An Introduction to Mathematics for Political Science

Problem Set 1

You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions or scans of handwritten solutions as a PDF. Note that starting with problem set 3 all solutions must be written in LaTeX.

Set Basics

1. Let $A = (40, 60)$ and $B = [50, 70]$.

a) Is $A \subset B$, $B \subset A$, both, or neither?

b) What is $A \cup B$?

c) What is $A \cap B$?

d) Write three elements of the Cartesian product $A \times B$.

Solution:

a) Neither

b) $(40, 70]$

c) $[50, 60)$

d) $(50, 50)$, $(55, 60)$, $(59, 70)$

2. Identify whether the following sets are (a) open, closed, or neither; (b) bounded; (c) compact; (d) convex:

a) $(0, 1)$

b) $[0, 1]$

c) $(0, 1]$

d) $[0, \infty)$

e) $(0, \infty)$

f) $[0, 1] \cup [2, 3]$

g) $[0, 1] \times [0, 1]$

h) $(0, 3] \cap [1, 4]$

i) $[0, 5] \backslash \{1, 2\}$

Solution:

a) open, bounded, not compact, convex

b) closed, bounded, compact, convex

c) neither, bounded, not compact, convex

d) closed, unbounded (above), not compact, convex

e) open, unbounded (above), not compact, convex

f) closed, bounded, compact, not convex

g) closed, bounded, compact, convex

h) closed, bounded, compact, convex

i) neither, bounded, not compact, not convex

3. Express the following sentences in mathematical notation:

   a) A is the set of all real numbers less than or equal to seven, excluding zero and four.

   b) B is the intersection of the natural numbers and the real numbers between π and 30.5.

   c) For all epsilon greater than zero, there exists a delta greater than zero.

   d) The set of all even integers between 5 and 21.

   Solution:

   a) $A = \{x \in \mathbb{R}/\{0, 4\} | x \leq 7\}$

   b) $B = \mathbb{N} \cap (\pi, 30.5) = \{x \in \mathbb{N} | 4 \leq x \leq 30\}$

c) $\forall\ \epsilon > 0, \exists\ \delta > 0$

d) $\{x \in \mathbb{Z} | 5 \leq x \leq 21, x \text{ even}\}$

Algebra

4. Simplify into one term or evaluate the following:

a) $y \cdot y \cdot y \cdot y$

b) $(-a)(-b)^3 - b^2 + a^3$

c) $(4b + 2)(a - 5)$

d) $\frac{5!}{2!}$

e) $\sum_{i=1}^{3} (\frac{1}{3})^i$

f) $\sum_{i=2}^{5} 2^i$

g) $\prod_{i=1}^{3} (\frac{1}{3})^i$

h) $\frac{48}{4} - 6 \cdot 9$

i) $(3^3 + (-5)) \cdot 3 - (-7)$

j) $[6 + (\frac{-66}{11})] \cdot (-2)^3$

k) $\frac{y-11}{5} + \frac{y+12}{3}$

Solution:

a) y^4

b) $ab - b^2 + a^3$

c) $4ab + 2a - 20b - 10$

d) 60

e) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{9+3+1}{27} = \frac{13}{27}$

f) $4 + 8 + 16 + 32 = 60$

g) $\frac{1}{3 \cdot 9 \cdot 27} = \frac{1}{729}$

h) -42

i) 73

j) 0

k) $\frac{8y+27}{15}$

5. Solve the following for x:

a) $5(-3x-2)-(2x-3)=-4(4x+5)+13$

b) $8x^2 = 15 - 14x$

c) $x^2 - 3x + 4 = 2(x-1)$

d) $-6 > 5x + 5 + 4$

e) $-2(x+1) + 4 < 10$

f) $2\ln(2x+1) - 10 = 0$

g) $3^x e^{4x} = e^7$

Solutions

a) $x = 0$

b) $x \in \{-2.5, .75\}$

c) $x \in \{3, 2\}$

d) $x < -3$

e) $x > 4$

f) $x = \frac{\exp(5)-1}{2}$

g) $x = \frac{7}{\ln(3)+4}$

Functions

6) Find the image/range of x^2 on the domain $[-3, 3]$.

Solution: $[0, 9]$

7) Let $f(x) = x^2 - 4x + 2$ and $g(x) = 3x - 7$. Find (a) $f(x) - g(x)$; (b) $f(x) + g(x)$; (c) $f(g(x))$; and (d) $g(f(x))$. Evaluate each for $x = 2$.

