CONSUMPTION

JYOTIRMOY BHATTACHARYA

1. Two-period case

1.1. **Budget constraint.** Consider a consumer who lives for two periods, has an endowment of y_1 and y_2 units of goods in the two periods respectively and can borrow and lend any amount that they like at the real rate of interest r.

Suppose the consumer consumes c_1 in the first period. Then she will have to take a loan of $c_1 - y_1$ to finance her consumption. (This number can be negative, in which case the consumer is lending rather than borrowing.) In the next period the consumer will therefore have to make loan repayments of $(1+r)(c_1 - y_1)$. Assume that the consumer does not want to make any bequests and cannot die with any outstanding loans, consumption in the second period must be,

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

Simplifying and rearranging we have

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{1}$$

This is the budget constraint faced by the consumer. We can interpret this to mean that the present value of the consumer's consumption stream must equal the present value of their incomes.

1.2. **Utility maximization.** Suppose the consumer maximises a quasiconcave utility function $U(c_1, c_2)$ subject to this budget constraint. Then the consumer's first-order conditions are

$$U_1(c_1, c_2) = \lambda \tag{2}$$

$$U_2(c_1, c_2) = \lambda/(1+r) \tag{3}$$

where λ is the Lagrange multiplier corresponding to the budget constraint and $U_i(c_1, c_2)$ denotes the partial derivative $\partial U/\partial c_i$. We have explicitly shown the dependence of the partial derivatives on the value of consumption in both periods. These first-order conditions

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along with the budget constraint (1) together determines the value of c_1, c_2 and λ.

1.3. Comparative statics.

Assuming that consumption in both periods is a normal good, an increase in either y_1 or y_2 increases both c_1 and c_2.

The effects of a change in r are ambiguous. An increase in r makes consumption in period 2 relatively cheap compared to consumption in period 1. Therefore the substitution effect causes c_1 to decrease and c_2 to increase. It is traditional to decompose the income effect into two parts. First, an increase in r reduces the present value of the consumer's endowments and hence decreases his real income. Second, an increase in r, by making the consumption in period 2 cheaper increases his real income.[1] The sign of the resultant of these two effects on consumption depends on whether the consumer is a net lender in period 1 and a net borrower in period 2 or vice-versa. In case the consumer is a net lender in period 1 and a net borrower in period 2 the net income effect is positive. Assuming the consumption in both periods in a normal good, this means that the substitution effect and the income effect act in opposite directions on c_1 in this case leading to an ambiguous effect.

2. Many periods

Assume that rather than just living for two periods the consumer lives for $T+1$ periods. Further assume that the real rate of interest takes a constant value r over the consumer's lifetime. For convenience we define $\delta = 1/(1+r)$. It is also convenient to start time from period 0 rather than period 1.

2.1. Budget constraint.

Arguing as before, the consumer's budget constraint is

$$\sum_{i=0}^{T} \delta^i c_i = \sum_{i=0}^{T} \delta^i y_i \tag{4}$$

2.2. Utility function.

We could proceed as before by assuming a utility function $U(c_0, \ldots, c_T)$ and deriving the first order conditions. However, because the marginal utility in each period depends on consumption in all periods it is hard to draw any sharp conclusions at this level of generality. Therefore we need to impose some restrictions on the form of the utility functions.

[1] For more about the Slutsky equation in the case of a consumer with fixed endowments of goods see section 9.1 in Varian's *Microeconomic Analysis*, 3rd ed.

Suppose, for example we assume that the utility function is additively separable, i.e.

$$U(c_0, \ldots, c_T) = v_0(c_0) + v_1(c_1) + \cdots + v_T(c_T) \tag{5}$$

Then the first-order conditions take the form

$$v_i'(c_i) = \delta^i \lambda \qquad i = 0, \ldots, T \tag{6}$$

where, as before, λ is the Lagrange multiplier corresponding to the budget constraint.

