Existence of Solutions and Fixed Points

Motivation

- In the last lecture we learned how to analyze how the solution to a parameterized optimization and its value change with a parameter.

Motivation

- In the last lecture we learned how to analyze how the solution to a parameterized optimization and its value change with a parameter.
- In doing so, we assumed the existence of a solution.

Motivation

- In the last lecture we learned how to analyze how the solution to a parameterized optimization and its value change with a parameter.
- In doing so, we assumed the existence of a solution.
- Now we drop this assumption and explore how we can guarantee that a solution to an optimization problem exists.

Weierstrass Theorem

- One of the most basic existence theorems we have is the **Weierstrass theorem**.

Weierstrass Theorem

Theorem

A continuous functional on a compact set achieves a maximum and a minimum.

Proof

- Let $M = \sup_{x \in X} f(x)$.

Proof

- Let $M = \sup_{x \in X} f(x)$.
- There exists a sequence $x^n \in X$ with $f(x^n) \to M$.

Proof

- Let $M = \sup_{x \in X} f(x)$.
- There exists a sequence $x^n \in X$ with $f(x^n) \to M$.
- Since the set X is compact, there exists a convergent subseqeuence $x^m \to x^*$ and $f(x^m) \to M$

Proof

- Let $M = \sup_{x \in X} f(x)$.
- There exists a sequence $x^n \in X$ with $f(x^n) \to M$.
- Since the set X is compact, there exists a convergent subseqeuence $x^m \to x^*$ and $f(x^m) \to M$
- Since f is continuous, $f(x^m) \to f(x^*)$.

Proof

- Let $M = \sup_{x \in X} f(x)$.
- There exists a sequence $x^n \in X$ with $f(x^n) \to M$.
- Since the set X is compact, there exists a convergent subseqeuence $x^m \to x^*$ and $f(x^m) \to M$
- Since f is continuous, $f(x^m) \to f(x^*)$.
- We conclude that $f(x^*) = M$.

Theorem of the Maximum

- In the last lecture we met the theorem of the maximum and focused on how the continuity of a problem ensures some degree of continuity in the the solution correspondence and value function.

Theorem of the Maximum

- In the last lecture we met the theorem of the maximum and focused on how the continuity of a problem ensures some degree of continuity in the the solution correspondence and value function.
- The theorem of the maximum will also provide conditions that imply a nonempty solution correspondence.

The Theorem of the Maximum

- Theorem

  Consider the constrained maximization problem

  $$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ *is continuous and the constraint correspondence* $G : \theta \rightrightarrows X$ *is continuous and compact-valued, then the value function* $V(\theta)$ *is continuous and the solution correspondence* $x^*(\theta)$ *is non-empty, compact-valued, and upper hemicontinuous.*

The Theorem of the Maximum

- Theorem
  Consider the constrained maximization problem

$$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ *is continuous and the constraint correspondence* $G : \theta \rightrightarrows X$ *is continuous and compact-valued, then the value function* $V(\theta)$ *is continuous and the solution correspondence* $x^*(\theta)$ *is non-empty, compact-valued, and upper hemicontinuous.*
  - Why is $x^*(\theta)$ non-empty when the conditions of the theorem of the maximum are satisfied?

The Theorem of the Maximum

- Theorem

  Consider the constrained maximization problem

  $$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ *is continuous and the constraint correspondence* $G : \theta \rightrightarrows X$ *is continuous and compact-valued, then the value function* $V(\theta)$ *is continuous and the solution correspondence* $x^*(\theta)$ *is non-empty, compact-valued, and upper hemicontinuous.*

  - Why is $x^*(\theta)$ non-empty when the conditions of the theorem of the maximum are satisfied?
  - A useful extension of the theorem of the maximum is the **convex maximum theorem**.

The Theorem of the Maximum

- Theorem

  Consider the constrained maximization problem

$$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ *is continuous and the constraint correspondence* $G : \theta \rightrightarrows X$ *is continuous and compact-valued, then the value function* $V(\theta)$ *is continuous and the solution correspondence* $x^*(\theta)$ *is non-empty, compact-valued, and upper hemicontinuous.*

  - Why is $x^*(\theta)$ non-empty when the conditions of the theorem of the maximum are satisfied?
  - A useful extension of the theorem of the maximum is the **convex maximum theorem**.
  - It states that if we add quasi-concavity of f and convexity of $G(\theta)$ to the conditions of the theorem of the maximum, then $x^*(\theta)$ will also be convex-valued.

Fixed Points

- In game theory, we exploit fixed point theorems to show the existence of an equilibrium.

Fixed Points

- In game theory, we exploit fixed point theorems to show the existence of an equilibrium.
- Recall that a **fixed point** of a self-mapping $f : X \to X$ is a point x^* such that $f(x^*) = x^*$.

Fixed Points

- In game theory, we exploit fixed point theorems to show the existence of an equilibrium.
- Recall that a **fixed point** of a self-mapping $f : X \to X$ is a point x^* such that $f(x^*) = x^*$.
- For a correspondence $G : X \to X$, a fixed point is a point x^* such that $x^* \in G(x^*)$.

