# Existence of Solutions

## Motivation

In the last lecture we learned how to analyze how the solution to a parameterized optimization problem and its value change with a parameter. In doing so, we assumed the existence of a solution. Now we drop this assumption and explore how we can guarantee that a solution to an optimization problem exists.

## Weierstrass Theorem

One of the most basic existence theorems we have is the **Weierstrass theorem**.

**Theorem 1 (Weierstrass)** *A continuous functional on a compact set achieves a maximum and a minimum.*

**Proof**: Let $M = \sup_{x \in X} f(x)$. There exists a sequence $x^n \in X$ with $f(x^n) \to M$. It follows from the Bolzano-Weierstrass theorem that since the set $X$ is compact, there exists a convergent subseqeuence $x^m \to x^*$ and $f(x^m) \to M$. Since $f$ is continuous, $f(x^m) \to f(x^*)$. We conclude that $f(x^*) = M$. ■

In the last lecture we met the theorem of the maximum and focused on how the continuity of a problem ensures some degree of continuity in the the solution correspondence and value function. The theorem of the maximum will also provide conditions that imply a nonempty solution correspondence.

**Theorem 2 (The Theorem of the Maximum)** *Consider the constrained maximization problem*

$$\max_{x \in G(x)} f(x, \theta)$$

*If $f : X \times \Theta \to \mathbb{R}$ is continuous and the constraint correspondence $G : \theta \rightrightarrows X$ is continuous and compact-valued, then the value function $V(\theta)$ is continuous and the solution correspondence $x^*(\theta)$ is non-empty, compact-valued, and upper hemicontinuous.*

It is easy to see how the Weierstrass theorem implies that $x^*(\theta)$ is non-empty when the conditions of the theorem of the maximum are satisfied. A useful extension of the theorem of the maximum is the **convex maximum theorem**. It states that if we add quasi-concavity of $f$ and convexity of $G(\theta)$ to the conditions of the theorem of the maximum, then $x^*(\theta)$ will also be convex-valued.

# Fixed Point Theorems

In game theory, we exploit fixed point theorems to show the existence of an equilibrium. Recall that a **fixed point** of a self-mapping $f : X \to X$ is a point $x^*$ such that $f(x^*) = x^*$. For a correspondence $G : X \to X$, a fixed point is a point $x^*$ such that $x^* \in G(x^*)$.

The typical problem we have in game theory is as follows. Let $x^*(x; y, \theta)$ solve

$$\max_{x} f_1(x; y, \theta)$$

and let $y^*(y; x, \theta)$ solve

$$\max_{y} f_2(y; x, \theta)$$

We want to find two points $x^*$ and $y^*$ such that $x^* \in x^*(x; y^*, \theta)$ and $y^* \in y^*(y; x^*, \theta)$. If these points exist, we know that an equilibrium to the model exists. Fixed point theorems will help us identify when an equilibrium exists when we cannot solve for one explicitly. You have already encountered one fixed point theorem earlier in the lecture on metric spaces, the

Contraction Mapping Theorem (also known as Banach's fixed point theorem). One of the simplest fixed point theorems is **Brouwer's fixed point theorem**.

**Theorem 3 (Brouwer's Fixed Point Theorem)** *Let $X$ be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every continuous function $f : X \to X$ has a fixed point.*

The theorem is difficult to prove in general but is very simple to show on $\mathbb{R}$. Draw a one by one square and then draw a continuous function. It is impossible not to intersect the 45-degree line. Note that there may exist multiple fixed points.

Returning to our example, we can create a new solution correspondence $b(x, y; \theta) \equiv x^*(x; y, \theta) \times y^*(y; x, \theta)$ where for a given $\theta$, $b : X \times Y \to X \times Y$. If $X \times Y$ is a nonempty, compact, convex subset of a finite dimensional normed linear space and if each of the solution correspondences is single-valued, continuous, and nonempty, then $b$ has a fixed point, $(x^*, y^*)$.

Brouwer's theorem is very useful but requires strict assumptions. In particular, it requires $b$ to be single-valued. Kakutani's theorem relaxes this assumption and generalizes Brouwer's theorem to correspondences.

**Theorem 4 (Kakutani's Fixed Point Theorem)** *Let $X$ be a nonempty, compact, convex subset of a finite dimensional normed linear space. Every closed, convex-valued correspondence $G : S \rightrightarrows S$ has a fixed point.*

Now if $x^*(x; y, \theta)$ and $y^*(y; x, \theta)$ are not singletons, we can use Kakutani's theorem to show that if $b$ is closed and convex-valued, then $b$ has a fixed point.

When solving games, we can combine the theorem of the maximum with Kakutani's theorem to show the existence of an equilibrium. Formally, a **strategic game** is comprised of $N$ players, for each player $i$ a nonempty set $S_i$ of strategies, and a utility function $u_i : S \to \mathbb{R}$ on the *strategy space* $S = S_1 \times S_2 \times ... \times S_n$. If we assume that each $S_i$ is nonempty, compact, and convex, then $S$ is also nomempty, compact, and convex. Let $B_i(s) \equiv \arg\max_{s_i \in S_i} u_i(s)$.

If we assume that all $u_i$ are continuous and quasi-concave, then each $B_i$ is compact valued and upper hemi-continuous by the theorem of the maximum and convex-valued by the convex maximum theorem. Now let $B(s) = B_1(s) \times ... \times B_n(s)$. Then $B(s)$ is a closed, convex-valued correspondence. By Kakutani's theorem, $B$ has a fixed point. We refer to this point as a **Nash equilibrium** of the game.

Brouwer's fixed point theorem requires that $f$ be continuous. As we saw in the previous lecture, continuity of an objective function may be too strict of an assumption. Best response functions will also be prone to having some discontinuities. We would still like to know if a solution exists when this is the case.

**Theorem 5 (Tarski's Fixed Point Theorem)** *Every increasing function $f : X \to X$ on a complete lattice $(X, \succsim)$ has a greatest and lowest fixed point.*

A compact subset of $\mathbb{R}$ is a complete lattice. Therefore any increasing function on a compact subset of $\mathbb{R}$ has a fixed point. This is not true of a decreasing function. Try this with the unit square on a piece of paper.

# Exercises

1) Prove that if the conditions of the theorem of the maximum are satisfied, then $x^*(\theta)$ is closed-valued. Then prove that $x^*(\theta)$ is compact-valued.
2) Let $f$ be a continuous function on $[a, b]$. Prove that for all $y \in (f(a), f(b))$, there exists an $c \in (a, b)$ such that $f(c) = y$.

# Source Material

Ashworth, S. & Bueno de Mesquita. (2006). "Monotone Comparative Statics for Models of Politics." *American Journal of Political Science 50*(1), 214-231.

Carter, M. (2001). *Foundations of Mathematical Economics.* Cambridge, MA: The MIT Press.

McCarty, N., & Meirowitz, A. (2007) *Political Game Theory: An Introduction.* New York, NY: Cambridge University Press.