Ordered Sets

Motivation

When we build theories in political science, we assume that political agents seek the best element in an appropriate set of feasible alternatives. Voters choose their favorite candidate from those listed on the ballot. Citizens may choose between exit and voice when unsatisfied with their government. Consequently, political science theory requires that we can rank alternatives and identify the best element in various sets of choices. Sets whose elements can be ranked are called **ordered sets**. They are the subject of this lecture.

Relations

Given two sets A and B, a **binary relation** is a subset $R \subset A \times B$. We use the notation $(a, b) \in R$ or more often aRb to denote the relation R holding for an ordered pair (a, b). This is read "a is in the relation R to b." If $R \subset A \times A$, we say that R is a **relation on** A.

Example. Let $A = \{$Austin, Des Moines, Harrisburg$\}$ and let $B = \{$Texas, Iowa, Pennsylvania$\}$. Then the relation $R = \{$(Austin, Texas), (Des Moines, Iowa), (Harrisburg, Pennsylvania)$\}$ expresses the relation "is the capital of." ■

Example. Let $A = \{a, b, c\}$. The relation $R = \{(a, b), (a, c), (b, c)\}$ expresses the relation "occurs earlier in the alphabet than." We read aRb as "a occurs earlier in the alphabet than b." ■

Properties of Binary Relations

A relation R on a nonempty set X is

reflexive if xRx for each $x \in X$

complete if xRy or yRx for all $x, y \in X$

symmetric if for any $x, y \in X$, xRy implies yRx

antisymmetric if for any $x, y \in X$, xRy and yRx imply $x = y$

transitive if xRy and yRz imply xRz for any $x, y, z \in X$

Any relation which is reflexive and transitive is called a **preorder**. A set on which a preorder is defined is called a **preordered set**. In our theory of choice, we refer to a preorder on X as a **preference relation** on X. Preorders fall into two fundamental categories, depending on whether or not the relation is symmetric. A symmetric preorder is called an **equivalence relation**. A preorder that is not symmetric is called an **order relation**.

Equivalence Relations

A relation $\sim$ on a nonempty set X is called an equivalence relation if it is reflexive, symmetric, and transitive. For any $x \in X$, the **equivalence class** of x relative to $\sim$ is defined as the set $[x]_\sim \equiv \{y \in X : y \sim x\}$. We often exploit indifference in our theory of choice and equivalence classes help us formalize this.

Example: Equality is an equivalence relation on $\mathbb{R}$. Clearly $x = x$ so the relation is reflexive. It is also symmetric: if $x = y$, then $y = x$. Transitivity is also easy to check: if $x = y$ and $y = z$, then $x = y = z$ so $x = z$. The equivalence class of $\sim$ is $[x]_\sim = \{y \in X : y = x\}$. Clearly for any x, its equivalence class is a singleton: on $\mathbb{R}$, each element is equal only to itself.[1] $\blacksquare$

Example: Let $X = \{(a, b) : a, b \in \mathbb{N}\}$ and define the relation $\sim$ on X by $(a, b) \sim (c, d)$

[1] If $u : X \to \mathbb{R}$ is a utility function representing preferences on a set X, then defining $x \sim y$ by $u(x) = u(y)$ gives the indifference equivalence relation.

if and only if $ad = bc$. First we check reflexiveness: $(a,b) \sim (a,b)$ is true iff $ab = ab$ which checks out. Now check symmetry: does $(a,b)R(c,d)$ imply $(c,d)R(a,b)$? Let's check: does $ad = bc$ imply $cb = da$? Yes. Finally check transitivity. Do $(a,b)R(c,d)$ and $(c,d)R(e,f)$ imply $(a,b)R(e,f)$? That is, do $ad = bc$ and $cf = de$ imply $af = be$? Note that $ad = bc$ implies $\frac{a}{b} = \frac{c}{d}$ and $cf = de$ implies $\frac{c}{d} = \frac{e}{f}$ so $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ which gives us $af = be$. The equivalence class is given by $[(a,b)]_\sim = \{(c,d) \in X : \frac{c}{d} = \frac{a}{b}\}$. With this relation, $\frac{4}{2}$ is equivalent to $\frac{8}{4}$. This is obvious to us precisely because the rational numbers are constructed as equivalence classes such that $(4,2),(8,4) \in [(2,1)]_\sim$. ■

There is a close relationship between equivalence relations on a set and partitions of that set. A **partition** of a set X is a collection of disjoint subsets of X whose union is the full set X. Any equivalence relation on a set X partitions X. Each element of X belongs to one and only one equivalence class. In the first examples above, equality partitions $\mathbb{R}$ into singletons. In the second example, $\sim$ partitions X into the rational numbers.

