Normed Linear Spaces Exercises

1) Prove the following: for any x, y in a normed linear space,

$$\|x\| - \|y\| \leq \|x - y\|$$

Solution: $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ by property (4). Rearranging yields $\|x\| - \|y\| \leq \|x - y\|$ ■

2) Prove that if $x_n \to x$ is a convergent sequence in a normed linear space, then $\|x_n\| \to \|x\|$

Solution: We are given that for all $\epsilon > 0$, there exists an N such that for all $n > N$, $\|x_n - x\| < \epsilon$. The sequence $\|x_n\|$ lies in the metric space $(\mathbb{R}, d)$. Therefore to show that $\|x_n\| \to \|x\|$, we need to show that for all $\epsilon > 0$, there exists an N such that for all $n > N$, $d(\|x_n\|, \|x\|) = |\|x_n\| - \|x\|| < \epsilon$. Note that $|\|x_n\| - \|x\|| \leq \|x_n - x\|$: for $a, b \in \mathbb{R}$, $\|y - x\| = \max\{(y - x), (x - y)\}$. By the reverse triangle inequality, $\|x_n\| - \|x\| \leq \|x_n - x\|$ and $\|x\| - \|x_n\| \leq \|x - x_n\|$. Since $\|x_n - x\| = \|x - x_n\|$, we have that $|\|x_n\| - \|x\|| \leq \|x_n - x\| < \epsilon$ for N sufficiently high. ■

3) Prove that $\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1-\delta}$ for $\delta \in (0, 1)$.

Solution: The partial sums are given by

$$s_n = \sum_{k=0}^{n} a\delta^k = a\frac{1-\delta^{n+1}}{1-\delta} :$$

$$(1-\delta)\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1-\delta} = \sum_{k=0}^{n} a\delta^k - \sum_{k=0}^{n} a\delta^{k+1}$$

$$= a + a\delta + a\delta^2 + ... + a\delta^n - \delta(a + a\delta + a\delta^2 + ... + a\delta^n)$$

$$= a - a\delta^{n+1}$$

For $\delta \in (0,1)$, $\lim_{n\to\infty} a\delta^{n+1} = 0$. Therefore $\lim_{n\to\infty} s_n = \frac{a}{1-\delta}$. ■