Continuous Functions

Motivation

The last several lectures introduced the spaces in which we carry out research in political science. Now we reintroduce functions as relationships between elements in two or more spaces. In this lecture we study functions that preserve the geometric structure of the sets they associate, continuous functions.

Functions

A **function** is a rule $f : X \to Y$ that assigns every element x of a set X to a single element of a set Y. The set X is called the **domain**. The set Y is called the **codomain**. A mapping is denoted $y = f(x)$ and y is referred to as the **image** of x under f. In our application of functions to research, we typically refer to x as an **independent variable** and $y = f(x)$ as a **dependent variable**. The **range** of $f : X \to Y$ is the set of all elements in Y that are images of elements in X, denoted

$$f(X) = \{y \in Y : y = f(x) \text{ for some } y \in Y\}$$

If $f(X) = Y$, then f maps X **onto** Y. If $f(x) = f(x')$ implies $x = x'$, then f is **one-to-one**. A function f has an **inverse** $f^{-1} : Y \to X$ if and only if f is one-to-one and onto. A

graph of a function

$$\text{graph}(f) = \{(x, y) \in X \times Y : y = f(x), x \in X\}$$

Example: Let $f(x) = x^2$ for $X = Y = \mathbb{R}$. The range of f is $f(X) = \mathbb{R}_+$. The function is not onto. It is also not one-to-one: $-x^2 = x^2$.

Example: Let $f(x) = x^3$ for $X = Y = \mathbb{R}$. The range of f is $f(X) = \mathbb{R}$. Therefore the function is onto. It is also one-to-one.

Example: Let $X = \{\text{members of the department}\}$ and $Y = \{\text{days of the year}\}$. Let $f : X \to Y$ be a rule that assigns each member of the department his or her birthday. Note that this is a function: each member of the department has a unique birthday. Under what conditions is f onto? One-to-one?

Example: A **polynomial** of degree n is a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$.

We will often work with a special type of function in political science that maps into $\mathbb{R}$. A **functional** is a real-valued function $f : X \to \mathbb{R}$. Utility functions such as $-(x - z)^2$ are functionals.

We also will often use compositions of functions in our research. A **composition of functions** $f : X \to Y$ and $g : Y \to Z$ is denoted $g \circ f$ or $g(f(x))$ for $x \in X$ and

$$g \circ f : X \to Z$$

Example: Let $f(x, y) = x^\alpha y^\beta$ and let $g(\cdot) = \ln(\cdot)$. Note $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_{++} \to \mathbb{R}$. The composition of these two functions is $g(f(x)) = \alpha \ln(x) + \beta \ln(y)$ and $g(f(x)) : \mathbb{R}^2_+ \to \mathbb{R}$.

An additional definition will prove useful in defining continuity. The **inverse image** of a function of a set $T \subseteq Y$ is the set of $x \in X$ that are mapped to a elements $y \in T$

$$f^{-1}(T) = \{x \in X | f(x) \in T\}$$

Example: Let $f(x) = x^2$ and let $T = [0,4] \subset \mathbb{R} = Y$. The inverse image $f^{-1}(T)$ is $[0,2] \subset \mathbb{R} = X$.

Continuous Functions

A function is continuous if small changes in x produce only small changes in y. Let

$$f : [0,1] \to \{0,1\}$$

map a candidate's vote share in an election to a dummy variable that identifies whether or not he or she won (1) or lost (0) the election. In a two-candidate race, $f(x) = 0$ for $x < 1/2$ and $f(x) = 1$ for $x > 1/2$. A small change in x in the neighborhood of $1/2$ results in a large change in y. This is a discontinuous function. Now consider the function

$$f(x,y,z) = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 z + \epsilon$$

for $\beta_i, \epsilon, x, y, z \in \mathbb{R}$. Note that a small change in any variable results in only a small change in $f(x)$. This is a continuous function.

Recall that our concept of distance (and hence a formal notion of "small difference") is defined for metric spaces. A continuous function preserves the geometric structure of the sets it associates i.e. the metric spaces it associates. For the rest of this lecture we will assume that X and Y are metric spaces. There are multiple equivalent definitions of continuity.

Continuity: Epsilon-Delta Characterization

A function $f : X \to Y$ is continuous at x_0 if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x \in X$,

$$d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \epsilon$$

Informally, the definition states that if you give me any $\epsilon > 0$, potentially extremely small, I can give you a δ such that if two points x and x_0 in the domain are less than δ away from each other, then the distance between the points $f(x)$ and $f(x_0)$ in the codomain are less than ϵ away from each other. If I can't find such a δ, then a small distance between x_0 and any point x in the domain implies a large difference between $f(x_0)$ and $f(x)$ in the codomain.

