

An Introduction to Mathematics for Political Science

Problem Set 1

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions or scans of handwritten solutions as a PDF. Note that starting with problem set 3 all solutions must be written in LaTeX.

Set Basics

1. Let $A = (40, 60)$ and $B = [50, 70]$.
 - a) Is $A \subset B$, $B \subset A$, both, or neither?
 - b) What is $A \cup B$?
 - c) What is $A \cap B$?
 - d) Write three elements of the Cartesian product $A \times B$.

Solution:

- a) Neither
 - b) $(40, 70]$
 - c) $[50, 60)$
 - d) $(50, 50)$, $(55, 60)$, $(59, 70)$
2. Identify whether the following sets are (a) open, closed, or neither; (b) bounded; (c) compact; (d) convex:

- a) $(0, 1)$
- b) $[0, 1]$
- c) $(0, 1]$
- d) $[0, \infty)$
- e) $(0, \infty)$
- f) $[0, 1] \cup [2, 3]$
- g) $[0, 1] \times [0, 1]$
- h) $(0, 3] \cap [1, 4]$
- i) $[0, 5] \setminus \{1, 2\}$

Solution:

- a) open, bounded, not compact, convex
- b) closed, bounded, compact, convex
- c) neither, bounded, not compact, convex
- d) closed, unbounded (above), not compact, convex
- e) open, unbounded (above), not compact, convex
- f) closed, bounded, compact, not convex
- g) closed, bounded, compact, convex
- h) closed, bounded, compact, convex
- i) neither, bounded, not compact, not convex

3. Express the following sentences in mathematical notation:

- a) A is the set of all real numbers less than or equal to seven, excluding zero and four.
- b) B is the intersection of the natural numbers and the real numbers between π and 30.5.
- c) For all epsilon greater than zero, there exists a delta greater than zero.
- d) The set of all even integers between 5 and 21.

Solution:

- a) $A = \{x \in \mathbb{R} \setminus \{0, 4\} \mid x \leq 7\}$
- b) $B = \mathbb{N} \cap (\pi, 30.5) = \{x \in \mathbb{N} \mid 4 \leq x \leq 30\}$

c) $\forall \epsilon > 0, \exists \delta > 0$

d) $\{x \in \mathbb{Z} | 5 \leq x \leq 21, x \text{ even}\}$

Algebra

4. Simplify into one term or evaluate the following:

a) $y \cdot y \cdot y \cdot y$

b) $(-a)(-b)^3 - b^2 + a^3$

c) $(4b + 2)(a - 5)$

d) $\frac{5!}{2!}$

e) $\sum_{i=1}^3 \left(\frac{1}{3}\right)^i$

f) $\sum_{i=2}^5 2^i$

g) $\prod_{i=1}^3 \left(\frac{1}{3}\right)^i$

h) $\frac{48}{4} - 6 \cdot 9$

i) $(3^3 + (-5)) \cdot 3 - (-7)$

j) $\left[6 + \left(\frac{-66}{11}\right)\right] \cdot (-2)^3$

k) $\frac{y-11}{5} + \frac{y+12}{3}$

Solution:

a) y^4

b) $ab - b^2 + a^3$

c) $4ab + 2a - 20b - 10$

d) 60

e) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{9+3+1}{27} = \frac{13}{27}$

f) $4 + 8 + 16 + 32 = 60$

g) $\frac{1}{3 \cdot 9 \cdot 27} = \frac{1}{729}$

h) -42

i) 73

j) 0

k) $\frac{8y+27}{15}$

5. Solve the following for x :

a) $5(-3x - 2) - (2x - 3) = -4(4x + 5) + 13$

b) $8x^2 = 15 - 14x$

c) $x^2 - 3x + 4 = 2(x - 1)$

d) $-6 > 5x + 5 + 4$

e) $-2(x + 1) + 4 < 10$

f) $2\ln(2x + 1) - 10 = 0$

g) $3^x e^{4x} = e^7$

Solutions

a) $x = 0$

b) $x \in \{-2.5, .75\}$

c) $x \in \{3, 2\}$

d) $x < -3$

e) $x > 4$

f) $x = \frac{\exp(5)-1}{2}$

g) $x = \frac{7}{\ln(3)+4}$

Functions

6) Find the image/range of x^2 on the domain $[-3, 3]$.

Solution: $[0, 9]$

7) Let $f(x) = x^2 - 4x + 2$ and $g(x) = 3x - 7$. Find (a) $f(x) - g(x)$; (b) $f(x) + g(x)$; (c) $f(g(x))$; and (d) $g(f(x))$. Evaluate each for $x = 2$.

