

# An Introduction to Mathematics For Political Science

## Problem Set 4

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. All solutions must be written in LaTeX.

### Optimization

1. Find all extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain. In your answer, report both the max/min and argmax/armin.

a)  $f(x) = x^2 - 4x + 2$  on  $[0, 3]$

The function attains global maximum at  $x = 0$  where  $f(0) = 2$ . Global minimum at  $x = 2$ ,  $f(2) = -2$ . Local maximum at  $x = 3$ ,  $f(3) = -1$ .

b)  $f(x) = 2x^3 - x$  on  $(-1, 1]$

Global max at  $x = 1$ ,  $f(1) = 1$ . Local min at  $x \approx .408$ ,  $f(.408) \approx -0.272$ . Local max at  $x \approx -0.408$ ,  $f(x) \approx 0.272$ . Global min at  $x = -1$ ,  $f(-1) = -1$ .

c)  $f(x) = \sqrt{x}$  on  $[0, 4)$

Global min at  $x = 0$ ,  $f(0) = 0$ . The function has no local or global max.

d)  $f(x) = -x^2 + 4$  on  $(-2, 2)$

Global max at  $x = 0$ ,  $f(0) = 4$ . The function has no local or global min.

2. Explain (in words) the difference between a global maximum and a supremum.

A function is defined at its global maximum but need not be defined on its supremum. A supremum is a function's least upper bound. A global maximum is always a supremum but a supremum can exist even if a maximum does not exist.

3. Find the second derivative with respect to  $x$  of the following functions:

a)  $2x^3 - 4x^2 + x$

$12x - 8$

b)  $x^4 + e^{2x}$

$12x^2 + 4e^{2x}$

c)  $\ln^2(x)$

$-\frac{2\ln(x)-2}{x^2}$

d)  $-(x - a)^2$

$-2$

e)  $xe^{-x}$

$(x - 2)e^{-x}$

4. Find all critical points and inflection points of the following functions. Identify whether each critical point is a local maximum, local minimum, or inflection point.

a)  $f(x) = x^3 - 3x^2$

Critical points are at  $x = 0$  (local max) and  $x = 2$  (local min).  $x = 1$  is an inflection point.

b)  $f(x) = x^3 - 6x^2 + 9x + 15$

Critical points are at  $x = 1$  (local max) and  $x = 3$  (local min).  $x = 2$  is an inflection point

c)  $f(x) = -(x - b)^2$

Critical point is at  $x = b$  (local max).

d)  $f(x) = -x^3$

Critical point is at  $x = 0$ , which is an inflection point.

5. Identify the regions of  $\mathbb{R}$  on which the following functions are weakly concave and/or convex:

a)  $f(x) = -x^3$

Convex on  $(-\infty, 0]$ , concave on  $[0, \infty)$

b)  $f(x) = \frac{1}{x}$

Concave on  $(-\infty, 0)$ , convex on  $(0, \infty)$  (undefined at  $x = 0$ ).

c)  $f(x) = x^3 - 3x^2$

Concave on  $(-\infty, 1]$ , convex on  $[1, \infty)$

d)  $f(x) = 4x - 5$

Weakly convex and concave everywhere.