Insurance analytics

We shrunk the parameters - Lasso, friends of Lasso and the actuary

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Acknowledgement

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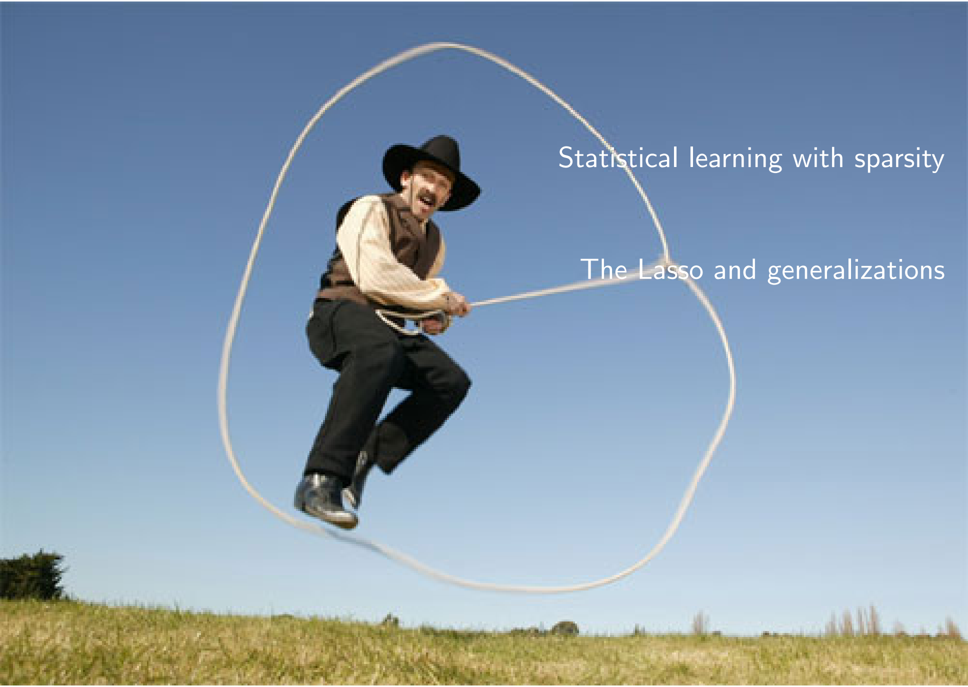
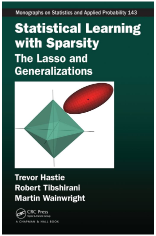

Statistical learning with sparsity
The Lasso and generalizations

Motivation
Sparsity



- Crucial need to sort through the mass of information and bring it down to its bare essentials.

- One form of simplicity is sparsity.

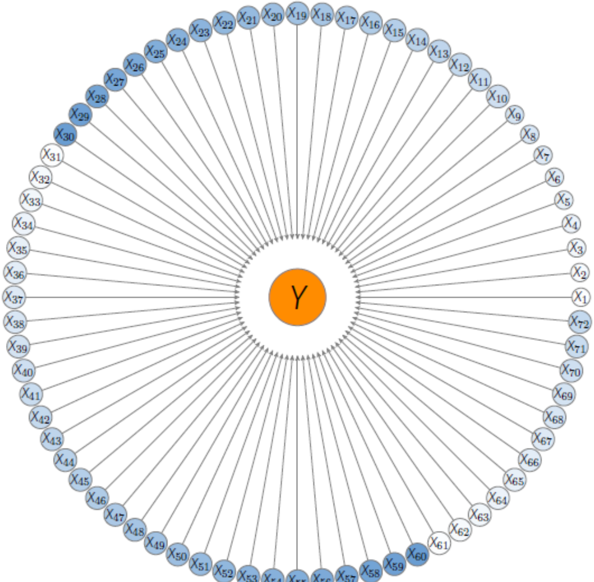
- In a sparse statistical model only a relatively small number of parameters (or predictors) play a role.

- The 'bet on sparsity' principle:

Use a procedure that does well in sparse problems, since no procedure does well in dense problems.

