Ordered Sets

Motivation

When we build theories in political science, we assume that political agents seek the best element in an appropriate set of feasible alternatives. Voters choose their favorite candidate from those listed on the ballot. Citizens may choose between exit and voice when unsatisfied with their government. Consequently, political science theory requires that we can rank alternatives and identify the best element in various sets of choices. Sets whose elements can be ranked are called **ordered sets**. They are the subject of this lecture.

Relations

Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$. We use the notation $(a, b) \in R$ or more often aRb to denote the relation R holding for an ordered pair (a, b). This is read "a is in the relation R to b." If $R \subset A \times A$, we say that R is a **relation on** A.

Example. Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$. Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of." ■

Example. Let $A = \{a, b, c\}$. The relation $R = \{(a, b), (a, c), (b, c)\}$ expresses the relation "occurs earlier in the alphabet than." We read aRb as "a occurs earlier in the alphabet than b." ■

Properties of Binary Relations

A relation R on a nonempty set X is

reflexive if xRx for each $x \in X$

complete if xRy or yRx for all $x, y \in X$

symmetric if for any $x, y \in X$, xRy implies yRx

antisymmetric if for any $x, y \in X$, xRy and yRx imply $x = y$

transitive if xRy and yRz imply xRz for any $x, y, z \in X$

Any relation which is reflexive and transitive is called a **preorder**. A set on which a preorder is defined is called a **preordered set**. In our theory of choice, we refer to a preorder on X as a **preference relation** on X. Preorders fall into two fundamental categories, depending on whether or not the relation is symmetric. A symmetric preorder is called an **equivalence relation**. A preorder that is not symmetric is called an **order relation**. We will focus on order relations in this lecture but you should familiarize yourself with equivalence relations in the Appendix.

Order Relations

An order relation is a relation, $\succsim$, that is reflexive and transitive but not symmetric. An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X. For any two elements $x, y \in X$, the statement $x \succsim y$ is read "x dominates y."

Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.

$$x \succ y \iff x \succsim y \ \wedge \ \neg[y \succsim x]$$

The statement "$x \succ y$" is read "x strictly dominates y." The relation $\succ$ is transitive but not reflexive.

$$x \sim y \iff x \succsim y \ \wedge \ y \succsim x$$

for all $x, y \in X$. The statement "$x \sim y$" is read "x and y" dominate each other.

Example The natural order $\geq$ on $\mathbb{R}$ induces the strict order $>$ and the equivalence relation $=$. ■

We typically use order relations in the context of choice. Therefore it is convenient to read $x \succsim y$ as "x is preferred to y." The statement $x \succ y$ can be read as "x is strictly preferred to y." The statement $x \sim y$ can be read "x is indifferent to y."

Maximal and Best Elements

Our goal in our theory of choice is identify what our actor will choose given a set of alternatives. In general we assume that the actor will choose an object they most prefer. The following definitions formalize the notion of "most prefer."

Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$. $x \in X$ is called the **best element** in X if $x \succsim y$ for all $y \in X$.[1]

An ordered set may have no maximal or best elements. For these sets the concept of upper bounds is useful. Let A be a nonempty ordered subset of X. An element $x \in X$ is called an **upper bound** for A if $x \succsim a$ for all $a \in A$. An upper bound is called a **least upper bound** or **supremum** for A if it precedes every other upper bound for A.

Example Consider the ordered set $([a, b], \geq)$. b is the maximal element, the best element, and the least upper bound. Now consider $((a, b), \geq)$. The set has no maximal or best element. b is its least upper bound in $\mathbb{R}$. ■

Example A 1-simplex, Δ^1, is a collection of points in $\mathbb{R}^2_+$ such that for all $(a, b) \in \Delta^1$, $a + b = 1$. Graphically, a 1-simplex is represented as a line segment connecting $(1, 0)$ on the x-axis to $(0, 1)$ on the y-axis. Let $X = \Delta^1$ and $\geq$ be the natural vector order. That is, for $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$. Every element in the ordered set $(\Delta^1, \geq)$ is a maximal element. To see this, note that there exists no $(a, b) \in \Delta^1$ such

