Monotone, Linear, and Convex Functions Exercises

1) Let $f_1, f_2, ..., f_n$ be convex functions and $\alpha_1, \alpha_2, ...\alpha_n \geq 0$. Prove that $f(x) = \alpha_1 f_1(x) + ... + \alpha_n f_n(x)$ is convex. Is $\alpha_1 f_1 - \alpha_2 f_2$ convex? Prove your answer.

$$f(\lambda x + (1-\lambda)y) = \alpha_1 f_1(\lambda x + (1-\lambda)y) + ... + \alpha_n f_n(\lambda x + (1-\lambda)y)$$

$$\leq \alpha_1(\lambda f_1(x) + (1-\lambda)f_1(y)) + ... + \alpha_n(\lambda f_n(x) + (1-\lambda)f_n(y))$$

$$= \lambda(\alpha_1 f_1(x) + ... + \alpha_n f_n(x)) + (1-\lambda)(\alpha_1 f_1(y) + ... + \alpha_n f_n(y))$$

$$= \lambda f(x) + (1-\lambda)f(y)$$

2) Prove the Cauchy-Schwarz inequality for $\mathbb{R}^n$.

3) Prove the following: $L : \mathbb{R}^l \to \mathbb{R}$ is a continuous, linear functional if and only if there exists a $y \in \mathbb{R}^l$ such that for all $x \in \mathbb{R}^l$, $L(x) = y^T x$.