

## Continuity Exercises Solutions

1) Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be defined by  $f(x) = \sqrt{x}$ . Use the  $\epsilon - \delta$  definition of continuity to show that  $f(x)$  is continuous at  $c > 0$ .

**Solution:** Note that for  $x \in X$ ,

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| = \frac{x - c}{\sqrt{x} + \sqrt{c}} \leq \frac{1}{\sqrt{c}}|x - c|$$

Therefore for any  $\epsilon > 0$ , let  $\delta = \sqrt{c}\epsilon$ .

2) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$\frac{x + x^3 + 5x^5}{1 + x^2}$$

Prove that  $f(x)$  is continuous.

**Solution:** Let  $g(x) = x + x^3 + 5x^5$  and  $h(x) = 1 + x^2$ . Consider a sequence in the domain of  $g$ ,  $x_n \rightarrow x_0$ . We have

$$\lim g(x_n) = \lim[x_n + x_n^3 + 5x_n^5] = \lim x_n + \lim x_n^3 + 5 \lim x_n^5 = x_0 + x_0^3 + 5x_0^5 = g(x_0)$$

Therefore  $g(x)$  is continuous. A similar argument establishes that  $h(x)$  is continuous. We know that if  $g$  and  $h$  are continuous, then  $g/h$  is continuous. Therefore  $f$  is continuous.

3) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$f(x) = \begin{cases} e^x & \text{for } x \leq 0 \\ 0 & \text{for } x > 0 \end{cases}$$

Show that  $f(x)$  is discontinuous.

**Solution:** Let  $u = \ln(1/2)$ . Consider the open subset of the codomain,  $(1/2, 2)$ . The inverse image under  $(1/2, 2)$  is  $(u, 0]$  which is not open. Therefore  $f$  is not continuous.

4) Let  $f$  and  $g$  be continuous at  $x_0$  in  $\mathbb{R}$ . Prove that  $\max(f, g)$  is continuous at  $x_0$  (Hint: first show that for any  $a, b \in \mathbb{R}$ ,  $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$ ).

**Solution:** Note that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

because  $f$  and  $g$  are real-valued. We know that the sum and difference of continuous functions is continuous. Therefore  $f + g$  and  $f - g$  are continuous. Now note that  $|\cdot|$  is a continuous function. Therefore the composition  $|f - g|$  is continuous. By the same rule,  $\frac{1}{2}(f + g)$  and  $\frac{1}{2}|f - g|$  are continuous. Finally, because this is the sum of two continuous functions,  $\max(f, g)$  is continuous at  $x_0$ .

5) Prove that if  $f$  and  $g$  are continuous at  $x_0$ , then their product  $fg$  is continuous at  $x_0$ .

**Solution:** Consider a sequence in the intersection of the domain of  $f$  and  $g$  that converges to  $x_0$ . Because each is continuous we have  $\lim f(x_n) = f(x_0)$  and  $\lim g(x_n) = g(x_0)$ . Therefore

$$\lim(fg)(x_n) = (\lim f(x_n))(\lim g(x_n)) = f(x_0)g(x_0) = (fg)(x_0)$$