[1] It is more common to refer to a best element as a "maximum element." I use best instead of maximum to more clearly differentiate it from a maximal element.

that $(a, b) > (1, 0)$. This follows from the fact that $1 > a$ for all $(a, b) \neq (1, 0)$. Now consider $(1 - \epsilon, \epsilon)$. Since $\epsilon > b$ for all $(a, b) \neq (1, 0)$, there is no $(a, b) > (1 - \epsilon, \epsilon)$. Although every element in Δ^1 is a maximal element, there is no best element. To see this, assume (a, b) is a best element. This implies that $(a, b) \succsim (c, d)$ for all (c, d). But for all (a, b), $(c, d) = (a - \epsilon, b + \epsilon)$ which implies that (a, b) and (c, d) cannot be compared. Therefore it cannot be true that $(a, b) \succsim (c, d)$ which implies that (a, b) is not a best element. ■

As the examples illustrate, an ordered set or subset may have multiple maximal and best elements or no maximal or best elements. This can be problematic in applications when we want to study what choices we believe our actors will make. By imposing additional properties on the set X or the order relation $\succsim$, we can ensure the existence of a maximal or best element or the uniqueness of the maximal or best element.

We begin by imposing finiteness on X. If X is finite, then $(X, \succsim)$ has at least one maximal element where $\succsim$ is any order relation. Proof of this is left as an exercise. In the rest of this lecture, we allow X to be infinite and instead impose additional properties of binary relations on order relations.

Partially Ordered Sets

Uniqueness f a best element may be achieved by imposing the additional property of anti-symmetry on an order relation. A **partial order** is a relation that is reflexive, transitive, and antisymmetric. A set paired with a partial order is called a **partially ordered set**.

Example The natural vector order on $\mathbb{R}^n$ is a partial order. Antisymmetry is easy to check. $(a, b) \geq (c, d)$ implies that $a \geq c$ and $b \geq d$. Similarly, $(c, d) \geq (a, b)$ implies $c \geq a$ and $d \geq b$. Therefore $a = c$ and $b = d$. Transitivity is straightforward to check (verify that this is true). Note that the order is not complete. For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable. ■

The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique. To see this, consider a least upper bound of X, x. By the definition of least upper bound, $x \succsim y$ for all other upper bounds y. Let y be a second least upper bound. Because y is a least upper bound, $y \succsim x$. But antisymmetry implies $x = y$. Therefore x is unique. If a best element exists, the best element is a least upper bound and is therefore unique. Antisymmetry, however, is not sufficient to guarantee existence, only uniqueness.

Example: Consider $\mathbb{R}^n_+$ ordered by the natural vector order. Now take the subset of points, X, such that for any $(a, b) \in X$, $a + b \leq 1$. The ordered subset has a continuum of maximal points: all points such that $a + b = 1$ are maximal. It does not have a best element as no points in the maximal set can be compared. The least upper bound of the set is unique: $(1, 1)$. ■

Example: Consider $\mathbb{R}^n_+$ ordered by the natural vector order. Consider the subset of points, X, such that for any $(a, b) \in X$, $a \leq 1$ and $b \leq 1$. The set has a unique maximal and best point, $(1, 1)$, which corresponds to the least upper bound of the set. ■

Totally Ordered Sets

A partial order is partial in the sense that not all elements are comparable. If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order** and refer to a totally ordered set as a **totally ordered set**.[2]

Example $(\mathbb{R}, \geq)$ is a totally ordered set. ■

Example $(\Delta^1, \geq)$ is not a totally ordered set where $\geq$ is the natural vector order. Neither $(1, 0) \geq (0, 1)$ nor $(0, 1) \geq (1, 0)$ is true. ■

We saw that for a partially ordered set, if a least upper bound exists, it is unique, implying that if a best element exists, it is unique. In the examples we saw that multiple maximal elements may exist in a partially ordered set. For a totally ordered set, if a maximal element

[2]It is common to refer to a totally ordered set as a "chain." A total order may also be referred to as a "linear order."

exists it is unique. This implies that if a best element exists, it is unique.

