

Introduction to Mathematics for Political Science

Problem Set 9: Matrix Inversion and Determinants

Instructions: You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$\begin{aligned}3x_1 - x_2 &= 10 \\ -x_1 + 4x_2 &= 4\end{aligned}$$

Write this system in $A\mathbf{x} = \mathbf{b}$ form and solve via matrix inversion.

$$\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$A^{-1} = \frac{1}{3(4) - (-1)(-1)} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

2. Let $C = AB$ where C is invertible and A and B are square matrices. Solve for A^{-1} .¹

If C is invertible and A and B are square then A and B are also invertible. Therefore,

$$C^{-1} = B^{-1}A^{-1}$$

$$BC^{-1} = (BB^{-1})A^{-1}$$

$$BC^{-1} = A^{-1}$$

¹Strang p. 90 #12

3. Let $M = ABC$ where M is invertible and A , B , and C are square matrices. Solve for B^{-1} .²

If M is invertible and A , B , and C are square then A , B , and C are also invertible. Therefore,

$$\begin{aligned}M^{-1} &= C^{-1}B^{-1}A^{-1} \\ CM^{-1}A &= (CC^{-1})B^{-1}(A^{-1}A) \\ CM^{-1}A &= B^{-1}\end{aligned}$$

4. If B is the inverse of A^2 , show that AB is the inverse of A .³

$$\begin{aligned}B &= (A^2)^{-1} \\ B &= (AA)^{-1} \\ (AA)B &= AA(AA)^{-1} \\ (AA)B &= I \\ A^{-1}(AA)B &= A^{-1} \\ (A^{-1}A)AB &= A^{-1} \\ AB &= A^{-1}\end{aligned}$$

5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

What are X^{-1} and Y^{-1} , assuming $ad \neq bc$.⁴ **Hint:** First, multiply the two matrices XY . Then, attempt to solve for X^{-1} and Y^{-1} .

$$XY = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & ad - bc \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}XY &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ (X^{-1}X)Y &= (ad - bc)X^{-1} \\ Y &= (ad - bc)X^{-1} \\ \frac{1}{ad - bc}Y &= X^{-1}\end{aligned}$$

$$\begin{aligned}XY &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ X(YY^{-1}) &= (ad - bc)Y^{-1} \\ \frac{1}{ad - bc}X &= Y^{-1}\end{aligned}$$

²Strang p. 90 #13

³Strang p. 90 #18

⁴Strang p. 90 #16

Notice that $\det X = ad - bc$ so this obeys our usual rules for inversion of 2×2 matrices.

6. A is an *idempotent* matrix if and only if $AA = A$. Show that if A is symmetric ($A^\top = A$) and idempotent then $(I - A) = (I - A)(I - A)^\top$, where I is the identity matrix.

$$\begin{aligned} I - A &= I - AA^T && \text{by idempotency of } A \\ &= I - \underbrace{(A - A^T)}_{=0} - AA^T && \text{by symmetry of } A \\ &= (I - A)(I - A^T) \\ &= (I - A)(I - A)^T \end{aligned}$$

7. Let $R_{m \times n}$ be a rectangular matrix ($m \neq n$) and $A_{m \times m}$ be a symmetric matrix. Show $R^T A R$ is also symmetric. What are the dimensions of this matrix?⁵

$$\begin{aligned} (R^T A R)^T &= R^T A^T (R^T)^T \\ (R^T A R)^T &= R^T A^T R \\ (R^T A R)^T &= R^T A R && \text{by symmetry of } A \end{aligned}$$

$$R_{n \times m}^T A_{m \times m} R_{m \times n} = M_{n \times n}$$

8. Show every orthogonal matrix A has determinant 1 or -1. Hint: Apply the product rule ($|AB| = |A||B|$) and the transpose rule ($|A| = |A^T|$) for determinants.⁶

By the orthogonality of A , $A^T A = I$. By the product rule,

$$\begin{aligned} |A^T A| &= |I| \\ |A^T| |A| &= 1 && \text{by the product rule and definition of the determinant} \\ (|A|)^2 &= 1 && \text{by the transpose rule} \end{aligned}$$

and $\sqrt{1} \in \{-1, 1\}$.

9. Let $\mathbf{x} = \{x_1, \dots, x_{50}\}$ denote the number of electoral votes for each state. Let $\mathbf{y}_i = \{y_{i1}, \dots, y_{i50}\}$ denote whether or not candidate $i \in \{R, D\}$ won the votes of a given state, where $y_{ij} \in \{0, 1\}$. Write an expression for the total number of votes for each candidate.

Candidate R :

$$\mathbf{x}^T \mathbf{y}_R$$

Candidate D :

$$\mathbf{x}^T \mathbf{y}_D$$

⁵Strang 117 #19

⁶Strang p. 252 #8