CONSUMPTION: CERTAINTY

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1. Two-period case

1.1. **Budget constraint.** Consider a consumer who lives for two periods, has an endowment of y_1 and y_2 units of goods in the two periods respectively and can borrow and lend any amount that they like at the real rate of interest r.

Suppose the consumer consumes c_1 in the first period. Then she will have to take a loan of $c_1 - y_1$ to finance her consumption. (This number can be negative, in which case the consumer is lending rather than borrowing.) In the next period the consumer will therefore have to make loan repayments of $(1+r)(c_1 - y_1)$. Assume that the consumer does not want to make any bequests and cannot die with any outstanding loans, consumption in the second period must be,

$$c_2 = y_2 - (1+r)(c_1 - y_1)$$

Simplifying and rearranging we have

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{1}$$

This is the budget constraint faced by the consumer. We can interpret this to mean that the present value of the consumer's consumption stream must equal the present value of their incomes.

1.2. **Utility maximization.** Suppose the consumer maximises a quasiconcave utility function $U(c_1, c_2)$ subject to this budget constraint. Then the consumer's first-order conditions are

$$U_1(c_1, c_2) = \lambda \tag{2}$$
$$U_2(c_1, c_2) = \lambda/(1+r) \tag{3}$$

where λ is the Lagrange multiplier corresponding to the budget constraint and $U_i(c_1, c_2)$ denotes the partial derivative $\partial U/\partial c_i$. We have explicitly shown the dependence of the partial derivatives on the value of consumption in both periods. These first-order conditions

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along with the budget constraint (1) together determines the value of c_1, c_2 and λ.

1.3. **Comparative statics.** Assuming that consumption in both periods is a normal good, an increase in either y_1 or y_2 increases both c_1 and c_2.

The effects of a change in r are ambiguous. An increase in r makes consumption in period 2 relatively cheap compared to consumption in period 1. Therefore the substitution effect causes c_1 to decrease and c_2 to increase. It is traditional to decompose the income effect into two parts. First, an increase in r reduces the present value of the consumer's endowments and hence decreases his real income. Second, an increase in r, by making the consumption in period 2 cheaper increases his real income.[1] The sign of the resultant of these two effects on consumption depends on whether the consumer is a net lender in period 1 and a net borrower in period 2 or vice-versa. In case the consumer is a net lender in period 1 and a net borrower in period 2 the net income effect is positive. Assuming the consumption in both periods in a normal good, this means that the substitution effect and the income effect act in opposite directions on c_1 in this case leading to an ambiguous effect.

2. Many periods

Assume that rather than just living for two periods the consumer lives for $T+1$ periods. Further assume that the real rate of interest takes a constant value r over the consumer's lifetime. For convenience we define $\delta = 1/(1+r)$. It is also convenient to start time from period 0 rather than period 1.

2.1. **Budget constraint.** Arguing as before, the consumer's budget constraint is

$$\sum_{i=0}^{T} \delta^i c_i = \sum_{i=0}^{T} \delta^i y_i \tag{4}$$

2.2. **Utility function.** We could proceed as before by assuming a utility function $U(c_0, \ldots, c_T)$ and deriving the first order conditions. However, because the marginal utility in each period depends on consumption in all periods it is hard to draw any sharp conclusions at this level of generality. Therefore we need to impose some restrictions on the form of the utility functions.

[1] For more about the Slutsky equation in the case of a consumer with fixed endowments of goods see section 9.1 in Varian's *Microeconomic Analysis*, 3rd ed.

Suppose, for example we assume that the utility function is additively separable, i.e.

$$U(c_0, \ldots, c_T) = v_0(c_0) + v_1(c_1) + \cdots + v_T(c_T) \tag{5}$$

Then the first-order conditions take the form

$$v_i'(c_i) = \delta^i \lambda \qquad i = 0, \ldots, T \tag{6}$$

where, as before, λ is the Lagrange multiplier corresponding to the budget constraint.

Sometimes we want to restrict the consumers preferences even further, by assuming that the different v_i differ from each other by only a geometric discounting factor.

$$U(c_0, \ldots, c_T) = \sum_{i=0}^{T} \beta^i u(c_i) \tag{7}$$

where β is a constant, referred to as the subjective rate of discount, such that $0 < \beta < 1$.

In this case the first-order conditions take the particularly simple form

$$u'(c_i) = \left(\frac{\delta}{\beta}\right)^i \lambda \qquad i = 0, \ldots, T \tag{8}$$

In case $\delta = \beta$, this implies that $u'(c_i)$ is the same for all i, which, assuming that $u'(\cdot)$ is a strictly decreasing function, means that c_i is constant for all i. The present period's income does not influence the present period's consumption at all. Consumption is determined solely by lifetime resources as given by (4).

