

Smooth Functions

1. Prove that every continuous linear functional; is differentiable with $Df[\mathbf{x}] = \boldsymbol{\alpha}$.¹
2. Prove that if a differentiable functional f is increasing, then $Df[\mathbf{x}_0](\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.²
3. Let f be a differentiable functional. Prove that the $\nabla f(\mathbf{x}_0)$ is orthogonal to the hyperplane tangent to the contour through $f(\mathbf{x}_0)$.
4. Let the policy production function discussed above be written

$$f(x, y) = x^\alpha y^\beta$$

Give a sufficient condition for this function to be concave on $\{\mathbb{R}_{++} \times \mathbb{R}_{++}\}$.

Hint: A 2×2 symmetric matrix A is negative definite if $A_{11} < 0$ and $A_{11}A_{22} - A_{12}A_{21} > 0$.

¹Carter 4.6

²Carter 4.15, recall the definition of increasingness from the lecture on monotonic functions.