bmatrix} \mathbf{a}_1 \mathbf{a}_1^\top & \mathbf{a}_1 \mathbf{a}_2^\top & \dots & \mathbf{a}_1 \mathbf{a}_n^\top \\ \mathbf{a}_2 \mathbf{a}_1^\top & \mathbf{a}_2 \mathbf{a}_2^\top & \dots & \mathbf{a}_2 \mathbf{a}_n^\top \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_n \mathbf{a}_1^\top & \mathbf{a}_n \mathbf{a}_2^\top & \dots & \mathbf{a}_n \mathbf{a}_n^\top \end{bmatrix} \\
 &= I
 \end{aligned}$$

The last equality implies that all diagonal elements,  $\mathbf{a}_i \mathbf{a}_i^\top = \|\mathbf{a}_i\|^2, i = 1, \dots, n$ , should be equal to 1. In addition, all off-diagonal elements,  $\mathbf{a}_i \mathbf{a}_j^\top = \mathbf{a}_i \cdot \mathbf{a}_j, i \neq j$ , should be equal to 0. Therefore, each row has a norm of 1 and the rows are orthogonal to each other.

## Question 8

Use the Cauchy-Schwartz Inequality

$$\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

to prove the Triangle Inequality

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

**Hint:** Convince yourself that  $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$

**Solution:**

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

And because norms are strictly positive we can conclude

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

as desired.