IMPS 2018: Final Exam (Practice)

11 September 2018

Instructions: This is a closed book examination. Calculators are not permitted. **There are 9 questions, from which you can choose 6 to answer.** Each question is worth ten points and should take about 30 minutes to complete. You have three hours.

1. Let S and T be sets. Show

$$(S \cup T)^c = S^c \cap T^c$$

2. Consider a sequence $\{x_n\} \in (\mathbb{R}, d_1)$ such that

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} + 1 & \text{if } n \text{ is even} \end{cases}$$

   Does this sequence converge? Prove your answer.

3. Let f be a continuous functional on a metric space (X, d). Prove αf is continuous for every $\alpha \in \mathbb{R}$.

4. Let $\boldsymbol{N} : \mathbb{R}^n \to \mathbb{R}$ be a norm. Use the properties of a norm prove that $\boldsymbol{N}$ is a convex function.

5. Consider the projection operator $\boldsymbol{p}_s : X \to S$ where S is a subspace of a inner product space X. Prove that if $\boldsymbol{x} - \boldsymbol{p}_S(\boldsymbol{x}) \perp S$ then

$$d(\boldsymbol{x}, \boldsymbol{p}_s(\boldsymbol{x})) = \min \{d(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} \in S\}$$

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable functional. Prove that if f is decreasing, then $Df[\boldsymbol{x}] \leq 0$ for all $\boldsymbol{x}$.

7. Solve

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = 1 - (x_1 - 1)^2 - (x_2 - 1)^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1 \end{aligned} \tag{1}$$

   Hint: Be sure to check the concavity of the objective function along the constraint set.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function. Consider the optimization problem

$$\max_x f(x;\theta)$$

The value function is given by $V(\theta) = f(x^\star(\theta);\theta)$. Prove that

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^\star(\theta);\theta)}{\partial \theta}$$

9. Prove for any square, invertible matrix A, and for all $n > 0$, $(A^n)^{-1} = (A^{-1})^n$.