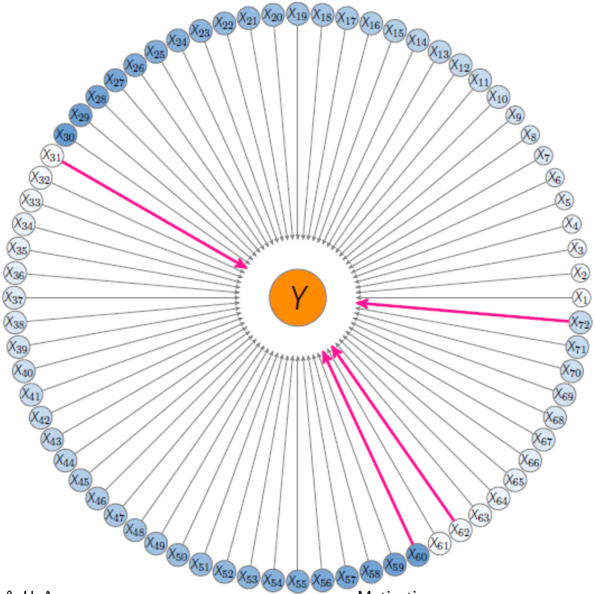
Motivation

Bet on sparsity



Motivation

Bet on sparsity



Motivation

Shrinkage methods

- Our pricing example initially applied a best subset selection strategy to select relevant predictors.
- Alternative strategy:
  - fit a model with all p predictors
  - constrain or regularize the coefficient estimates $\rightsquigarrow$ shrink the coefficient estimates to zero.
- Shrinking the coefficient estimates can significantly reduce their variance. Some types of shrinkage put some of the coefficients exactly equal to zero!

Ridge (least squares) regression

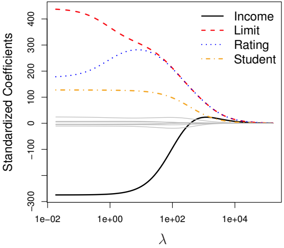
- The least-squares optimization problem

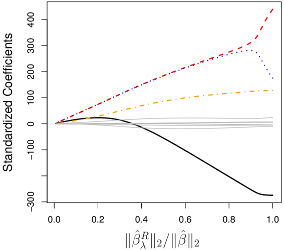
$$\min_{\beta_0, \boldsymbol{\beta}} \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 = \min_{\beta_0, \boldsymbol{\beta}} \text{ RSS}$$

subject to a 'budget' t constraint

$$\sum_{j=1}^{p} \beta_j^2 \leq t \text{ or } \|\boldsymbol{\beta}\|_2^2 \leq t.$$

Ridge regression





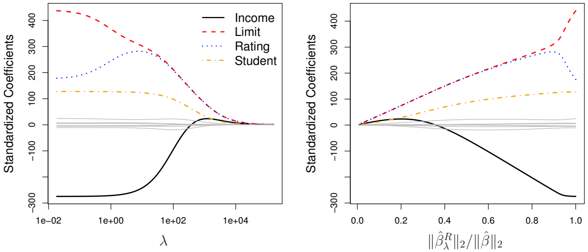
- Dual problem formulation

$$\min_{\beta_0, \boldsymbol{\beta}} \ \text{RSS} + \lambda \sum_{j=1}^{p} \beta_j^2,$$

with

- $\lambda \geq 0$ a tuning parameter and $\lambda \sum_{j=1}^{p} \beta_j^2$ a shrinkage penalty
- with $\lambda = 0$ the least squares estimates result (all $\neq 0$!)
- with $\lambda \to \infty$ coefficients will approach zero.

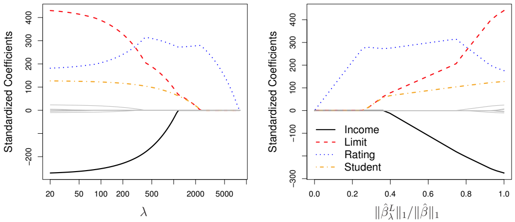
Ridge regression



- Points of attention:
  - a set of coefficient estimates $\hat{\boldsymbol{\beta}}_\lambda^R$ for each value of λ!
  - shrink the estimated association of each predictor with the response, but do not shrink the intercept
  - with centered to mean zero predictors, then $\hat{\beta}_0 = \bar{y} = \sum_{i=1}^n y_i/n$
  - standard least squares coefficients are scale invariant, not the case for ridge regression coefficients!
  - therefore, best to apply ridge regression after standardizing the predictors

$$\tilde{x}_{ij} = \frac{x_{ij}}{\sqrt{\frac{1}{n}\sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}}.$$

Lasso



- The ridge penalty shrinks all coefficients to zero, but does not set any of them exactly to zero.