Fixed Points

- The typical problem we have in game theory is as follows.

Fixed Points

- The typical problem we have in game theory is as follows.
- Let $x^*(x; y, \theta)$ solve

$$\max_x \; f_1(x; y, \theta)$$

and let $y^*(y; x, \theta)$ solve

$$\max_y \; f_2(y; x, \theta)$$

Fixed Points

- The typical problem we have in game theory is as follows.
- Let $x^*(x; y, \theta)$ solve

$$\max_x \; f_1(x; y, \theta)$$

  and let $y^*(y; x, \theta)$ solve

$$\max_y \; f_2(y; x, \theta)$$

- We want to find two points x^* and y^* such that $x^* \in x^*(x; y^*, \theta)$ and $y^* \in y^*(y; x^*, \theta)$.

Fixed Points

- The typical problem we have in game theory is as follows.
- Let $x^*(x; y, \theta)$ solve

$$\max_x \ f_1(x; y, \theta)$$

  and let $y^*(y; x, \theta)$ solve

$$\max_y \ f_2(y; x, \theta)$$

- We want to find two points x^* and y^* such that $x^* \in x^*(x; y^*, \theta)$ and $y^* \in y^*(y; x^*, \theta)$.
- If these points exist, we know that an equilibrium to the model exists.

Fixed Point Theorems

- Fixed point theorems will help us identify when an equilibrium exists when we cannot solve for one explicitly.

Fixed Point Theorems

- Fixed point theorems will help us identify when an equilibrium exists when we cannot solve for one explicitly.
- You have already encountered one fixed point theorem earlier in the lecture on metric spaces, the Contraction Mapping Theorem (also known as Banach's fixed point theorem).

Fixed Point Theorems

- Fixed point theorems will help us identify when an equilibrium exists when we cannot solve for one explicitly.
- You have already encountered one fixed point theorem earlier in the lecture on metric spaces, the Contraction Mapping Theorem (also known as Banach's fixed point theorem).
- One of the simplest fixed point theorems is **Brouwer's fixed point theorem**.

Brouwer's Fixed Point Theorem

- Theorem
  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every continuous function $f : X \to X$ has a fixed point.

Brouwer's Fixed Point Theorem

- Theorem
  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every continuous function $f : X \to X$ has a fixed point.
  - The theorem is difficult to prove in general but is very simple to show on $\mathbb{R}$.

Brouwer's Fixed Point Theorem

- Theorem
  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every continuous function $f : X \to X$ has a fixed point.
  - The theorem is difficult to prove in general but is very simple to show on $\mathbb{R}$.
  - Note that there may exist multiple fixed points.

Brouwer's Fixed Point Theorem

- Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given θ, $b : X \times Y \to X \times Y$.

Brouwer's Fixed Point Theorem

- Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given θ, $b : X \times Y \to X \times Y$.
- If $X \times Y$ is a nonempty, compact, convex subset of a finite dimensional normed linear space and if each of the solution correspondences is single-valued, continuous, and nonempty, then b has a fixed point, (x^*, y^*).

Brouwer's Fixed Point Theorem

- Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given θ, $b : X \times Y \to X \times Y$.
- If $X \times Y$ is a nonempty, compact, convex subset of a finite dimensional normed linear space and if each of the solution correspondences is single-valued, continuous, and nonempty, then b has a fixed point, (x^*, y^*).
- Brouwer's theorem is very useful but requires strict assumptions.

Brouwer's Fixed Point Theorem

- Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given θ, $b : X \times Y \to X \times Y$.
- If $X \times Y$ is a nonempty, compact, convex subset of a finite dimensional normed linear space and if each of the solution correspondences is single-valued, continuous, and nonempty, then b has a fixed point, (x^*, y^*).
- Brouwer's theorem is very useful but requires strict assumptions.
- For example, it requires b to be single-valued.

Brouwer's Fixed Point Theorem

- Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given θ, $b : X \times Y \rightarrow X \times Y$.
- If $X \times Y$ is a nonempty, compact, convex subset of a finite dimensional normed linear space and if each of the solution correspondences is single-valued, continuous, and nonempty, then b has a fixed point, (x^*, y^*).
- Brouwer's theorem is very useful but requires strict assumptions.
- For example, it requires b to be single-valued.
- Kakutani's theorem relaxes this assumption and generalizes Brouwer's theorem to correspondences.

Kakutani's Fixed Point Theorem

- Theorem

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

Kakutani's Fixed Point Theorem

- Theorem
  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.
  - Now if $x^*(x; y, \theta)$ and $y^*(y; x, \theta)$ are not singletons, we can use Kakutani's theorem to show that if b is closed and convex-valued, then b has a fixed point.

