# Metric Spaces

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- We also often think about distance in our theories.
- To think spatially in a rigorous manner, we need **metric spaces**

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- Shortest distance between two points should be the direct route.

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- Elements of a metric space are usually called **points**.
- The most familiar metric space is $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$.

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- Sup metric:

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

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- We then say that $(Y, d|_{Y \times Y})$ is a metric subspace of $(X, d)$.
- $[0, 1]$ can be through of as a metric subspace of $\mathbb{R}^2$ where the distance between $x, y \in [0, 1]$ is calculated by viewing $x$ and $y$ as points in $\mathbb{R}$ : $d_1(x, y) = |x - y|$

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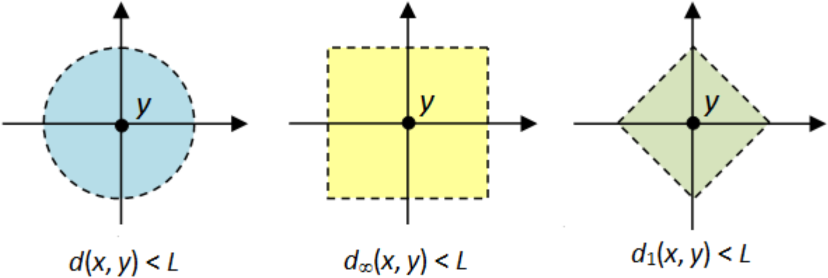
- Open balls are not necessarily spherical: their shape depends on the metric.

# Open Balls



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- The **boundary** is the set of all boundary points of $S$, $b(S)$.
- The **closure** $\overline{S}$ of $S$ is the union of $S$ with its boundary, i.e. $\overline{S} = S \cup b(S)$.

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- We often encounter sequences in political science.
- Time series data
- Sequence of moves in a game
- Asymptotic properties of an estimator identified by analyzing sequence of estimates of population parameter

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- We denote convergence by $x^n \to x$ or $\lim x^n = x$

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- Hence $\frac{19}{7(7n-4)} = \frac{3n+1}{7n-4} - \frac{3}{7} < \epsilon$
- Therefore for $n > N$, $|\frac{3n+1}{7n-4} - \frac{3}{7}| < \epsilon$

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- This implies $x = y$.

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- Every convergent sequence is bounded.

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- We often denote a subsequence of $(x^n)$ with $(x^{n_k})$
- *Example*: Let $(x^n) = 1, 1/2, 1/3, 1/4, 1/5, .....$
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- Sequences allow an alternative characterization of closed sets from the definition given earlier.
- A set $S$ in a metric space $X$ is closed if and only if every sequence in $S$ that converges in $X$ converges to a point in $S$

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- But since $d(x^n, x) \to 0$, there must exist an $N$ such that $x^N \in B_r(x)$
- This contradicts the assertion that all terms of the sequence $(x^n)$ are in $S$

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- Therefore we can find an $x \in X \backslash S$ such that every open ball around $x$ intersects $S$
- Thus for any $n = 1, 2, ...$, there is an $x^n \in B_{\frac{1}{n}}(x) \cap S$
- But then $(x^n) \in S$ and $\lim x^n = x$ yet $x \notin S$
- Thus if $S$ is not closed, there would exist at least one sequence in $S$ that converges to a point outside of $S$

# Compactness

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- You've probably encountered a definition of compact that is loosely something like "a set that is closed and bounded."

# Compactness

- Compactness is one of the most fundamental concepts in real analysis and plays an important role in optimization theory.
- You've probably encountered a definition of compact that is loosely something like "a set that is closed and bounded."
- This is true in some metric spaces but is not a proper definition of compact.

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- A metric space $X$ is **compact** if *every* open cover of $X$ has a finite subcover.

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- Therefore $(0,1)$ is not compact.

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- We conclude that $[0,1]$ is compact

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  1. $S$ is closed and bounded
  2. $S$ is compact, i.e. every open cover of $S$ has a finite subcover
- In general, compactness implies closed and bounded but the reverse is guaranteed to hold only in Euclidean space.

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- If each player $i = 1, 2, ...n$ has a compact strategy space, the strategy space is compact.
- A compact strategy space will be a key component of the fixed point theorems we use to prove the existence of a Nash equilibrium.

