# Introduction to Mathematics for Political Science

## Problem Set 9: Matrix Inversion and Determinants

**Instructions:** You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$\begin{aligned} 3x_1 - x_2 &= 10 \\ -x_1 + 4x_2 &= 4 \end{aligned}$$

   Write this system in $A\boldsymbol{x} = \boldsymbol{b}$ form and solve via matrix inversion.

2. Let $C = AB$ where $C$ is invertible and $A$ and $B$ are square matrices. Solve for $A^{-1}$.[1]

3. Let $M = ABC$ where $M$ is invertible and $A$, $B$, and $C$ are square matrices. Solve for $B^{-1}$.[2]

4. If $B$ is the inverse of $A^2$, show that $AB$ is the inverse of $A$.[3]

5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   What are $X^{-1}$ and $Y^{-1}$, assuming $ad \neq bc$.[4] **Hint:** First, multiply the two matrices $XY$. Then, attempt to solve for $X^{-1}$ and $Y^{-1}$.

6. $A$ is an *idempotent* matrix if and only if $AA = A$. Show that if $A$ is symmetric ($A^\top = A$) and idempotent then $(I - A) = (I - A)(I - A)^\top$, where $I$ is the identity matrix.

7. Let $R_{m \times n}$ be a rectangular matrix ($m \neq n$) and $A_{m \times m}$ be a symmetric matrix. Show $R^T A R$ is also symmetric. What are the dimensions of this matrix?[5]

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[1] Strang p. 90 #12
[2] Strang p. 90 #13
[3] Strang p. 90 #18
[4] Strang p. 90 #16
[5] Strang 117 #19

8. Show every orthogonal matrix $A$ has determinant 1 or -1. Hint: Apply the product rule ($|AB| = |A||B|$) and the transpose rule ($|A| = |A^T|$) for determinants.[6]

9. Let $\boldsymbol{x} = \{x_1, ..., x_{50}\}$ denote the number of electoral votes for each state. Let $\boldsymbol{y}_i = \{y_{i1}, ..., y_{ij}, ..., y_{i50}\}$ denote whether or not candidate $i \in \{R, D\}$ won the votes of a given state, where $y_{ij} \in \{0, 1\}$. Write an expression for the total number of votes for each candidate.

[6] Strang p. 252 #8