Linear Spaces

Motivation

In political science research, we often concern ours with objects that can be added and scaled in natural ways. This is especially true in quantitative research when working with quantities such as GDP, vote share, or tax rates. If candidate one receives v_1 votes and candidate two receives v_2 votes, then the number of votes cast is $v_1 + v_2$. Lists of political quantities can similarly be added and scaled item by item. If candidate one's strategy for advertising spending in 50 states is $x = (x_1, x_2, ..., x_{50})$, then $2x = (2x_1, 2x_2, ...2x_{50})$ is another advertising strategy in which he or she doubles advertising spending in each state. Similarly $x + (y, 0, ..., 0) = (y + x_1, x_2, ...x_{50})$ is another advertising strategy in which an additional y is spent in state one. Note that these addition and scaling operations do not change the fundamental nature of the objects. After applying operations to an advertising budget, we are left with another advertising budget. This distinguishes linear spaces from sets for which it is nonintuitive how to scale and add objects that remain in the set such as the set of all countries. This lecture introduces the spaces in which we can combine and scale objects into new objects of the same type.

Linear Space

A linear space is a set X whose elements have the following properties:

Additivity

For every $x, y \in X$, there exists another element $x + y \in X$ called the **sum** of x and y such that

1. $x + y = y + x$ (commutativity)
2. $(x + y) + z = x + (y + z)$ (associativity)
3. There exists an element $0 \in X$ such that $x + 0 = x$
4. For every $x \in X$, there exists a unique element $-x \in X$ such that $x + (-x) = 0$

Homogeneity

For every $x \in X$ and $\alpha \in \mathbb{R}$, there exists and element $\alpha x \in X$ called the **scalar multiple** of x such that

5. $(\alpha\beta)x = \alpha(\beta x)$ (associativity)
6. $1x = x$

The addition and scalar multiplication operators obey distributive rules of arithmetic:

7. $\alpha(x + y) = \alpha x + \alpha y$
8. $(\alpha + \beta)x = \alpha x + \beta x$

A linear space is alternatively referred to as a "vector space" and its elements called "vectors."

In political science we do almost all of our work in one of the most common linear spaces, $\mathbb{R}^n$. Linear spaces generalize the familiar algebra of $\mathbb{R}^n$. The universe of linear spaces is much more general than $\mathbb{R}^n$, however. Sets of real sequences and polynomials, for example, are linear spaces. Some common fields, however, are not linear spaces.

Example: Is $\mathbb{Z}$ a linear space? Note that $\mathbb{Z}$ satisfies additivity. For any $x, y \in \mathbb{Z}$, $x+y \in \mathbb{Z}$. It does not, however, have the property of homogeneity. Let $x = 1$ and $\alpha = 1/2$. Clearly $\alpha x \notin \mathbb{Z}$.

Linear Subspaces

A **linear combination** of elements in a set $S \subseteq X$ is a finite sum of the form

$$\sum_{i=1}^{n} \alpha_i x_i$$

where $x_i \in S$ and $\alpha_i \in \mathbb{R}$ for all i. The **span** of a set of elements S is the set of all linear combinations of elements in S:

$$\text{span}(S) = \{\sum_{i=1}^{n} \alpha_i x_i \ : x_i \in S, \alpha_i \in \mathbb{R} \ \text{for all} \ \ i\}$$

Example: The span of $\{(1,0),(0,2)\}$ in $\mathbb{R}^2$ is $\mathbb{R}^2$.

Example: The span of $\{(1,0),(0,2),(1,1)\}$ in $\mathbb{R}^2$ is $\mathbb{R}^2$.

Example: The span of $\{(1,0),(2,0)\}$ in $\mathbb{R}^2$ is the set of all elements of $\mathbb{R}^2$ such that the second term is zero.

Example: Is $(1,0)$ in the span of $\{(1,1),(2,1)\}$? To answer, we need to find real numbers a_1 and a_2 such that $a_1(1,1) + a_2(2,1) = (1,0)$. This is a simple system of equations:

$$a_1 + 2a_2 = 0$$

$$a_1 + a_2 = 0$$

The solution to this system is $a_1 = -1$ and $a_2 = 1$. Now we have $(-1,-1) + (2,1) = (1,0)$.