# Cauchy Sequences

- A sequence $(x^n)$ in a metric space $X$ is called a **Cauchy sequence** if for any $\epsilon > 0$, there exists an $N \in \mathbb{R}$ such that $d(x^k, x^l) < \epsilon$ for all $k, l \geq N$.

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- Then if $n, m > N$, we have

$$|x^n - x^m| = |1/2^n - 1/2^m| \leq 1/2^n + 1/2^m \leq 1/2^N + 1/2^N < \epsilon \quad \blacksquare$$

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- The other property, **completeness**, generalizes the idea of richness.
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- A **complete metric space** is defined as a metric space $(X, d)$ in which every Cauchy sequence in $X$ converges to a point in $X$.
- We saw earlier that $(0, 1]$ with the $d_1$ metric is not complete.

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- Let $X$ be a metric space and $Y$ a metric subspace of $X$.
- If $Y$ is complete, then it is closed in $X$.
- Conversely, if $Y$ is closed in $X$ and $X$ is complete, then $Y$ is complete.

# Proof: Complete Subspace $\implies$ Subspace Closed

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- Therefore $Y$ is complete.

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- First, a metric subspace of a complete metric space is complete if and only if it is closed in $X$.
- This is a corollary of the proposition we just proved.
- Second, every compact metric space is complete.

# Intro to Fixed Point Theory

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- Fixed points will be extremely useful in political science theory for ensuring that a model has an equilibrium.

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- To verify that this is a contraction, consider two arbitrary points, $x, y \in \mathbb{R}$
- It is easy to check that $d(f(x), f(y)) \leq Kd(x, y)$ for some $K \in (0, 1)$:

$$d(f(x), f(y)) = |f(x) - f(y)| = |\frac{x}{2} - \frac{y}{2}|$$

$$= |\frac{1}{2}(x - y)| \leq \frac{1}{2}|x - y| = \frac{1}{2}d(x, y)$$

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- In fact, we can show that it must have a *unique* fixed point!

# Intro to Fixed Point Theory

- When a contraction is defined on a complete metric space, a contraction must map a point to itself.
- In other words, it *must* have a fixed point.
- In fact, we can show that it must have a *unique* fixed point!
- Formally, the **Banach Fixed Point Theorem** or **Contraction Mapping Theorem** states that if $(X, d)$ is a complete metric space and $f$ is a contraction defined on $X$, then there exists a unique $x^* \in X$ such that $f(x^*) = x^*$.

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- We can show that the sequence is Cauchy
- By construction,

$$d(x^{n+1}, x^{n+2}) \leq Kd(x^n, x^{n+1}) \leq K^2 d(x^{n-1}, x^n) \leq$$

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- That is, $d(x^{n+1}, x^n) \leq K^n d(x^1, x^0)$ for all $n = 1, 2, ...$

# Proof (cont.)

- Thus for any $k > l + 1$,

$$d(x^k, x^l) \leq d(x^k, x^{k-1}) + ... + d(x^{l+1}, x^l)$$

$$\leq (K^{k-1} + ... + K^l) d(x^1, x^0)$$

$$= \frac{K^l(1 - K^{k-l})}{1 - K} d(x^1, x^0)$$

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- Therefore $d(x^k, x^l) < \frac{K^l}{1-K} d(x^1, x^0)$

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- Therefore $d(x^k, x^l) < \frac{K^l}{1-K} d(x^1, x^0)$
- This implies that $(x^n)$ is a Cauchy sequence.

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$$d(f(x^*), x^*) \leq d(f(x^*), x^{n+1}) + d(x^{n+1}, x^*)$$

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&= d(f(x^*), f(x^n)) + d(x^{n+1}, x^*) \\
&\leq Kd(x^*, x^n) + d(x^{n+1}, x^*) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}
$$

- Since $\epsilon > 0$ is arbitrary, we must have $d(f(x^*), x^*) = 0$ which is possible only if $f(x^*) = x^*$.

# Proof (cont.)

- To prove uniqueness, note that if $x \in X$ was another fixed point, we would have

$$d(x, x^*) = d(f(x), f(x^*)) \leq Kd(x, x^*)$$

which is possible only if $x = x^*$.