Normed Linear Spaces

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- A normed linear space accomplishes this.
- In the quantitative sequence, you will be working almost exclusively in normed linear spaces.

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- A normed linear space is a subclass of **metric linear spaces** which is a linear space equipped with a metric.
- It is a special type of metric linear space with rich interaction of the algebraic and linear structures.

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- What if we simply added up the net outputs of all the goods and services i.e. $\sum y_i$?
- Since some of the net outputs are negative, it would be more appropriate to take their absolute value first.
- Therefore one possible measure of size is

$$\|y\|_1 = \sum_{i=1}^{n} |y_i|$$

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- No: $|y_r| = 0$ does not imply $\|y\| = 0$
- It is possible for a government program to consume inputs and produce no outputs.
- A related measure that does meet the requirements of a norm uses the largest component of y as the measure of the size or the plan:

$$\|y\|_\infty = \max_{i=1}^{n} |y_i|$$

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- The space $(\mathbb{R}^n, \|\cdot\|_2)$ is referred to as a **Euclidean space**.

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- Euclidean space is an example of a Banach space.
- It turns out that any finite dimensional normed linear space is a Banach space.

Series

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- If $s_1, s_2, \ldots$ converges to some s, we say that the series $x_1, x_2, \ldots$ is **convergent** and we call $s = x_1 + x_2 + \ldots$ the sum of the infinite series.

Example

$$(\frac{1}{2^k})_{k=1}^{\infty} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_1, s_2, s_3, ... = \frac{1}{2}, \frac{3}{4}, \frac{7}{8} ... \to 1$$

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- A sequence that converges does not necessarily induce a sequence of partial sums that converges.

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- The sequence of partial sums of the series however diverges:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= 1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + \ldots + \frac{1}{16})$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty$$

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- *Example*: $1 + \frac{1}{2} + \frac{1}{4} + ... = \sum_{i=1}^{\infty} (\frac{1}{2})^{i-1} = \frac{1}{1-1/2} = 2$
- Geometric series will play an important role in models where players discount the future.

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- We originally took this powerful result for granted.
- Now we will prove it.

Limit Points

- A **limit point** (or "accumulation" point or "cluster" point) of a set A is a point a such that for all $r > 0$, there exists at least one point $c \in A \backslash \{a\}$ such that $c \in B_r(a)$.

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- Limit points need not be in the set.
- *Example*: Every point in $[0, 1]$ is a limit point of $(0, 1)$.

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- Since U does not contain any point of K, the set K is already covered by $C'_K = C'_T \backslash U$ i.e. a finite subcollection of the original collection C_K.
- It is therefore possible to extract from any open cover C_K of K a finite subcover.

Heine-Borel Theorem

Theorem

For a subset S of Euclidean space $\mathbb{R}^n$, S is compact if and only if it is closed and bounded.

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- Then C fails to cover S.

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- Then C fails to cover S.
- Indeed $\cap V_U$ is a neighborhood W of a in $\mathbb{R}^n$.
- Since a is a limit point of S, W must contain a point $x \in S$.
- This point is not covered by C since every U in C is disjoint from V_U and hence disjoint from W which contains x.

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- If S is compact but not closed, then it has a limit point $a \notin S$.

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For a subset S of Euclidean space $\mathbb{R}^n$, S is compact if and only if it is closed and bounded.

- If S is compact but not closed, then it has a limit point $a \notin S$.
- Consider a collection C' consisting of an open neighborhood $N(x)$ for every $x \in S$ chosen small enough not to intersect some neighborhood V_x of a.

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- This contradicts the compactness of S.
- Therefore every limit point of S is in S so S is closed.

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- This can cover any set, because all points in the set are some distance away from that point.
- Any finite subcover of this cover must be bounded because all balls in the subcover are contained in the largest open ball within that subcover.
- Therefore any set covered by this subcover must also be bounded.

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For a subset S of Euclidean space $\mathbb{R}^n$, S is compact if and only if it is closed and bounded.