Solutions:

a) $x^2 - 7x + 9$, $f(2) - g(2) = -1$

b) $x^2 - x - 5, f(2) + g(2) = -3$

c) $9x^2 - 54x + 77, f(g(2)) = 5$

d) $3x^2 - 12x - 1, g(f(2)) = -13$

8) Identify whether each of the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$ is (a) surjective/onto; (b) injective/one-to-one; (c) bijective.

a) $f(x) = x^2$

b) $f(x) = x^3 - x$

c) $f(x) = e^x$

d) $f(x) = x^3$

Solutions:

a) Neither; negative reals are not mapped into and positives are mapped into twice

b) Onto, not one-to-one; all reals are mapped into, reals near $f(0)$ mapped into twice

c) Not onto, one-to-one; negative reals not mapped into, each positive real mapped into only once

d) Both.

9) Find an equation for the inverse for each of the following functions:

a) $f(x) = (5x - 1)^3$

b) $f(x) = \frac{x+4}{3x-5}$

c) $f(x) = e^{5x-1}$

Solutions:

a) $f^{-1}(x) = \frac{1}{5}(x^{\frac{1}{3}} + 1)$

b) $f^{-1}(x) = \frac{5x+4}{3x-1}$ [Multiply by $3x - 5$, distribute, add $5y$, subtract x , factor x , divide by $3y - 1$, swap]

c) $f^{-1}(x) = \frac{\ln(x)+1}{5}$

10) Evaluate each of the following limits or show that they do not exist:

a) $\lim_{x \rightarrow 5} \frac{x^2-25}{x^2+x-30}$

b) $\lim_{x \rightarrow -1} \frac{x^3}{(x+1)^2}$

c) $\lim_{x \rightarrow 2} \frac{x^2+4x-12}{|x-2|}$

d) $\lim_{x \rightarrow \infty} \frac{x^2-1}{2x^2+1}$

e) $\lim_{x \rightarrow \infty} (\frac{x^3}{x^2+2} - x)$

f) $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x}$

Solutions:

a) $\frac{x^2-25}{x^2+x-30} = \frac{(x-5)^2}{(x-5)(x+6)} = \frac{x+5}{x+6} \cdot \lim_{x \rightarrow 5} \frac{x+5}{x+6} = 10/11$

b) Limit does not exist; function approaches $-\infty$ as x approaches -1 from each side.

c) To evaluate the RHS limit, find $\lim_{x \rightarrow 2^+} \frac{x^2+4x-12}{x-2} = \lim_{x \rightarrow 2^+} \frac{(x-2)(x+6)}{(x-2)} = \lim_{x \rightarrow 2^+} x + 6 = 8$. To evaluate the LHS limit, find $\lim_{x \rightarrow 2^-} \frac{x^2+4x-12}{-(x-2)} = \lim_{x \rightarrow 2^-} \frac{(x-2)(x+6)}{-(x-2)} = \lim_{x \rightarrow 2^-} -(x+6) = -8$. Because the left and right limits are not equivalent, the limit does not exist.

d) $\lim_{x \rightarrow \infty} \frac{x^2-1}{2x^2+1} = \lim_{x \rightarrow \infty} \frac{x^2}{2x^2} = 1/2$

e) $(\frac{x^3}{x^2+2} - x) = \frac{-2x}{x^2+2} \cdot \lim_{x \rightarrow \infty} \frac{-2x}{x^2+2} = -2 \lim_{x \rightarrow \infty} \frac{x}{x^2+2} = -2 \lim_{x \rightarrow \infty} \frac{1/x}{1+2/x^2} = -2 \frac{\lim_{x \rightarrow \infty} 1/x}{\lim_{x \rightarrow \infty} (1+2/x^2)} = -2 \cdot \frac{0}{1} = 0$.

11) Determine whether the following functions are continuous at the specified value. Use limits where appropriate. You may find sketching the graph of each function to be helpful.

a) $f(x) = \frac{x^2+1}{x^3+1}$ at $x = -1$

b) $f(x) = \begin{cases} 3x-5 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \quad \text{at } x = 1$

c) $f(x) = \begin{cases} \frac{x-6}{x-3} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{4+x^2} & x > 0 \end{cases} \quad \text{at } x = 0$

Solutions:

a) Function is undefined at $x = -1$ and is therefore discontinuous at -1 .

b) The function is defined at 1 since $f(1) = 2$. To determine whether it is continuous, consider $\lim_{x \rightarrow 1} 3x-5 = 3-5 = -2$. Since $\lim_{x \rightarrow 1} f(x) \neq f(1)$, the function is not continuous at $x = 1$.

c) The function is clearly defined at $x = 0$. Now check limits on either side of 0.
 $\lim_{x \rightarrow 0^-} \frac{x-6}{x-3} = -6/-3 = 2$. $\lim_{x \rightarrow 0^+} \sqrt{4+x^2} = \sqrt{4} = 2$. Conclude that the function is continuous at $x = 0$.