

# IMPS 2018: Final Exam (Practice)

11 September 2018

**Instructions:** This is a closed book examination. Calculators are not permitted. **There are 9 questions, from which you can choose 6 to answer.** Each question is worth ten points and should take about 30 minutes to complete. You have three hours.

1. Let  $S$  and  $T$  be sets. Show

$$(S \cup T)^c = S^c \cap T^c$$

Take  $x \in (S \cup T)^c$ . Then  $x \notin S \cup T \implies x \notin S \wedge x \notin T \implies x \in S^c \wedge x \in T^c \implies x \in S^c \cap T^c$ . So  $(S \cup T)^c \subseteq S^c \cap T^c$ . Now take  $x \in S^c \cap T^c$ . Then  $x \in S^c \wedge x \in T^c \implies x \notin S \wedge x \notin T \implies x \notin S \cup T \implies x \in (S \cup T)^c$ . So  $S^c \cap T^c \subseteq (S \cup T)^c$ . If  $(S \cup T)^c \subseteq S^c \cap T^c$  and  $S^c \cap T^c \subseteq (S \cup T)^c$  then  $(S \cup T)^c = S^c \cap T^c$ . ■

2. Consider a sequence  $\{x_n\} \in (\mathbb{R}, d_1)$  such that

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} + 1 & \text{if } n \text{ is even} \end{cases}$$

Does this sequence converge? Prove your answer.

**Solution** The sequence converges to one. To see why, first consider the case in which  $n$  is odd. Let  $N^* = \lceil \frac{1}{\epsilon} \rceil$ . Then, for  $n_{\text{odd}} > N^*$ ,

$$\begin{aligned} d(x_n, 1) &= \left| \left( \frac{1}{n} + 1 \right) - 1 \right| \\ &< \left| \frac{1}{N^*} \right| \\ &\leq \epsilon \end{aligned}$$

If  $n$  is even, then  $x_n = 1$  and

$$d(x_n, 1) = d(1, 1) = 0 < \epsilon$$

■

3. Let  $f$  be a continuous functional on a metric space  $(X, d)$ . Prove  $\alpha f$  is continuous for every  $\alpha \in \mathbb{R}$ .

**Solution**

If  $f$  is continuous then for every  $\epsilon > 0$  there exists a  $\delta$  such that

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\epsilon}{\alpha}$$

Then,

$$\begin{aligned} \alpha |f(x_1) - f(x_2)| &< \epsilon \\ |\alpha f(x_1) - \alpha f(x_2)| &< \epsilon \end{aligned}$$

which demonstrates the continuity of  $\alpha f$ . ■

4. Let  $N : \mathbb{R}^n \rightarrow \mathbb{R}$  be a norm. Use the properties of a norm prove that  $N$  is a convex function.

**Solution**

Using the properties of the norms, take  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\begin{aligned} N(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) &\leq N(\lambda \mathbf{x}) + N((1 - \lambda)\mathbf{y}) && \text{Triangle Inequality} \\ &= |\lambda|N(\mathbf{x}) + |(1 - \lambda)|N(\mathbf{y}) && \text{Homogeneity} \\ &= \lambda N(\mathbf{x}) + (1 - \lambda)N(\mathbf{y}) \end{aligned}$$

■

5. Consider the projection operator  $\mathbf{p}_s : X \rightarrow S$  where  $S$  is a subspace of a inner product space  $X$ . Prove that if  $\mathbf{x} - \mathbf{p}_s(\mathbf{x}) \perp S$  then

$$d(\mathbf{x}, \mathbf{p}_s(\mathbf{x})) = \min \{d(\mathbf{x}, \mathbf{y}) | \mathbf{y} \in S\}$$

**Solution**

Take an arbitrary  $\mathbf{y} \in S$ . It must be true that

$$\|\mathbf{x} - \mathbf{p}_s(\mathbf{x})\|^2 \leq \|\mathbf{x} - \mathbf{p}_s(\mathbf{x})\|^2 + \|\mathbf{p}_s(\mathbf{x}) - \mathbf{y}\|^2$$

Because  $\mathbf{x} - \mathbf{p}_s(\mathbf{x}) \perp \mathbf{p}_s(\mathbf{x}) - \mathbf{y}$ , Pythagorus' Theorem applies

$$\begin{aligned} \|\mathbf{x} - \mathbf{p}_s(\mathbf{x})\|^2 &\leq \|\mathbf{x} - \mathbf{p}_s(\mathbf{x})\|^2 + \|\mathbf{p}_s(\mathbf{x}) - \mathbf{y}\|^2 \\ &= \|\mathbf{x} - \mathbf{p}_s(\mathbf{x}) + \mathbf{p}_s(\mathbf{x}) - \mathbf{y}\|^2 \\ &= \|\mathbf{x} - \mathbf{y}\|^2 \end{aligned}$$

Since  $X$  and  $S$  are normed linear spaces,  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ . We have therefore shown that

$$d(\mathbf{x}, \mathbf{p}_s(\mathbf{x})) \leq d(\mathbf{x}, \mathbf{y})$$

for all  $\mathbf{y} \in S$  demonstrating the proposition. ■

6. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable functional. Prove that if  $f$  is decreasing, then  $Df[\mathbf{x}] \leq 0$  for all  $\mathbf{x}$ .

