Monotone, Linear, and Convex Functions

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- Linear functions preserve the *algebraic structure* of the spaces they link.
- These different types of functions have different properties we can exploit when building theories and analyzing data.

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- A function f is monotone if it is either increasing or decreasing.

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- The log of a Cobb-Douglass function is $\ln f(x) = a \ln(x) + b \ln(y)$

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- $\ln(x)$ is not a linear mapping: $\ln(1) = 0$, $\ln(2) \approx .7$, $\ln(3) \approx 1.1$

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- f is an example of an **affine function** which relate to linear functions in the same way as subspaces relate to affine sets
- Affine functions preserve affine sets (lines and planes).

Matrices as Linear Functions

- Any $m \times n$ matrix $A = (a_{ij})$ defines a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by

$$f(x) = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \dots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

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- A linear function (such as a matrix) that has an inverse $f^{-1} : Y \to X$ is said to be **nonsingular**. The inverse of a nonsingular linear function is also linear.

Bilinear Functions

- A function $f : X \times Y \to Z$ between linear spaces X, Y, and Z is **bilinear** if it is linear in each factor separately:

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$$

$$f(\alpha x, y) = \alpha f(x, y) = f(x, \alpha y)$$

for all $\alpha \in \mathbb{R}$

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- Any $m \times n$ matrix $A = (a_{ij})$ defines a bilinear functional on $\mathbb{R}^m \times \mathbb{R}^n$ by
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$$f(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_i$$
- In the quantitative sequence will often have an $n \times n$ weighting matrix W and use the operation $x^T W x$

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- We spend most of our time in Euclidean space $\mathbb{R}^n$ with the inner product $x^T y = \sum_{i=1}^{n} x_i y_i$.

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- det itself is not a linear functional

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- Quadratic forms are among the simplest nonlinear functionals we encounter.

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$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$
defines a symmetric linear operator on $\mathbb{R}^2$:
$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$
$$[x, y] \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix} = x^2 + 4xy + y^2 = Q(x, y)$$

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- A positive (negative) definite matrix has a positive (negative) diagonal.

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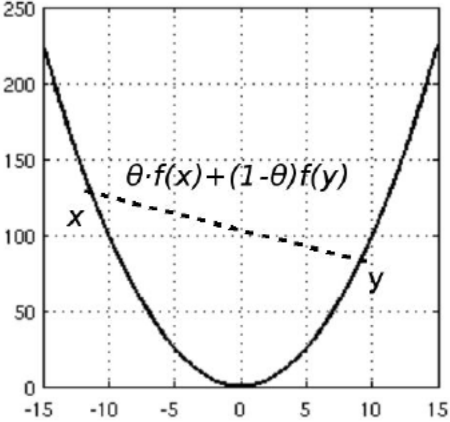
- A function is **strictly convex** if the inequality is strict and **concave** if the inequality is reversed.

Example



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- Any monotone function is both quasiconcave and quasiconvex.

Example