

# Metric Spaces Exercises

1) Show that  $d_\infty(x, y) = \max_{i=1}^n |x_i - y_i|$  is a metric for  $\mathbb{R}^n$ .

## Solution

1. Assume to the contrary that  $d(x, y) < 0$ . This implies that for all  $i$ ,  $|x_i - y_i| < 0$  which contradicts the fact that the image of the absolute value function is  $\mathbb{R}_+$ .

$$2. d(x, x) = \max_{i=1}^n |x_i - x_i| = \max_{i=1}^n |0| = 0.$$

$$3. d(x, y) = \max_{i=1}^n |x_i - y_i| = \max_{i=1}^n |y_i - x_i| = d(y, x)$$

4. Let  $a = \max_{i=1}^n |x_i - z_i|$  and  $b = \max_{i=1}^n |z_i - y_i|$ .  $a + b \geq |x_i - z_i| + |z_i - y_i| \geq |x_i - y_i|$  for all  $i$ . Therefore  $a + b \geq \max_{i=1}^n |x_i - y_i| = d(x, y)$ . ■

2) Consider the metric space  $(\mathbb{R}, d_1)$  and the sequence  $(\frac{1}{n^2})$ . Prove that  $\lim \frac{1}{n^2} = 0$ .

## Solution

We want  $|\frac{1}{n^2} - 0| < \epsilon$  and we want to know how big  $n$  must be. That is, we want  $\frac{1}{n^2} < \epsilon$  or  $\frac{1}{\sqrt{\epsilon}} < n$ . If our steps are reversible, we see  $n > \frac{1}{\sqrt{\epsilon}}$  implies  $|\frac{1}{n^2} - 0| < \epsilon$ . This suggests we put  $N = \frac{1}{\sqrt{\epsilon}}$ . Using this logic, a formal proof reads as follows:

Let  $\epsilon > 0$  and let  $N = \frac{1}{\sqrt{\epsilon}}$ . Then  $n > N$  implies  $n > \frac{1}{\sqrt{\epsilon}}$  which implies  $n^2 > \frac{1}{\epsilon}$  and hence  $\epsilon > \frac{1}{n^2}$ . Thus  $n > N$  implies  $|\frac{1}{n^2} - 0| < \epsilon$ . This proves  $\lim \frac{1}{n^2} = 0$ . ■

3) Prove that every convergent sequence in a metric space is bounded.

## Solution

Let  $(x^n) \rightarrow x$  be a sequence in metric space  $(X, d)$ . In order to prove that  $(x^n)$  is bounded it is sufficient to find a  $K \in \mathbb{R}$  such that for all  $n$ ,  $d(x^n, x) \leq K$ . Because the sequence converges, we can find an  $N$  such that  $n \geq N$  implies  $d(x_n, x) < 1$ . Therefore  $d(x_n, x) < 1$  for  $n \geq N$ . Now consider  $n < N$ . This is a finite set of terms which induces a finite set  $\{d(x_1, x), d(x_2, x), \dots, d(x_{N-1}, x)\}$ . Because the set is finite, it has a maximum term. Now add 1 to the set and let  $K = \max \{d(x_1, x), d(x_2, x), \dots, d(x_{N-1}, x), 1\}$ . We have just found a  $K$  such that  $d(x^n, x) \leq K$  for all  $n$ . ■

4) Prove that every convergent sequence in a metric space is Cauchy.

**Solution:** Suppose  $(x^n) \rightarrow x$  in  $(X, d)$ . By the triangle inequality we know that  $d(x^n, x^m) \leq d(x^n, x) + d(x, x^m)$ . Let  $\epsilon > 0$ . Because the sequence converges, there exists some  $N_1$  such that  $n \geq N_1$  implies  $d(x^n, x) < \frac{\epsilon}{2}$ . Similarly, there exists some  $N_2$  such that  $m \geq N_2$  implies  $d(x, x^m) < \frac{\epsilon}{2}$ . Let  $N = \max \{N_1, N_2\}$ . Then  $n, m \geq N$  implies  $d(x^n, x^m) < \epsilon$ . Thus  $(x^n)$  is a Cauchy sequence. ■

5) Prove that every Cauchy sequence is bounded.

**Solution:** Because the sequence is Cauchy, we can find an  $N$  such that  $n, m \geq N$  implies  $d(x^n, x^m) < 1$ . In particular,  $d(x^n, x^N) < 1$ . To show that  $(x^n)$  is bounded, we need to find some  $p \in X$  and  $K \in \mathbb{R}$  such that  $d(x^n, p) \leq K$  for all  $x^n \in (x^n)$ . Let  $L = \max_{i < N} \{d(x^i, x^N)\}$ . This is a finite set and therefore has a maximum. Now for  $n \geq N$ ,  $d(x^N, x^n) < 1$ . For  $n < N$ ,  $d(x^N, x^n) \leq L$ . Let  $p = x^N$  and  $K = \max \{L, 1\}$ . It follows that for all  $n$ ,  $d(x^n, p) \leq K$ . Therefore  $(x^n)$  is bounded. ■

6) The Bolzano-Weierstrass theorem states that every bounded sequence of real numbers has a convergent subsequence. Use the theorem to prove that  $\mathbb{R}$  is complete.

**Solution** To show the completeness of  $\mathbb{R}$ , we need to show that every Cauchy sequence in  $\mathbb{R}$  converges to a point in  $\mathbb{R}$ . Let  $(x^n)$  be an arbitrary Cauchy sequence in  $\mathbb{R}$ . Since  $(x^n)$

is Cauchy, it is bounded. Since  $(x^n)$  is a bounded real sequence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence  $(x^{n_k})$  in  $\mathbb{R}$ . Since  $(x^n)$  is Cauchy and has a subsequence that converges in  $\mathbb{R}$ ,  $(x^n)$  converges in  $\mathbb{R}$ . Because we chose  $(x^n)$  arbitrarily, we have shown that all Cauchy sequences in  $\mathbb{R}$  converge in  $\mathbb{R}$ . Therefore  $\mathbb{R}$  is complete. ■.