MCMC Algorithms

Markov chain definition

Markov chain: discrete time stochastic process

Definition: a series of random variables X_0, X_1, X_2, ... (all valued over the same state space) with the "memoryless" **Markov property**:

$$p(X_i = x | X_0 = x_0, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) = p(X_i = x | X_{i-1} = x_{i-1})$$

The set E of all possible values of X_i is called the **state space**

2 parameters:

1. initial distribution $p(X_0)$
2. tansition probabilities $T(x, A) = p(X_i \in A | X_{i-1} = x)$

NB: only **homogeneous** Markov chains considered here:
$p(X_{i+1} = x | X_i = y) = p(X_i = x | X_{i-1} = y)$

Markov chains properties

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Property: a Markov chain is **aperiodic** if nothing induces periodic behavior of the trajectories

Stationary law & ergodic theorem

Definition: A probability distribution $\tilde{p}$ is called **invariant law** (or **stationary law**) for a Markov chain if it verifies the following property:

if $X_i \sim \tilde{p}$, then $X_{i+j} \sim \tilde{p} \ \forall j \geq 1$

Remark: a Markov chain can admit several stationary laws

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Ergodic theorem (infinite space): A positive irreducible and recurrent Markov chain admits a single invariant probability distribution $\tilde{p}$ and converges towards it

Markov chain example (discrete state space)– I

Doudou (a hamster) follows a Markov chain every minute with 3 states:

- S sleep
- E eat
- W work out

⇒ its activity in 1min only depends on its current activity

Matrix of transition probabilities:

$$P = \begin{pmatrix} X_i/X_{i+1} & S & E & W \\ S & 0.9 & 0.05 & 0.05 \\ E & 0.7 & 0 & 0.3 \\ W & 0.8 & 0 & 0.2 \end{pmatrix}$$

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1) Is the Markov chain irreducible ? recurrent ? aperiodic ?

2) Suppose Doudou is now asleep. What about in 2 min ? in 10 min ?

3) Suppose now that Doudou is working out. What about in 10 min ?

Markov chain example (discrete state space) – II

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Markov chain example (discrete state space) – II

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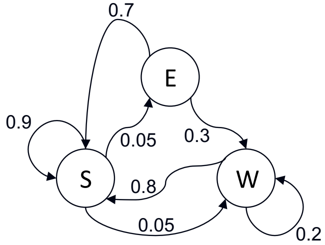

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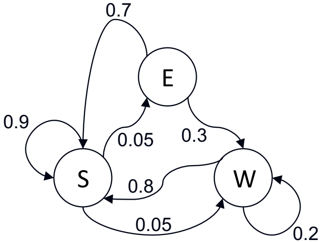

2) Suppose Doudou is now asleep. What about in 2 min ? in 10 min ?

$$x_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}^T \quad x_2 = x_0P^2 = \begin{pmatrix}0.885\\0.045\\0.070\end{pmatrix}^T \quad x_{10} = x_2P^8 = x_0P^{10} = \begin{pmatrix}0.884\\0.044\\0.072\end{pmatrix}^T$$

Markov chain example (discrete state space) – II

3) Suppose now that Doudou is working out. What about in 10 min ?

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Markov chain example (discrete state space) – II

3) Suppose now that Doudou is working out. What about in 10 min ?

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \qquad x_{10} = x_0 P^{10} = \begin{pmatrix} 0.884 \\ 0.044 \\ 0.072 \end{pmatrix}^T$$

Here, the Markov chain being aperiodic, recurrent and irreducible, there is a stationary law: $\tilde{p} = \tilde{p}P$.