Order Relations

An order relation is a relation, $\succsim$, that is reflexive and transitive but not symmetric. An **ordered set** $(X, \succsim)$ consists of a set X with an order relation $\succsim$ defined on X.

Every order relation $\succsim$ on X induces two additional relations, $\succ$ and $\sim$.

$$x \succ y \iff x \succsim y \ \wedge \neg[y \succsim x]$$

The relation $\succ$ is transitive but not reflexive.

$$x \sim y \iff x \succsim y \ \wedge \ y \succsim x$$

for all $x, y \in X$.

Example The natural order $\geq$ on $\mathbb{R}$ induces the strict order $>$ and the equivalence relation

=. ■

Maximal and Maximum Elements

Given $(X, \succsim)$, an element x is a **maximal element** if there is no $y \in X$ such that $y \succ x$. $x \in X$ is called the **maximum element** in X if $x \succsim y$ for all $y \in X$. Clearly for any ordered set $(X, \succsim)$, every maximum of a nonempty subset of X is maximal in that set.

Let A be a nonempty ordered subset of X. An element $x \in X$ is called an **upper bound** for A if $x \succsim a$ for all $a \in A$. An upper bound is called a **least upper bound** or **supremum** for A if it precedes every other upper bound for A.

Example Consider the ordered set $([a, b], \geq)$. b is the maximal element, the maximum element, and the least upper bound. Now consider $((a, b), \geq)$. The set has no maximal or best element. b is its least upper bound in $\mathbb{R}$. ■

Example Let X be a 1-simplex and $\geq$ be the natural vector order. That is, for $x, y \in \mathbb{R}^2$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2$. Every element in the set is a maximal element. To see this, note that there exists no $(a, b) \in \Delta^1$ such that $(a, b) > (1, 0)$. This follows from the fact that $1 > a$ for all $(a, b) \neq (1, 0)$. Now consider $(1-\epsilon, \epsilon)$. Since $\epsilon > b$ for all $(a, b) \neq (1, 0)$, there is no $(a, b) > (1-\epsilon, \epsilon)$. Although every element in Δ^1 is a maximal element, there is no maximum element. To see this, assume (a, b) is a maximum. This implies that $(a, b) \succsim (c, d)$ for all (c, d). But for all (a, b), $(c, d) = (a-\epsilon, b+\epsilon)$ which implies that (a, b) and (c, d) cannot be compared. Therefore it cannot be true that $(a, b) \succsim (c, d)$ which implies that (a, b) is not a maximum. ■

Partially Ordered Sets, Chains, and Lattices

In general an ordered set may have many maximal and maximum elements and its subsets may have multiple least upper bounds.

Example Let X be a 1-simplex and $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$. The points (1,0) and (0,1) are both maximal and maximum elements. ■

Uniqueness may be achieved by imposing the additional requirement of antisymmetry. The result is a **partially ordered set**. A **partial order** is a relation that is reflexive, transitive, and antisymmetric.

Example The natural order on $\mathbb{R}^n$ is only partial. It is not complete. For example, (1,1) $\geq$ (1,0), but (0,1) and (1,0) are not comparable. Antisymmetry is easy to check. $(a, b) \geq (c, d)$ implies that $a \geq c$ and $b \geq d$. Similarly, $(c, d) \geq (a, b)$ implies $c \geq a$ and $d \geq b$. Therefore $a = c$ and $b = d$. ■

The significance of antisymmetry is that, if it exists, the least upper bound of any subset of a partially ordered set is unique. To see this, consider a least upper bound of X, x. By the definition of least upper bound, $x \succsim y$ for all other upper bounds y. Let y be a second least upper bound. Because y is a least upper bound, $y \succsim x$. But antisymmetry implies $x = y$. Therefore x is unique. If a maximum exists, the maximum is a least upper bound and is therefore unique. Antisymmetry, however, is not sufficient to guarantee existence, only uniqueness.

