Metric Spaces Exercises

1) Show that $d_\infty(x, y) = \max_{i=1}^{n} |x_i - y_i|$ is a metric for $\mathbb{R}^n$.

Solution

1. Assume to the contrary that $d(x, y) < 0$. This implies that for all i, $|x_i - y_i| < 0$ which contradicts the fact that the image of the absolute value function is $\mathbb{R}_+$.

2. $d(x, x) = \max_{i=1}^{n} |x_i - x_i| = \max_{i=1}^{n} |0| = 0$.

3. $d(x, y) = \max_{i=1}^{n} |x_i - y_i| = \max_{i=1}^{n} |y_i - x_i| = d(y, x)$

4. Let $a = \max_{i=1}^{n} |x_i - z_i|$ and $b = \max_{i=1}^{n} |z_i - y_i$. $a + b \geq |x_i - z_i| + |z_i - y_i| \geq |x_i - y_i|$ for all i. Therefore $a + b \geq \max_{i=1}^{n} |x_i - y_i| = d(x, y)$. ■

2) Consider the metric space $(\mathbb{R}, d_1)$ and the sequence $(\frac{1}{n^2})$. Prove that $\lim \frac{1}{n^2} = 0$.

Solution

We want $|\frac{1}{n^2} - 0| < \epsilon$ and we want to know how big n must be. That is, we want $\frac{1}{n^2} < \epsilon$ or $\frac{1}{\sqrt{\epsilon}} < n$. If our steps are reversible, we see $n > \frac{1}{\sqrt{\epsilon}}$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This suggests we put $N = \frac{1}{\sqrt{\epsilon}}$. Using this logic, a formal proof reads as follows:

Let $\epsilon > 0$ and let $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $|\frac{1}{n^2} - 0| < \epsilon$. This proves $\lim \frac{1}{n^2} = 0$. ■

3) Prove that every convergent sequence in a metric space is bounded.

Solution

Let $(x^n) \to x$ be a sequence in metric space (X, d). In order to prove that (x^n) is bounded it is sufficient to find a $K \in \mathbb{R}$ such that for all n, $d(x^n, x) \leq K$. Because the sequence converges, we can find an N such that $n \geq N$ implies $d(x_n, x) < 1$. Therefore $d(x_n, x) < 1$ for $n \geq N$. Now consider $n < N$. This is a finite set of terms which induces a finite set $\{d(x_1, x), d(x_2, x), ...d(x_{N-1}, x)\}$. Because the set is finite, it has a maximum term. Now add 1 to the set and let $K = \max \{d(x_1, x), d(x_2, x), ...d(x_{N-1}, x), 1\}$. We have just found a K such that $d(x^n, x) \leq K$ for all n. ■

4) Prove that every convergent sequence in a metric space is Cauchy.

Solution: Suppose $(x^n) \to x$ in (X, d). By the triangle inequality we know that $d(x^n, x^m) \leq d(x^n, x) + d(x, x^m)$. Let $\epsilon > 0$. Because the sequence converges, there exists some N_1 such that $n \geq N_1$ implies $d(x^n, x) < \frac{\epsilon}{2}$. Similarly, there exists some N_2 such that $m \geq N_2$ implies $d(x, x^m) < \frac{\epsilon}{2}$. Let $N = \max \{N_1, N_2\}$. Then $n, m \geq N$ implies $d(x^n, x^m) < \epsilon$. Thus (x^n) is a Cauchy sequence. ■

5) Prove that every Cauchy sequence is bounded.

Solution: Because the sequence in Cauchy, we can find an N such that $n, m \geq N$ implies $d(x^n, x^m) < 1$. In particular, $d(x^n, x^N) < 1$. To show that (x^n) is bounded, we need to find some $p \in X$ and $K \in \mathbb{R}$ such that $d(x^n, p) \leq K$ for all $x^n \in (x^n)$. Let $L = \max_{i<N} \{d(x^i, x^N)\}$. This is a finite set and therefore has a maximum. Now for $n \geq N$, $d(x^N, x^n) < 1$. For $n < N$, $d(x^N, x^n) \leq L$. Let $p = x^N$ and $K = \max \{L, 1\}$. It follows that for all n, $d(x^n, p) \leq K$. Therefore (x^n) is bounded. ■

6) The Bolzano-Weierstrass theorem states that every bounded sequence of real numbers has a convergent subsequence. Use the theorem to prove that $\mathbb{R}$ is complete.

Solution To show the completeness of $\mathbb{R}$, we need to show that every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$. Let (x^n) be an arbitrary Cauchy sequence in $\mathbb{R}$. Since (x^n)

is Cauchy, it is bounded. Since (x^n) is a bounded real sequence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence (x^{n_k}) in $\mathbb{R}$. Since (x^n) is Cauchy and has a subsequence that converges in $\mathbb{R}$, (x^n) converges in $\mathbb{R}$. Because we chose (x^n) arbitrarily, we have shown that all Cauchy sequences in $\mathbb{R}$ converge in $\mathbb{R}$. Therefore $\mathbb{R}$ is complete. $\blacksquare$.