Introduction to Mathematics for Political Science: Continuous Functions

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Functions Review

You encountered functions at the very beginning of this course. Now that we've spent time studying sets and spaces, we can understand functions as simply *relationships between sets*. Take an element from a set X (the function's *domain*) and a function f that relates elements in X to elements in Y (the function's *co-domain*). We write

$$f : X \to Y$$

and $f(x)$ gives us some $y \in Y$ for every $x \in X$.

Definition: A function is a rule that relates every $x \in X$ to a single $y \in Y$.[1]

[1] In POL 502 you will encounter *correspondences*, which can map elements in X to *multiple* elements in Y.

Most of the functions you encountered in your summer self-study were mappings from the real numbers to other real numbers

$$f : \mathbb{R} \to \mathbb{R}$$

but functions can be far more general.[2] To give another familiar example, the dot product turns n-dimensional vectors of real numbers into real scalars, so

$$f : \mathbb{R}^n \to \mathbb{R}$$

[2] What should the function $f : \{\text{frog}, \text{tiger}, \text{flamingo}\} \to \{\text{orange}, \text{pink}, \text{green}\}$ look like?

Definition: A *functional* f is a real-valued function

$$f : X \to \mathbb{R}$$

Definition: The *inverse image* of a function of a set $T \subset Y$ is the set of $x \in X$ that are mapped to a elements $y \in T$

$$f^{-1}(T) = \{x \in X | f(x) \in T\}$$

Definition: A function f has an *inverse function* $f^{-1} : Y \to X$ iff ($\Longrightarrow$) f is *one-to-one* and *onto*.[3]

[3] What do each of these statements mean about f?

Definition: A *composition* of functions $f : X \to Y$ and $g : Y \to Z$ is denoted $g \circ f$ or $g(f(x))$ for $x \in X$ and

$$g \circ f : X \to Z$$

Continuity

Consider an *election function* $g : [0,1] \rightarrow \{0,1\}$. The function takes the vote share of a given candidate $v_i \in [0,1]$ and returns $g(v_i) = 1$ if candidate i wins the election and 0 otherwise.[4] This function is *discontinuous*. Small changes in vote share produce large changes in outcomes.[5]

Now consider a *contest function* $h : \{\mathbb{R}_+ \times \mathbb{R}_+\} \rightarrow [0,1]$ which determines the probability that country i wins a war with country j given ω_i and ω_j, the military capabilities of each country.[6] Conflict researchers often assume the contest function takes the following form

$$h(\omega_i, \omega_j) = \frac{\omega_i}{\omega_i + \omega_j}$$

This function is *continuous* in ω_i – small changes in military capability produce small changes in the likelihood of victory. This conforms to our intuition – adding one soldier to a large state's military is unlikely to fundamentally change its war prospects.

Continuity clearly plays a (sometimes implicit) role in political science. Today, we'll formalize our intiution about how continuity works and relate continuous functions to our study of metric spaces. Continuity will play an important role when we study optimization later this week.

We intuitively defined a function to be continuous if small changes in input produce small changes in outputs. What is "small" anyway? In order to know what's small, we need the concept of distance, so we'll study functions whose domain and co-domain are metric spaces.[7] We'll start with the definition that corresponds most closely to our intuition, then study less intuitive consequences of continuity.

Definition 1: Take two metric spaces, (X, d_X) and (Y, d_Y). A function

$$f : X \rightarrow Y$$

is *continuous* at $x \in X$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all other elements $y \in X$

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

That was a mouthful, but notice that the definition is simply formalizing our intiutive definition discussed above. If two points $x, y \in X$ are less than δ apart, there must be an ϵ that bounds the distance between $f(x)$ and $f(y)$.[8] We can think of this as a game. I give you an x and an ϵ and you give me $\delta(x, \epsilon)$ such that

[4] Assume a candidate wins the election if $v_i \geq .5$. Plot this function.

[5] This democratic discontinuity turns out to be incredibly useful in studying all sorts of political questions. See de la Cuesta, Brandon and Kosuke Imai. 2016. "Misunderstandings About the Regression Discontinuity Design in the Study of Close Elections." *Annual Review of Political Science* 19: 375-396.

[6] See Skaperdas, Stergios. 1996. "Contest Success Functions." *Economic Theory* 7: 283-290.

[7] Recall a metric space is a set and an associated metric, (X, d_X).

[8] Think about the election function in this context. Take ϵ smaller than 1 and $x = .5$. Is it possible to find a δ that ensures $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$ for all y?

$d_X(x,y) < \delta(x,\epsilon) \implies d_Y(f(x), f(y)) < \epsilon$. Let's flesh this out a bit with an example.

