

# Smooth Functions

1. Prove that every continuous linear functional; is differentiable with  $Df[\mathbf{x}] = \boldsymbol{\alpha}$ .<sup>1</sup>

We need to find a  $g(\mathbf{x})$  such  $f(\mathbf{x}_0 + \mathbf{x}) - f(\mathbf{x}_0) \approx g(\mathbf{x})$ . Since  $f$  is a linear function, we can write

$$f(\mathbf{x}) = \boldsymbol{\alpha}^T \mathbf{x}$$

for some  $\boldsymbol{\alpha}$ . Then,

$$\begin{aligned} f(\mathbf{x}_0 + \mathbf{x}) - f(\mathbf{x}_0) &= \boldsymbol{\alpha}^T (\mathbf{x}_0 + \mathbf{x}) - \boldsymbol{\alpha}^T \mathbf{x}_0 \\ &= \boldsymbol{\alpha}^T \mathbf{x} = g(\mathbf{x}) \end{aligned}$$

Then  $\mathbf{x} \rightarrow \mathbf{0} \implies f(\mathbf{x}_0 + \mathbf{x}) - f(\mathbf{x}_0) - g(\mathbf{x}) \rightarrow \mathbf{0} \implies \eta(\mathbf{x}) \rightarrow \mathbf{0}$  as desired.

2. Prove that if a differentiable functional  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is increasing, then  $Df[\mathbf{x}_0](\mathbf{x}) \geq 0$  for all  $\mathbf{x} \in X$ , or  $\frac{\partial f}{\partial x_i} \geq 0$  for all  $i \in \{x_1, \dots, x_n\}$ .

Suppose for sake of contradiction that

$$\frac{\partial f}{\partial x_i} < 0$$

for some  $i$  and take  $d\mathbf{x} = \{0, \dots, dx_i, \dots, 0\}$ . By the mean value theorem, there exists some  $\bar{\mathbf{x}} = \{0, \dots, \bar{x}_i, \dots, 0\}$  with  $0 < \bar{x}_i < dx_i$  such that

$$f(\mathbf{x}_0 + d\mathbf{x}) - f(\mathbf{x}_0) = Df[\bar{\mathbf{x}}](d\mathbf{x})$$

Our hypothesis implies

$$f(\mathbf{x}_0 + d\mathbf{x}) - f(\mathbf{x}_0) < 0$$

But then  $f$  is not increasing. We conclude  $\frac{\partial f}{\partial x_i} \geq 0$ . ■

3. Let  $f$  be a differentiable functional. Prove that the  $\nabla f(\mathbf{x}_0)$  is orthogonal to the hyperplane tangent to the contour through  $f(\mathbf{x}_0)$ .

Let  $c = f(\mathbf{x}_0)$ . The contour through  $f(\mathbf{x}_0)$  is

$$f^{-1}(c) = \{\mathbf{x} : f(\mathbf{x}) = c\}$$

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<sup>1</sup>Carter 4.6

Define an implicit function  $h(t)$  where

$$h(t) = f(\mathbf{x}(t)) = c$$

for all  $t$ . Since  $f$  is differentiable, we have by the chain rule that

$$Dh(t) = Df(\mathbf{x}(t))^T D\mathbf{x}(t) = 0$$

because  $c$  is constant. Since  $f$  is a functional this can be written

$$Dh(t) = \nabla f(\mathbf{x}(t))^T D\mathbf{x}(t) = 0$$

Since  $\nabla f(\mathbf{x})$  is the gradient of  $f$  and  $\nabla f(\mathbf{x})^T D\mathbf{x}(t) = 0$  we know  $\nabla f(\mathbf{x}_0)$  and  $D\mathbf{x}(t)$  are orthogonal. Note that  $D\mathbf{x}(t)$  is a linear approximation of the function  $\mathbf{x}(t)$  at  $t$  – a tangent hyperplane to the contour. ■

4. Let the policy production function discussed above be written

$$f(x, y) = x^\alpha y^\beta$$

Give a sufficient condition for this function to be concave on  $\{\mathbb{R}_{++} \times \mathbb{R}_{++}\}$ .

**Hint:** A  $2 \times 2$  symmetric matrix  $A$  is negative definite if  $A_{11} < 0$  and  $A_{11}A_{22} - A_{12}A_{21} > 0$ .

We need to find conditions under which

$$\mathbf{z}^T H_f(x, y) \mathbf{z} \leq 0$$

for arbitrary  $\mathbf{z}$ . The Hessian is given by

$$H_f(x, y) = \begin{pmatrix} \frac{\partial^2 f(x, y)}{\partial x^2} & \frac{\partial^2 f(x, y)}{\partial x \partial y} \\ \frac{\partial^2 f(x, y)}{\partial x \partial y} & \frac{\partial^2 f(x, y)}{\partial y^2} \end{pmatrix}$$

Computing second derivatives and cross partials,

$$H_f(x, y) = \begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2}y^\beta & \alpha x^{\alpha-1}\beta y^{\beta-1} \\ \alpha x^{\alpha-1}\beta y^{\beta-1} & x^\alpha \beta(\beta-1)y^{\beta-2} \end{pmatrix} = \begin{pmatrix} \frac{\alpha(\alpha-1)f(x, y)}{x^2} & \frac{\alpha\beta f(x, y)}{xy} \\ \frac{\alpha\beta f(x, y)}{xy} & \frac{\beta(\beta-1)f(x, y)}{y^2} \end{pmatrix} = f(x, y) \begin{pmatrix} \frac{\alpha(\alpha-1)}{x^2} & \frac{\alpha\beta}{xy} \\ \frac{\alpha\beta}{xy} & \frac{\beta(\beta-1)}{y^2} \end{pmatrix}$$

Applying the hint, we need

$$H_{11} < 0 \quad H_{11}H_{22} > H_{12}^2$$

$H_{11} < 0$  and  $H_{22} < 0$  require  $\alpha, \beta < 1$ . Rearranging the final condition,

$$\begin{aligned}
H_{11}H_{22} &> H_{12}^2 \\
\frac{\alpha(\alpha-1)}{x^2} \frac{\beta(\beta-1)}{y^2} &> \left( \frac{\alpha\beta}{xy} \right)^2 \\
\frac{\alpha(\alpha-1)\beta(\beta-1)}{y^2x^2} &> \frac{\alpha^2\beta^2}{x^2y^2} \\
\frac{\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{(\alpha^2 - \alpha)(\beta^2 - \beta) - \alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{\alpha^2\beta^2 - \alpha\beta^2 - \beta\alpha^2 + \alpha\beta - \alpha^2\beta^2}{x^2y^2} &> 0 \\
\frac{\alpha\beta(1 - \beta - \alpha)}{x^2y^2} &> 0
\end{aligned}$$

which requires  $\alpha + \beta < 1$ .