- Finally, we show that closed and bounded implies compact.
- If a set $S \in \mathbb{R}^n$ is bounded, then it can be enclosed in an n-box $T_0 = [-a, a]^n$ where $a > 0$.

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- Finally, we show that closed and bounded implies compact.
- If a set $S \in \mathbb{R}^n$ is bounded, then it can be enclosed in an n-box $T_0 = [-a, a]^n$ where $a > 0$.
- By the lemma, if we can show that T_0 is compact, then because S is a closed subset of T_0, that will prove that S is compact.

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- Assume FSOC that that T_0 is not compact.

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- Finally, we show that closed and bounded implies compact.
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- By the lemma, if we can show that T_0 is compact, then because S is a closed subset of T_0, that will prove that S is compact.
- Assume FSOC that that T_0 is not compact.
- Then there exists an infinite open cover C of T_0 that does not admit any finite subcover.

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For a subset S of Euclidean space $\mathbb{R}^n$, S is compact if and only if it is closed and bounded.

- By bisecting each of the sides of the T_0 box, T_0 can be broken into 2^n sub n-boxes, each of which has a diameter equal to half the diameter of T_0.

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- Then at least one of the 2^n sections of T_0 must require an infinite subcover of C.
- Otherwise C itself would have a subcover formed by taking the union of the finite subcovers of the sections.
- Call this T_1. The sides of T_1 can be bisected as well which yields 2^n smaller sections of T_1, at least one of which must require an infinite subcover of C.

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- The side length of T_k is $\frac{2a}{2^k} \to 0$.

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- Define a sequence (x_k) such that each x_k is in T_k.
- The sequence is Cauchy.

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- But this contradicts our claim that we need an infinite number of members of C in order to cover T_k.
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- Since S is a closed subset of T_0, S is compact.

Sequential Compactness

- Closely related to the Heine-Borel theorem is the Bolzano-Weierstrass theorem.

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Sequential Compactness

- Closely related to the Heine-Borel theorem is the Bolzano-Weierstrass theorem.
- In metric spaces, a set is compact if and only if it is **sequentially compact**,
- A metric space is sequentially compact if every infinite sequence has a convergent subsequence.
- The Bolzano Weierstrass theorem will allow us to use this equivalence to write a simpler proof of the Heine-Borel theorem.

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Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

- Let (x_n) be a bounded sequence.

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- Let (x_n) be a bounded sequence.
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- Denote that interval I_1 which has length $\frac{1}{2}$.
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- Now divide I_1 into two halves each of length $\frac{1}{4}$.

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- At least one of the halves contains infinitely many terms of (x_n).
- Denote that interval I_1 which has length $\frac{1}{2}$.
- Let (x_{n_k}) be the subsequence of (x_n) consisting of every term that lies in I_1.
- Now divide I_1 into two halves each of length $\frac{1}{4}$.
- At least one of these contains infinitely many terms of the (x_{n_k}) and denote that half by I_2.

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Theorem

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

- Continue this way to construct a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \ldots$ where the length of I_n is $(\frac{1}{2})^n$ and each interval contains an infinite number of terms of the original sequence (x_n).

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- Now construct a subsequence (z_n) of (x_n) made up of one term for each interval I_n.
- This is a Cauchy sequence: for all $m, n > N$, $|z_m - z_n| < (\frac{1}{2})^N$.

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- This is a Cauchy sequence: for all $m, n > N$, $|z_m - z_n| < (\frac{1}{2})^N$.
- Since $\mathbb{R}^n$ is a complete metric space, (z_n) converges.

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- Consider a closed bounded subset S of $\mathbb{R}^n$.
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- By BW, x_n has a convergent subsequence in $\mathbb{R}^n$.

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- It is now much easier to prove that closed and bounded implies compact in $\mathbb{R}^n$.
- Consider a closed bounded subset S of $\mathbb{R}^n$.
- Let x_n be a sequence in S.
- Because S is bounded, so is x_n.
- By BW, x_n has a convergent subsequence in $\mathbb{R}^n$.
- Since S is closed, the limit belongs to S.
- Therefore S is sequentially compact and therefore compact.