IMPS 2018: Final Exam

11 September 2018

Instructions: This is a closed book examination. Calculators are not permitted. **There are 8 questions, from which you can choose 6 to answer.** Each question is worth ten points and should take about 30 minutes to complete. You have three hours.

1. Let S and T be sets. Show

$$(S \cup T)^c = S^c \cap T^c$$

Take $x \in (S \cup T)^c$. Then $x \notin S \cup T \implies x \notin S \wedge x \notin T \implies x \in S^c \wedge x \in T^c \implies x \in S^c \cap T^c$. So $(S \cup T)^c \subseteq S^c \cap T^c$. Now take $x \in S^c \cap T^c$. Then $x \in S^c \wedge x \in T^c \implies x \notin S \wedge x \notin T \implies x \notin S \cup T \implies x \in (S \cup T)^c$. So $S^c \cap T^c \subseteq (S \cup T)^c$. If $(S \cup T)^c \subseteq S^c \cap T^c$ and $S^c \cap T^c \subseteq (S \cup T)^c$ then $(S \cup T)^c = S^c \cap T^c$. ■

2. Consider a sequence $\{x_n\} \in (\mathbb{R}, d_1)$ such that

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} + 1 & \text{if } n \text{ is even} \end{cases}$$

Does this sequence converge? Prove your answer.

Solution The sequence converges to one. To see why, first consider the case in which n is odd. Let $N^\star = \lceil \frac{1}{\epsilon} \rceil$ Then, for $n_{\text{odd}} > N^\star$,

$$\begin{aligned} d(x_n, 1) &= |\left(\frac{1}{n} + 1\right) - 1| \\ &< |\frac{1}{N^\star}| \\ &\leq \epsilon \end{aligned}$$

If n is even, then $x_n = 1$ and

$$d(x_n, 1) = d(1, 1) = 0 < \epsilon$$

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3. Let f be a continuous functional on a metric space (X, d). Prove αf is continuous for every $\alpha \in \mathbb{R}$.

Solution

If f is continuous then for every $\epsilon > 0$ there exists a δ such that

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\epsilon}{\alpha}$$

Then,

$$\begin{aligned}\alpha|f(x_1) - f(x_2)| &< \epsilon \\ |\alpha f(x_1) - \alpha f(x_2)| &< \epsilon\end{aligned}$$

which demonstrates the continuity of αf. ■

4. Let $\boldsymbol{N} : \mathbb{R}^n \to \mathbb{R}$ be a norm. Use the properties of a norm prove that $\boldsymbol{N}$ is a convex function.

Solution

Using the properties of the norms, take $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

$$\begin{aligned}\boldsymbol{N}(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &\leq \boldsymbol{N}(\lambda \boldsymbol{x}) + \boldsymbol{N}((1-\lambda)\boldsymbol{y}) && \text{Triangle Inequality} \\ &= |\lambda|\boldsymbol{N}(\boldsymbol{x}) + |(1-\lambda)|\boldsymbol{N}(\boldsymbol{y}) && \text{Homogeneity} \\ &= \lambda\boldsymbol{N}(\boldsymbol{x}) + (1-\lambda)\boldsymbol{N}(\boldsymbol{y})\end{aligned}$$

■

5. Consider the projection operator $\boldsymbol{p}_s : X \to S$ where S is a subspace of a inner product space X. Prove that if $\boldsymbol{x} - \boldsymbol{p}_S(\boldsymbol{x}) \perp S$ then

$$d(\boldsymbol{x}, \boldsymbol{p}_s(\boldsymbol{x})) = \min\{d(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} \in S\}$$

Solution

Take an arbitrary $\boldsymbol{y} \in S$. It must be true that

$$\|\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x})\|^2 \leq \|\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x})\|^2 + \|\boldsymbol{p}_s(\boldsymbol{x}) - \boldsymbol{y}\|^2$$

Because $\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x}) \perp \boldsymbol{p}_s(\boldsymbol{x}) - \boldsymbol{y}$, Pythagorus' Theorem applies

$$\begin{aligned}\|\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x})\|^2 &\leq \|\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x})\|^2 + \|\boldsymbol{p}_s(\boldsymbol{x}) - \boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x} - \boldsymbol{p}_s(\boldsymbol{x}) + \boldsymbol{p}_s(\boldsymbol{x}) - \boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x} - \boldsymbol{y}\|^2\end{aligned}$$

Since X and S are normed linear spaces, $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. We have therefore shown that

$$d(\boldsymbol{x}, \boldsymbol{p}_s(\boldsymbol{x})) \leq d(\boldsymbol{x}, \boldsymbol{y})$$

for all $\boldsymbol{y} \in S$ demonstrating the proposition. ■

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable functional. Prove that if f is decreasing, then $Df[\boldsymbol{x}] \leq 0$ for all $\boldsymbol{x}$.