- The lasso shrinks coefficient estimates to zero, and performs variable selection

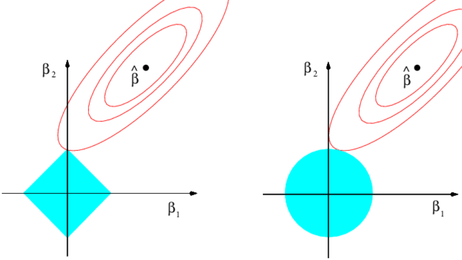
$$\min_{\beta_0,\boldsymbol{\beta}} \text{ RSS subject to } \sum_{j=1}^{p} |\beta_j| \leq t \text{ or } \min_{\beta_0,\boldsymbol{\beta}} \text{ RSS} + \lambda \sum_{j=1}^{p} |\beta_j|.$$

  Thus, lasso uses the ℓ_1 penalty instead of ℓ_2 penalty.

- Lasso is for Least absolute shrinkage and selection operator.

Lasso

Variable selection property



- When $p = 2$:
  - lasso coefficient estimates have smallest RSS out of all points in the diamond

$$|\beta_1| + |\beta_2| \leq t$$

  - ridge coefficient estimates have smallest RSS out of all points in the circle

$$\beta_1^2 + \beta_2^2 \leq t$$

  - ellipses (around least-squares $\hat{\boldsymbol{\beta}}$) represent regions of constant RSS
  - since lasso has corners at each of the axes, ellipse will often intersect the constraint region at an axis.

Lasso

Variable selection property

- Recall the best subset selection problem

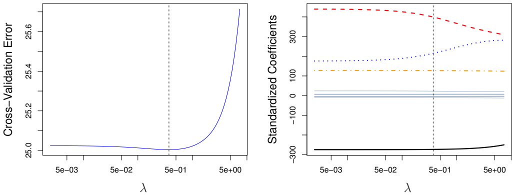
$$\min_{\beta_0, \boldsymbol{\beta}} \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 \text{ subject to } \sum_{j=1}^{p} I(\beta_j \neq 0) \leq t.$$

  Solving this problem is computationally infeasible when p is large!

- In general: with the ℓ_q norm of $\boldsymbol{\beta}$ as penalty
  - $q < 1$ the solution is sparse, but the problem is not convex
  - $q > 1$ the problem is convex, but the solution is not sparse.
- The value $q = 1$ is the smallest value that yields a convex problem.
- Convexity, as well as the sparsity assumption, greatly simplifies the computation.

Ridge and Lasso

Selecting the tuning parameter



- Use cross-validation to select a value for λ (or, equivalently, for the budget t).
- Choose a grid of λ values:
  - compute the cross-validation (CV) error for each value of λ
  - select the tuning parameter value for which the CV error is smallest.
- Refit the model using all available observations and the selected value of λ.

Ridge and Lasso

A glimpse at the computation

- To gain intuition about ridge and lasso regression, consider a simplified problem:
  - $n = p$
  - $\boldsymbol{X}$ a unit matrix
  - no intercept.
- This basic setting sheds light on the computation of the Lasso estimates in more general problems.

Ridge and Lasso

A glimpse at the computation

- Usual least squares problem:

$$\min_{\beta_1,\ldots,\beta_p} \sum_{j=1}^{p} (y_j - \beta_j)^2.$$

- Take derivative wrt β_j and solve:

$$2 \cdot (y_j - \beta_j) \; = \; 0.$$

With solution: $\hat{\beta}_j = y_j$.