To see this, assume that x and y are maximal elements in a totally ordered set. By definition, there is no z such that $z \succ x$ and no w such that $w \succ y$. For x this implies that for all z, either $x \succsim z$ or the two elements are not comparable. Similarly for all w, either $y \succsim w$ or the two elements are not comparable. Because the total order is complete, all elements are comparable. Therefore for all z, $x \succsim z$ and for al w, $y \succsim w$. Therefore $x \succsim y$ and $y \succsim x$. Because the order relation is antisymmetric, this implies that $x = y$. Therefore the maximal element is unique.

Weakly Ordered Sets

A **weak order** is an order relation that is complete, reflexive, and transitive. A weak order is often referred to as a **rational order** or a **rational preference relation**. In a **weakly ordered set**, every element is related to every other element. For our theory of choice this is desirable. We want the actors in our models to be able to compare any action or outcome in the model in terms of their preferences. Dropping the requirement of antisymmetry also allows us to more easily express indifference in the set of choices that actors make.

Example As we saw, $(\mathbb{R}^2, \geq)$ is not a weakly ordered set. The elements $(0,1)$ and $(1,0)$ for example cannot be compared. Now let $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$. $(\mathbb{R}^2, \succsim)$ is a weakly ordered set. Note that $(0,1) \succsim (1,0)$ and $(1,0) \succsim (0,1)$ or $(1,0) \sim (0,1)$. ■

Note the difference between a weakly ordered set and a total order. A total order imposes antisymmetry on a complete order set while a weak order does not. A consequence of this is that a weakly ordered set may have multiple best elements while a totally ordered set can only have one best element. In political science a weak or rational order often makes more sense than a total order. We constantly deal in tradeoffs across multiple dimensions of policy. This induces indifference classes across our choice space. We may think, for example, that it is a reasonable assumption that a Republic politician considers a combination of liberal social policy and conservative economic policy to be no better or worse than a conservative social policy and liberal economic policy. If these are the only two options available for her to vote on, then our theory of choice allows both of these to be best elements.

Another useful property of weakly ordered sets is that maximal and best elements coincide. That is, if x is maximal, then x is also best and vice versa. To see this, let $(X, \succsim)$ be a weakly ordered set and let x be a best element. It is a property of a best element that it is also a maximal element (proof of this is left as an exercise). Now let x be a maximal element. By the definition of maximal element, there is no $y \in X$ such that $y \succ x$. Recall the definition of $\succ$. If $y \succ x$, then $y \succsim x$ and $\neg[x \succsim y]$. Because x is maximal, this statement is false. That is, either $\neg[y \succsim x]$ and/or $x \succsim y$. If $x \succsim y$ for all y, then x is best. Now consider $\neg[y \succsim x]$. This can be true in two ways. Either $x \succsim y$ or neither $x \succsim y$ nor $y \succsim x$. Because $\succsim$ is a weak order and is complete, it must be the case that $x \succsim y$. This is true for all $y \in X$. Therefore maximal implies best in a weakly ordered set.

Note that "rational" in our theory of choice refers to something very specific and mathematical. Presumably you have encountered or perhaps uttered a sentence such as "that dictator is irrational." The way we use the term "rational," this would imply that his or her preferences over some set of political outcomes is incomplete or intransitive. Perhaps he or she prefers apples to bananas, bananas to carrots, and carrots to apples. Or he or she cannot compare economic growth to political survival or that she neither prefers coffee to tea nor is indifferent between the two.

Exercises

1) Prove that for the set of positive integers, the relation "m is a multiple of n" is an order relation.

2) Let $X = \{1, 2, ..., 9\}$, ordered by the relation "m is a multiple of n". Find all maximal

and best elements of this ordered set and its least upper bound in $\mathbb{Z}$.

3) Show that $x \sim y$ is an equivalence relation if $\succsim$ is rational.

4) Prove or disprove the following statements

i) Every best element is a maximal element.

ii) Every maximal element is a best element.

iii) An element is a best element if and only if it is a maximal element.

