Logic and Proofs

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- If x is a banana and X is the set of all types of fruit, then A is true.

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- Necessity: $\neg A \implies \neg B$

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- $A \wedge B$ is read "A and B" and corresponds to the concept of intersection in set theory.

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- **Noncontradiction**: $(A \wedge \neg A)$ is always false

Proofs

- Mathematics has well-defined procedures for verifying that a given statement is true or false.

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- B can be any statement, not necessarily one that we are trying to prove or disprove.

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- This implies that b^2 is even which implies that b is also even.
- But we just deduced that b is odd. Therefore we have a contradiction: b is both even and odd.
- Therefore our presumption that $\sqrt{2}$ is rational must be false. $\blacksquare$

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- This gives us a specific x for which A is false which is often enough to produce a contradiction.

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- This implies that $n + 1 + 17/n$ is not an integer for all n.
- This therefore implies that $17/n$ is not an integer for all n which implies that 1 is not an integer which is false. ■

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- To prove $A \implies B$, we assume $\neg(A \implies B)$.
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- Assume $(A \wedge \neg B)$ and show that $(C \wedge \neg C)$ for some statement C.

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- Claim: Assume $a \in \mathbb{Z}$. If a^2 is even, then a is even.
- Proof: Assume a is odd and a^2 is even.
- Since a is odd, there exists an integer c for which $a = 2c + 1$.
- Then $a^2 = 2(2c^2 + 2c) + 1$ which implies that a^2 is odd, a contradiction.

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- Some statements describe a property of an index number n and may be written as $A(n)$.
- One way to prove that $A(n)$ is true for all natural numbers n is to demonstrate that $A(1)$ is true and that if $A(n)$ is true then $A(n+1)$ must be true.

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- Now assume $n = k$ is true (inductive hypothesis)
- That is, we assume $1 + 2 + ... + k = \frac{k(k+1)}{2}$
- Now we just need show that $n = k + 1$ holds:

$$1 + 2 + ... + k + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}$$

Example (cont.)

- We need to show $1 + 2 + ... + k + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$

Example (cont.)

- We need to show $1+2+...+k+(k+1) = \frac{(k+1)((k+1)+1)}{2}$
- Start with the left side of the equation. By the inductive hypothesis,

$$
\begin{aligned}
1+2+...+k+(k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2} \quad \blacksquare
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- Algorithm for proof by induction is simple.
- First prove the statement for a base case.
- Then assume the statement is true for some n.
- Then show that given the inductive hypothesis (step 2), the statement holds for $n+1$.
- While the algorithm is simple, intuition for why inductive proofs are valid may take a while to understand.

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- Proof by contraposition exploits this fact to prove $A \implies B$
- Often $A \implies B$ is too hard to prove by deduction, contradiction, or induction while $\neg B \implies \neg A$ is relatively simple to prove by one of these techniques.

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- Proof: We will prove that m is even implies $7m$ is even.
- If m is even, then $m = 2k$ for some integer $k \implies 7m = 7(2k) \implies 7m = 2(7k) \implies 7m = 2n$ for some integer $n \implies 7m$ is even. ■

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- To prove a conditional statement of the form $A \iff B$, we have to prove $A \implies B$ and $B \implies A$.
- We can use different proof techniques to prove both sides of the statement.