An Introduction to Mathematics For Political Science

Problem Set 6

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. All solutions must be written in LaTeX.

Random Variables and Distributions

1. A popular state lottery game requires participants to select a three-digit number (leading 0s allowed). Then three balls, each with one digit, are chosen at random from well-mixed bowls. The sample space here consists of all triples (i_1, i_2, i_3) where $i_j \in \{0, ..., 9\}$ for $j = 1, 2, 3$. If $s = (i_1, i_2, i_3)$, define $X(s) = 100i_1 + 10i_2 + i_3$. For example, $X(0, 1, 5) = 15$. Find $Pr(X = x)$ for each integer $x \in \{0, 1, ..., 999\}$.

$Pr(X = x) = 0.001$

2. Now consider a lottery in which participants select a real number, x, between 0 and 1. A real number, X, is then randomly selected from the interval with equal probability, i.e. X is a draw from the uniform distribution on $[0, 1]$. Find $Pr(X = x)$ for some real number x.

$Pr(X = x) = 0.$

3. Suppose that the p.d.f. of a certain random variable X has the following form:

$$f(x) = \begin{cases} cx & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find c.

For every p.d.f., it must be true that $\int_{-\infty}^{\infty} f(x) = 1$. Therefore $\int_0^4 cx\, dx = 8c = 1$ so $c = 1/8$

4. Suppose that the p.d.f. of a certain random variable X has the following form:

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $Pr(1 \leq X \leq 2)$ and $Pr(X > 2)$.

$Pr(1 \leq X \leq 2) = \int_1^2 \frac{1}{8}x\, dx = \frac{3}{16}$ and $Pr(X > 2) = \int_2^4 \frac{1}{8}\, dx = \frac{3}{4}$

5. The c.d.f. of a random variable X is given by $F(x) = 1 - e^{-\lambda x}$ on $\mathbb{R}_+$. Find $Pr(x \geq \lambda)$ and $Pr(x \leq \lambda^2)$.

$Pr(x \geq \lambda) = 1 - (1 - e^{-\lambda^2}) = e^{-\lambda^2})$. $Pr(x \leq \lambda^2) = 1 - e^{-\lambda^3}$.

6. Find the p.d.f. of the random variable X from the previous problem.

$f(x) = F'(x) = \lambda e^{-\lambda x}$

7. Let the c.d.f. of a random variable X be given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^{\frac{2}{3}} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Find the p.d.f. of X.

$$f(x) = F'(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{2}{3}x^{-\frac{1}{3}} & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

8. Suppose that a point (X, Y) is selected at random from inside the circle, S, defined by $x^2 + y^2 \le 9$. Find the joint p.d.f. of (X, Y).

Hint: the pdf will have the form

$$f(x, y) = \begin{cases} c & \text{for } (x, y) \in S \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

We must have $\int_S \int f(x, y)\, dx\, dy \; = c \cdot (\text{area of S}) = 1$. The area of S is 9π. The value of the constant must therefore be $\frac{1}{9\pi}$.

9. An investor is considering whether or not to invest 18 dollars in a stock for one year. The value of the stock after one year in dollars will be $18 + X$ where X is a random variable. Suppose X is distributed such that $Pr(X = -2) = .1$, $Pr(X = 0) = .4$, $Pr(X = 1) = .3$, and $Pr(X = 4) = .2$. Alternatively, the investor can place her 18 dollars in the bank at 4 percent interest. Which use of her money is optimal if the investor is risk neutral?

The expected payoff of her investment is $-2(.1) + 0(.4) + 1(.3) + 4(.2) = .9$. Her payoff from investing the money in the bank is $18 \cdot .004 = .72$. The investment is the better option than the bank.

10. Let X be a random variable with p.d.f. $f(x) = 2x$ with support only on $(0, 1)$. Find $E[X]$.

$E[X] = \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}$

11. A product has a warranty of one year. Let X be the time at which the product fails. Suppose that X has a continuous distribution with the p.d.f.

$$f(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{2}{x^3} & \text{for } x \geq 1 \end{cases}$$

Find the expected time to failure.

$E[X] = \int_1^\infty x \frac{2}{x^3}\, dx = \int_1^\infty \frac{2}{x^2}\, dx = 2$

12. A random variable Y is a linear function of random variables X_1 and X_2 and ϵ. In particular, let $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$. Let X_1 be distributed according to the p.d.f from question 10 and let X_2 be a Bernoulli random variable with parameter $p = \frac{1}{4}$. Let $\epsilon \sim N(0,1)$. The terms β_i are real-valued scalars. Find $E[Y]$.

$E[Y] = E[\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon] = E[\beta_0] + E[\beta_1 X_1] + E[\beta_2 X_2] + E[\epsilon] = E[\beta_0] + \beta_1 E[X_1] + \beta_2 E[X_2] + E[\epsilon] = \beta_0 + \beta_1 \frac{2}{3} + \beta_2 \frac{1}{4}$.

13. Suppose a politician's utility function is described by $U(x) = -(z - x)^2$ where z is her preferred policy $z \in \mathbb{R}$ and x is a random policy shock with p.d.f. $f(x)$. Let $f(x)$ be the uniform distribution on $[0, 1]$. Find her expected utility.

$E[U(x)] = \int_0^1 U(x) f(x)\, dx = \int_0^1 U(x)\, dx = \int_0^1 -(z - x)^2\, dx = -z^2 + z - \frac{1}{3}$

14. Find the variance for the following series of numbers:

a) 12, 6, 7, 3, 15, 10, 18, 5

b) 2, 3, 6, 8, 11

c) 3, 5, 2, 7, 6, 4, 9, 1

a) 23.75

b) 10.8

c) 6.234

15. X is a random variable with equal probability of taking any one of five values, $-2, 0, 1, 3,$ and 4. Compute the variance of X.

$E[X^2] = \frac{1}{5}[(-2)^2 + 0^2 + 1^2 + 3^2 + 4^2] = 6$. $E[X] = 1.2$. $Var(X) = 6 - (1.2)^2$.