Example: We will show the function $f(x) = 1/x$ is continuous on $(\mathbb{R}_{++}, d_1)$. We need to find a $\delta(x,\epsilon) > 0$ such that

$$|x - y| < \delta(x,\epsilon) \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon$$

Notice that for $x, y > 0$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy}$$

Let $x > y$ (without loss of generality) and

$$\delta(\epsilon, x) = \frac{\epsilon x^2}{1+x} \implies \epsilon = \frac{\delta}{x(x-\delta)}$$

Then

$$|x - y| < \delta(\epsilon, x) = \frac{\epsilon x^2}{1+x}$$

implies

$$\frac{|x-y|}{xy} < \frac{\delta}{xy} = \frac{\delta}{x(x-\delta)} = \epsilon$$

Notice that $|x - y| < \delta$ and $\delta \in (0, x) \implies y = x - \delta$

That proof was presented as a bit of divine inspiration. How we came up with $\delta(\epsilon, x) = \frac{\epsilon x^2}{1+x}$ is discussed in more detail in the sidebar.[9]

Sometimes it's easier to think about continuity in terms of open ϵ and δ-balls.[10] A function is continuous at x if every *neighborhood* in the co-domain, we can find a corresponding neighborhood in the domain. When f maps neighbors in the domain into the co-domain, they remain neighbors.

Definition 2: A function f is continuous at x if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(B_\delta(x)) \subset B_\epsilon(f(x))$$

The definition requires that we can find an open neighborhood in the preimage whose image is contained within an ϵ-neighborhood around $f(x)$.[11]

In $\mathbb{R}$, we say a function is continuous at a if points near a approach it from both sides:

$$\lim_{x \to a} f(x) = f(a)$$

[9] We start by setting $|x - y| = \delta$ and noticing $y = x - \delta$ for $x > y$. Now we can substitute δ and y into

$$d_Y(x,y) = \frac{|x-y|}{xy}$$

giving

$$\epsilon = \frac{\delta}{x(x-\delta)}$$

Solving for $\delta(\epsilon, x)$ gives

$$\delta(\epsilon, x) = \frac{\epsilon x^2}{1+x}$$

This is our choice of δ. Now we make sure that everything goes through as desired with the strong inequality, as shown in the body of the notes. Focus on getting the concept of continuity straight in your head for now. The algorithm will come naturally with practice.

[10] Recall an *open ball* about x_0 with radius r is the set of points

$$B_r(x) = \{x \in X | d(x, x_0) < r\}$$

[11] What is $f(B_\delta(x))$ of the election function at .5? What is $B_\epsilon(f(x))$?

We can generalize this definition to arbitrary metric spaces as follows.

Definition 3: A function f is continuous if for any element $x \in X$ and sequence x^m,

$$x^m \to x \implies f(x^m) \to f(x)$$

This definition says that for a function to be continuous, the image of any convergent sequence in X must converge to $f(x)$.

Finally, we call a function continuous if it preserves the topological properties of its preimage.

Definition 4: A function $f : X \to Y$ is continuous iff ($\iff$) for any open subset $T \in Y$, $f^{-1}(T)$ is an open subset of X.

Under this definition of continuity, functions are continuous if they map all open sets of the preimage into open sets in the image. This definition might be hard to get your head around. But it turns out we can recover our more intuitive limit definition of continuity by starting with this topological premise.[12]

[12] We will show $\impliedby$ here, $\implies$ is left as an exercise.

Proposition: If for every open subset of a metric space Y $S \subset Y$, $f^{-1}(S)$ is open in a metric space X, then for any element $x \in X$ and sequence x^m,

$$x^m \to x \implies f(x^m) \to f(x)$$

Proof: Take some sequence $x^m \to x$ with $x \in X$ and some $\epsilon > 0$. $B_\epsilon(f(x))$ is open, so $f^{-1}\left(B_\epsilon(f(x))\right)$ is also open by our premise and $x \in f^{-1}\left(B_\epsilon(f(x))\right)$. For $\lim x^m \to x$, we must have $x^m \in f^{-1}\left(B_\epsilon(f(x))\right)$ for some $m \geq M$.[13]

[13] Recall the definition of convergence from the lectures on metric spaces.

$$x^m \in f^{-1}\left(B_\epsilon(f(x))\right) \implies f(x^m) \in B_\epsilon(f(x))$$

which implies $f(x^m)$ converges

$$f(x^m) \to f(x)$$

as desired.

Applications

Often in applied formal theory, some decision maker will be trying to make a choice that leaves her best off. We'll study these optimization problems in greater detail in the coming days. What is best? And how do we know whether or not it is achiveable? Continuity can sometimes help us out here.

Theorem (Weierstrass): A continuous functional $f : X \to Y$ on a compact set achieves a maximum and a minimum.

We need to show that the supremum of the function's image is contained within the image. Let $M = \sup_{x \in X} f(x)$. Take a sequence $f(x^n) \in Y$ with $f(x^n) \to M$. Then $x^n \in X$. With X compact, we can find a convergent subsequence $x^{n_k} \to x^\star$ where $f(x^{n_k}) \to M$. Applying the limit definition of continuity, we have $f(x^{n_k}) \to f(x^\star) = M$. So $f(x^\star) = M$. The same proof can be applied to the infimum of the set, demonstrating the Theorem.

References

1. Moore, Will H. and David A. Siegel. *A Mathematics Course for Political and Social Research*. Chapter 3.
2. Ok, Efe A. *Real Analysis with Economic Applications*. Chapter D.
3. Carter, Michael. *Foundations of Mathematical Economics*. Chapter 2.1, 2.3.