Introduction to Mathematics for Political Science

Problem Set 9: Matrix Inversion and Determinants

Instructions: You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$\begin{aligned} 3x_1 - x_2 &= 10 \\ -x_1 + 4x_2 &= 4 \end{aligned}$$

   Write this system in $A\boldsymbol{x} = \boldsymbol{b}$ form and solve via matrix inversion.

$$\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ A^{-1}A\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \boldsymbol{x} &= A^{-1}\boldsymbol{b} \end{aligned}$$

$$A^{-1} = \frac{1}{3(4) - (-1)(-1)} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{3}{11} \end{bmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

2. Let $C = AB$ where C is invertible and A and B are square matrices. Solve for A^{-1}.[1]

If C is invertible and A and B are square then A and B are also invertible. Therefore,

$$\begin{aligned} C^{-1} &= B^{-1}A^{-1} \\ BC^{-1} &= (BB^{-1})A^{-1} \\ BC^{-1} &= A^{-1} \end{aligned}$$

[1] Strang p. 90 #12

3. Let $M = ABC$ where M is invertible and A, B, and C are square matrices. Solve for B^{-1}.[2]

If M is invertible and A, B, and C are square then A, B, and C are also invertible. Therefore,

$$\begin{aligned} M^{-1} &= C^{-1}B^{-1}A^{-1} \\ CM^{-1}A &= (CC^{-1})B^{-1}(A^{-1}A) \\ CM^{-1}A &= B^{-1} \end{aligned}$$

4. If B is the inverse of A^2, show that AB is the inverse of A.[3]

$$\begin{aligned} B &= (A^2)^{-1} \\ B &= (AA)^{-1} \\ (AA)B &= AA(AA)^{-1} \\ (AA)B &= I \\ A^{-1}(AA)B &= A^{-1} \\ (A^{-1}A)AB &= A^{-1} \\ AB &= A^{-1} \end{aligned}$$

5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

What are X^{-1} and Y^{-1}, assuming $ad \neq bc$.[4] **Hint:** First, multiply the two matrices XY. Then, attempt to solve for X^{-1} and Y^{-1}.

$$XY = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & ad - bc \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad{-}bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} XY &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ (X^{-1}X)Y &= (ad - bc)X^{-1} \\ Y &= (ad - bc)X^{-1} \\ \frac{1}{ad - bc}Y &= X^{-1} \end{aligned}$$

$$\begin{aligned} XY &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ X(YY^{-1}) &= (ad - bc)Y^{-1} \\ \frac{1}{ad - bc}X &= Y^{-1} \end{aligned}$$

[2] Strang p. 90 #13
[3] Strang p. 90 #18
[4] Strang p. 90 #16

Notice that $\det X = ad - bc$ so this obeys our usual rules for inversion of 2×2 matrices.

6. A is an *idempotent* matrix if and only if $AA = A$. Show that if A is symmetric ($A^\top = A$) and idempotent then $(I - A) = (I - A)(I - A)^\top$, where I is the identity matrix.

$$\begin{aligned} I - A &= I - AA^T && \text{by idempotency of } A \\ &= I - \underbrace{(A - A^T)}_{=0} - AA^T && \text{by symmetry of } A \\ &= (I - A)(I - A^T) \\ &= (I - A)(I - A)^T \end{aligned}$$

7. Let $R_{m\times n}$ be a rectangular matrix ($m \neq n$) and $A_{m\times m}$ be a symmetric matrix. Show $R^T AR$ is also symmetric. What are the dimensions of this matrix?[5]

$$\begin{aligned} (R^T AR)^T &= R^T A^T (R^T)^T \\ (R^T AR)^T &= R^T A^T R \\ (R^T AR)^T &= R^T AR && \text{by symmetry of } A \end{aligned}$$

$$R^T_{n\times m} A_{m\times m} R_{m\times n} = M_{n\times n}$$

8. Show every orthogonal matrix A has determinant 1 or -1. Hint: Apply the product rule ($|AB| = |A||B|$) and the transpose rule ($|A| = |A^T|$) for determinants.[6]

By the orthogonality of A, $A^T A = I$. By the product rule,

$$\begin{aligned} |A^T A| &= |I| \\ |A^T||A| &= 1 && \text{by the product rule and definition of the determinant} \\ (|A|)^2 &= 1 && \text{by the transpose rule} \end{aligned}$$

and $\sqrt{1} \in \{-1, 1\}$.

9. Let $\boldsymbol{x} = \{x_1, ..., x_{50}\}$ denote the number of electoral votes for each state. Let $\boldsymbol{y}_i = \{y_{i1}, ..., y_{ij}, ..., y_{i50}\}$ denote whether or not candidate $i \in \{R, D\}$ won the votes of a given state, where $y_{ij} \in \{0, 1\}$. Write an expression for the total number of votes for each candidate.

Candidate R:

$$\boldsymbol{x}^T \boldsymbol{y}_R$$

Candidate D:

$$\boldsymbol{x}^T \boldsymbol{y}_D$$

[5] Strang 117 #19
[6] Strang p. 252 #8