Comparative Statics

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- "H1: A rise in X leads to a fall in Y"
- Before asserting "H1" and going to the data, it is necessary first to demonstrate that a hypothesized relationship between two variables follows logically from a set of explicit assumptions.
- This exercise is referred to as "comparative statics."

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- These assumptions yield a **parameterized objective function**, $f(x;\theta)$
- To find out how much time a legislator spends fundraising given θ, we solve

$$\max_x f(x;\theta)$$

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- That is, we want to know the sign of $\frac{\partial}{\partial \theta} x^*(\theta)$.
- The set of maximizers, $x^*(\theta)$, is referred to as the **solution correspondence**.

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- We see clearly then that $x^*(\theta)$ is increasing in θ.

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- Fortunately, we have an extremely powerful tool to help us with this problem.

Implicit Function Theorem

- The **implicit function theorem** states that as long as a certain condition is satisfied, we can use differential calculus to characterize how $x^*(\theta)$ varies in response to small changes in θ.

Implicit Function Theorem (Light)

Theorem

Let $x^* \in \mathbb{R}$ solve $g(x, y) = 0$ at $y^* \in \mathbb{R}$. If $g(\cdot, \cdot)$ is continuously differentiable and $\frac{\partial}{\partial x} g(x^*, y^*) \neq 0$ then for some open set A containing x^* and an open set B containing y^*, there exists a continuously differentiable function $h : B \to A$ with $g(h(y), y) = 0$. The derivative of this function at y^* is given by

$$\frac{\partial h(y^*)}{\partial y} = -\left(\frac{\partial g(x^*, y^*)}{\partial y}\right)\left(\frac{\partial g(x^*, y^*)}{\partial x}\right)^{-1}$$

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- The problem must have a solution to begin with.
- In the next lecture we will explore how verify that a general program has a solution.
- Here we assume that all problems have solutions, i.e. that $x^*(\theta)$ is always nonempty.

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- If we assume that marginal cost is non-decreasing in x, then $\frac{\partial^2 c(x)}{\partial x^2} \geq 0$
- The sign of $\frac{\partial^2 v(x;\theta)}{\partial x \partial \theta}$ will therefore tell us how $x^*(\theta)$ changes with θ.

Implicit Function Theorem

The implicit function theorem generalizes to higher dimensions with n choice variables and m parameters.

Theorem

Let $x^* \in \mathbb{R}^n$ solve $f(x, y) = 0$ at $y \in \mathbb{R}^m$. If $f_1(\cdot)$ through $f_n(\cdot)$ are continuously differentiable in each coordinate of x and y and the Jacobian matrix with respect to the endogenous variables is nonsingular, then for some open set A containing x^* and an open set B containing y^*, there exists a continuously differentiable function $\phi : B \to A$ with $f(\phi(y), y) = 0$. The derivative of this function at y^* is given by the $n \times m$ matrix

$$D_y\phi(y^*) = -[D_x f(x^*, y^*)]^{-1} D_y f(x^*, y^*)$$

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- The implicit function theorem tells us that

$$D_\theta x^*(\theta) = -[H_x f(x, \theta)]^{-1} D_\theta f(x, \theta)$$

where $H_x f(x^*, \theta)$ is the $n \times n$ Hessian matrix of $f(x^*, \theta)$ with respect to the choice variables and $D_\theta f(x^*, \theta)$ is the $n \times m$ Jacobian of $f(x^*, \theta)$ with respect to the parameters.

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- Note that the implicit function theorem requires that H be nonsingular.

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- Let
$$f(x, y; \theta) = v(x) + \theta w(y) - c(x, y)$$
where $v(x)$ and $w(y)$ are increasing an concave and $c(x, y)$ is increasing and convex in both arguments.