Solutions:

a) $x^2 - 7x + 9,\ f(2) - g(2) = -1$

b) $x^2 - x - 5$, $f(2) + g(2) = -3$

c) $9x^2 - 54x + 77$, $f(g(2)) = 5$

d) $3x^2 - 12x - 1$, $g(f(2)) = -13$

8) Identify whether each of the following functions $f : \mathbb{R} \to \mathbb{R}$ is (a) surjective/onto; (b) injective/one-to-one; (c) bijective.

a) $f(x) = x^2$

b) $f(x) = x^3 - x$

c) $f(x) = e^x$

d) $f(x) = x^3$

Solutions:

a) Neither; negative reals are not mapped into and positives are mapped into twice

b) Onto, not one-to-one; all reals are mapped into, reals near $f(0)$ mapped into twice

c) Not onto, one-to-one; negative reals not mapped into, each positive real mapped into only once

d) Both.

9) Find an equation for the inverse for each of the following functions:

a) $f(x) = (5x - 1)^3$

b) $f(x) = \frac{x+4}{3x-5}$

c) $f(x) = e^{5x-1}$

Solutions:

a) $f^{-1}(x) = \frac{1}{5}(x^{\frac{1}{3}} + 1)$

b) $f^{-1}(x) = \frac{5x+4}{3x-1}$ [Multiply by $3x - 5$, distribute, add $5y$, subtract x, factor x, divide by $3y - 1$, swap]

c) $f^{-1}(x) = \frac{\ln(x)+1}{5}$

10) Evaluate each of the following limits or show that they do not exist:

a) $\lim_{x \to 5} \frac{x^2-25}{x^2+x-30}$

b) $\lim_{x \to -1} \frac{x^3}{(x+1)^2}$

c) $\lim_{x\to 2} \frac{x^2+4x-12}{|x-2|}$

d) $\lim_{x\to\infty} \frac{x^2-1}{2x^2+1}$

e) $\lim_{x\to\infty}(\frac{x^3}{x^2+2} - x)$

f) $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$

Solutions:

a) $\frac{x^2-25}{x^2+x-30} = \frac{(x-5)^2}{(x-5)(x+6)} = \frac{x+5}{x+6}$. $\lim_{x\to 5} \frac{x+5}{x+6} = 10/11$

b) Limit does not exist; function approaches $-\infty$ as x approaches -1 from each side.

c) To evaluate the RHS limit, find $\lim_{x\to 2^+} \frac{x^2+4x-12}{x-2} = \lim_{x\to 2^+} \frac{(x-2)(x+6)}{(x-2)} = \lim_{x\to 2^+} x + 6 = 8$. To evaluate the LHS limit, find $\lim_{x\to 2^-} \frac{x^2+4x-12}{-(x-2)} = \lim_{x\to 2^-} \frac{(x-2)(x+6)}{-(x-2)} = \lim_{x\to 2^-} -(x+6) = -8$. Because the left and right limits are not equivalent, the limit does not exist.

d) $\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{x^2}{2x^2} = 1/2$

e) $(\frac{x^3}{x^2+2} - x) = \frac{-2x}{x^2+2}$. $\lim_{x\to\infty} \frac{-2x}{x^2+2} = -2\lim_{x\to\infty} \frac{x}{x^2+2} = -2\lim_{x\to\infty} \frac{1/x}{1+2/x^2} = -2\frac{\lim_{x\to\infty} 1/x}{\lim_{x\to\infty}(1+2/x^2)} = -2 \cdot \frac{0}{1} = 0$.

11) Determine whether the following functions are continuous at the specified value. Use limits where appropriate. You may find sketching the graph of each function to be helpful.

a) $f(x) = \frac{x^2+1}{x^3+1}$ at $x = -1$

b) $f(x) = \begin{cases} 3x - 5 \text{ if } x \neq 1 \\ 2 \text{ if } x = 1 \end{cases}$ at $x = 1$

c) $f(x) = \begin{cases} \frac{x-6}{x-3} \text{ if } x < 0 \\ 2 \text{ if } x = 0 \\ \sqrt{4 + x^2} \;\; x > 0 \end{cases}$ at $x = 0$

Solutions:

a) Function is undefined at $x = -1$ and is therefore discontinuous at -1.

b) The function is defined at 1 since $f(1) = 2$. To determine whether it is continuous, consider $\lim_{x\to 1} 3x-5 = 3-5 = -2$. Since $\lim_{x\to 1} f(x) \neq f(1)$, the function is not continuous at $x = 1$.

c) The function is clearly defined at $x = 0$. Now check limits on either side of 0. $\lim_{x\to 0^-} \frac{x-6}{x-3} = -6/-3 = 2$. $\lim_{x\to 0^+} \sqrt{4+x^2} = \sqrt{4} = 2$. Conclude that the function is continuous at $x = 0$.