Ordered Sets Exercises

1) Prove that for the set of positive integers, the relation "m is a multiple of n" is an order relation.

Solution: First show reflexiveness. Reflexiveness requires that $a \succsim a$ for all $a \in \mathbb{Z}_+$. Note that $a \succsim b$ iff $\frac{a}{b} \in \mathbb{Z}_+$. It is straightforward to check that $\frac{a}{a} = 1 \in \mathbb{Z}$. Therefore reflexiveness holds. Now check transitivity. $a \succsim b$ implies $\frac{a}{b} = \alpha \in \mathbb{Z}_+$. $b \succsim c$ implies $\frac{b}{c} = \beta \in \mathbb{Z}_+$. This yields $b = \beta c$, $\frac{a}{\beta c} = \alpha$, $\frac{a}{c} = \alpha\beta$. Since $\alpha, \beta \in \mathbb{Z}_+$, $\alpha\beta \in \mathbb{Z}_+$. Therefore $a \succsim c$. Finally, we need to show that symmetry does not hold, that is, $a \succsim b$ does not imply $b \succsim a$. It is sufficient to find a single counter example to prove this. Let $a = 2$ and $b = 1$. $a \succsim b$ is true: $2/1 = 1$. $b \succsim a$, however, is not true: $1/2 \notin \mathbb{Z}$. ■

2) Let $X = \{1, 2, ..., 9\}$, ordered by the relation "m is a multiple of n". Find all maximal and best elements of this ordered set and its least upper bound in $\mathbb{Z}$.

Solution: To find the maximal elements, we need to find the set of all y such that there is no $x \in X$ with $x \succ y$ where $\succ$ is the relation "x is a proper multiple of y." First check for $x \succ 1$. Clearly for all $x > 1$, this holds. Now check for $x \succ 2$. All even $x > 2$, this holds. Similarly for 3 and 4: $9 \succ 3$ and $8 \succ 4$. For 5, the next proper multiple of 5 is $10 \notin X$. Therefore 5 is a maximal element. Similarly, the next proper multiple of 6 is 12, 7 is 14, 8 is 16, and 9 is 18. None of these are in X. We conclude that the maximal set is the set $\{5, 6, 7, 8, 9\}$. To find a best element, we need to find a member x of the maximal set such that $x \succsim y$ for all $y \in X$. Let's try 5. Is it true that $5 \succsim 9$? No. Therefore 5 cannot be a best element. What about 6? $6 \succsim 9$ is also false. Similarly, $7 \succsim 9$ and $8 \succsim 9$ are both false.

What about 9? $9 \succsim 9$ is true. But $9 \succsim 2$ and $9 \succsim 4$ are false. Therefore we conclude that the ordered set has no best element. Finally, to find a lower bound, we need to find an integer $z \in \mathbb{Z}$ such that $z \succsim x$ for all $x \in X$. That is, we need to find a multiple of every element of X. In particular, we need to find the *least common multiple* of $1, ...9$. It turns out that 2520 is the least common multiple and therefore the least upper bound of X. $\blacksquare$

3) Show that $x \sim y$ is an equivalence relation if $\succsim$ is rational.

Solution: $\succsim$ rational means that $\succsim$ is complete, reflexive, and transitive. $\sim$ is defined as $x \sim y \iff x \succsim y \wedge y \succsim x$. We need to show that $\sim$ is reflexive, symmetric, and transitive. Let's start with reflexiveness. $x \sim x$ implies $x \succsim x$ (and $x \succsim x$). Because $\succsim$ is reflexive, $x \sim x$ is reflexive. Now for transitivity. $x \sim y$ implies $x \succsim y$ and $y \succsim x$. $y \sim z$ implies $y \succsim z$ and $z \succsim y$. Because $\succsim$ is transitive, we have that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$. Therefore $x \sim y$ and $y \sim z$ imply $x \sim z$. Finally we check symmetry. $x \sim y$ implies $x \succsim y$ and $y \succsim x$. We also know from the definition of $\sim$ that $y \succsim x$ and $x \succsim y$ iff $y \sim x$. Therefore because $y \succsim x \wedge x \succsim y \equiv x \succsim y \wedge y \succsim x$, $x \sim y$ implies $y \sim x$. $\blacksquare$

4) Prove or disprove the following statements

i) Every best element is a maximal element.

ii) Every maximal element is a best element.

iii) An element is a best element if and only if it is a maximal element.

Solution:

i) True. By the definition of best element, $x \succsim y$ for all $y \in X$. If x is not maximal, this implies that for some $z \in X$, $z \succ x$ i.e. $z \succsim x$ and $\neg[x \succsim z]$. Therefore every best element is a maximal element.

ii) False. See counterexamples in lecture notes.

iii) False. Proof follows immediately from ii.

5) Let $X = \Delta^1$ and $\succsim$ be defined such that for any $(a, b), (c, d) \in X$, $(a, b) \succsim (c, d)$ if and only if $\max\{a, b\} \geq \max\{c, d\}$.

i) Find all maximal elements and best elements if they exist.

ii) Find all least upper bounds of the set in $\mathbb{R}^2$.

iii) Use the properties of binary relations to identify whether the set is partially ordered, totally ordered, and/or weakly ordered.

Solution

i) Maximal elements and best elements are the same: $(0,1)$ and $(1,0)$.

ii) The set of least upper bounds is the set of all points $(1,a)$ and $(b,1)$ for $a,b \leq 1$.

iii) The order relation is not antisymmetric: $(0,1) \succsim (1,0)$ and $(1,0) \succsim (0,1)$ but $(1,0) \neq (0,1)$. Therefore the ordered set is not a partially ordered set or a totally ordered set. It is straightforward to check that the order relation on X is complete and transitive.

6) Prove that if X is finite, $(X, \succsim)$ has at least one maximal element for all order relations.

Solution If X is a singleton, x is trivially a maximal element. Now consider a non-singleton finite X and assume that there is no maximal element. This implies that for all $x \in X$, there exists a $z \in X$ such that $z \succ x$. Consider an arbitrary $x_0 \in X$. We know that there must be some element in X that is strictly preferred to x_0. By the definition of $\succ$ and the reflexiveness of $\succsim$, this must be distinct from x_0. Label this x_1. We now have $x_1 \succ x_0$. Because there is no maximal element, there must be some element in X that is strictly preferred to x_1. By the transitivity of $\succsim$, this cannot be x_0. By reflexiveness and the definition of $\succ$, this element can also not be x_1. Label this new element x_2. Let N denote the cardinality of the set X. Continue this process until element x_{N-1}. Now there must be some x_N such that $x_N \succ x_{N-1}$. By transitivity and reflexivity, x_N must be the last remaining element that has not been shown to be strictly preferred to any other. Because there is no maximal x, there must be some $x \succ x_N$. But if such an x exists, $x_N \succ x$ by the transitivity of the order relation. By the definition of $\succ$, this is a contradiction. Therefore there exists a maximal element.