A subset S of a linear space X is a **subspace** if for every $x, y \in S$, every linear combination of x and y, $\alpha x + \beta y$, is in S. To verify that a subset S is a subspace, it is sufficient to show that it satisfies additivity and homogeneity (why?). That is, we only need to verify that a subset S is closed under addition and scalar multiplication. A set S is **closed under addition** if for all $x, y \in S$, $x + y \in S$. The set S is **closed under scalar multiplication** if $\alpha x \in S$ for all $x \in S$ and $\alpha \in \mathbb{R}$.

Example Consider the set $S = \{x \in \mathbb{R}^2 | x = \alpha(1,0); \alpha \in \mathbb{R}\}$. Is S a subspace of $\mathbb{R}^2$? First check that it is closed under addition. Take any $x, y \in S$. Then $x = \alpha(1,0)$ and $y = \alpha'(1,0)$. Now $x + y = (\alpha, 0) + (\alpha', 0) = (\alpha + \alpha', 0) = (\alpha + \alpha', 0)$. So S is closed under addition. Now check if the subspace is closed under scalar multiplication. Let $x = \alpha(1,0)$. Now take any real number, β. Note that for any β, $\beta x = (\alpha\beta, 0) \in S$. Therefore S is a subspace.

Example Let X be $\mathbb{R}^2$ and let S be the space of all elements (a,b) for which $b = 0$. S is a subspace. For any $(a,0)$ and $(a',0)$, $\alpha(a,0)+\alpha'(a',0) = (\alpha a, 0)+(\alpha' a', 0) = (\alpha a+\alpha' a', 0) \in S$.

Example The subspaces of $\mathbb{R}^3$ are the origin, all lines through the origin, all planes through the origin, and $\mathbb{R}^3$.

A subset S is a subspace if and only if $S = \text{span}(S)$. To see this, let $S = \text{span}(S)$. Recall that the span of S is the set of all linear combinations of elements of S. Let $w = \sum_{i=1}^{n} \alpha_i x_i$ and $z = \sum_{i=1}^{n'} \alpha_i' x_i'$. $w + z$ is a sum of elements of S weighted by a real number i.e. a linear combination of elements of S. Therefore $w + z \in S$. It is simple to check that for any $\sum_{i=1}^{n} \alpha_i x_i \in \text{span}(S)$, $\beta \sum_{i=1}^{n} \alpha_i x_i \in \text{span}(S) = S$ for any real number β. Therefore $S = \text{span}(S)$ implies that β is a subspace. Now let S be a subspace. Because S is a subspace, for any x and y in S, $(\alpha x + \beta y) \in S$. Note that $\alpha x + \beta y$ is a linear combination of elements of S. Therefore $\alpha x + \beta y$ is in the span of S. Now take any element in the span of S and assume it is not contained in S. This implies that there is some $\sum_{i=1}^{n} \alpha_i x_i \notin S$ where $x_i \in S$. This implies that S is not closed under addition or scalar multiplication. But this contradicts the presumption that S is a subspace. Therefore all elements of the span of S are contained in S. Therefore if S is a subspace then $S = \text{span}(S)$.

Linear Independence

An element $x \in X$ is **linearly dependent** on a set S if $x \in \text{span}(S)$. If x is not linearly dependent, then x is **linearly independent** of S. If there exists some element $x \in S$ such that x is linearly independent on the other elements of S, i.e. $x \in \text{span}(S \backslash \{x\})$, then S is a linearly dependent set. Otherwise S is a linearly independent set.

Example Let $S = \{(1,0),(0,2)\}$. The element $x = (1,1)$ is linearly dependent on S: $x = 1 \cdot (1,0) + 1/2 \cdot (0,2)$. The set S is linearly independent: there is no scalar α such that

$\alpha(1,0) = (0,2)$. Similarly there is no scaler β such that $\beta(0,2) = (1,0)$.

Example Let $S = \{(1,1),(1,0),(0,1)\}$. The set S is not linearly independent. $(1,1)$ can be expressed as a linear combination of $(1,0)$ and $(0,1)$.