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- We want to find $\begin{bmatrix} \frac{\partial x^*(\theta)}{\partial \theta} \\ \frac{\partial y^*(\theta)}{\partial \theta} \end{bmatrix}$
- By the implicit function theorem,

$$\begin{bmatrix} \frac{\partial x^*(\theta)}{\partial \theta} \\ \frac{\partial y^*(\theta)}{\partial \theta} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 f(x,y;\theta)}{\partial x^2} & \frac{\partial^2 f(x,y;\theta)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y;\theta)}{\partial x \partial y} & \frac{\partial^2 f(x,y;\theta)}{\partial y^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^2 f(x,y;\theta)}{\partial x \partial \theta} \\ \frac{\partial^2 f(x,y;\theta)}{\partial y \partial \theta} \end{bmatrix}$$

$$= - \begin{bmatrix} \frac{\partial^2 v(x)}{\partial x^2} - \frac{\partial^2 c(x,y)}{\partial x^2} & -\frac{\partial^2 c(x,y)}{\partial x \partial y} \\ -\frac{\partial^2 c(x,y)}{\partial x \partial y} & \frac{\partial^2 \theta w(y)}{\partial y^2} - \frac{\partial^2 c(x,y)}{\partial y^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{\partial w(y)}{\partial y} \end{bmatrix}$$

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- Therefore $y^*(\theta) > 0$.
- We have not yet made any assumptions about $\frac{\partial^2 c(x,y)}{\partial x \partial y}$.
- Since $-\frac{\partial w(y)}{\partial y} < 0$, if $\frac{\partial^2 c(x,y)}{\partial x \partial y} \geq 0$, then $x^*(\theta) \leq 0$. Otherwise, $x^*(\theta) > 0$.

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- For discontinuous or continuous but nondifferentiable objective functions, we need a different tool.

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- The sign of $\frac{\partial V(\theta)}{\partial \theta}$ tells us how an actor's utility changes for a small rise in θ given that she is maximizing.

Value Functions

- So far we have been analyzing the behavior of the maximizers of a parameterized optimization problem.
- We often will also want to analyze how the value of a parameterized optimization problem changes with θ.
- The function

  $$V(\theta) \equiv f(x^*(\theta);\theta)$$

  is referred to as the **value function** or **indirect utility function** for the problem

  $$\max_x f(x;\theta)$$

- The sign of $\frac{\partial V(\theta)}{\partial \theta}$ tells us how an actor's utility changes for a small rise in θ given that she is maximizing.
- It is not obvious how to do this as θ has two effects on $V(\theta)$, a direct effect and an indirect effect that operates through $x^*(\theta)$.

The Envelope Theorem

Theorem

For some θ, suppose the unique maximizer $x^*(\theta)$ is locally characterized by first order conditions

$$\frac{\partial f(x^*(\theta), \theta)}{\partial x} = 0$$

Suppose that the conditions of the implicit function theorem hold locally so that $x^*(\cdot)$ is a differentiable function of θ. Then

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^*(\theta), \theta)}{\partial \theta}$$

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- What the envelope theorem says is that the indirect effect does not matter!
- Only the direct effect of the exogenous parameter needs to be considered even though the exogenous parameter may enter the value function indirectly.

Example

- Let's revisit the earlier fundraising example where $f(x;\theta) = \theta \ln(x) - x^2$ and $x^*(\theta) = \sqrt{\frac{\theta}{2}}$.

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$$\frac{\partial V(\theta)}{\theta} = \ln(\sqrt{\frac{\theta}{2}}) - \frac{1}{2} + \theta \frac{\partial}{\partial \theta} \ln(\sqrt{\frac{\theta}{2}})$$

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- Therefore

$$\frac{\partial V(\theta)}{\partial \theta} = \ln(\sqrt{\frac{\theta}{2}}) = \frac{\partial f(x,\theta)}{\partial \theta}\Big|_{x^*(\theta)}$$