Ridge and Lasso

A glimpse at the computation

- In this setting, ridge regression amounts to

$$\min_{\beta_1,\ldots,\beta_p} \sum_{j=1}^{p}(y_j - \beta_j)^2 + \lambda \sum_{j=1}^{p} \beta_j^2,$$

with solution

$$\hat{\beta}_j^R \quad = \quad \frac{y_j}{(1+\lambda)}.$$

Ridge and Lasso

A glimpse at the computation

- In this setting, Lasso regression amounts to

$$\min_{\beta_1,\ldots,\beta_p} \sum_{j=1}^{p}(y_j-\beta_j)^2+\lambda\sum_{j=1}^{p}|\beta_j|,$$

with solution

$$\hat{\beta}_j^L = \begin{cases} y_j-\lambda/2 & y_j>\lambda/2 \\ y_j+\lambda/2 & y_j<-\lambda/2 \\ 0 & |y_j|\leq\lambda/2. \end{cases}$$

- In compact notation, the $\hat{\beta}_j^L=\mathcal{S}_{\frac{\lambda}{2}}(\hat{\beta}_j)$ where $\mathcal{S}$ is the soft-thresholding operator with $\mathcal{S}_{\frac{\lambda}{2}}(x)=\text{sign}(x)(|x|-\frac{\lambda}{2})_+$.

Ridge and Lasso

A glimpse at the computation

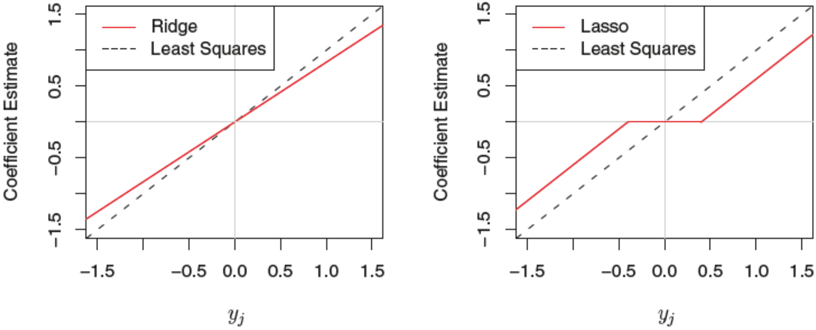


FIGURE 6.10. *The ridge regression and lasso coefficient estimates for a simple setting with $n = p$ and $\mathbf{X}$ a diagonal matrix with 1's on the diagonal.* Left: *The ridge regression coefficient estimates are shrunken proportionally towards zero, relative to the least squares estimates.* Right: *The lasso coefficient estimates are soft-thresholded towards zero.*

Ridge and Lasso with linear models

Implementation in `R`

- OLS and ridge regression have analytic solutions.
- GLMs and GAMs have only numerical solutions with iterative methods.
- The lasso (with linear models) falls somewhere in-between these two cases:
  - has a direct numerical solution via the Least Angle Regression (LAR) algorithm (in `R`, the `lars` package)
  - the `glmnet` package implements pathwise (cyclical) coordinate descent

    can be faster than LAR in large problems.

Ridge and Lasso

Generalized Linear Model setting

- Minimize

$$\min_{\beta_0, \boldsymbol{\beta}} -\frac{1}{n}\mathcal{L}(\beta_0, \boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X}) + \lambda\|\boldsymbol{\beta}\|_1.$$

  Here $\mathcal{L}$ is the log-likelihood of a GLM.

- Some examples:

Gaussian	$\frac{1}{2\sigma^2}\|\boldsymbol{y} - \beta_0\mathbf{1} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2$
logistic	$\sum_{i=1}^{n} y_i(\beta_0 + \boldsymbol{\beta}^t x_i) - \log\left(1 + e^{\beta_0 + \boldsymbol{\beta}^t x_i}\right)$
Poisson	$\sum_{i=1}^{n} y_i(\beta_0 + \boldsymbol{\beta}^t x_i) - e^{\beta_0 + \boldsymbol{\beta}^t x_i}.$

Ridge and Lasso

The `glmnet` package in `R`

- Family members: (a.o.) `gaussian`, `binomial`, `poisson`.
- Penalties:

$$\lambda P_\alpha(\boldsymbol{\beta}) = \lambda \cdot \sum_{j=1}^{p} \left\{ \frac{(1-\alpha)}{2}\beta_j^2 + \alpha|\beta_j| \right\},$$

  with
  - $\alpha \in [0,1]$ the elastic-net parameter (to mix ridge and lasso).