A partial order is partial in the sense that not all elements are comparable. If all elements in a partially ordered set are comparable so that $\succsim$ is also complete, we refer to $\succsim$ is a **total order** and refer to a totally ordered set as a **chain**.

Example $(\mathbb{R}, \geq)$ is a chain. ■

Example $(\Delta^1, \geq)$ is not a chain since neither $(1, 0) \geq (0, 1)$ nor $(0, 1) \geq (1, 0)$ is true. ■

A **lattice** is a partially ordered set in which every pair of elements has a least upper bound. If x and y are two elements of a lattice, L, their least upper bound $x \vee y$ is an element of L called a **join**. Their greatest lower bound $x \wedge y$, called their **meet**, is also an element of L.

Lattices have some desirable properties. We will see later on that lattices admit a powerful tool for theoretical analysis called "monotone comparative statics." They also allow us to identify when a subset of $\mathbb{R}^n$ will have a maximum element (or least upper bound).

Example: We saw above that a 1-simplex ordered by $\geq$ does not have a maximum element. Consider another subset $X \subset \mathbb{R}^2_+$ with the natural vector order $\geq$ and let X be a lattice. In particular, let X be the unit square. X is clearly a lattice: $(1,0) \vee (0,1) = (1,1)$ and $(1,0) \wedge (0,1) = (0,0)$. X also has a maximal element: $(1,1)$. ■

Example Now let X be the subset of the unit square that contains points less than or equal to Δ^1 when ordered by $\geq$. This set is not a lattice: $(1,0) \vee (0,1) = (1,1) \notin X$. The set of maximal elements of X is Δ^1. As we saw above, Δ^1 has no maximum element. ■

The above examples illustrates a useful shorthand to keep in mind later on in your research career when working in $(\mathbb{R}^n, \geq)$: "squares" are lattices, "triangles" are not.

Weakly Ordered Sets

In the previous section we imposed antisymmetry on a preorder to assist our theory of choice. We also often impose completeness rather than antisymmetry. A **weak order** is an order relation that is complete, reflexive, and transitive. A weak order is often referred to as a **rational order** or a **rational preference relation**. In a weakly ordered set, every element is related to every other element. For our theory of choice this is desirable. We want the actors in our models to be able to compare any action or outcome in the model in terms of their preferences.

Example $(\mathbb{R}^2, \geq)$ is not a weakly ordered set. Now let $\succsim$ be defined so that $x \succsim y$ if and only if $\max\{x_1, x_2\} \geq \max\{y_1, y_2\}$. $(\mathbb{R}^2, \succsim)$ is a weakly ordered set. ■

Note the difference between a weakly ordered set and a chain. A total order imposes antisymmetry on a set while a weak order does not. A consequence of this is that a weakly ordered set may have multiple maximum elements while a totally ordered set can only have one maximum element. In political science a weak or rational order often makes more sense than a total order. This comes from the fact that we constantly deal in tradeoffs across multiple dimensions of policy. This induces indifference classes across our choice space. We may think, for example, that it is a reasonable assumption that a Republic politician considers a combination of liberal social policy and conservative economic policy to be no better or worse than a conservative social policy and liberal economic policy. If these are the only two options available for her to vote on, then our theory of choice allows both of these to be maximum elements in her choice set.

It is worth noting here that "rational" here refers to something very specific and mathematical. Presumably you have encountered or perhaps uttered a sentence such as "that dictator is irrational." Colloquially this is could be all well and good but the way we use the term "rational," this would imply that his or her preferences over some set of political outcomes is incomplete or intransitive. I'd be skeptical if you told me he or she prefers apples to bananas, bananas to carrots, and carrots to apples. I'd be similarly skeptical if you told me he or she cannot compare economic growth to political survival or that she neither prefers coffee to tea nor is indifferent between the two.

Exercises

1) Prove that for the set of positive integers, the relation "m is a multiple of n" is an order relation.