If a function is continuous for all x_0 in its domain, we call the function continuous.

Example: Let $f(x) = 2x^2 + 1$ for $X = \mathbb{R}$. To show that this function is continuous, take an arbitrary x_0 and $\epsilon > 0$. We need to show that $d(f(x_0), f(x)) < \epsilon$ for $d(x_0, x)$ sufficiently small (less than some δ) where x is an element of the domain. Note that $d(x, x_0) = |x - x_0|$ and observe that

$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)| = |2x^2 - 2x_0^2| = 2|x - x_0| \cdot |x + x_0|$$

We need a bound for $|x + x_0|$ that does not depend on x. Note that if $|x - x_0| < k$, then $|x| - |x_0| < k$ or $|x| < |x_0| + k$. By the triangle inequality, $|x + x_0| \leq |x| + |x_0| < 2|x_0| + k$. Thus

$$|f(x) - f(x_0)| \leq 2|x - x_0|(2|x_0| + k)$$

if $|x - x_0| < k$. For $2|x - x_0|(2|x_0| + k) < \epsilon$ to hold, it is sufficient that $|x - x_0| < \frac{\epsilon}{2(2|x_0|+k)}$ and $|x - x_0| < k$. So let

$$\delta = \min\{k, \frac{\epsilon}{2(2|x_0| + k))}\}$$

Therefore $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

Example: Let $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. Is $f(x)$ continuous at $x = 0$? Let $\epsilon > 0$. Notice that $d(f(x), f(0)) = |f(x) - f(0)| = |f(x)|$. Note that $f(x) \leq x^2$ for all x because the range of $\sin(\cdot)$ is $[-1, 1]$. Let $\delta = \sqrt{\epsilon}$. Now note that $|x - 0| < \delta$ implies $x^2 < \delta^2 = \epsilon$. Therefore $|x - 0| < \delta$ implies $|f(x) - f(0)| < \epsilon$ so f is continuous at $x = 0$.

Example: Let $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$. To show that $f(x)$ is discontinuous at $x = 0$, note that $d(f(0), f(x)) = d(0, f(x)) = |f(x)|$. Suppose I claim that if you give me an ϵ, I can find a $\delta(\epsilon)$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$. In particular, suppose you give me $\epsilon = 1/2$. My claim is that if $|x| < \delta(\epsilon)$, then $|f(x)| < 1/2$. This is equivalent to claiming that I can identify an interval $(-\delta, \delta)$ such that for all $x \in (-\delta, \delta)$, then $|f(x)| < 1/2$. But note that for any $x < 0$, $|f(x)| = 1$. I cannot find a δ such that there are only positive x in the interval. Therefore my claim is false: there is an ϵ such that I cannot find a δ such that $d(x, 0) < \delta \implies d(f(0), f(x)) < \epsilon$.

Continuity: Sequence Convergence Characterization

The $\epsilon-\delta$ definition of continuity is difficult to work with. Often an alternative and equivalent characterization of continuity is easier:

A function $f : X \to Y$ is continuous if and only if $f(x) = \lim_{n\to\infty} f(x^n)$ for every sequence $x^n \to x$.

This alternative definition implies that values of $f(x)$ are close to $f(x_0)$ when values of x are close to x_0.

Example: Let $f(x) = 2x^2 + 1$ for $X = \mathbb{R}$. Suppose we take an arbitrary sequence in X such that $\lim x^n = x_0$. We have

$$\lim(2x^2 + 1) = 2(\lim x)^2 + 1 = 2x_0^2 + 1 = f(x_0)$$

Therefore $f(x)$ is continuous.

Example: Let $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$. To show that f is discontinuous, it is sufficient to find single sequence $x^n \to x_0$ in X such that $\lim f(x^n) \neq f(x_0)$. Consider the sequence $x^n = -\frac{1}{n}$. Note that $\lim x^n = 0$ and $x^n < 0$ for all x^n. We therefore have $\lim f(x^n) = \lim 1 = 1 \neq f(x^0) = 0$. Therefore f is discontinuous at $x = 0$.

Continuity: Inverse Image Characterization

A third characterization of continuity is often useful in proofs that require or seek to establish continuity.

A function $f : X \to Y$ is continuous if and only if the inverse image of any open (closed) subset of Y is an open (closed) subset of X.

Example: Let $f(x) = x^3$. Consider the subset of its codomain $(0, 2)$. The inverse image $f^{-1}((0, 8))$ is the set of all points x in the domain of x^3 such that $x^3 \in (0, 8)$. Therefore $f^{-1}((0, 2)) = (0, 2)$. Note that this is an open subset of $X = \mathbb{R}$.

