Smooth Functions

1. Prove that every continuous linear functional; is differentiable with $Df[\boldsymbol{x}] = \boldsymbol{\alpha}$.[1]

2. Prove that if a differentiable functional $f : \mathbb{R}^n \to \mathbb{R}$ is increasing, then $Df[\boldsymbol{x}_0](\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in X$, or $\frac{\partial f}{\partial x_i} \geq 0$ for all $i \in \{x_1, ..., x_n\}$.

3. Let f be a differentiable functional. Prove that the $\nabla f(\boldsymbol{x}_0)$ is orthogonal to the hyperplane tangent to the contour through $f(\boldsymbol{x}_0)$.

4. Let the policy production function discussed above be written

$$f(x, y) = x^\alpha y^\beta$$

   Give a sufficient condition for this function to be concave on $\{\mathbb{R}_{++} \times \mathbb{R}_{++}\}$. **Hint:** A 2×2 symmetric matrix A is negative definite if $A_{11} < 0$ and $A_{11}A_{22} - A_{12}A_{21} > 0$.

[1] Carter 4.6