The following fact is often quite useful in proofs:

Proposition 1 *A subset S of a linear space X is linearly independent if and only if $0 = \sum_{i=1}^{n} \alpha_i x_i$ for $x_i \in S$ only for $\alpha_i = 0$ for all i. In other words, a subset S of a linear space X is linearly independent if and only if 0 cannot be expressed as a linear combination of elements of S with non-zero coefficients.*

Proof:

Suppose S is linearly independent. For contradiction suppose that $\sum_{i=1}^{n} \alpha_i x_i = 0$ where at least one $\alpha_i \neq 0$. Without loss of generality let α_1 be non-zero. Now subtract $\alpha_1 x_1$ from both sides and divide by $-\alpha_1$ to get

$$\frac{1}{\alpha_1} \sum_{i=2}^{n} \alpha_i x_i = x_1$$

This implies that x_i is a linear combination of $x_2, ..., x_n$. But this implies S is linearly dependent, a contradiction. Therefore 0 cannot be expressed as a linear combination of elements of S with non-zero coefficients.

Now we still need to show that if 0 cannot be expressed as a linear combination of S with non-zero coefficients, then S is linearly independent. Suppose for contraposition that S is not linearly independent. Then there is some $x \in S$ such that

$$x = \sum_{n=1}^{n} \alpha_i x_i$$

Subtracting x from both sides yields

$$0 = (-1)x + \sum_{n=1}^{n} \alpha_i x_i$$

This is a linear combination of elements in S that sums to 0 with at least one non-zero coefficient. $\square$

Basis and Dimension

Every subspace S of a linear space is linearly dependent. This is a fundamental characteristic of a linear space. All elements in a linear space are related to one another in a precise manner so that any element can be represented by other elements. This distinguishes a linear space from other arbitrary sets such as the set of all fruits. It is not intuitive that a banana plus a kiwi is contained in this set or that a strawberry can be represented by scalar multiplying some other type of fruit.

A **basis** for a linear space X is a linearly independent subset S that spans X i.e. $\text{span}(S) = X$. That is, every element of X can be represented by a linear combination of elements in S. Because S is linearly independent, this representation is unique. A basis is a minimally spanning set in that it encapsulates the entire vector space uniquely.

When working in $\mathbb{R}^n$, we commonly span the linear space with **standard basis vectors** or unit vectors. For $\mathbb{R}^3$, the standard basis vectors are $(1,0,0),(0,1,0),(0,0,1)$. For $\mathbb{R}^2$, the standard basis vectors are $(1,0)$ and $(0,1)$.

Example Is $S = \{(1,0,1),(2,0,0)\}$ a basis for $\mathbb{R}^3$? No. There are no scalars a_1 and a_2 such that $a_1(1,0,1) + a_2(2,0,0) = (0,1,0)$.

Example Is $S = \{(1,1),(1,2)\}$ a basis for $\mathbb{R}^2$? To establish that S is a basis, we need to show that we can express any arbitrary vector (x,y) as a linear combination $a_1(1,1)+a_2(1,2)$. Set up a system of equations and solve for a_1 and a_2 in terms of x and y:

$$a_1 + a_2 = x$$

$$a_1 + a_2 = y$$

Solving this system yields $a_1 = 2x - y$ and $a_2 = y - x$. Therefore S is a basis for $\mathbb{R}^2$.

Example Is $S = \{(1,1),(1,2),(u,v)\}$ a basis for $\mathbb{R}^2$? We just saw that any element in $\mathbb{R}^2$ can be represented as a linear combination of $(1,1)$ and $(1,2)$. Therefore (u,v) can be represented by a linear combination of $(1,1)$ and $(1,2)$ which implies that S is not linearly independent. Therefore S is not a basis.

A linear space that has a basis with a finite number of elements is **finite dimensional**. Otherwise the linear space is called **infinite dimensional**. In a finite dimensional space, every basis has the same number of elements. This number is called the **dimension** of the linear space.

Example: $\mathbb{R}^n$ is an n-dimensional linear space.

Returning to the examples above, note that $S = \{(1,0,1),(2,0,0)\}$ is linearly independent but is not a basis for $\mathbb{R}^3$ because it does not span $\mathbb{R}^3$. We also saw that $S = \{(1,1),(1,2),(u,v)\}$ spans $\mathbb{R}^2$ but is not a basis because it is not linearly independent. Finally, we saw that $S = \{(1,1),(1,2)\}$ is a basis for $\mathbb{R}^2$. It is linearly independent and spans $\mathbb{R}^2$.