Ridge and Lasso

The `glmnet` package in `R`

- `glmnet` implements coordinate-descent algorithms for fitting (elastic net) penalized GLMs:
  - apply coordinate descent to quadratic approximation (cfr. PIRLS)
  - nested algorithm with

    (outer loop) decrement λ

    (middle loop) update quadratic approximation using current parameter estimates $(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}})$

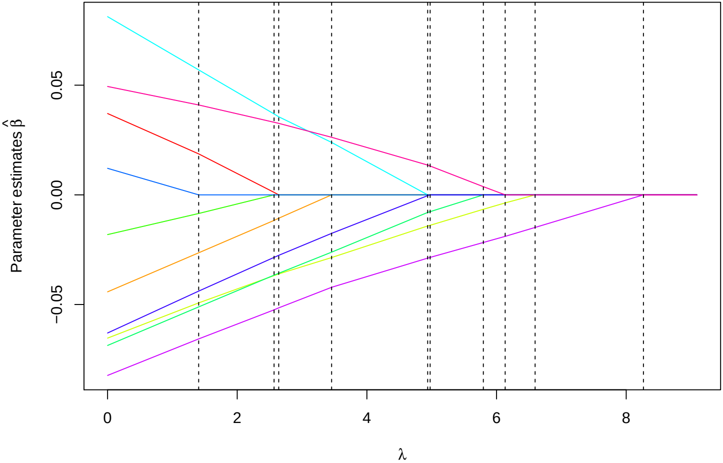
    (inner loop) run coordinate descent on penalized weighted-least-squares problem

$$\min_{\beta_0, \boldsymbol{\beta}} -\ell_Q(\beta_0, \boldsymbol{\beta}) + \lambda P_\alpha(\boldsymbol{\beta}).$$

Ridge and Lasso

A typical Lasso plot with `glmnet`

overfitting $\longleftarrow$ λ $\longrightarrow$ underfitting



Lasso and friends

- We turn to some useful variations of the basic lasso ℓ_1-penalty:
  - groups of correlated features

    $\rightsquigarrow$ lasso does not perform well, elastic net is better and selects correlated features (or not) together
  - structurally grouped features

    $\rightsquigarrow$ select or omit all within a group together via group lasso
  - neighbouring coefficients to be the same

    $\rightsquigarrow$ fused lasso.

Lasso and friends

Elastic net

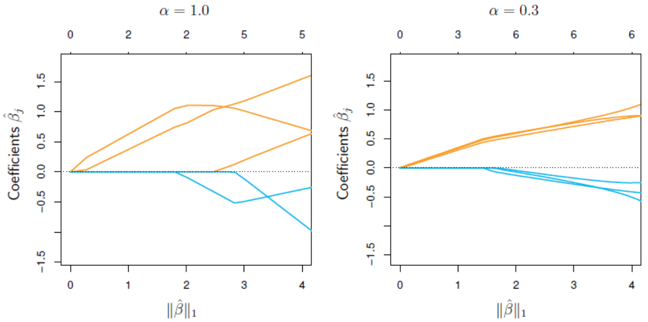


Figure 4.1 *Six variables, highly correlated in groups of three. The lasso estimates ($\alpha = 1$), as shown in the left panel, exhibit somewhat erratic behavior as the regularization parameter λ is varied. In the right panel, the elastic net with ($\alpha = 0.3$) includes all the variables, and the correlated groups are pulled together.*

Lasso and friends

And the actuary ...

- Adjust lasso regularization to the type of risk factor:
  - determine type (nominal / numeric $\sim$ ordinal / spatial)
  - allocate logical penalty.
- Thus, for J risk factors, each with convex regularization term $g_j(.)$, we want to optimize:

$$-\frac{1}{n}\log\mathcal{L}\left(\beta_0,\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_J\right)+\lambda\cdot\sum_{j=1}^{J}g_j\left(\boldsymbol{\beta}_j\right).$$

A multi-type regularized predictive model!

