

Midterm Examination

POL 500 - Introduction to Mathematics for Political Science

September 6, 2019

This is a closed book examination. Calculators are not permitted. Attempt to answer all questions. Each question is worth ten points and should take about ten minutes. The exam ends sharply at 11:50.

Question 1

Evaluate the following limits:

a.

$$\lim_{x \rightarrow 5} \frac{\sqrt{x^2 - 9} - 4}{x - 5}$$

Solution: Reduces to $\frac{x+5}{\sqrt{x^2-9}+4}$ where we can plug in $x = 5$ to get $10/8$.

b.

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^6 + 9}}{2x^3 + 6x + 1}$$

Solution: Note that $\sqrt{4x^6}$ simplifies to $-2x^3$ on this side of zero. The standard method will then show that the limit is -1 .

c.

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

Solution: Limit DNE.

Question 2

Let $g(x) = \ln(x)$. Use the limit definition of the derivative to find $g'(x)$. **Hint:** $\lim_{k \rightarrow 0} -(1 + \frac{k}{x})^{1/k} = -e^{1/x}$

Solution:

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) = \lim_{h \rightarrow 0} \ln\left(\left(1 + \frac{h}{x}\right)^{1/h}\right)$$

Note that $\lim_{h \rightarrow 0} (1 + \frac{h}{x})^{1/h} = e^{1/x}$ so

$$\lim_{h \rightarrow 0} \ln\left(\left(1 + \frac{h}{x}\right)^{1/h}\right) = \ln(e^{1/x}) = \frac{1}{x} \ln(e) = \frac{1}{x}$$

Question 3

Compute the following derivatives with respect to x :

a) $f(x) = \sqrt{x^3 - 7x}$

Solution: $\frac{3x^2 - 7}{2\sqrt{x^3 - 7x}}$.

b) $f(x) = x^2 e^{-x}$

Solution: $x(2 - x)e^{-x}$

c) $f(x) = x^2 \sqrt{1 - x^2}$

Solution: $2x\sqrt{1 - x^2} - \frac{x^3}{\sqrt{1 - x^2}}$

Question 4

Let $f(x) = x^2$ and $g(x) = x$. Let $A = \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$ and $B = \{(x, y) \in \mathbb{R}^2 : y \leq g(x)\}$.

a) $A \cap B$ is a region in \mathbb{R}^2 . Find the area of this region.

Solution: $f(x)$ and $g(x)$ intersect at 0 and 1. The area is therefore given by

$$\int_0^1 (x - x^2) dx = \int_0^1 x dx - \int_0^1 x^2 dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

square units.

b) Now assume that a point is drawn completely at random from $A \cap B$. Find the joint probability density function that represents this process.

Solution: $f(x, y) = 6$ if $(x, y) \in A \cap B$, 0 otherwise.

Question 5

Evaluate the following integrals

a) $\int_0^2 (x^2 - xb) dx$

Solution: $-\frac{6b-8}{3}$

b) $\int x \ln(x) dx$

Solution: $\int x \ln(x) dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln(x) - (1/2) \int x dx = \frac{x^2}{2} \ln(x) - (1/2) \frac{x^2}{2} +$

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Question 6

A family has two children. Given that one of the children is a boy and that he was born on a Tuesday, what is the probability that both children are boys?

Solution: Let B be the event that the family has one boy born on Tuesday and A be the event that both children are boys. Note that there are 49 permutations for days of the week the boys were born on and 13 of these have a boy born on a Tuesday so $Pr(B|A) = \frac{13}{49}$. $Pr(A) = \frac{1}{4}$. There are $14^2 = 196$ ways to select the gender and day of the week the child was born on. Of these, $13^2 = 169$ ways do not have a boy born on Tuesday so $P(B) = \frac{27}{196}$. By Bayes' rule then $Pr(A|B) = \frac{13}{27}$.

Question 7

Prove the following statement: If A is orthogonal, then the rows of A are orthogonal to each other and each row has a norm of 1. **Hint: A matrix is orthogonal if $A^T A = I$**

Solution: Let n denote the number of rows and \mathbf{a}_i denote the i th row of A . Note that \mathbf{a}_i is a row vector, not a column vector. So its dot product is written as $\mathbf{a}_i \mathbf{a}_i^\top$.

$$\begin{aligned}
 AA^\top &= \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_n \end{bmatrix} [\mathbf{a}_1^\top \ \dots \ \mathbf{a}_n^\top] \\
 &= \begin{bmatrix} \mathbf{a}_1 \mathbf{a}_1^\top & \mathbf{a}_1 \mathbf{a}_2^\top & \dots & \mathbf{a}_1 \mathbf{a}_n^\top \\ \mathbf{a}_2 \mathbf{a}_1^\top & \mathbf{a}_2 \mathbf{a}_2^\top & \dots & \mathbf{a}_2 \mathbf{a}_n^\top \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_n \mathbf{a}_1^\top & \mathbf{a}_n \mathbf{a}_2^\top & \dots & \mathbf{a}_n \mathbf{a}_n^\top \end{bmatrix} \\
 &= I
 \end{aligned}$$

The last equality implies that all diagonal elements, $\mathbf{a}_i \mathbf{a}_i^\top = \|\mathbf{a}_i\|^2, i = 1, \dots, n$, should be equal to 1. In addition, all off-diagonal elements, $\mathbf{a}_i \mathbf{a}_j^\top = \mathbf{a}_i \cdot \mathbf{a}_j, i \neq j$, should be equal to 0. Therefore, each row has a norm of 1 and the rows are orthogonal to each other.

Question 8

Use the Cauchy-Schwartz Inequality

$$\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

to prove the Triangle Inequality

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Hint: Convince yourself that $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$

Solution:

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2 \\ &\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

And because norms are strictly positive we can conclude

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

as desired.