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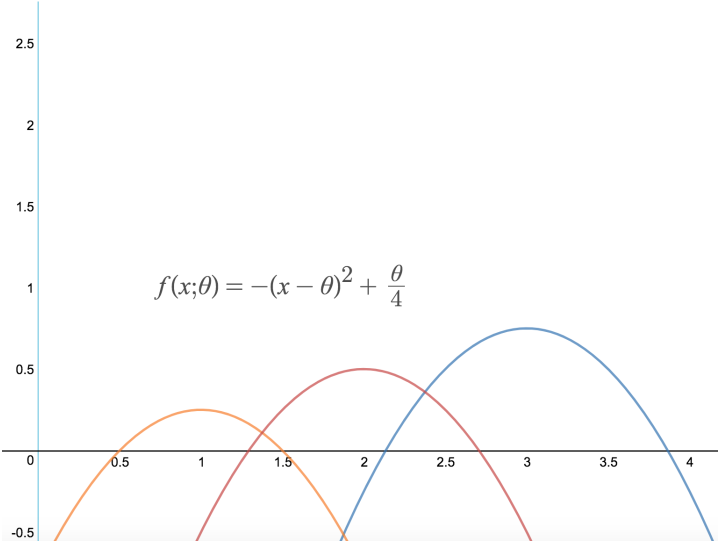
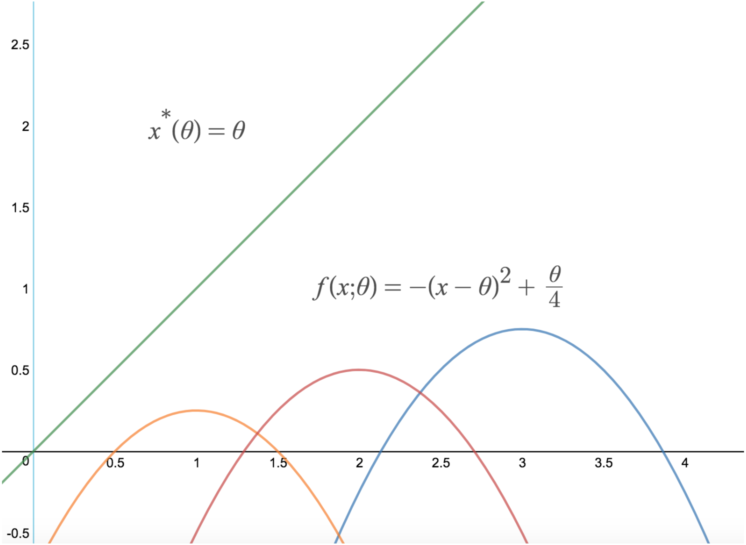
- $V(\theta) = v(x^*(\theta)) + \theta w(y^*(\theta)) - c(x^*(\theta), y^*(\theta))$
- By the envelope theorem, $\frac{\partial V(\theta)}{\partial \theta} = w(y^*(\theta)) \geq 0$.



$f(x;\theta) = -(x-\theta)^2 + \frac{\theta}{4}$
2.5
2
1.5
1
0.5
0
-0.5
0.5
1
1.5
2
2.5
3
3.5
4



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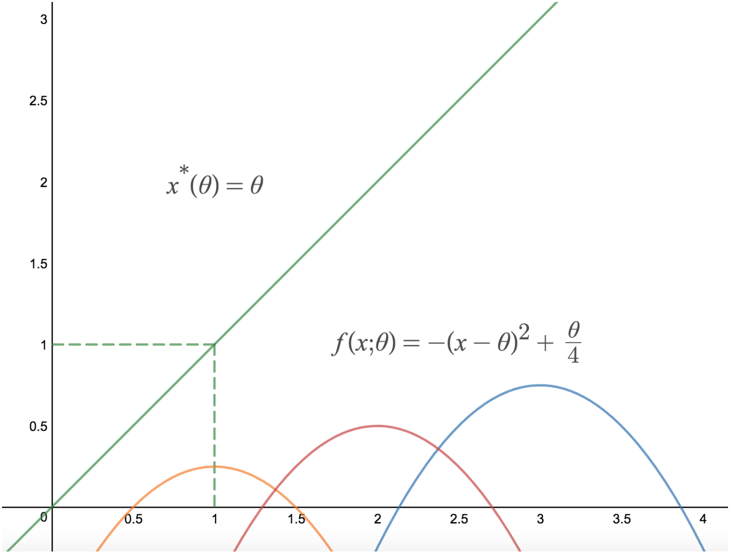