Sometimes we want to restrict the consumers preferences even further, by assuming that the different v_i differ from each other by only a geometric discounting factor.

$$U(c_0, \ldots, c_T) = \sum_{i=0}^{T} \beta^i u(c_i) \tag{7}$$

where β is a constant, referred to as the subjective rate of discount, such that $0 < \beta < 1$.

In this case the first-order conditions take the particularly simple form

$$u'(c_i) = \left(\frac{\delta}{\beta}\right)^i \lambda \qquad i = 0, \ldots, T \tag{8}$$

In case $\delta = \beta$, this implies that $u'(c_i)$ is the same for all i, which, assuming that $u'(\cdot)$ is a strictly decreasing function, means that c_i is constant for all i. The present period's income does not influence the present period's consumption at all. Consumption is determined solely by lifetime resources as given by (4).

The case $\delta \neq \beta$ is also instructive. Suppose $\delta > \beta$. In this case it follows from (8) that consumption decreases over time. Formally, this is because if $\delta > \beta$ then by (8) $u'(c_i)$ increases over time, and since $u'(c)$ is a decreasing function of consumption, this implies that c decreases over time.

The economic logic behind this result is that δ is the number of units of consumption we have to give up at present in order to purchase one more unit of consumption next period, whereas β is the number of units of marginal utility we are willing to give up at present in order to have one more unit of marginal utility in the next period. Suppose we start with the same consumption c in this period and the next. If we reduce consumption in the next period by a small amount Δc then at the prevailing market prices we can increase present consumption by $\delta \Delta c$. The increase in utility from the increase in present consumption is approximately $u'(c)(\delta\Delta c)$.[2] The decrease in utility from the reduction in next period's consumption is approximately $\beta u'(c)(\Delta c)$. The net change in utility would be $(\delta - \beta)u'(c)(\Delta c)$ which is positive when $\delta > \beta$. Thus it is beneficial to increase present consumption and reduce future consumption if we are starting from a position of equality. Indeed, it will be optimal to increase consumption in the present period (say period i) and decrease consumption in the next period (period $i+1$) till the following equality between the MRS and the price ratio is satisfied,

$$\frac{u'(c_{i+1})}{u'(c_i)} = \frac{\delta}{\beta}$$

2.3. **Exogenous variables.** It is possible to unify (5) and (7) by writing

$$v_i(c_i) = \beta^i u(c_i, \xi_i)$$

where ξ_i is an exogenous variable such a the consumer's age or the number of members in the household. In this case the first-order conditions become

$$u'(c_i, \xi_i) = \left(\frac{\delta}{\beta}\right)^i \lambda \qquad i = 0, \ldots, T$$

Knowing how ξ affects the marginal utility would now let us make some predictions regarding the path of consumption.

2.4. **Comparative statics.** Assuming that consumption in every period is a normal good, an increase in y_i increases every c_i.

The effect of an increase in r, or equivalently, a decrease in δ remains ambiguous because of the same income and substitution effects as discussed earlier. But for the utility function given by (7), we can say a little more. From (8) we can see that a decrease in δ means that the *growth rate* of consumption speeds up. Remember that even in this case we do not have any information regarding the *level* of consumption in any period since the level would depend on λ which in turn depends on δ.

3. Dynamic programming

3.1. **Example: log utility.** Suppose that the consumer maximises

$$\sum_{i=0}^{T} \log(c_i)$$

[2] We are using Taylor's theorem: $u(c + \delta\Delta c) - u(c) \approx u'(c)(\delta\Delta c)$

subject to

$$\begin{aligned}
w_0 &= \overline{w_0} \\
w_{t+1} &= R(w_t - c) \qquad \text{for } t = 0, \ldots, T \\
w_{T+1} &= 0
\end{aligned}$$

What is the value function and policy function for this problem?