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of
  - N players

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of
  - N players
  - for each player i a nonempty set S_i of strategies

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of
  - N players
  - for each player i a nonempty set S_i of strategies
  - a utility function $u_i : S \to \mathbb{R}$ on the *strategy space* $S = S_1 \times S_2 \times \ldots \times S_n$

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of
  - N players
  - for each player i a nonempty set S_i of strategies
  - a utility function $u_i : S \to \mathbb{R}$ on the *strategy space* $S = S_1 \times S_2 \times \ldots \times S_n$
- If we assume that each S_i is nomempty, compact, and convex, then S is also nonempty, compact, and convex.

Strategic Games

- When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium.
- Formally, a **strategic game** is comprised of
  - N players
  - for each player i a nonempty set S_i of strategies
  - a utility function $u_i : S \to \mathbb{R}$ on the *strategy space* $S = S_1 \times S_2 \times ... \times S_n$
- If we assume that each S_i is nomempty, compact, and convex, then S is also nonempty, compact, and convex.
- Let $B_i(s) \equiv \arg\max_{s_i \in S_i} u_i(s)$

Strategic Games

- Theorem (The Theorem of the Convex Maximum)

  Consider the constrained maximization problem

$$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ is continuous and quasi-concave and the constraint correspondence $G : \theta \rightrightarrows X$ is continuous, compact-valued, and convex, then the value function $V(\theta)$ is continuous and the solution correspondence $x^*(\theta)$ is non-empty, compact-valued, upper hemicontinuous, and convex.

Strategic Games

- Theorem (The Theorem of the Convex Maximum)
  Consider the constrained maximization problem

$$\max_{x \in G(x)} f(x, \theta)$$

  If $f : X \times \Theta \to \mathbb{R}$ is continuous and quasi-concave and the constraint correspondence $G : \theta \rightrightarrows X$ is continuous, compact-valued, and convex, then the value function $V(\theta)$ is continuous and the solution correspondence $x^*(\theta)$ is non-empty, compact-valued, upper hemicontinuous, and convex.
  - If we assume that all u_i are continuous and quasi-concave, then each B_i is compact valued, convex valued, and upper hemicontinuous by the theorem of the convex maximum (recall that we assume S_i is nonempty, compact, and convex)

Strategic Games

- Theorem (Kakutani)

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

Strategic Games

- Theorem (Kakutani)

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

  - Now let $B(s) = B_1(s) \times ... \times B_n(s)$

Strategic Games

- Theorem (Kakutani)

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

  - Now let $B(s) = B_1(s) \times ... \times B_n(s)$
  - Then $B(s)$ is a closed, convex-valued correspondence.

Strategic Games

- Theorem (Kakutani)

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

  - Now let $B(s) = B_1(s) \times ... \times B_n(s)$
  - Then $B(s)$ is a closed, convex-valued correspondence.
  - Because S is nonempty, compact, and convex and B is closed and convex valued, by Kakutani's theorem, B has a fixed point.

Strategic Games

- Theorem (Kakutani)

  Let X be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.

  - Now let $B(s) = B_1(s) \times ... \times B_n(s)$
  - Then $B(s)$ is a closed, convex-valued correspondence.
  - Because S is nonempty, compact, and convex and B is closed and convex valued, by Kakutani's theorem, B has a fixed point.
  - We refer to this point as a **Nash equilibrium** of the game.

Discontinuous Functions

- Recall that Brouwer's fixed point theorem requires that f be continuous.

Discontinuous Functions

- Recall that Brouwer's fixed point theorem requires that f be continuous.
- As we saw in the previous lecture, continuity of an objective function may be too strict of an assumption.

Discontinuous Functions

- Recall that Brouwer's fixed point theorem requires that f be continuous.
- As we saw in the previous lecture, continuity of an objective function may be too strict of an assumption.
- Best response functions will also be prone to having some discontinuities.

Discontinuous Functions

- Recall that Brouwer's fixed point theorem requires that f be continuous.
- As we saw in the previous lecture, continuity of an objective function may be too strict of an assumption.
- Best response functions will also be prone to having some discontinuities.
- We would still like to know if a solution exists when this is the case.

Tarski's Fixed Point Theorem

- Theorem
  Every increasing function $f : X \to X$ *on a complete lattice* $(X, \succsim)$ *has a greatest and lowest fixed point.*

Tarski's Fixed Point Theorem

- Theorem
  Every increasing function $f : X \to X$ on a complete lattice $(X, \succsim)$ has a greatest and lowest fixed point.
  - A compact subset of $\mathbb{R}$ is a complete lattice.

Tarski's Fixed Point Theorem

- Theorem
  Every increasing function $f : X \to X$ on a complete lattice $(X, \succsim)$ has a greatest and lowest fixed point.
  - A compact subset of $\mathbb{R}$ is a complete lattice.
  - Therefore any increasing function on a compact subset of $\mathbb{R}$ has a fixed point.

Tarski's Fixed Point Theorem

- Theorem
  Every increasing function $f : X \to X$ on a complete lattice $(X, \succsim)$ has a greatest and lowest fixed point.
  - A compact subset of $\mathbb{R}$ is a complete lattice.
  - Therefore any increasing function on a compact subset of $\mathbb{R}$ has a fixed point.
  - This is not true of a decreasing function.