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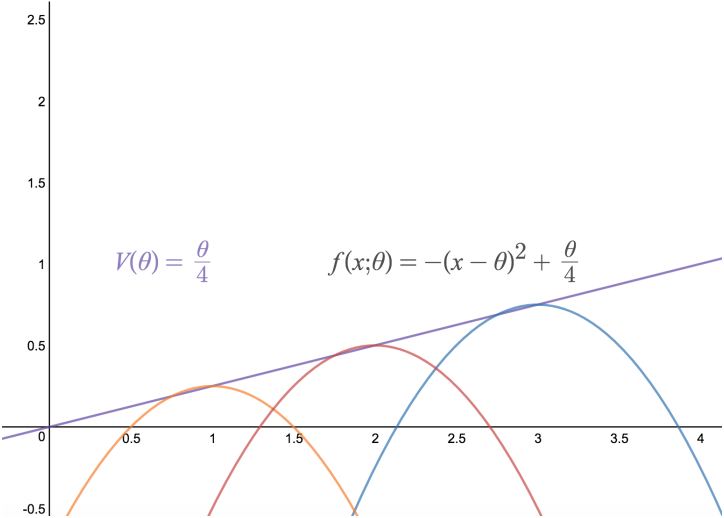

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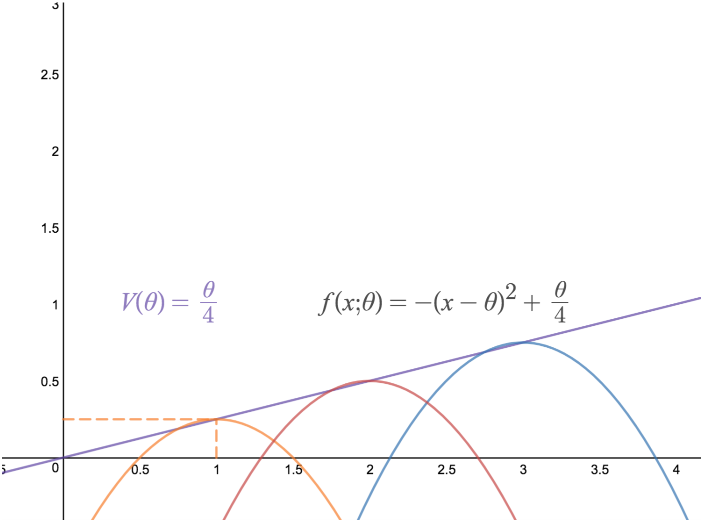

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- We typically assume that $x \in G(\theta)$ where G is called a **feasible set**.
- If $G(\theta)$ depends on θ, then unless $G(\theta)$ is a singleton, then we need a more general concept than function to think about the properties of $G(\theta)$ and $x^*(\theta)$ (why?)

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- A common utility function in spatial models is the **absolute loss function**, $u(x; z) = -|x - z|$.
- We cannot use the implicit function for comparative statics here.
- Intuitively, however, we should be able to exploit the continuity of the objective function to learn about the solution correspondence.

The Theorem of the Maximum (Light)

Theorem

For a parameterized optimization problem, if $f : X \times \Theta \to \mathbb{R}$ is continuous and the feasible set G is compact, then $V(\theta)$ is continuous and $x^*(\theta)$ is nonempty, compact-valued, and upper hemicontinuous.

- The result of the theorem is that if the elements of an optimization problem are sufficiently continuous, then some, but not all, of that continuity is preserved in the solutions.