Since there can neither be bequests nor outstanding debt at period T, the policy in that period is simple: consume all your wealth. Denoting the policy function by $g(\cdot)$ and the value function in period t by $V_t(\cdot)$, we have,

$$g_T(w) = w, \qquad V_T(w_T) = \log(g_T(w)) = \log(w) \tag{9}$$

Now consider period $T-1$. Bellman's principle of optimality tells us,

$$\begin{aligned}
V_{T-1}(w) &= \max_c [\log(c) + V_T(R(w-c))] \\
&= \max_c [\log(c) + \log(R(w-c))] \qquad \text{[using (9)]}
\end{aligned} \tag{10}$$

The first-order condition for this maximisation problem is:

$$\begin{aligned}
\frac{1}{c} + \frac{-R}{R(w-c)} &= 0 \\
w - c &= c \\
c &= w/2
\end{aligned}$$

Since the objective function in (10) is concave in c (check this!), the first-order condition is sufficient and gives us our policy function:

$$g_{T-1}(w) = w/2$$

Substituting this into (10) we get the value function,

$$\begin{aligned}
V_{T-1}(w) &= \log(g_{T-1}(w)) + V_T(R(w - g_{T-1}(w))) \\
&= \log(w/2) + \log(Rw/2) \\
&= \log(R) + 2\log(w/2)
\end{aligned} \tag{11}$$

Now that we know V_{T-1} we could use the Bellman equation relating V_{t-2} to V_{t-1} to derive g_{T-2} and V_{t-2}. If you do this you will find,

$$g_{T-2}(w) = w/3, \qquad V_{T-2}(w) = (1+2)\log(R) + 3\log(w/3) \tag{12}$$

We could continue like this to find $V_{T-3}, \ldots, V_0$. In general this is precisely what we do. In fact, in most applications of dynamic programming it is not possible to express the value function by a formula in the state variables and the best that we can do is to use a computer to calculate the value function at a number of possible values of the state variable using Bellman's equation.

But our present problem is a particularly simple one. Looking at (11) and (12) suggests to us the guess,

$$V_{T-n}(w) = \frac{n(n+1)}{2}\log(R) + (n+1)\log\left(\frac{w}{n+1}\right) \tag{13}$$

[Remember $1+2+\cdots+n = n(n+1)/2$]

How do we check that our guess is right? We will use the principle of mathematical induction. By comparing to (11) we see that (13) is correct for $n = 1$. Suppose that the equation is true for $n = k$. What then would be $V_{T-(k+1)}$? We once again write down the Bellman equation

$$\begin{aligned}
V_{T-(k+1)} &= \max_c[\log(c) + V_{T-k}(R(w-c))] \\
&= \max_c\left[\log(c) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{R(w-c)}{k+1}\right)\right] \\
&\text{[assuming (13)]}
\end{aligned} \tag{14}$$

The first-order condition is:

$$\begin{aligned}
\frac{1}{c} + (k+1)\left(\frac{k+1}{R(w-c)}\right)\left(\frac{-R}{k+1}\right) &= 0 \\
(k+1)\frac{1}{(w-c)} &= \frac{1}{c} \\
c &= w/(k+2)
\end{aligned} \tag{15}$$

Substituting this into (14) we have

$$\begin{aligned}
V_{T-(k+1)} &= \log(c) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{R(w-c)}{k+1}\right) \\
&\text{substituting (15),} \\
&= \log\left(\frac{w}{k+2}\right) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{Rw}{k+2}\right) \\
&= \log\left(\frac{w}{k+2}\right) + \frac{k(k+1)}{2}\log(R) + (k+1)\log(R) + (k+1)\log\left(\frac{w}{k+2}\right) \\
&= \frac{(k+2)(k+1)}{2}\log(R) + (k+2)\log\left(\frac{w}{k+2}\right)
\end{aligned} \tag{16}$$

But this is the same as (13) for $n = k + 1$. We therefore conclude that if (13) is true for $n = k$ it is also true for $n = k + 1$. We have already checked that (13) is true for $n = 1$. Hence we conclude by the principle of mathematical induction that the value function for our dynamic programming problem is given by (13) for $n = 1, \ldots, T$.

