Introduction to Mathematics for Political Science

Problem Set 8: Solving Systems of Equations

Instructions: You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. You may use a calculator for basic arithmetic, but do not search the internet for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Solve the following systems of equations using substitution or elimination.

   - $$\begin{aligned} 2x + 3y &= 10 \\ 2y &= 4 \end{aligned}$$

     $$y = 2 \implies 2x + 6 = 10 \implies x = 2$$

     $$y = 2 \qquad x = 2$$

   - $$\begin{aligned} 3x + 5y - 2z &= 1 \\ 3y + z &= 8 \\ 5z &= 25 \end{aligned}$$

     $$z = 5 \implies 3y + 5 = 8 \implies y = 1 \implies 3x + 5 - 10 = 1 \implies x = 2$$

     $$x = 2 \qquad y = 1 \qquad z = 5$$

   - $$\begin{aligned} 2x + 2y &= 8 \\ 6x + 2y &= 4 \end{aligned}$$

     $$-4x = 4 \implies x = -1 \implies -2 + 2y = 8 \implies y = 5$$

     $$x = -1 \qquad y = 5$$

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$$\begin{aligned} x - 3y - 2z &= 6 \\ 2x - 4y - 3z &= 8 \\ -3x + 6y + 8z &= -5 \end{aligned}$$

$$\begin{array}{r} x - 3y - 2z = 6 \\ \underline{+2x - 4y - 3z = 8} \\ 3x - 7y - 5z = 14 \\ \underline{-3x + 6y + 8z = -5} \\ -y + 3z = -9 \end{array}$$

$$\begin{array}{r} 2(x - 3y - 2z = 6) \\ \underline{-2x - 4y - 3z = 8} \\ -2y - z = 4 \end{array}$$

$$\begin{array}{r} -2y - z = 4 \\ \underline{-2(-y + 3z = -9)} \\ -7z = -14 \end{array}$$

This leaves the upper triangular system

$$\begin{aligned} x - 3y - 2z &= 6 \\ -2y - z &= 4 \\ -7z &= -14 \end{aligned}$$

which can be solved by back substitution leaving

$$x = 1 \qquad y = -3 \qquad z = 2$$

2. Provide a set of *basis* or *spanning* vectors for the following vector spaces.
   - $\mathbb{R}^2$

$$\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\end{bmatrix}$$

   - $\mathbb{R}^3$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

   - A 45 degree line intersecting the origin in $\mathbb{R}^2$

$$\begin{bmatrix}1\\1\end{bmatrix}$$

3. Consider the following systems $\boldsymbol{Ax} = \boldsymbol{b}$. Find the *rank* of $\boldsymbol{A}$. If the solution ($\boldsymbol{x}$) to the system is unique, solve for it. If not, explain why not.
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$$\underbrace{\begin{bmatrix}3 & 2 & 1\\2 & 1 & 3\\7 & 4 & 7\end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix}10\\12\\34\end{bmatrix}}_{\boldsymbol{b}}$$

The rank of $\boldsymbol{A}$ is 2. To see why, let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ denote the rows of $\boldsymbol{A}$. We can write

$$\boldsymbol{a}_3 = \boldsymbol{a}_1 + 2\boldsymbol{a}_2$$

meaning $\boldsymbol{a}_3$ is linearly dependent on $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. $\boldsymbol{A}$ therefore has 2 linearly independent row vectors and has a rank of 2. Since the rank of $\boldsymbol{A}$ is less than the number of rows, $\boldsymbol{Ax} = \boldsymbol{b}$ does not have a unique solution.

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$$\underbrace{\begin{bmatrix}3 & 1 & 4\\9 & -3 & -2\\1 & 1 & 1\end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix}8\\4\\3\end{bmatrix}}_{\boldsymbol{b}}$$

The rows are linearly independent and we can solve by elimination as follows

$$\begin{array}{r} 3x_1 + x_2 + 4x_3 = 8 \\ -3(x_1 + x_2 + x_3 = 3) \\ \hline \\ -2x_2 + x_3 = -1 \end{array}$$

$$\begin{array}{r} -3(3x_1 + x_2 + 4x_3 = 8) \\ 9x_1 - 3x_2 - 2x_3 = 4 \\ \hline \\ -6x_2 - 14x_3 = -20 \end{array}$$

$$\begin{array}{r} -3(-2x_2 + x_3 = -1) \\ -6x_2 - 14x_3 = -20 \\ \hline \\ -17x_3 = -17x_3 \end{array}$$

This yields the upper triangular matrix

$$\begin{aligned} 3x_1 + x_2 + 4x_3 &= 8 \\ -6x_2 - 14x_3 &= -20 \\ -17x_3 &= -17 \end{aligned}$$

that can be solved by back substitution giving

$$x_1 = 1 \qquad x_2 = 1 \qquad x_3 = 1$$

4. Provide a non-zero column vector $\boldsymbol{a}_3$ such that the following matrices $\boldsymbol{A}$ are *singular*.

   - $$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & \\ 4 & 2 & \boldsymbol{a}_3 \\ 1 & 1 & \end{bmatrix}$$
     $$\boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix}$$

   - $$\boldsymbol{A} = \begin{bmatrix} 9 & 12 & \\ 7 & 8 & \boldsymbol{a}_3 \\ 3 & 4 & \end{bmatrix}$$
     $$\boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$$

5. For what values of the parameter k does the following system have a) no solution, b) one solution, and c) more than one solution?[1]

$$x_1 + x_2 = 1$$
$$x_1 - kx_2 = 1$$

Consider $k \neq -1$. Subtracting the equations gives

$$(1 + k)x_2 = 0$$

which requires $x_2 = 0$ for all $k \neq -1$. Then, $x_1 = 1$. So when $k \neq -1$, the system has exactly one solution. Now consider $k = -1$. The equations are now redundant with $x_1 + x_2 = 1$. Infinitely many solutions lie along this line.

[1] **Source:** Moore and Siegel 13.4 #3