Introduction to Mathematics for Political Science: Linear Spaces

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Suppose we observe some features of N countries related to their level of social and economic development. Each observation (a country) can be represented as a vector, where each element corresponds to a unique feature (life expectancy, per capita gdp, literacy rate, etc). Together, these countries constitute a set. Now suppose, due to poor data quality, we need to predict a country's per capita gdp after observing its life expectancy and literacy rate. By assuming that these data live in a broader *space*, we can extrapolate out of our set and "fill in the blanks." We only observe N countries, but we can consider a broader universe of countries that might have different features than the ones we observe. What allows us to do this extrapolation is the assumption that our data our drawn from a *linear space*, whose elements obey the familiar laws of arithmetic and algebra. Transformations of elements within and between linear spaces is at the core of applied data analysis. We seek to understand how these spaces work today. This lecture will work to unify your self-study of linear algebra with the more abstract notions of sets and spaces we've been studying in the last few days. In the next lecture, we'll attach a metric to these spaces to build "normed linear spaces."

Exercises

1. Prove the following: If a set S is convex, then all sets αS are also convex for all $\alpha \in \mathbb{R}$.[1]

2. Prove that if S_1 and S_2 are subspaces of a linear space X, then $S_1 + S_2$ is also a linear subspace of X.[2]

3. Prove that if S_1 and S_2 are subspaces of a linear space X, then $S_1 + S_2$ is the linear hull of $S_1 \cup S_2$.[3]

4. Prove the following: Any nonzero vector is in the nullspace of a set S iff ($\iff$) there exists a linearly dependent vector $y \in S$.

[1] $\boldsymbol{x} \in S \implies \alpha\boldsymbol{x} \in \alpha S$ for all $\alpha \in \mathbb{R}$. Carter 1.166

[2] $\boldsymbol{x}_1 \in S_1$ and $\boldsymbol{x}_2 \in X_2$ implies $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in S_1 + S_2$. Carter 1.131

[3] Strang 3.1 #30.