Also, now that we have verified that (13) is indeed the value function of the problem, (15) gives the policy function, i.e.

$$g_{T-n}(w) = w/(n+1) \tag{17}$$

4. The Euler equation

As we discussed in the last section, for most dynamic programming problems it is not possible to compute the value and policy functions in terms of simple formulae. The best we can do is to calculate numerical values. But even if we cannot find an exact formula for the solution to our optimisation problem, it may still be possible to get some qualitative information about the problem by studying the consequences of the Bellman equation. That is the subject of this section.

Let's recall the Bellman equation,

$$V_t(w_t) = \max_{c_t}[u(c_t) + V_{t+1}(R(w_t - c_t))$$

The first order condition for this maximisation problem is:

$$u'(c_t) = RV'_{t+1}(R(w_t - c_t)) \tag{18}$$

By itself (18) does not seem very useful unless we know $V_{t+1}(\cdot)$ and can calculate its derivative. But there is a trick that we can use to eliminate this unknown derivative from (18).[3]

Let $c_t^*(w_t)$ be the optimal consumption in period t when period t wealth is w_t. From (18) we already know that,

$$u'[c_t^*(w_t)] = RV'_{t+1}[R(w_t - c_t^*(w_t))] = RV'_{t+1}(w_{t+1}) \tag{19}$$

[3]The 'trick' is a particular case of a general result known as the envelope theorem. See section M.L of Mas-Colell, Whinston and Green or some mathematical methods book for more detail.

But from the definition of the value function

$$V_t(w_t) = u[c_t^*(w_t)] + V_{t+1}[R(w_t - c_t^*(w_t))]$$

Differentiating with respect to w_t we have,

$$\begin{aligned} V_t'(w_t) &= u'[c_t^*(w)]c_t^{*\prime}(w_t) + V_{t+1}'[R(w_t - c_t^*(w_t))][R(1 - c_t^{*\prime}(w_t))] \\ &= c_t^{*\prime}(w_t)[u'(\cdot)) - RV_{t+1}'(\cdot)] + RV_{t+1}'[R(w_t - c_t^*(w_t))] \end{aligned}$$

From (19) the first terms equals 0, so,

$$V_t'(w_t) = RV_{t+1}'(w_{t+1})$$

Using (19)

$$V_t'(w_t) = u'(c_t)$$

The equation above was derived for arbitrary t. So it is equally good for $t+1$, i.e.

$$V_{t+1}'(w_{t+1}) = u'(c_{t+1})$$

Substituting this in (19) we have,

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{R} = \delta \tag{20}$$

This condition is known as the Euler[4] equation for our dynamic programming problem. We can alternatively derive it by starting out with an optimal consumption plan, increasing consumption in period t by a small amount Δc and reducing consumption in period $t+1$ by $R\Delta c$ so that wealth at the end of the period $t+1$ is once again the same as what it would have been under the optimal plan. The first-order change in utility from this deviation is

$$\Delta u = u'(c_t)[\Delta c] - u'(c_{t+1})[R\Delta c]$$

Now for the original plan to have been optimal Δu must be 0 since if $\Delta u > 0$ the deviation considered above increases total utility whereas if $\Delta u < 0$ then the opposite of the deviation considered above increases total utility. But $\Delta u = 0$ implies

$$u'(c_t) - Ru'(c_{t+1}) = 0$$

which is again our Euler equation (20).

The Euler equation also follows from the first-order conditions (6) of the Lagrange-multiplier approach, showing that we have come full circle.

[4]Pronounced "oiler". Named after a eighteenth-century mathematician who was among the earliest to study dynamic optimisation problems.

The Euler equation tells us how consumption should grow or decline. It does not tell us the level of the consumption. But we can characterise the entire consumption path if we keep track of the path of wealth implied by the path of consumption and impose, in addition to the Euler equation, the conditions

$$w_0 = \overline{w_0}$$

which comes to us as a given data and

$$w_T = 0$$

which comes to us from our no bequest, no terminal borrowing, monotonic utility assumptions about the terminal period.

