IMPS 2019: Final Exam

Instructions: This is a closed book examination. Calculators are not permitted. **There are 8 questions, from which you can choose 6 to answer.** Each question is worth ten points and should take about 30 minutes to complete. You have three hours.

1. Let S and T be sets. Show
$$(S \cap T)^c = S^c \cup T^c$$
2. Consider a sequence $\{x_n\} \in \left(\mathbb{R}^2, d_2\right)$ with
$$x_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$$
for all n. Does this sequence converge? Prove your answer.
3. Let f, g be continuous functionals on a metric space (X, d). Prove $f + g$ is continuous.
4. Use the definition of a strictly convex function to show that if $f(x) = x^2$ then $f(x)$ is strictly convex. Proofs employing derivatives will not be accepted.
5. Let X be an innner product space. Show that if
$$\boldsymbol{x}_1^T \boldsymbol{x}_2 \leq \|\boldsymbol{x}_1\| \|\boldsymbol{x}_2\|$$
for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$, then
$$\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$$
6. Use the definition of a differentiable function to show that if $f, g : X \to Y$ are differentiable at $\boldsymbol{x}_0$ then $f + g$ is also differentiable with derivative
$$D(f+g)[\boldsymbol{x}_0] = Df[\boldsymbol{x}_0] + Dg[\boldsymbol{x}_0]$$
7. Consider the n equation system
$$y_1 = \beta x_1 + \epsilon_1$$
$$\cdots$$
$$y_n = \beta x_n + \epsilon_n$$

where β is a scalar. Show that the $\hat{\beta}$ that minimizes the sum of squared errors ($\sum_{i=1}^{n} \epsilon_i^2$) is given by

$$\hat{\beta} = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$$

8. Consider the optimization problem

$$\max_x f(x;\theta) = (1-x)x + \theta(1-x)$$

Show that the function is strictly concave in x and write the value function $V(\theta) = f(x^\star(\theta);\theta)$. How does $x^\star(\theta)$ change with θ?