Solution: First show reflexiveness. Reflexiveness requires that $a \succsim a$ for all $a \in \mathbb{Z}_+$. Note that $a \succsim b$ iff $\frac{a}{b} \in \mathbb{Z}_+$. It is straightforward to check that $\frac{a}{a} = 1 \in \mathbb{Z}$. Therefore reflexiveness holds. Now check transitivity. $a \succsim b$ implies $\frac{a}{b} = \alpha \in \mathbb{Z}_+$. $b \succsim c$ implies $\frac{b}{c} = \beta \in \mathbb{Z}_+$. This yields $b = \beta c$, $\frac{a}{\beta c} = \alpha$, $\frac{a}{c} = \alpha\beta$. Since $\alpha, \beta \in \mathbb{Z}_+$, $\alpha\beta \in \mathbb{Z}_+$. Therefore $a \succsim c$. Finally, we need to show that symmetry does not hold, that is, $a \succsim b$ does not imply $b \succsim a$. It is sufficient to find a single counter example to prove this. Let $a = 2$ and $b = 1$. $a \succsim b$ is true: $2/1 = 1$. $b \succsim a$, however, is not true: $1/2 \notin \mathbb{Z}$. ■

2) Let $X = \{1, 2, ..., 9\}$, ordered by the relation "m is a multiple of n". Find all maximal

and maximum elements of this ordered set and its least upper bound in $\mathbb{Z}$.

Solution: To find the maximal elements, we need to find the set of all y such that there is no $x \in X$ with $x \succ y$ where $\succ$ is the relation "x is a proper multiple of y." First check for $x \succ 1$. Clearly for all $x > 1$, this holds. Now check for $x \succ 2$. All even $x > 2$, this holds. Similarly for 3 and 4: $9 \succ 3$ and $8 \succ 4$. For 5, the next proper multiple of 5 is $10 \notin X$. Therefore 5 is a maximal element. Similarly, the next proper multiple of 6 is 12, 7 is 14, 8 is 16, and 9 is 18. None of these are in X. We conclude that the maximal set is the set $\{5, 6, 7, 8, 9\}$. To find a maximum, we need to find a member x of the maximal set such that $x \succsim y$ for all $y \in X$. Let's try 5. Is it true that $5 \succsim 9$? No. Therefore 5 cannot be a maximum. What about 6? $6 \succsim 9$ is also false. Similarly, $7 \succsim 9$ and $8 \succsim 9$ are both false. What about 9? $9 \succsim 9$ is true. But $9 \succsim 2$ and $9 \succsim 4$ are false. Therefore we conclude that the ordered set has no maximum. Finally, to find a lower bound, we need to find an integer $z \in \mathbb{Z}$ such that $z \succsim x$ for all $x \in X$. That is, we need to find a multiple of every element of X. In particular, we need to find the *least common multiple* of $1, ...9$. It turns out that 2520 is the least common multiple and therefore the least upper bound of X. $\blacksquare$

3) Show that $x \sim y$ is an equivalence relation if $\succsim$ is rational.

Solution: $\succsim$ rational means that $\succsim$ is complete, reflexive, and transitive. $\sim$ is defined as $x \sim y \iff x \succsim y \wedge y \succsim x$. We need to show that $\sim$ is reflexive, symmetric, and transitive. Let's start with reflexiveness. $x \sim x$ implies $x \succsim x$ (and $x \succsim x$). Because $\succsim$ is reflexive, $x \sim x$ is reflexive. Now for transitivity. $x \sim y$ implies $x \succsim y$ and $y \succsim x$. $y \sim z$ implies $y \succsim z$ and $z \succsim y$. Because $\succsim$ is transitive, we have that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$. Therefore $x \sim y$ and $y \sim z$ imply $x \sim z$. Finally we check symmetry. $x \sim y$ implies $x \succsim y$ and $y \succsim x$. We also know from the definition of $\sim$ that $y \succsim x$ and $x \succsim y$ iff $y \sim x$. Therefore because $y \succsim x \wedge x \succsim y \equiv x \succsim y \wedge y \succsim x$, $x \sim y$ implies $y \sim x$. $\blacksquare$

Source Material

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