5. Uncertainty

5.1. **Euler equation.** In the case of uncertainty in labour incomes, but with certain interest rates, the Euler equation becomes

$$v_t'(c_t) = RE_t[v_{t+1}'(c_{t+1})] \tag{21}$$

where E_t denotes the mathematical expectation conditional on information at time t.

5.2. **Quadratic felicity.**

5.2.1. *Martingale property.* Suppose the felicity (i.e. per-period utility) function is

$$v_t(c_t) = \beta^t(ac_t - 0.5c_t^2)$$

where a is some constant.

In this case (21) specialises to

$$a - c_t = R\beta(a - E_t c_{t+1})$$

If we further assume that $R = 1/\beta$ then

$$E_t c_{t+1} = c_t \tag{22}$$

that is, consumption is a martingale process.

Since c_t is part of the information set at time t, $E_t c_t = c_t$ Therefore, another way to write (22) is

$$E_t(c_{t+1} - c_t) = 0$$

which says that the change in consumption between time t and $t + 1$ has no predictable direction based on information at time t.

This result is a consequence of the very special assumptions that we have made. Assuming the same felicity function for each period (apart from the discount factor β) and then assuming that the market rate of discount $(1/R)$ equals this subjective discount factor creates a situation where the consumer has no desire to have a higher consumption in any particular period of her life either to meet greater consumption needs or to take advantage of the difference between market and subjective discount rates. Unconstrained lending and borrowing mean that the consumer can actually move around her income across periods so as to achieve this perfect symmetry in her consumption in the sense of equating expected marginal utility across periods. But with quadratic felicity expected marginal utility is the same thing as expected consumption and we have our martingale result.

The long list of assumptions leading up to the martingale result means that this precise result is not very robust or realistic. Therefore rather than taking it as a property that is likely to be literally true, we should understand it as a demonstration of the tendency of the lending and borrowing behaviour of consumers to delink current consumption from current income. This tendency will be there as long as consumers have access to asset markets, though in more realistic settings it will be overlaid with factors which impart a systematic pattern to the trajectory of consumption such as a changing pattern of lifetime consumption needs or differences between the subjective and market rate of discount.

5.2.2. *A martingale lemma.* The definition of a martingale tells us about the expectation of a process at a period conditional on information available in the immediately preceding period. However, sometimes we need to find the expectation of a process at a period conditional on information available much further back in the past. The following lemma helps us.

Lemma 1. *If X_t is a martingale then for any m and any $n > 0$,*

$$E[X_{m+n}|m] = X_m.$$

Proof. The proof is by mathematical induction on n.

For $n = 1$ the proof follows directly from the definition of a martingale.

Suppose the lemma is true for $n = k$. Consider the case $n = k + 1$. Noting that the information available at time $m + k$ is a superset of the information available at time m, we have from the law of iterated expectations

$$E[X_{m+k+1}|m] = E[E[X_{m+k+1}|m + k]|m]$$

The martingale property, applied at time $m + k$ tells us that

$$E[X_{m+k+1}|m + k] = X_{m+k}$$

. The assumption that the lemma is true for $n = k$ gives us,

$$E[X_{m+k}|m] = X_m$$

Putting everything together, we have

$$E[X_{m+k+1}|m] = E[E[X_{m+k+1}|m + k]|m] = E[X_{m+k}|m] = X_m$$

thus establishing the result for $n = k + 1$.

Since we have shown that the result is true for $n = 1$ and it is true for $n = k + 1$ whenever it is true for $n = k$, it follows from the principle of mathematical induction that it is true for all $n > 0$. □

5.2.3. *The level of consumption.* The martingale property of consumption only tells us how consumption evolves from one point to the next, not the *level* of consumption. The level of consumption would depend on the consumer's resources in terms of her initial wealth and expected labour income. We now show that this is so mathematically by deriving an explicit formula for the level of consumption in the case where consumption is a martingale.

