Introduction to Mathematics for Political Science

Problem Set 7: Scalars, Vectors, and Matrices

Instructions: You are encouraged to work in groups and actively participate on the course discussion page. Submitted solutions must be your individual work. You may use a calculator for basic arithmetic, but do not search the internet for solutions. You do not need to replicate drawings in your answers if you describe them verbally. Show all of your work. Submit typed solutions using the link on the course page.

Vectors, Dot Products, Norms

1. Calculate $\boldsymbol{u} \cdot \boldsymbol{v}$

- $\boldsymbol{u}$= $(1, 2, 3, 4, 5)$ and $\boldsymbol{v}$=$(1, 2, 3, 4, 5)$

$$1 + 4 + 9 + 16 + 25 = 55$$

- $\boldsymbol{u}$= $(1, 2)$ and $\boldsymbol{v}$=$(-1, 2)$

$$-1 + 4 = 3$$

- $\boldsymbol{u}$= $(0, 0, 0, 0, 1)$ and $\boldsymbol{v}$=$(1, 2, 3, 4, 7)$

$$7$$

- $\boldsymbol{u}$= $(3, 4, 5)$ and $\boldsymbol{v}$=$(4, 2, 1)$

$$12 + 8 + 5 = 25$$

2. Calculate $\|\boldsymbol{u}\|$

- $\boldsymbol{u}$=$(1, 0, 0)$

$$\sqrt{1} = 1$$

- $\boldsymbol{u}$=$(5, 2, 4, 2)$

$$\sqrt{25+4+16+4} = 7$$

- $\boldsymbol{u}$=$(5, 5, 5, 5)$

$$\sqrt{25+25+25+25} = 10$$

- $\boldsymbol{u}$=$(3, 4)$

$$\sqrt{9+16} = 5$$

3. Solve for θ, the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ (degrees or radians are acceptable). **Hint: If you can visualize the vectors, you may not have to calculate lengths and dot products.**

$$\cos(\theta) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

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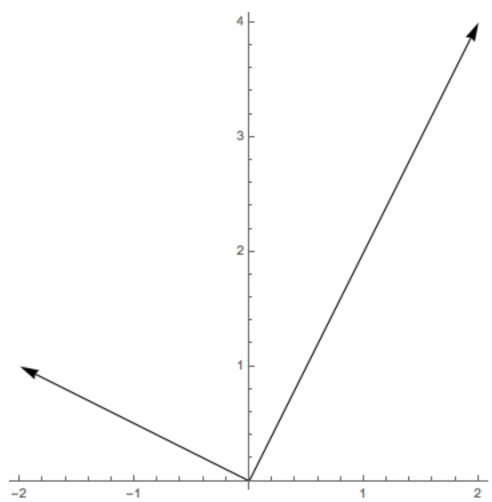
$$\boldsymbol{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\cos(\theta) = \frac{-4+4}{\sqrt{5}\sqrt{20}} = 0 \implies \theta = 90^\circ \quad \theta = \frac{\pi}{2}$$



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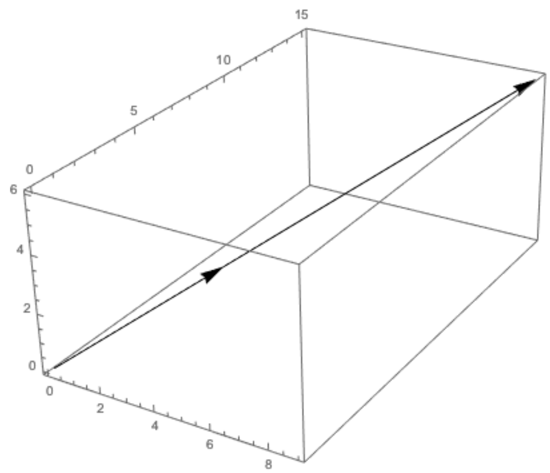
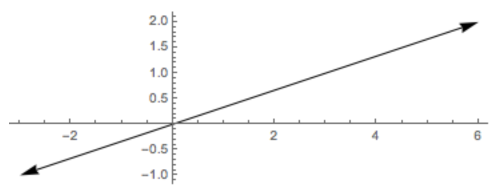
$$\boldsymbol{u} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 9 \\ 15 \\ 6 \end{bmatrix}$$

$$\cos(\theta) = \frac{27+75+12}{\sqrt{38}\sqrt{342}} = 1 \implies \theta = 0^\circ \quad \theta = 0$$



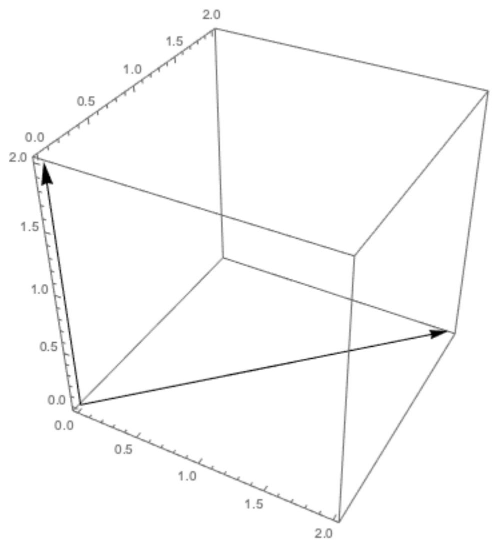
- $$\boldsymbol{u} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\cos(\theta) = \frac{-18 - 2}{\sqrt{10}\sqrt{40}} = -1 \implies \theta = 180^{\circ} \quad \theta = \pi$$



- $$\boldsymbol{u} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\cos(\theta) = \frac{0}{\sqrt{8}\sqrt{4}} = 0 \implies \theta = 90^{\circ} \quad \theta = \frac{\pi}{2}$$



4. Show that the Cauchy-Schwarz inequality ($|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$) holds for the following two dimensional vectors.