Solution We can approximate changes in the function f with the derivative

$$f(\boldsymbol{x}_0 + \boldsymbol{dx}) - f(\boldsymbol{x}_0) = Df[\boldsymbol{x}_0](\boldsymbol{dx})$$

Since f is a decreasing function $f(\boldsymbol{x}') < f(\boldsymbol{x})$ for all $\boldsymbol{x}' > \boldsymbol{x}$. Then for any $\boldsymbol{dx} \geq 0$, $f(\boldsymbol{x}_0 + \boldsymbol{dx}) - f(\boldsymbol{x}_0) \leq 0 \implies Df[\boldsymbol{x}_0](\boldsymbol{dx}) \leq 0$ as desired.

7. Solve

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = 1 - (x_1 - 1)^2 - (x_2 - 1)^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1 \end{aligned} \tag{1}$$

Hint: Be sure to check the concavity of the objective function along the constraint set.

Solution

We start by showing the global concavity of the objective function, which is sufficient to guarantee that the function is concave along the constraint set. Taking first order partials, we have

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = -2(x_1 - 1) \qquad \frac{\partial f(x_1, x_2)}{\partial x_2} = -2(x_2 - 1)$$

Then, taking second order partials, we have

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = -2 \qquad \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = -2 \qquad \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2 = 0}$$

This allows us to construct the Hessian.

$$H_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

We then check the Hessian's definiteness. We must have $\boldsymbol{y}^T H_f \boldsymbol{y} \leq 0$ for all $\boldsymbol{y} \neq 0$ for f to be concave.

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

giving $-2y_1^2 - 2y_2^2 < 0$ which holds for all $\boldsymbol{y}$. So f is globally concave.

Now we can write the Legrangian

$$\mathcal{L} = 1 - (x_1 - 1)^2 - (x_2 - 1)^2 - \lambda(1 - x_1^2 - x_2^2)$$

and take first order conditions.

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2(x_1 - 1) + 2\lambda x_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_1} = -2(x_2 - 1) + 2\lambda x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(1 - x_1^2 - x_2^2) = 0$$

The first two conditions imply

$$x_1 = \frac{1}{1 - \lambda} = x_2$$

Let $x = x_1 = x_2$. Then by the last first order condition we must have

$$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x_1^\star = x_2^\star = x^\star &= \frac{1}{\sqrt{2}} \end{aligned}$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function. Consider the optimization problem

$$\max_x f(x; \theta)$$

The value function is given by $V(\theta) = f(x^\star(\theta); \theta)$. Prove that

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star(\theta); \theta)}{\partial \theta}$$

Solution

Because f is concave, we must have

$$\frac{\partial f(x^\star(\theta); \theta)}{\partial x} = 0$$

by the first order condition. The derivative of the value function with respect to θ is given by

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star(\theta); \theta)}{\partial \theta} = \underbrace{\frac{\partial f(x^\star(\theta); \theta)}{\partial x^\star(\theta)}}_{=0} \frac{\partial x^\star(\theta)}{\partial \theta} + \frac{\partial f(x^\star(\theta); \theta)}{\partial \theta}$$

giving

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star(\theta); \theta)}{\partial \theta}$$

as desired.

9. Prove for any square, invertible matrix A, and for all $n > 0$, $(A^n)^{-1} = (A^{-1})^n$.

Solutions

We prove the statement by induction. For $n = 1$, this is just $(A^1)^{-1} = A^{-1} = (A^{-1})^1$. Therefore, the statement holds for the base case. Now let us assume that the statement is true for a given n, that is $(A^n)^{-1} = (A^{-1})^n$. We need to show that the statement is also true for $n+1$. We can prove this statement by showing that $A^{n+1}(A^{-1})^{n+1} = I$

$$\begin{aligned}
A^{n+1}(A^{-1})^{n+1} &= AA^n(A^{-1})^n A^{-1} \\
&= AA^n(A^n)^{-1}A^{-1} && \text{By the induction assumption} \\
&= AIA^{-1} \\
&= AA^{-1} \\
&= I
\end{aligned}$$