Consider a consumer who stands at period t with wealth w_t and is planning her future consumption for the periods $t, t + 1, \dots, T$. Since she cannot leave any bequests or outstanding debt in period T, it must be the case that her *realized* stream of consumption (c_t) and labour income (y_t) must satisfy,

$$\sum_{i=t}^{T} \delta^{i-t} c_i = \sum_{i=t}^{T} \delta^{i-t} y_i + w_t \tag{23}$$

Taking expectations as of time t,

$$\sum_{i=t}^{T} \delta^{i-t} E_t c_i = \sum_{i=t}^{T} \delta^{i-t} E_t y_i + w_t \tag{24}$$

From Lemma 1 on page 10, $E_t c_i = c_t$ for all $i > t$. And for $i = t$, $E_t c_t = c_t$ because c_t is known at time t. Hence,

$$c_t \sum_{i=t}^{T} \delta^{i-t} = \sum_{i=t}^{T} \delta^{i-t} E_t y_i + w_t \tag{25}$$

$$c_t = \frac{1}{\sum_{i=t}^{T} \delta^{i-t}} \left[\sum_{i=t}^{T} \delta^{i-t} E_t y_i + w_t \right] \tag{26}$$

Thus the level of consumption in a given period depends on the expected discounted value of the future stream of labour income over the entire remaining lifetime as well as initial wealth. This once again reiterates the idea of the permanent income hypothesis that the consumption in each period depends not just on income in that period but on the entire expected path of future income.

5.2.4. *Increments in consumption.* With an explicit formula for the level of consumption in hand, we can now try to understand the martingale result better by seeing what it is exactly that drives changes in consumption.

Rewriting (25) for period $t+1$ we have,

$$c_{t+1} \sum_{i=t+1}^{T} \delta^{i-t-1} = \sum_{i=t+1}^{T} \delta^{i-t-1} E_{t+1} y_i + w_{t+1}$$

Substituting $w_{t+1} = (w_t + y_t - c_t)/\delta$,

$$c_{t+1} \sum_{i=t+1}^{T} \delta^{i-t-1} = \sum_{i=t+1}^{T} \delta^{i-t-1} E_{t+1} y_i + (w_t + y_t - c_t)/\delta$$

Multiplying throughout by δ,

$$c_{t+1} \sum_{i=t+1}^{T} \delta^{i-t} = \sum_{i=t+1}^{T} \delta^{i-t} E_{t+1} y_i + w_t + y_t - c_t$$

Subtracting (25) from this equation

$$(c_{t+1} - c_t) \sum_{i=t+1}^{T} \delta^{i-t} - c_t = \sum_{i=t+1}^{T} \delta^{i-t} (E_{t+1} y_i - E_t y_i) + y_t - E_t y_t - c_t$$

But $E_t y_t = y_t$ since income at time t is known at that time,

$$\begin{aligned} (c_{t+1} - c_t) \sum_{i=t+1}^{T} \delta^{i-t} &= \sum_{i=t+1}^{T} \delta^{i-t} (E_{t+1} y_i - E_t y_i) \\ c_{t+1} - c_t &= \frac{1}{\sum_{i=t+1}^{T} \delta^{i-t}} \left[\sum_{i=t+1}^{T} \delta^{i-t} (E_{t+1} y_i - E_t y_i) \right] \end{aligned}$$

We can divide both the numerator and denominator by δ to have the factor multiplying the first term in the sums equal to one.[5] This gives

[5]This is a purely aesthetic change and does not change any results.

us our final formula,

$$c_{t+1} - c_t = \frac{1}{\sum_{i=t+1}^{T} \delta^{i-t-1}} \left[\sum_{i=t+1}^{T} \delta^{i-t-1} (E_{t+1} y_i - E_t y_i) \right] \tag{27}$$

What the formula above says is that changes in consumption are a result of *revisions* in expectations of future income based on the difference in information at time t and $t+1$. Therefore changes in income that were predictable at time t do not contribute to the change in consumption between time t and $t+1$. This is consistent with our assumption that consumption is a martingale but goes further by predicting the actual size of the consumption change rather than just asserting that the expected value of this change would be zero.

