Inner Product Spaces, Orthogonality, Projection

Exercises

1. Let X and Y be normed linear spaces. Let $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ be a basis for X and $\{\boldsymbol{y}_1, ..., \boldsymbol{y}_m\}$ a basis for Y. Prove that if $x_i \perp y_j$ for all $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, then X and Y are orthogonal spaces.

Because $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ is a basis for X, all $\boldsymbol{x} \in X$ can be expressed as linear combinations of $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$. We therefore need to show that all linear combinations of $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ are orthogonal to all linear combinations of of $\{\boldsymbol{y}_1, ..., \boldsymbol{y}_n\}$. More formally, we need

$$\alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j \perp \alpha_y \boldsymbol{y}_i + \beta_y \boldsymbol{y}_j$$

for arbitrary $\alpha_x, \beta_x, \alpha_y, \beta_y, \boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{y}_i, \boldsymbol{y}_j$. Equivalently,

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i + \beta_y \boldsymbol{y}_j \rangle$$

By the bilinearity (additivity) of the inner product, we can write

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i + \beta_y \boldsymbol{y}_j \rangle = \underbrace{\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i \rangle}_{A} + \underbrace{\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \beta_y \boldsymbol{y}_j}_{B}$$

Focusing on A, we also know by the symmetry of inner products that

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i \rangle = \langle \alpha_x \boldsymbol{x}_i, \alpha_y \boldsymbol{y}_i \rangle + \langle \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i \rangle$$

Again by the bilinearity (homogeneity) of the inner product, this can be written

$$\langle \alpha_x \boldsymbol{x}_i, \alpha_y \boldsymbol{y}_i \rangle + \langle \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i \rangle = \alpha_x \alpha_y \underbrace{\langle \boldsymbol{x}_i, \boldsymbol{y}_i \rangle}_{=0} + \beta_x \alpha_y \underbrace{\langle \boldsymbol{x}_j, \boldsymbol{y}_i \rangle}_{=0}$$

where $\langle \boldsymbol{x}_i, \boldsymbol{y}_i \rangle$ and $\langle \boldsymbol{x}_j, \boldsymbol{y}_i \rangle = 0$ by the orthogonality of the basis vectors. We conclude

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i \rangle = 0$$

Repeating the same argument for B gives

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \beta_y \boldsymbol{y}_j \rangle = 0$$

which gives

$$\langle \alpha_x \boldsymbol{x}_i + \beta_x \boldsymbol{x}_j, \alpha_y \boldsymbol{y}_i + \beta_y \boldsymbol{y}_j \rangle = 0$$

as desired. ■

2. Prove: If a vector $\boldsymbol{\alpha}$ is in the null space of a set of vectors $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$, then it is orthogonal to the space spanned by $\{\boldsymbol{y}_1, ..., \boldsymbol{y}_m\}$ where

$$\boldsymbol{y}_i = \{x_{1i}, ..., x_{ni}\}$$

Let

$$X = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix}$$

store the $\boldsymbol{x}$ vectors. If $\boldsymbol{\alpha} \in N(X)$, then

$$X\boldsymbol{\alpha} = \mathbf{0}$$

or

$$\boldsymbol{\alpha}^T X^T = \mathbf{0}$$

Now let

$$X^T = \begin{bmatrix} \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix}$$

We can represent the space spanned by these vectors with $X^T\boldsymbol{\beta}$ with $\boldsymbol{\beta}$ taking arbitrary linear combinations of the columns of X^T. We want to show

$$\boldsymbol{\alpha}^T X^T \boldsymbol{\beta} = \mathbf{0}$$

Because $\boldsymbol{\alpha}^T X^T = \mathbf{0}$, this must be the case. ■.

3. Donald Trump tweeted 100 times in April, 150 times in May, and 110 times in June.[1] Let $\boldsymbol{b} = (100, 150, 110)$ represent the number of tweets in each month. Project $\boldsymbol{b}$ onto the linear space spanned by $\boldsymbol{a} = (1, 1, 1)$. Interpret your result.

We have $\boldsymbol{p} = \hat{x}\boldsymbol{a}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$. We need $\boldsymbol{e}$ to be orthogonal to $\boldsymbol{a}$, or

$$\begin{aligned} \boldsymbol{a}^T(\boldsymbol{b} - \hat{x}\boldsymbol{a}) &= 0 \\ \boldsymbol{a}^T\boldsymbol{b} - \hat{x}\boldsymbol{a}^T\boldsymbol{a} &= 0 \\ \boldsymbol{a}^T\boldsymbol{b} &= \hat{x}\boldsymbol{a}^T\boldsymbol{a} \\ \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} &= \hat{x} \end{aligned}$$

Substituting our values, this becomes

$$\frac{\sum_i b_i}{\sum_i 1} = \frac{100 + 150 + 110}{3} = 120$$

Notice that for n months of tweeting, this is

$$\frac{1}{n}\sum_i b_i$$

or simply the mean number of tweets.

[1]Disclaimer: these data are of suspect quality.