This illustrates a dual feature of basis. A basis is a maximal linearly independent set in the sense that the addition of one more element implies that the set is linearly dependent.

Proposition 2 *Any set of $n+1$ elements in an n-dimensional linear space is linearly dependent.*

Proof: Left as exercise. $\Box$

A basis is also a minimal spanning set in the sense that removing a single element implies that the set does not span the linear space X.

Proposition 3 *No set of $m < n$ elements in an n-dimensional linear space can span X.*

Proof: Suppose S is a subset of X with $m < n$ elements. Assume that S spans X. Because S spans X, we can find a linearly independent subset of S, S' that also spans X.

Because $\text{span}(S') = X$ and S' is linearly independent, then S' is a basis for X. But this contradicts the fact that X is an n-dimensional basis. Therefore S does not span X. $\square$

The following two facts further highlight this duality between minimally spanning and maximally linearly independent sets.

Proposition 4 *A set of n elements in an $n-$dimensional linear space X is a basis if and only if it spans X.*

Proposition 5 *A set of n elements in an n-dimensional is a basis if and only if it is linearly independent.*

Note the implication of the statement. Any collection of n linearly independent elements is a basis. That is, any element in a linear space X can be represented as a linear combination of any n linearly independent elements.

Affine Sets

We saw that subspaces are generalizations of lines and planes passing through the origin. Another class of subsets of linear spaces are lines and planes that do not go through the origin. A subset S of a linear space is called an **affine set** if for all x and y in S, $\alpha x + (1-\alpha)y \in S$ for all $\alpha \in \mathbb{R}$. For a fixed x and y, the set of points $\alpha x + (1-\alpha)y$ for all $\alpha \in \mathbb{R}$ is called a **line through** x and y. A set is affine if the straight line through any two points remains entirely within the set.

Convex Sets

A subset S of a linear space X is called a **convex set** if for every $x, y \in S$,

$$\alpha x + (1-\alpha)y \in S$$

for $\alpha \in [0,1]$. You should already be familiar with convex sets and examples of such sets from earlier in the course. Convex sets will play a very important role in optimization theory. Note that convexity is an algebraic notion while compactness is a geometric concept. Hence we define compactness in metric spaces and convexity in linear spaces. Here we establish some properties of convex sets that will be useful.

If $S_1, S_2, ..., S_n$ are convex subsets of linear spaces $X_1, X_2, ...X_n$, their product $S_1 \times S_2 \times ... \times S_n$ is a convex subset of the product space $X_1 \times X_2 \times ... \times X_n$.

Example: Let $X_1 = X_2 = \mathbb{R}$ and let $S_1 = (0,1)$ and $S_2 = (2,3)$. Then the open square $(0,1) \times (2,3) \in \mathbb{R}^2$ is a convex subset of $\mathbb{R}^2$.

If S is convex, then αS is convex for all $\alpha \in \mathbb{R}$.

Example: Let $S = (0,1)$. The set $\alpha S = (0,\alpha)$ for $\alpha > 0$. For $\alpha < 0$, $\alpha S = (\alpha, 0)$.

One additional definition will prove useful later. A **convex combination** of elements of a set $S \subseteq X$ is a linear combination of elements with the property that for the scalar weights α_i, (i) $\alpha_i \in [0,1]$ and (ii) $\sum_{i=1}^{n} \alpha_i = 1$. The **convex hull** of a set of elements is the set of all convex combinations of vectors in S i.e.

$$\text{conv}(S) = \{\sum_{i=1}^{n} |x_i \in S, \alpha_i \in [0,1], \sum_{i=1}^{n} \alpha_i = 1\}$$

Note that the convex hull of a set of vectors is the smallest convex subset of X that contains S. An alternative definition of a convex set is that a set is convex if and only if it contains all convex combinations of its elements.

Exercises

1) Let S be a basis for X so that for every $x \in X$, there exist elements $x_1, x_2, ..., x_n \in X$ and scalars $\alpha_1, \alpha_2, ..., ...\alpha_n \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

Prove that α_i is unique for all i. (Hint: use Proposition 1).

2) Prove or disprove the following statement: any vector space X has a unique basis.

3) Prove that if X is an n-dimensional linear space, then any set $S \subset X$ of $n+1$ elements is linearly dependent. (Hint: use Propositions 1 and 5).