**Solution** We can approximate changes in the function  $f$  with the derivative

$$f(\mathbf{x}_0 + d\mathbf{x}) - f(\mathbf{x}_0) = Df[\mathbf{x}_0](d\mathbf{x})$$

Since  $f$  is a decreasing function  $f(\mathbf{x}') < f(\mathbf{x})$  for all  $\mathbf{x}' > \mathbf{x}$ . Then for any  $d\mathbf{x} \geq 0$ ,  $f(\mathbf{x}_0 + d\mathbf{x}) - f(\mathbf{x}_0) \leq 0 \implies Df[\mathbf{x}_0](d\mathbf{x}) \leq 0$  as desired.

7. Solve

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = 1 - (x_1 - 1)^2 - (x_2 - 1)^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 = 1 \end{aligned} \tag{1}$$

**Hint:** Be sure to check the concavity of the objective function along the constraint set.

**Solution**

We start by showing the global concavity of the objective function, which is sufficient to guarantee that the function is concave along the constraint set. Taking first order partials, we have

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = -2(x_1 - 1) \quad \frac{\partial f(x_1, x_2)}{\partial x_2} = -2(x_2 - 1)$$

Then, taking second order partials, we have

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = -2 \quad \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = -2 \quad \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 0$$

This allows us to construct the Hessian.

$$H_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

We then check the Hessian's definiteness. We must have  $\mathbf{y}^T H_f \mathbf{y} \leq 0$  for all  $\mathbf{y} \neq 0$  for  $f$  to be concave.

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

giving  $-2y_1^2 - 2y_2^2 < 0$  which holds for all  $\mathbf{y}$ . So  $f$  is globally concave.

Now we can write the Lagrangian

$$\mathcal{L} = 1 - (x_1 - 1)^2 - (x_2 - 1)^2 - \lambda(1 - x_1^2 - x_2^2)$$

and take first order conditions.

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2(x_1 - 1) + 2\lambda x_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -2(x_2 - 1) + 2\lambda x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(1 - x_1^2 - x_2^2) = 0$$

The first two conditions imply

$$x_1 = \frac{1}{1 - \lambda} = x_2$$

Let  $x = x_1 = x_2$ . Then by the last first order condition we must have

$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x_1^* = x_2^* = x^* = \frac{1}{\sqrt{2}}$$

8. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a concave function. Consider the optimization problem

$$\max_x f(x; \theta)$$

The value function is given by  $V(\theta) = f(x^*(\theta); \theta)$ . Prove that

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^*(\theta); \theta)}{\partial \theta}$$

### Solution

Because  $f$  is concave, we must have

$$\frac{\partial f(x^*(\theta); \theta)}{\partial x} = 0$$

by the first order condition. The derivative of the value function with respect to  $\theta$  is given by

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^*(\theta); \theta)}{\partial \theta} = \underbrace{\frac{\partial f(x^*(\theta); \theta)}{\partial x^*(\theta)}}_{=0} \frac{\partial x^*(\theta)}{\partial \theta} + \frac{\partial f(x^*(\theta); \theta)}{\partial \theta}$$

giving

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial f(x^*(\theta); \theta)}{\partial \theta}$$

as desired.

9. Prove for any square, invertible matrix  $A$ , and for all  $n > 0$ ,  $(A^n)^{-1} = (A^{-1})^n$ .

### Solutions

We prove the statement by induction. For  $n = 1$ , this is just  $(A^1)^{-1} = A^{-1} = (A^{-1})^1$ . Therefore, the statement holds for the base case. Now let us assume that the statement is true for a given  $n$ , that is  $(A^n)^{-1} = (A^{-1})^n$ . We need to show that the statement is also true for  $n + 1$ . We can prove this statement by showing that  $A^{n+1}(A^{-1})^{n+1} = I$

$$\begin{aligned} A^{n+1}(A^{-1})^{n+1} &= AA^n(A^{-1})^n A^{-1} \\ &= AA^n(A^n)^{-1} A^{-1} && \text{By the induction assumption} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$