Introduction to Mathematics for Political Science

Problem Set 9: Matrix Inversion and Determinants

Due: August 16

Instructions: You are encouraged to work in groups and actively participate on the Piazza page. Submitted solutions must be your individual work. Do not use a calculator or search for solutions. Show all of your work. Submit typed solutions using the link on the course page.

1. Consider the following system of equations:

$$\begin{aligned} 3x_1 - x_2 &= 10 \\ -x_1 + 4x_2 &= 4 \end{aligned}$$

   Write this system in $A\boldsymbol{x} = \boldsymbol{b}$ form and solve via matrix inversion.

2. Let $C = AB$ where C is invertible and A and B are square matrices. Solve for A^{-1}.[1]

3. Let $M = ABC$ where M is invertible and A, B, and C are square matrices. Solve for B^{-1}.[2]

4. If B is the inverse of A^2, show that AB is the inverse of A.[3]

5. Let

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

   Multiply the two matrices. What are X^{-1} and Y^{-1}, assuming $ad \neq bc$.[4]

6. A is an *idempotent* matrix if and only if $AA = A$. Show that if A is symmetric ($A^\top = A$) and idempotent then $(I - A) = (I - A)(I - A)^\top$, where I is the identity matrix.

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[1] Strang p. 90 #12
[2] Strang p. 90 #13
[3] Strang p. 90 #18
[4] Strang p. 90 #16

7. Let $R_{m\times n}$ be a rectangular matrix ($m \neq n$) and $A_{m\times m}$ be a symmetric matrix. Show $R^T AR$ is also symmetric. What are the dimensions of this matrix?[5]

8. Show every orthogonal matrix A has determinant 1 or -1. Hint: Apply the product rule ($|AB| = |A||B|$) and the transpose rule ($|A| = |A^T|$) for determinants.[6]

9. Let $\boldsymbol{x} = \{x_1, ..., x_{50}\}$ denote the number of electoral votes for each state. Let $\boldsymbol{y}_i = \{y_{i1}, ..., y_{ij}, ..., y_{i50}\}$ denote whether or not candidate $i \in \{R, D\}$ won the votes of state j, where $y_{ij} \in \{0, 1\}$. Write an expression for the total number of votes for each candidate.

[5]Strang 117 #19
[6]Strang p. 252 #8