Regularization with multi-type penalty

- Continuous or binary risk factors: lasso

$$g_{\text{Lasso}}(\boldsymbol{\beta}_j) = \sum_i w_{j,i} |\beta_{j,i}|.$$

- Ordinal risk factors: fused lasso

$$g_{\text{fLasso}}(\boldsymbol{\beta}_j) = \sum_i w_{j,i} |\beta_{j,i+1} - \beta_{j,i}| = ||\boldsymbol{D}(\boldsymbol{w}_j)\boldsymbol{\beta}_j||_1.$$

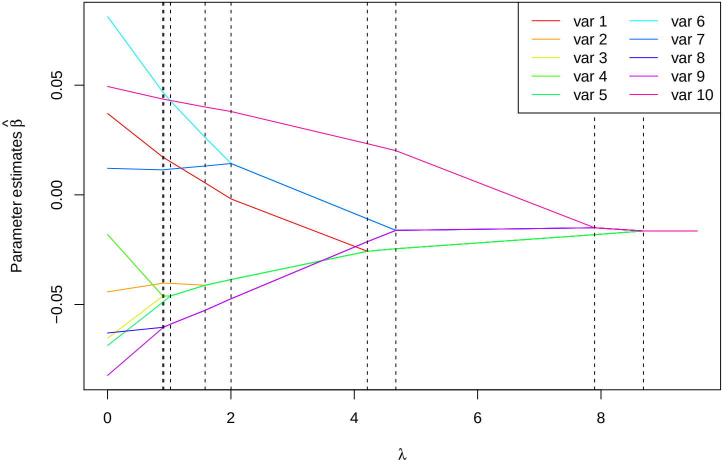
- Nominal risk factors: generalized fused lasso

$$g_{\text{gflasso}} = \sum_{(i,l)\in\mathcal{G}} w_{j,il} |\beta_{j,i} - \beta_{j,l}| = ||\boldsymbol{G}(\boldsymbol{w}_j)\boldsymbol{\beta}_j||_1.$$

Lasso and friends

Fused Lasso with `genlasso`

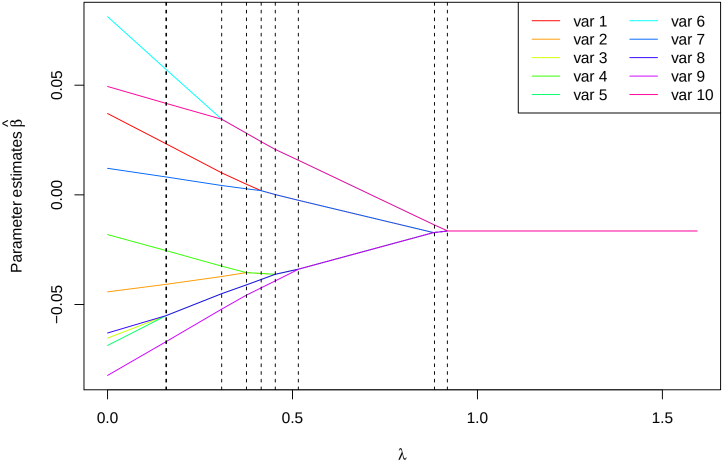
overfitting $\longleftarrow$ λ $\longrightarrow$ underfitting



Lasso and friends

Generalized Fused Lasso with `genlasso`

overfitting $\longleftarrow$ λ $\longrightarrow$ underfitting



SMuRF

Sparse Multi-type Regularized Feature modeling

- SMuRF unifies penalty-specific (machine learning) literature with statistical (or: actuarial) literature!
- Efficient algorithm (with proximal operators).
- Scalable to large (big) data (splits into smaller sub-problems).
- Flexible regularization
  - penalty takes type of risk factor into account
  - works for all popular penalties.

MTPL data: Poisson with multi-type penalty

- Model claim frequencies with regularized Poisson GLM

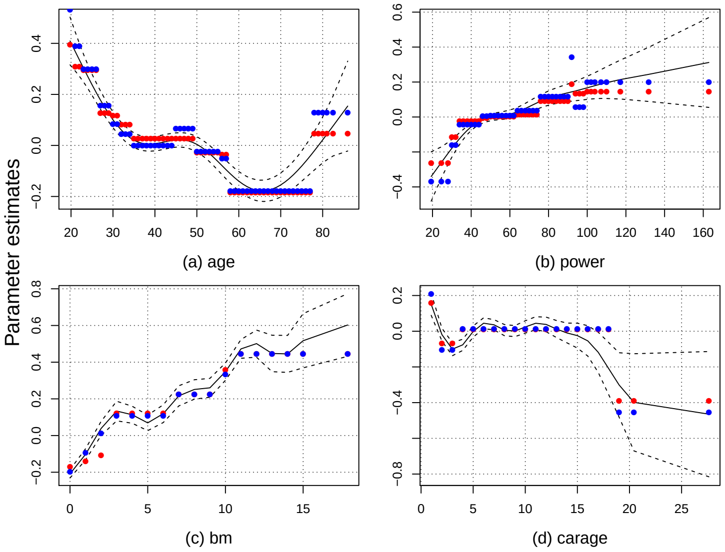
$$
\begin{aligned}
&-\frac{1}{n}\log \mathcal{L}(\boldsymbol{\beta}; \boldsymbol{X}, \boldsymbol{y}) \\
&\qquad + \lambda \left(\sum_{j \in \text{bin}} |w_j \beta_j| + \sum_{j \in \text{ord}} ||\boldsymbol{D}(\boldsymbol{w}_j)\boldsymbol{\beta}_j||_1 + ||\boldsymbol{G}(\boldsymbol{w}_{\text{muni}})\boldsymbol{\beta}_{\text{muni}}||_1 \right).
\end{aligned}
$$