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- A powerful and underutilized tool in political research, **monotone comparative statics**, allows us to study the properties of $x^*(\theta)$ even when f is discontinuous.
- We say that $f(x, \theta)$ satisfies the **single-crossing property** if for all $x > x'$ and $\theta > \theta'$,

$$f(x, \theta') - f(x', \theta') \geq 0 \implies f(x, \theta) - f(x', \theta) \geq 0$$

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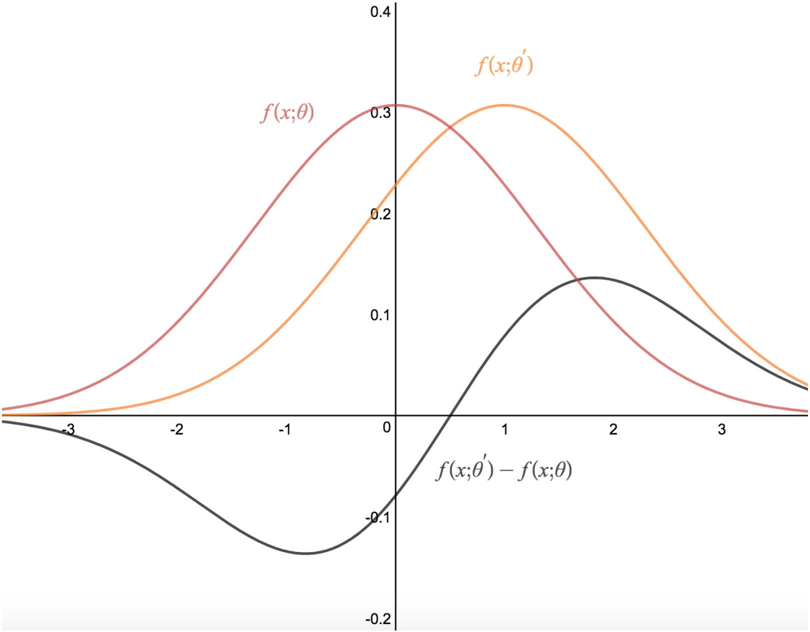
- Intuitively if the two objective functions for different θ cross only once, the function satisfies the single crossing property.

Single-Crossing



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- If $f(x, \theta') - f(x', \theta') \geq 0$, then

$$f(x, \theta) - f(x', \theta) \geq f(x, \theta') - f(x', \theta') \geq 0$$

  and f is single crossing

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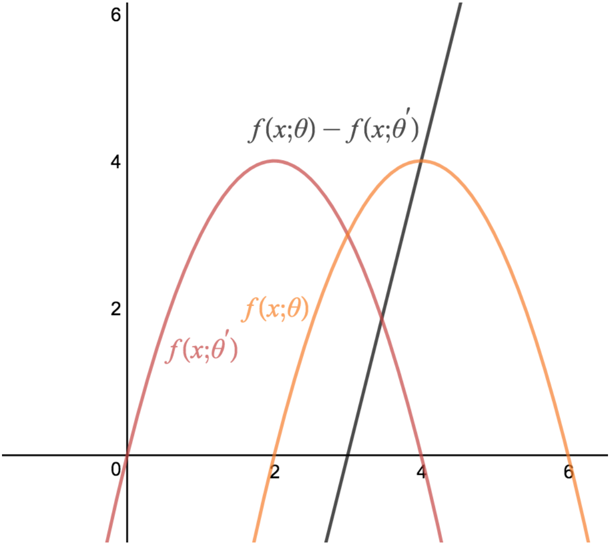

Monotone Comparative Statics

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- This result generalizes to higher dimensions.

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- Supermodularity generalizes the notion of "complementarity."

Topkis' Theorem

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Consider the problem

$$\max_{x \in X} f(x, \theta)$$

where $\theta \in \Theta$ *is a parameter. If* f *is supermodular, then every component of* $x^*(\theta)$ *is weakly increasing in* θ

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- Because the x_i are all complementary to one another, all indirect effects are also positive.
- Without supermodularity, indirect effects may cancel out direct effects.
- Hence the "monotone" in monotone comparative statics.