Monotone, Linear, and Convex Functions Exercises (Solutions)

1) Let $f_1, f_2, ..., f_n$ be convex functions and $\alpha_1, \alpha_2, ...\alpha_n \geq 0$. Prove that $f(x) = \alpha_1 f_1(x) + ... + \alpha_n f_n(x)$ is convex. Is $\alpha_1 f_1 - \alpha_2 f_2$ convex? Prove your answer.

Solution: Let $f_1, f_2, ..., f_n$ be convex functions and $\alpha_1, \alpha_2, ...\alpha_n \geq 0$, $x, y \in \mathbb{R}^n$, and $\lambda \in [0, 1]$. Then

$$f(\lambda x + (1-\lambda)y) = \alpha_1 f_1(\lambda x + (1-\lambda)y) + ... + \alpha_n f_n(\lambda x + (1-\lambda)y)$$

$$\leq \alpha_1(\lambda f_1(x) + (1-\lambda)f_1(y)) + ... + \alpha_n(\lambda f_n(x) + (1-\lambda)f_n(y))$$

$$= \lambda(\alpha_1 f_1(x) + ... + \alpha_n f_n(x)) + (1-\lambda)(\alpha_1 f_1(y) + ... + \alpha_n f_n(y))$$

$$= \lambda f(x) + (1-\lambda)f(y)$$

Now let $f_1(x) = x$ and let $f_2(x) = x^2$. $f(x) = x(1-x)$. Let $x = 0$ and $y = 2$. Let $\lambda = 1/2$. $\lambda f(0) + (1-\lambda)f(2) = -2 \cdot 1/2 = -1$. $f(\lambda 0 + (1-\lambda)2) = f(1) = 0$. Therefore f is not convex. ■

2) Prove the Cauchy-Schwarz inequality for $\mathbb{R}^n$.

Solution: Easy and available e.g. wikipedia offers at least three proofs. ■

3) Prove the following: $L : \mathbb{R}^l \to \mathbb{R}$ is a continuous, linear functional if and only if there exists a $y \in \mathbb{R}^l$ such that for all $x \in \mathbb{R}^l$, $L(x) = y^T x$.

Solution: If $L(x) = y^T x$, then L is continuous and linear. Now assume L is linear and let e_i denote the unit vector in the ith direction. Every $x \in \mathbb{R}^l$ has a unique representation as $x = \sum_{i=1}^{l} x_i e_i$, $x_i \in \mathbb{R}$. The linearity of L implies that $L(x) = \sum_{i=1}^{l} x_i L(e_i)$. Note that the function $g(x_i) = a_i x_i$ for some $a_i \in \mathbb{R}$ is continuous in x_i and recall that the sum of a finite number of continuous functions is continuous. Therefore $L(x)$ is continuous. If y is the vector having an ith component $y_i = L(e_i)$, $L(x) = \sum_{i=1}^{l} x_i y_i = y^T x$. ■