An Introduction to Mathematics For Political Science

Problem Set 5

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. All solutions must be written in LaTeX.

Probability

1. A patient arrives at a doctors office with a sore throat and low-grade fever. After an exam, the doctor decides that the patient has either a bacterial infection or a viral infection or both. The doctor decides that there is a probability of 0.7 that the patient has a bacterial infection and a probability of 0.4 that the person has a viral infection. What is the probability that the patient has both infections?

Let B be the event that the patient has a bacterial infection and let V be the event that the patient has a viral infection. We are asked to find $Pr(B \cap V)$ given $Pr(B) = .7$ and $Pr(V) = .4$. We know that $Pr(B \cup V) = Pr(B) + Pr(V) - Pr(B \cap V)$. We therefore know that $1 = .7 + .4 - Pr(B \cap V)$ or $Pr(B \cap V) = .1$.

2. A box is filled with candies in different colors. We have 40 white candies, 24 green ones, 12 red ones, 24 yellow ones and 20 blue ones. If we have selected one candy from the box without peeking into it, find the probability of getting a green or red candy.

Let A be the event of getting a green candy and B the event of getting a red candy.

$Pr(A) = \frac{24}{120} = \frac{1}{5}$, $Pr(B) = \frac{12}{120} = \frac{1}{10}$. The two events are mutually exclusive so $P(A \cup B) = P(A) + P(B) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}$.

3. Suppose two dice are rolled and a coin is flipped. How many outcomes are possible for this experiment?

There are six outcomes for each die and two outcomes for the coin: $6 \cdot 6 \cdot 2 = 72$

4. A standard combination lock has a dial with tick marks for 40 numbers from 0 to 39. The combination consists of a sequence of three numbers that must be dialed in the correct order to open the lock. Each of the 40 numbers may appear in each of the three positions of the combination regardless of what the other two positions contain. How many possible combinations exist? What is the minimum number of tick marks necessary to generate more than 2,000 combinations? Can you think of a better name for a "combination" lock?

$40^3 = 64,000$, 13 ticks, permutation lock.

5. Suppose that a club consists of 25 members and that a president and a secretary are to be chosen from the membership. What is the total possible number of ways in which these two positions can be filled?

$P_{25,2} = 25 \cdot 24 = 600$

6. Suppose that a committee composed of eight people is to be selected from a group of 20 people. How many different groups of people may be on the committee?

$C_{20,8} = \frac{20!}{8!12!} = 125,970$

7. Now suppose that the eight people in the committee each are given a unique job to perform on the committee. How many ways are there to choose the eight members and assign them to the eight different jobs?

$P_{20,8} = C_{20,8} \cdot 8! = 5,078,110,400$

8. From a deck of 52 we draw 2 cards one by one without replacement. Find the probability that both cards are aces.

Let A be the event of drawing the first ace. $Pr(A) = \frac{4}{52}$. Let B be the event of drawing an ace given that the first card drawn was an ace. $Pr(b) = \frac{3}{51}$. $Pr(A \cap B) = Pr(A)Pr(B) =$

$\frac{4}{52}\frac{3}{51} = \frac{1}{221}$.

9. You toss a fair coin three times. What is the probability of three heads? What is the probability that you observe exactly one heads? Given that you have observed at least one heads, what is the probability that you observe at least two heads?

$Pr(HHH) = Pr(H)Pr(H)Pr(H) = .5^3 = \frac{1}{8}$

$Pr(\text{One heads}) = Pr(HTT \cup THT \cup TTH) = Pr(HTT) + Pr(THT) + Pr(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Let A_1 be the event that you observe at least one heads and let A_2 be the event that you observe at least two heads. Then $A_1 = S - (TTT)$ so $Pr(A_1) = \frac{7}{8}$. $A_2 = \{HHT, HTH, THH, HHH\}$ so $Pr(A_2) = \frac{4}{8}$. We can now calculate $Pr(A_2|A_1) = \frac{Pr(A_2 \cap A_1)}{Pr(A_1)} = \frac{Pr(A_2)}{Pr(A_1)} = \frac{4}{7}$

10. An urn B_1 contains 2 white and 3 black balls and another urn B_2 contains 3 white and 4 black balls. One urn is selected at random and a ball drawn from it. If the ball drawn is black, find the probability that the urn chosen was B_1.

Let E_i denote the event that urn B_i is selected. $Pr(E_i) = \frac{1}{2}$. Let b denote the event that a black ball is selected. $Pr(b|E_1) = \frac{3}{5}$, $Pr(b|E_2) = \frac{4}{7}$. By Bayes' rule,

$$Pr(E_1|b) = \frac{Pr(b|E_1)Pr(E_1)}{Pr(b|E_1)Pr(E_1) + Pr(b|E_2)Pr(E_2)}$$

11. A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5 percent of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the following probabilities: (a) that the test result will be positive; (b) that, given a positive result, the person is a sufferer; (c) that, given a negative result, the person is a non-sufferer; (d) that the person will be misclassified.

Let T denote a positive test, S denote sufferer, M denote misclassified. $Pr(T|S).95$, $Pr(T| \sim S) = .10$, $Pr(S) = .005$.

a) $Pr(T) = Pr(T|S)Pr(S) + Pr(T| \sim S)Pr(\sim S) = (.95 \cdot .005) + (0.1 \cdot .995) = .10425$

b) $Pr(S|T) = \frac{Pr(T|S)Pr(S)}{Pr(T|S)Pr(S)Pr(T|\sim S)Pr(\sim S)} = .0455.$

c) $Pr(\sim S| \sim T) = \frac{Pr(\sim T|\sim S)Pr(\sim S)}{Pr(\sim T)} = .9997$

d) $Pr(M) = Pr(T \cap \sim S) + Pr(\sim T \cap S) = Pr(T| \sim S)Pr(\sim S) + Pr(\sim T|S)Pr(S) = 0.09975$