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$$\overbrace{X_0 \to X_1 \to X_2 \to \cdots \to X_n}^{\text{Markov chain convergence}} \to \overbrace{X_{n+1} \to X_{n+2} \to \cdots \to X_{n+N}}^{\text{Monte Carlo sample}}$$

General framework of MCMC algorithms

MCMC algorithms uses an acceptance-rejection framework

1. Initialise $x^{(0)}$
2. For $t = 1 \dots n + N$:
   a. Propose a new candidate $y^{(t)} \sim q(y^{(t)}|x^{(t-1)})$
   b. Accept $y^{(t)}$ with probability $\alpha(x^{(t-1)}, y^{(t)})$:
      $x^{(t)} := y^{(t)}$
      if $t > n$, "save" $x^{(t)}$ (as part of the final Monte Carlo sample)

where q is the instrumental distribution for proposing new samples and α is the acceptance probability.

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NB: *ideally* q is **easy** and **fast** to **compute**