

# An Introduction to Mathematics For Political Science

## Problem Set 6

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. All solutions must be written in LaTeX.

### Random Variables and Distributions

1. A popular state lottery game requires participants to select a three-digit number (leading 0s allowed). Then three balls, each with one digit, are chosen at random from well-mixed bowls. The sample space here consists of all triples  $(i_1, i_2, i_3)$  where  $i_j \in \{0, \dots, 9\}$  for  $j = 1, 2, 3$ . If  $s = (i_1, i_2, i_3)$ , define  $X(s) = 100i_1 + 10i_2 + i_3$ . For example,  $X(0, 1, 5) = 15$ . Find  $Pr(X = x)$  for each integer  $x \in \{0, 1, \dots, 999\}$ .

$$Pr(X = x) = 0.001$$

2. Now consider a lottery in which participants select a real number,  $x$ , between 0 and 1. A real number,  $X$ , is then randomly selected from the interval with equal probability, i.e.  $X$  is a draw from the uniform distribution on  $[0, 1]$ . Find  $Pr(X = x)$  for some real number  $x$ .

$$Pr(X = x) = 0.$$

3. Suppose that the p.d.f. of a certain random variable  $X$  has the following form:

$$f(x) = \begin{cases} cx & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

where  $c$  is a constant. Find  $c$ .

For every p.d.f., it must be true that  $\int_{-\infty}^{\infty} f(x) = 1$ . Therefore  $\int_0^4 cx \, dx = 8c = 1$  so  $c = 1/8$

4. Suppose that the p.d.f. of a certain random variable  $X$  has the following form:

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find  $Pr(1 \leq X \leq 2)$  and  $Pr(X > 2)$ .

$$Pr(1 \leq X \leq 2) = \int_1^2 \frac{1}{8}x \, dx = \frac{3}{16} \text{ and } Pr(X > 2) = \int_2^4 \frac{1}{8} \, dx = \frac{3}{4}$$

5. The c.d.f. of a random variable  $X$  is given by  $F(x) = 1 - e^{-\lambda x}$  on  $\mathbb{R}_+$ . Find  $Pr(x \geq \lambda)$  and  $Pr(x \leq \lambda^2)$ .

$$Pr(x \geq \lambda) = 1 - (1 - e^{-\lambda^2}) = e^{-\lambda^2}. \quad Pr(x \leq \lambda^2) = 1 - e^{-\lambda^3}.$$

6. Find the p.d.f. of the random variable  $X$  from the previous problem.

$$f(x) = F'(x) = \lambda e^{-\lambda x}$$

7. Let the c.d.f. of a random variable  $X$  be given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^{\frac{2}{3}} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Find the p.d.f. of  $X$ .

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{2}{3}x^{-\frac{1}{3}} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

8. Suppose that a point  $(X, Y)$  is selected at random from inside the circle,  $S$ , defined by  $x^2 + y^2 \leq 9$ . Find the joint p.d.f. of  $(X, Y)$ .

Hint: the pdf will have the form

$$f(x, y) = \begin{cases} c & \text{for } (x, y) \in S \\ 0 & \text{otherwise} \end{cases}$$

where  $c$  is a constant.

We must have  $\int_S \int f(x, y) dx dy = c \cdot (\text{area of } S) = 1$ . The area of  $S$  is  $9\pi$ . The value of the constant must therefore be  $\frac{1}{9\pi}$ .

9. An investor is considering whether or not to invest 18 dollars in a stock for one year. The value of the stock after one year in dollars will be  $18 + X$  where  $X$  is a random variable. Suppose  $X$  is distributed such that  $Pr(X = -2) = .1$ ,  $Pr(X = 0) = .4$ ,  $Pr(X = 1) = .3$ , and  $Pr(X = 4) = .2$ . Alternatively, the investor can place her 18 dollars in the bank at 4 percent interest. Which use of her money is optimal if the investor is risk neutral?

The expected payoff of her investment is  $-2(.1) + 0(.4) + 1(.3) + 4(.2) = .9$ . Her payoff from investing the money in the bank is  $18 \cdot .004 = .72$ . The investment is the better option than the bank.

10. Let  $X$  be a random variable with p.d.f.  $f(x) = 2x$  with support only on  $(0, 1)$ . Find  $E[X]$ .

$$E[X] = \int_0^1 x(2x) dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

11. A product has a warranty of one year. Let  $X$  be the time at which the product fails. Suppose that  $X$  has a continuous distribution with the p.d.f.

$$f(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{2}{x^3} & \text{for } x \geq 1 \end{cases}$$

Find the expected time to failure.

$$E[X] = \int_1^\infty x \frac{2}{x^3} dx = \int_1^\infty \frac{2}{x^2} dx = 2$$

12. A random variable  $Y$  is a linear function of random variables  $X_1$  and  $X_2$  and  $\epsilon$ . In particular, let  $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$ . Let  $X_1$  be distributed according to the p.d.f from question 10 and let  $X_2$  be a Bernoulli random variable with parameter  $p = \frac{1}{4}$ . Let  $\epsilon \sim N(0, 1)$ . The terms  $\beta_i$  are real-valued scalars. Find  $E[Y]$ .

$$\begin{aligned} E[Y] &= E[\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon] = E[\beta_0] + E[\beta_1 X_1] + E[\beta_2 X_2] + E[\epsilon] = E[\beta_0] + \beta_1 E[X_1] + \\ &\beta_2 E[X_2] + E[\epsilon] = \beta_0 + \beta_1 \frac{2}{3} + \beta_2 \frac{1}{4}. \end{aligned}$$

13. Suppose a politician's utility function is described by  $U(x) = -(z - x)^2$  where  $z$  is her preferred policy  $z \in \mathbb{R}$  and  $x$  is a random policy shock with p.d.f.  $f(x)$ . Let  $f(x)$  be the uniform distribution on  $[0, 1]$ . Find her expected utility.

$$E[U(x)] = \int_0^1 U(x)f(x) dx = \int_0^1 U(x) dx = \int_0^1 -(z - x)^2 dx = -z^2 + z - \frac{1}{3}$$

14. Find the variance for the following series of numbers:

a) 12, 6, 7, 3, 15, 10, 18, 5

b) 2, 3, 6, 8, 11

c) 3, 5, 2, 7, 6, 4, 9, 1

a) 23.75

b) 10.8

c) 6.234

15.  $X$  is a random variable with equal probability of taking any one of five values,  $-2, 0, 1, 3$ , and  $4$ . Compute the variance of  $X$ .

$$E[X^2] = \frac{1}{5}[(-2)^2 + 0^2 + 1^2 + 3^2 + 4^2] = 6. \quad E[X] = 1.2. \quad Var(X) = 6 - (1.2)^2.$$