An Introduction to Mathematics For Political Science

Problem Set 4

You are encouraged to work in groups and actively participate on the Piazza page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. All solutions must be written in LaTeX.

Optimization

1. Find all extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain. In your answer, report both the max/min and argmax/armin.

a) $f(x) = x^2 - 4x + 2$ on $[0, 3]$

Solution:

The function attains global maximum at $x = 0$ where $f(0) = 2$. Global minumum at $x = 2$, $f(2) = -2$. Local maximum at $x = 3$, $f(x) = -1$.

b) $f(x) = 2x^3 - x$ on $(-1, 1]$

Solution:

Global max at $x = 1$, $f(1) = 1$. Local min at $x = \frac{1}{\sqrt{6}}$, $f(\frac{1}{\sqrt{6}}) = \frac{-2}{3\sqrt{6}}$. Local max at $x = -\frac{1}{\sqrt{6}}$, $f(x) = \frac{2}{3\sqrt{6}}$. The function does not have a global max at $x = -1$ because the function is undefined.

c) $f(x) = \sqrt{x}$ on $[0, 4)$

Solution:

Global max at $x = 0$, $f(0) = 0$. The function has no local or global min.

d) $f(x) = -x^2 + 4$ on $(-2, 2)$

Solution:

Global max at $x = 0$, $f(0) = 4$. The function has no local or global max.

2. Explain (in words) the difference between a global maximum and a supremum.

Solution:

A global maximum is the largest element in the image of a function. If the image is open, this will not exist. A supremum is the least upper bound of the image of a function. A bounded set in $\mathbb{R}$ has a least upper bound whether it is open or closed. Therefore a function can have a supremum but not have a maximum.

3. Find the second derivative with respect to x of the following functions:

a) $2x^3 - 4x^2 + x$

Solution:

$12x - 8$

b) $x^4 + e^{2x}$

Solution:

$12x^2 + 4e^2x$

c) $\frac{2(1-\ln(x))}{x^2}$

Solution:

$\frac{2\ln(x)}{x^2}$

d) $-(x - a)^2$

Solution:

-2

e) xe^{-x}

Solution:

$(x - 2)e^{-x}$

4. Find all critical points and inflection points of the following functions. Identify whether each critical point is a local maximum, local minimum, or inflection point.

a) $f(x) = x^3 - 3x^2$

Solution:

Critical points are at $x = 0$ (local max) and $x = 2$ (local min). $x = 1$ is an inflection point.

b) $f(x) = x^3 - 6x^2 + 9x + 15$

Solution:

Critical points are at $x = 1$ (local max) and $x = 3$ (local min). $x = 2$ is an inflection point

c) $f(x) = -(x - b)^2$

Solution:

Critical point is at $x = b$ (local max).

d) $f(x) = -x^3$

Solution:

Critical point is at $x = 0$, which is an inflection point.

5. Identify the regions of $\mathbb{R}$ on which the following functions are weakly concave and/or convex:

a) $f(x) = -x^3$

Solution:

Convex on $(-\infty, 0)$, concave on $(0, \infty)$

b)$f(x) = \frac{1}{x}$

Solution:

Concave on $(-\infty, 0)$, convex on $(0, \infty)$ (undefined at $x = 0$).

c) $f(x) = x^3 - 3x^2$

Solution:

Concave on $(-\infty, 1)$, convex on $(1, \infty)$

d) $f(x) = 4x - 5$

Solution:

Weakly convex and concave everywhere.