

# Ordered Sets Exercises

1) Prove that for the set of positive integers, the relation “ $m$  is a multiple of  $n$ ” is an order relation.

**Solution:** First show reflexivity. Reflexiveness requires that  $a \succsim a$  for all  $a \in \mathbb{Z}_+$ . Note that  $a \succsim b$  iff  $\frac{a}{b} \in \mathbb{Z}_+$ . It is straightforward to check that  $\frac{a}{a} = 1 \in \mathbb{Z}$ . Therefore reflexivity holds. Now check transitivity.  $a \succsim b$  implies  $\frac{a}{b} = \alpha \in \mathbb{Z}_+$ .  $b \succsim c$  implies  $\frac{b}{c} = \beta \in \mathbb{Z}_+$ . This yields  $b = \beta c$ ,  $\frac{a}{\beta c} = \alpha$ ,  $\frac{a}{c} = \alpha\beta$ . Since  $\alpha, \beta \in \mathbb{Z}_+$ ,  $\alpha\beta \in \mathbb{Z}_+$ . Therefore  $a \succsim c$ . Finally, we need to show that symmetry does not hold, that is,  $a \succsim b$  does not imply  $b \succsim a$ . It is sufficient to find a single counter example to prove this. Let  $a = 2$  and  $b = 1$ .  $a \succsim b$  is true:  $2/1 = 2 \in \mathbb{Z}$ .  $b \succsim a$ , however, is not true:  $1/2 \notin \mathbb{Z}$ . ■

2) Let  $X = \{1, 2, \dots, 9\}$ , ordered by the relation “ $m$  is a multiple of  $n$ ”. Find all maximal and best elements of this ordered set and its least upper bound in  $\mathbb{Z}$ .

**Solution:** To find the maximal elements, we need to find the set of all  $y$  such that there is no  $x \in X$  with  $x \succ y$  where  $\succ$  is the relation “ $x$  is a proper multiple of  $y$ .” First check for  $x \succ 1$ . Clearly for all  $x > 1$ , this holds. Now check for  $x \succ 2$ . All even  $x > 2$ , this holds. Similarly for 3 and 4:  $9 \succ 3$  and  $8 \succ 4$ . For 5, the next proper multiple of 5 is  $10 \notin X$ . Therefore 5 is a maximal element. Similarly, the next proper multiple of 6 is 12, 7 is 14, 8 is 16, and 9 is 18. None of these are in  $X$ . We conclude that the maximal set is the set  $\{5, 6, 7, 8, 9\}$ . To find a best element, we need to find a member  $x$  of the maximal set such that  $x \succsim y$  for all  $y \in X$ . Let's try 5. Is it true that  $5 \succsim 9$ ? No. Therefore 5 cannot be a best element. What about 6?  $6 \succsim 9$  is also false. Similarly,  $7 \succsim 9$  and  $8 \succsim 9$  are both false.

What about 9?  $9 \succsim 9$  is true. But  $9 \succsim 2$  and  $9 \succsim 4$  are false. Therefore we conclude that the ordered set has no best element. Finally, to find a lower bound, we need to find an integer  $z \in \mathbb{Z}$  such that  $z \succsim x$  for all  $x \in X$ . That is, we need to find a multiple of every element of  $X$ . In particular, we need to find the *least common multiple* of  $1, \dots, 9$ . It turns out that 2520 is the least common multiple and therefore the least upper bound of  $X$ . ■

3) Show that  $x \sim y$  is an equivalence relation if  $\succsim$  is rational.