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

We need to show

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

Substituting our vectors and rearranging

$$\begin{aligned}
|u_1v_1 + u_2v_2| &\leq \sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2} \\
(u_1v_1 + u_2v_2)^2 &\leq \left(u_1^2 + u_2^2\right)\left(v_1^2 + v_2^2\right) \\
u_1^2v_1^2 + 2u_1v_1u_2v_2 + v_2^2u_2^2 &\leq u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2 \\
2u_1v_1u_2v_2 &\leq u_1^2v_2^2 + u_2^2v_1^2 \\
0 &\leq u_1^2v_2^2 - 2u_1v_1u_2v_2 + u_2^2v_1^2 \\
0 &\leq (u_1v_2 - u_2v_1)^2
\end{aligned}$$

Regardless of the value of $u_1v_2 - u_2v_1$, $(u_1v_2 - u_2v_1)^2$ will be positive, as desired.

5. Solve for a

- $$a \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 9 \end{bmatrix}$$

$$a = 3$$

- $$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1-a) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$a = \frac{3}{4}$$

6. Let $\boldsymbol{y} = \boldsymbol{x}\beta + \boldsymbol{\epsilon}$ where

$$\boldsymbol{y} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ 10 \\ 10 \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad \beta = 2$$

   a. Find $\boldsymbol{\epsilon}$.

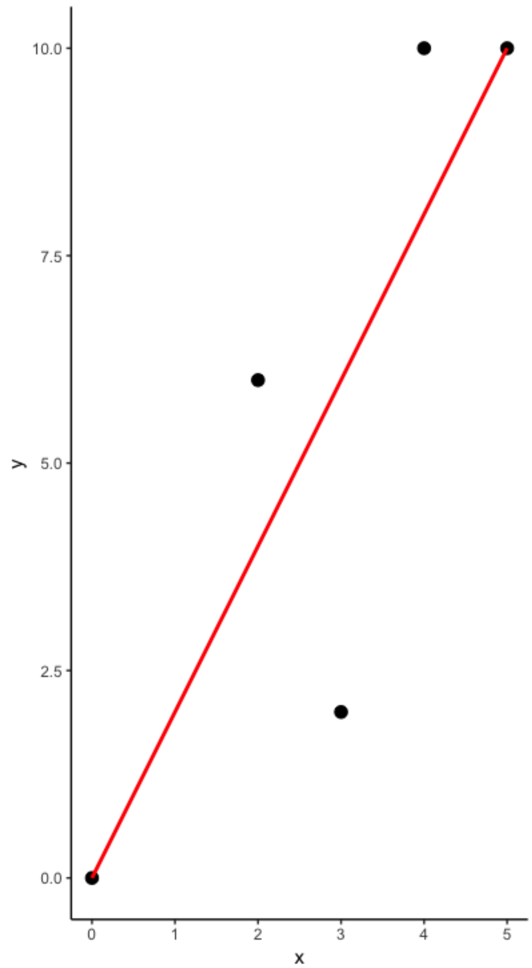
$$\boldsymbol{\epsilon} = \boldsymbol{y} - \boldsymbol{x}\beta$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 2 \\ 0 \end{bmatrix}$$

   b. Calculate $(\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y}$. Interpret your result. (Hint: Try drawing a picture.)

$$\begin{aligned} (\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y} &= \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{x} \cdot \boldsymbol{x}} \\ &= \frac{108}{54} \\ &= 2 \end{aligned}$$

We have showed $(\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y} = \beta$, where β is the slope of the line of best fit through the points (x_i, y_i). More generally, $(\boldsymbol{x}^T\boldsymbol{x})^{-1}\boldsymbol{x}^T\boldsymbol{y}$ *minimizes the sum of squared residuals*, $\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}$.



Matrix Properties

1. Calculate AB.

- $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 11 \\ 4 \\ 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 \\ 4 \\ 6 \end{bmatrix}$$

- $$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- $$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & 8 \\ 7 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 \\ 26 \\ 7 \end{bmatrix}$$

- $$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 1 & 0 & 5 \\ 2 & 1 & 2 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 3 & 4 & 15 \\ 10 & 3 & 2 & 15 \end{bmatrix}$$

2. Calculate the dimensions of the resulting matrix.

- $A_{3\times2}B_{2\times3}$

$$3 \times 3$$

- $A_{3\times2}B_{2\times3}C_{3\times1}$

$$3 \times 1$$

- $A_{2\times3}A^T_{2\times3}B_{2\times6}$

$$2 \times 6$$

- $A_{1\times3}A_{3\times1} + B_{1\times1}$

$$1 \times 1$$

3. What does it mean for a sequence of matrices to be *conformable*? Why are A^TA and AA^T always conformable?

A set of matrices is conformable if their product is defined. A matrix A with dimension $m \times n$ can always be multiplied by its transpose A^T with dimension $n \times m$, because A has the same number of columns as A^T has rows (n). Similarly, A^T has the same number of columns as A has rows (m), allowing these matrices to be multiplied.