- Incorporate multi-type penalty, with:
  - standard Lasso for binary `use`, `fleet`, `mono`, `four`, `sports`, `sex` and `fuel`
  - fused Lasso for ordinal `payfreq`, `coverage`, `ageph`, `bm`, `power`, `agec`
  - generalized fused Lasso for spatial `muni`.

MTPL data: Poisson with multi-type penalty

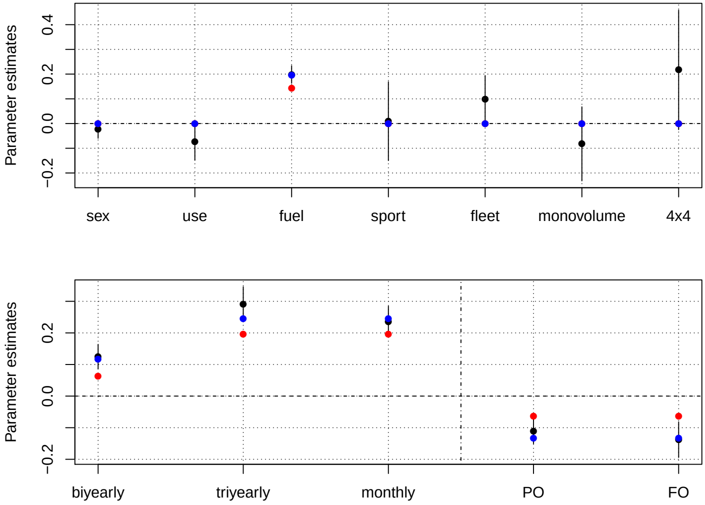
- Settings:
  - incorporate adaptive (GLM) and standardization weights for better consistency and predictive performance
  - tune λ with 10-fold stratified cross-validation where the deviance is used as error measure and the one-standard-error rule is applied
- Re-estimate the final sparse GLM with standard GLM routines (from 422 to 71 params.).

MTPL data: Poisson with multi-type penalty



GAM fit, penalized GLM fit, GLM refit with new bins

MTPL data: Poisson with multi-type penalty



GAM fit, penalized GLM fit, GLM refit with new bins

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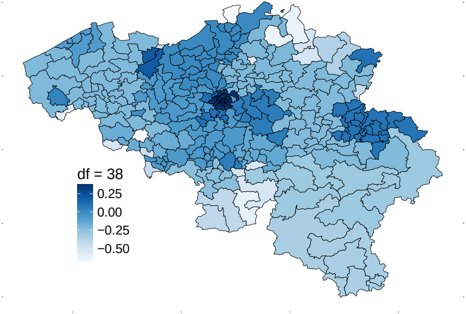
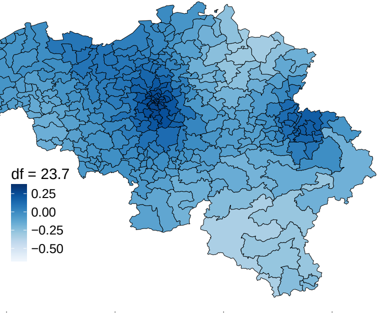


(a) SMuRF estimates



(b) GAM estimates

Wrap-up

- From multi-step (published in SAJ, `R` code upon request) to less is more.
- Flexible regularization can help predictive modeling tasks.
- `SMuRF` package, vignette and working paper forthcoming.

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