Introduction to Mathematics for Political Science

Problem Set 9: Matrix Inversion and Determinants

Instructions: You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$3x_1 - x_2 = 10$$
$$-x_1 + 4x_2 = 4$$

Write this system in $Ax = b$ form and solve via matrix inversion.

$$\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

$$A^{-1} = \frac{1}{3(4) - (-1)(-1)} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

2. Let $C = AB$ where C is invertible and A and B are square matrices. Solve for A^{-1}.[1]

If C is invertible and A and B are square then A and B are also invertible. Therefore,

$$C^{-1} = B^{-1}A^{-1}$$
$$BC^{-1} = (BB^{-1})A^{-1}$$
$$BC^{-1} = A^{-1}$$

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[1] Strang p. 90 #12

3. Let $M = ABC$ where M is invertible and A, B, and C are square matrices. Solve for B^{-1}.[2]

If M is invertible and A, B, and C are square then A, B, and C are also invertible. Therefore,

$$M^{-1} = C^{-1}B^{-1}A^{-1}$$
$$CM^{-1}A = (CC^{-1})B^{-1}(A^{-1}A)$$
$$CM^{-1}A = B^{-1}$$

4. If B is the inverse of A^2, show that AB is the inverse of A.[3]

$$B = (A^2)^{-1}$$
$$B = (AA)^{-1}$$
$$(AA)B = AA(AA)^{-1}$$
$$(AA)B = I$$
$$A^{-1}(AA)B = A^{-1}$$
$$(A^{-1}A)AB = A^{-1}$$
$$AB = A^{-1}$$

5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

What are X^{-1} and Y^{-1}, assuming $ad \neq bc$.[4] **Hint:** First, multiply the two matrices XY. Then, attempt to solve for X^{-1} and Y^{-1}.

$$XY = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & ad - bc \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$XY = (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$(X^{-1}X)Y = (ad - bc)X^{-1}$$
$$Y = (ad - bc)X^{-1}$$
$$\frac{1}{ad - bc}Y = X^{-1}$$

$$XY = (ad - bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$X(YY^{-1}) = (ad - bc)Y^{-1}$$
$$\frac{1}{ad - bc}X = Y^{-1}$$

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[2] Strang p. 90 #13
[3] Strang p. 90 #18
[4] Strang p. 90 #16

Notice that $\det X = ad - bc$ so this obeys our usual rules for inversion of 2×2 matrices.

6. A is an *idempotent* matrix if and only if $AA = A$. Show that if A is symmetric ($A^\top = A$) and idempotent then $(I - A) = (I - A)(I - A)^\top$, where I is the identity matrix.

$$
\begin{aligned}
I - A &= I - AA^T && \text{by idempotency of } A \\
&= I - \underbrace{(A - A^T)}_{=0} - AA^T && \text{by symmetry of } A \\
&= (I - A)(I - A^T) \\
&= (I - A)(I - A)^T
\end{aligned}
$$

7. Let $R_{m \times n}$ be a rectangular matrix ($m \neq n$) and $A_{m \times m}$ be a symmetric matrix. Show $R^T A R$ is also symmetric. What are the dimensions of this matrix?[5]

$$
\begin{aligned}
(R^T A R)^T &= R^T A^T (R^T)^T \\
(R^T A R)^T &= R^T A^T R \\
(R^T A R)^T &= R^T A R && \text{by symmetry of } A
\end{aligned}
$$

$$
R^T_{n \times m} A_{m \times m} R_{m \times n} = M_{n \times n}
$$

8. Show every orthogonal matrix A has determinant 1 or -1. Hint: Apply the product rule ($|AB| = |A||B|$) and the transpose rule ($|A| = |A^T|$) for determinants.[6]

By the orthogonality of A, $A^T A = I$. By the product rule,

$$
\begin{aligned}
|A^T A| &= |I| \\
|A^T||A| &= 1 && \text{by the product rule and definition of the determinant} \\
(|A|)^2 &= 1 && \text{by the transpose rule}
\end{aligned}
$$

and $\sqrt{1} \in \{-1, 1\}$.

9. Let $\boldsymbol{x} = \{x_1, ..., x_{50}\}$ denote the number of electoral votes for each state. Let $\boldsymbol{y}_i = \{y_{i1}, ..., y_{ij}, ..., y_{i50}\}$ denote whether or not candidate $i \in \{R, D\}$ won the votes of a given state, where $y_{ij} \in \{0, 1\}$. Write an expression for the total number of votes for each candidate.

Candidate R:
$$
\boldsymbol{x}^T \boldsymbol{y}_R
$$

Candidate D:
$$
\boldsymbol{x}^T \boldsymbol{y}_D
$$

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[5]Strang 117 #19
[6]Strang p. 252 #8