**Solution:**  $\succsim$  rational means that  $\succsim$  is complete, reflexive, and transitive.  $\sim$  is defined as  $x \sim y \iff x \succsim y \wedge y \succsim x$ . We need to show that  $\sim$  is reflexive, symmetric, and transitive. Let's start with reflexivity.  $x \sim x$  implies  $x \succsim x$  (and  $x \succsim x$ ). Because  $\succsim$  is reflexive,  $x \sim x$  is reflexive. Now for transitivity.  $x \sim y$  implies  $x \succsim y$  and  $y \succsim x$ .  $y \sim z$  implies  $y \succsim z$  and  $z \succsim y$ . Because  $\succsim$  is transitive, we have that  $x \succsim y$  and  $y \succsim z$  imply  $x \succsim z$ . Therefore  $x \sim y$  and  $y \sim z$  imply  $x \sim z$ . Finally we check symmetry.  $x \sim y$  implies  $x \succsim y$  and  $y \succsim x$ . We also know from the definition of  $\sim$  that  $y \succsim x$  and  $x \succsim y$  iff  $y \sim x$ . Therefore because  $y \succsim x \wedge x \succsim y \equiv x \succsim y \wedge y \succsim x$ ,  $x \sim y$  implies  $y \sim x$ . ■

4) Prove or disprove the following statements

- i) Every best element is a maximal element.
- ii) Every maximal element is a best element.
- iii) An element is a best element if and only if it is a maximal element.

**Solution:**

i) True. By the definition of best element,  $x \succsim y$  for all  $y \in X$ . If  $x$  is not maximal, this implies that for some  $z \in X$ ,  $z \succ x$  i.e.  $z \succsim x$  and  $\neg[x \succsim z]$ . Therefore every best element is a maximal element.

ii) False. See counterexamples in lecture notes.

iii) False. Proof follows immediately from ii.

5) Let  $X = \Delta^1$  and  $\succsim$  be defined such that for any  $(a, b), (c, d) \in X$ ,  $(a, b) \succsim (c, d)$  if and only if  $\max\{a, b\} \geq \max\{c, d\}$ .

- i) Find all maximal elements and best elements if they exist.

ii) Find all least upper bounds of the set in  $\mathbb{R}^2$ .

iii) Use the properties of binary relations to identify whether the set is partially ordered, totally ordered, and/or weakly ordered.

**Solution**

i) Maximal elements and best elements are the same:  $(0, 1)$  and  $(1, 0)$ .

ii) The set of least upper bounds is the set of all points  $(1, a)$  and  $(b, 1)$  for  $a, b \leq 1$ .

iii) The order relation is not antisymmetric:  $(0, 1) \succsim (1, 0)$  and  $(1, 0) \succsim (0, 1)$  but  $(1, 0) \neq (0, 1)$ . Therefore the ordered set is not a partially ordered set or a totally ordered set. It is straightforward to check that the order relation on  $X$  is complete and transitive.

6) Prove that if  $X$  is finite,  $(X, \succsim)$  has at least one maximal element for all order relations.

**Solution** If  $X$  is a singleton,  $x$  is trivially a maximal element. Now consider a non-singleton finite  $X$  and assume that there is no maximal element. This implies that for all  $x \in X$ , there exists a  $z \in X$  such that  $z \succ x$ . Consider an arbitrary  $x_0 \in X$ . We know that there must be some element in  $X$  that is strictly preferred to  $x_0$ . By the definition of  $\succ$  and the reflexivity of  $\succsim$ , this must be distinct from  $x_0$ . Label this  $x_1$ . We now have  $x_1 \succ x_0$ . Because there is no maximal element, there must be some element in  $X$  that is strictly preferred to  $x_1$ . By the transitivity of  $\succsim$ , this cannot be  $x_0$ . By reflexivity and the definition of  $\succ$ , this element can also not be  $x_1$ . Label this new element  $x_2$ . Let  $N$  denote the cardinality of the set  $X$ . Continue this process until element  $x_{N-1}$ . Now there must be some  $x_N$  such that  $x_N \succ x_{N-1}$ . By transitivity and reflexivity,  $x_N$  must be the last remaining element that has not been shown to be strictly preferred to any other. Because there is no maximal  $x$ , there must be some  $x \succ x_N$ . But if such an  $x$  exists,  $x_N \succ x$  by the transitivity of the order relation. By the definition of  $\succ$ , this is a contradiction. Therefore there exists a maximal element.