Example: Let $f(x) = 1$ for all x. Consider a subset of the codomain, $(-1, 1)$. The inverse image under $(-1, 1)$ is $\emptyset$ which is open. Now consider $(0, 2)$. The inverse image under $(0, 2)$ is $\mathbb{R}$.

Using this result to show that a function is continuous can be difficult because we have to evaluate every subset of the codomain. We typically use the result in proofs. It is easier to use the definition to show discontinuity because we only need to find a single example of a set that violates the result.

Example: Let $f(x) = x$ for $x \leq 0$ and let $f(x) = x + 1$ for $x > 0$. Consider the open subset of the codomain $(-1, 1) \subset Y$. The inverse image under $(-1, 1)$ is $(-1, 1]$ which is not open. Therefore $f(x)$ is not continuous.

Properties of Continuous Functions

The following properties of continuous functions will be useful.

Proposition 1 *Let f be a real-valued function with domain $X \subseteq \mathbb{R}$. If f is continuous at x_0 in X, then $|f|$ and kf, $k \in \mathbb{R}$ are continuous at x_0.*

Proof: Consider the sequence $x_n \to x_0$ in the domain of f. Because f is continuous at x_0, $\lim f(x_n) = f(x_0)$. Note that $\lim kf(x_n) = k \lim f(x_n) = kf(x_0)$. Therefore kf is continuous at x_0.

Now consider $|f|$ at x_0. We need to show $\lim |f(x_n)| = |f(x_0)|$. By the reverse triangle inequality,

$$||f(x_n)| - |f(x_0)|| \leq |f(x_n) - f(x_0)|$$

Now consider $\epsilon > 0$. Because $\lim f(x_n) = f(x_0)$, there exists an N such that $n > N$ implies $||f(x_n)| - |f(x_0)|| < \epsilon$ and thus $\lim |f(x_n)| = |f(x_0)|$. $\square$

Proposition 2 *If f and g are real-valued functions and continuous at x_0, then*

i) $f + g$ is continuous at x_0

ii) fg is continuous at x_0

iii) $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.

Proof: We know that x_0 is in the domain of f and g. Let x_n be a sequence in the intersection of their domains that converges to x_0. Because each is continuous, we have $\lim f(x_n) = f(x_0)$ and $\lim g(x_n) = g(x_0)$. Therefore

$$\lim(f+g)(x_n) = \lim[f(x_n) + g(x_n)] = \lim f(x_n) + \lim g(x_n) = f(x_0) + g(x_0) = (f+g)(x_0)$$

so $f + g$ is continuous at x_0. A similar proof establishes that fg and f/g are continuous at x_0. $\square$

Proposition 3 *If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g(f(x))$ is continuous at x_0.*

Proof: We know that x_0 is in the domain of f and $f(x_0)$ is in the domain of g. Let x_n be a sequence such that x_n is in the domain of f and $f(x_n)$ is in the domain of g converging to x_0. Because f is continuous at x_0, $\lim f(x_n) = f(x_0)$. Because $f(x_n)$ converges to $f(x_0)$ and g is continuous at $f(x_0)$, we have $\lim g(f(x_n)) = g(f(x_0))$.

Proposition 4 (Weierstrass) *A continuous functional $f : X \to Y$ on a compact set achieves a maximum and a minimum.*

Proof: First note that because X is bounded, $f(x)$ has a supremum. We need to show that the supremum of the function's image is contained within the image. Let $M = \sup_{x \in X} f(x)$. Take a sequence $f(x_n) \in Y$ with $f(x_n) \to M$. Then $x_n \in X$. Recall the Bolzano-Weierstrass Theorem: every sequence in a compact set $S \subseteq \mathbb{R}^n$ has a convergent subsequence which converges to a point in S. Therefore because X is compact, we can find a convergent subsequence $x_{n_k} \to x_0$ where $f(x_{n_k}) \to M$. Applying the limit definition of continuity, we have $\lim f(x_{n_k}) = f(x_0) = M$. Therefore $f(x_0) = M$. The same proof can be applied to the infimum of the set.

Exercises

1) Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Use the $\epsilon - \delta$ definition of continuity to show that $f(x)$ is continuous at $c > 0$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$\frac{x + x^3 + 5x^5}{1 + x^2}$$

Prove that $f(x)$ is continuous.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^x & \text{for} \quad x \leq 0 \\ 0 & \text{for} \quad x > 0 \end{cases}$$

Show that $f(x)$ is discontinuous.

4) Let f and g be continuous at x_0 in $\mathbb{R}$. Prove that $\max(f, g)$ is continuous at x_0 (Hint: first show that for any $a, b \in \mathbb{R}$, $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$).

5) Prove that if f and g are continuous at x_0, then their product fg is continuous at x_0.

Source Material

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