5.2.5. *Specific income processes.* In general the difference in expectations which occur on the right of (27) depends on all the information which becomes available to the consumer between time t and $t+1$. One special case which we now consider is when the only source of new information is the realisation of the labour income y_{t+1}. What revision this new information causes in the expectation of future labour income depends on how labour income in different periods are related.

As an example consider the labour income process given by the following stochastic difference equation,

$$y_{t+1} - \mu = \rho(y_t - \mu) + \epsilon_{t+1} \tag{28}$$

where ϵ_t is a white-noise process, μ and ρ are constants and $0 < \rho < 1$. This is a special case of what is known as a first-order autoregressive process (sometimes denoted as a AR(1) process). The coefficient ρ measures how persistent the deviations in y from its long-run average μ are.

Writing (28) for the period $t+2$ we have,

$$y_{t+2} - \mu = \rho(y_{t+1} - \mu) + \epsilon_{t+2}$$

Substituting (28),

$$y_{t+2} - \mu = \rho^2(y_t - \mu) + \rho\epsilon_{t+1} + \epsilon_{t+2}$$

Carrying out successive substitutions like this, we find for any $i > t$

$$y_i - \mu = \rho^{i-t}(y_t - \mu) + \sum_{j=t+1}^{i} \rho^{i-j}\epsilon_j$$

Taking expectations conditional on the information at time t,

$$E_t(y_i - \mu) = \rho^{i-t}(y_t - \mu) + \sum_{j=t+1}^{i} \rho^{i-j} E_t \epsilon_j$$

Here we have used the fact that y_t is known at time t. We further note that since ϵ_t is IID, ϵ_j is independent of all information at time t when $j > t$ and we can replace $E_t \epsilon_j$ by $E\epsilon_j$ which is 0 by the definition of white noise. Hence we conclude,

$$E_t(y_i - \mu) = \rho^{i-t}(y_t - \mu) \qquad \text{for } i \geq t \tag{29}$$

(We have established this above for $i > t$ and it is trivially true for $i = t$.)

Using $t + 1$ in the place of t,

$$E_{t+1}(y_i - \mu) = \rho^{i-t-1}(y_{t+1} - \mu) \qquad \text{for } i \geq t + 1 \tag{30}$$

For $i \geq t + 1$ both (29) and (30) hold. Subtracting the former from the latter we have,

$$E_{t+1} y_i - E_t y_i = \rho^{i-t-1}(y_{t+1} - \mu) - \rho^{i-t}(y_t - \mu)$$

Using (28),

$$\begin{aligned} &= \rho^{i-t-1}[\rho(y_t - \mu) + \epsilon_{t+1}] - \rho^{i-t}(y_t - \mu) \\ &= \rho^{i-t-1}\epsilon_{t+1} \end{aligned}$$

Substituting this in (27) we get,

$$c_{t+1} - c_t = \left[\frac{\sum_{i=t+1}^{T} (\delta\rho)^{i-t-1}}{\sum_{i=t+1}^{T} \delta^{i-t-1}} \right] \epsilon_{t+1}$$

So we see that for a given innovation in consumption, ϵ_{t+1}, the increment in consumption is higher the higher is the degree of persistence ρ in the income process.

In empirical applications of the model we can estimate ρ (or its analogues for more complex income processes) from data on consumers' incomes and then check if changes in consumption satisfy the forumla above. This yields a sharper test of our theory compared to just checking if consumption is a martingale.

AMBEDKAR UNIVERSITY DELHI
E-mail address: `jyotirmoy@jyotirmoy.net`