5) Let $X = \Delta^1$ and $\succsim$ be defined such that for any $(a, b), (c, d) \in X$, $(a, b) \succsim (c, d)$ if and only if $\max\{a, b\} \geq \max\{c, d\}$.

i) Find all maximal elements and best elements if they exist.

ii) Find all least upper bounds of the set in $\mathbb{R}^2$.

iii) Use the properties of binary relations to identify whether the set is partially ordered, totally ordered, and/or weakly ordered.

6) Prove that if X is finite, $(X, \succsim)$ has at least one maximal element for all order relations.

Appendix 1: Equivalence Relations

A relation $\sim$ on a nonempty set X is called an equivalence relation if it is reflexive, symmetric, and transitive. For any $x \in X$, the **equivalence class** of x relative to $\sim$ is defined as the set $[x]_\sim \equiv \{y \in X : y \sim x\}$. We often exploit indifference in our theory of choice and equivalence classes help us formalize this.

Example: Equality is an equivalence relation on $\mathbb{R}$. Clearly $x = x$ so the relation is reflexive. It is also symmetric: if $x = y$, then $y = x$. Transitivity is also easy to check: if $x = y$ and $y = z$, then $x = y = z$ so $x = z$. The equivalence class of $\sim$ is $[x]_\sim = \{y \in X : y = x\}$. Clearly for any x, its equivalence class is a singleton: on $\mathbb{R}$, each element is equal only to itself.[3] ■

Example: Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$

[3] If $u : X \to \mathbb{R}$ is a utility function representing preferences on a set X, then defining $x \sim y$ by $u(x) = u(y)$ gives the indifference equivalence relation.

if and only if $ad = bc$. First we check reflexiveness: $(a, b) \sim (a, b)$ is true iff $ab = ab$ which checks out. Now check symmetry: does $(a, b)R(c, d)$ imply $(c, d)R(a, b)$? Let's check: does $ad = bc$ imply $cb = da$? Yes. Finally check transitivity. Do $(a, b)R(c, d)$ and $(c, d)R(e, f)$ imply $(a, b)R(e, f)$? That is, do $ad = bc$ and $cf = de$ imply $af = be$? Note that $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which gives us $af = be$. The equivalence class is given by $[(a, b)]_{\sim} = \{(c, d) \in X : \frac{c}{d} = \frac{a}{b}\}$. With this relation, $\frac{4}{2}$ is equivalent to $\frac{8}{4}$. This is obvious to us precisely because the rational numbers are constructed as equivalence classes such that $(4, 2), (8, 4) \in [(2, 1)]_{\sim}$. ■

There is a close relationship between equivalence relations on a set and partitions of that set. A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X. Any equivalence relation on a set X partitions X. Each element of X belongs to one and only one equivalence class. In the first examples above, equality partitions $\mathbb{R}$ into singletons. In the second example, $\sim$ partitions X into the rational numbers.

Appendix 2: Lattices

A **lattice** is a partially ordered set in which every pair of elements has a least upper bound. If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**. Their greatest lower bound $x \wedge y$, called their **meet**, is also an element of L.

Lattices have some desirable properties. We will see later on that lattices admit a powerful tool for theoretical analysis called "monotone comparative statics." They also allow us to identify when a subset of $\mathbb{R}^n$ will have a best element (or least upper bound).

Example: We saw above that a 1-simplex ordered by $\geq$ does not have a best element. Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice. In particular, let X be the unit square. X is clearly a lattice: $(1, 0) \vee (0, 1) = (1, 1)$ and $(1, 0) \wedge (0, 1) = (0, 0)$. X also has a maximal element: $(1, 1)$. ■

Example Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$. This set is not a lattice: $(1, 0) \vee (0, 1) = (1, 1) \notin X$. The set of maximal elements of X is Δ^1. As we saw above, Δ^1 has no best element. ■

The above examples illustrates a useful shorthand to keep in mind later on in your research career when working in $(\mathbb{R}^n, \geq)$: "squares" are lattices, "triangles" are not.

Source Material

Carter, M. (2001). *Foundations of Mathematical Economics.* Cambridge, MA: The MIT Press.

Corbae, D., Stinchcombe, M.B., & Zeman, J. (2009). *An Introduction to Mathematical Analysis for Economic Theory and Econometrics.* Princeton, NJ: Princeton University Press.