The case $\delta \neq \beta$ is also instructive. Suppose $\delta > \beta$. In this case it follows from (8) that consumption decreases over time. Formally, this is because if $\delta > \beta$ then by (8) $u'(c_i)$ increases over time, and since $u'(c)$ is a decreasing function of consumption, this implies that c decreases over time.

The economic logic behind this result is that δ is the number of units of consumption we have to give up at present in order to purchase one more unit of consumption next period, whereas β is the number of units of marginal utility we are willing to give up at present in order to have one more unit of marginal utility in the next period. Suppose we start with the same consumption c in this period and the next. If we reduce consumption in the next period by a small amount Δc then at the prevailing market prices we can increase present consumption by $\delta \Delta c$. The increase in utility from the increase in present consumption is approximately $u'(c)(\delta\Delta c)$.[2] The decrease in utility from the reduction in next period's consumption is approximately $\beta u'(c)(\Delta c)$. The net change in utility would be $(\delta - \beta)u'(c)(\Delta c)$ which is positive when $\delta > \beta$. Thus it is beneficial to increase present consumption and reduce future consumption if we are starting from a position of equality. Indeed, it will be optimal to increase consumption in the present period (say period i) and decrease consumption in the next period (period $i+1$) till the following equality between the MRS and the price ratio is satisfied,

$$\frac{u'(c_{i+1})}{u'(c_i)} = \frac{\delta}{\beta}$$

2.3. **Exogenous variables.** It is possible to unify (5) and (7) by writing

$$v_i(c_i) = \beta^i u(c_i, \xi_i)$$

where ξ_i is an exogenous variable such a the consumer's age or the number of members in the household. In this case the first-order conditions become

$$u'(c_i, \xi_i) = \left(\frac{\delta}{\beta}\right)^i \lambda \qquad i = 0, \ldots, T$$

Knowing how ξ affects the marginal utility would now let us make some predictions regarding the path of consumption.

2.4. **Comparative statics.** Assuming that consumption in every period is a normal good, an increase in y_i increases every c_i.

The effect of an increase in r, or equivalently, a decrease in δ remains ambiguous because of the same income and substitution effects as discussed earlier. But for the utility function given by (7), we can say a little more. From (8) we can see that a decrease in δ means that the *growth rate* of consumption slows down. Remember that even in this case we do not have any information regarding the *level* of consumption in any period since the level would depend on λ which in turn depends on δ.

3. Dynamic programming

3.1. **Example: log utility.** Suppose that the consumer maximises

$$\sum_{i=0}^{T} \log(c_i)$$

[2]We are using Taylor's theorem: $u(c + \delta\Delta c) - u(c) \approx u'(c)(\delta\Delta c)$

subject to

$$\begin{aligned} w_0 &= \overline{w_0} \\ w_{t+1} &= R(w_t - c) \qquad \text{for } t = 0, \dots, T \\ w_{T+1} &= 0 \end{aligned}$$

What is the value function and policy function for this problem?

Since there can neither be bequests nor outstanding debt at period T, the policy in that period is simple: consume all your wealth. Denoting the policy function by $g(\cdot)$ and the value function in period t by $V_t(\cdot)$, we have,

$$g_T(w) = w, \qquad V_T(w_T) = \log(g_T(w)) = \log(w) \tag{9}$$

Now consider period $T-1$. Bellman's principle of optimality tells us,

$$\begin{aligned} V_{T-1}(w) &= \max_c [\log(c) + V_T(R(w-c))] \\ &= \max_c [\log(c) + \log(R(w-c))] \qquad \text{[using (9)]} \end{aligned} \tag{10}$$

The first-order condition for this maximisation problem is:

$$\begin{aligned} \frac{1}{c} + \frac{-R}{R(w-c)} &= 0 \\ w - c &= c \\ c &= w/2 \end{aligned}$$

Since the objective function in (10) is concave in c (check this!), the first-order condition is sufficient and gives us our policy function:

$$g_{T-1}(w) = w/2$$

Substituting this into (10) we get the value function,

$$\begin{aligned} V_{T-1}(w) &= \log(g_{T-1}(w)) + V_T(R(w - g_{T-1}(w))) \\ &= \log(w/2) + \log(Rw/2) \\ &= \log(R) + 2\log(w/2) \end{aligned} \tag{11}$$

Now that we know V_{T-1} we could use the Bellman equation relating V_{t-2} to V_{t-1} to derive g_{T-2} and V_{t-2}. If you do this you will find,

$$g_{T-2}(w) = w/3, \qquad V_{T-2}(w) = (1+2)\log(R) + 3\log(w/3) \tag{12}$$

We could continue like this to find $V_{T-3}, \dots, V_0$. In general this is precisely what we do. In fact, in most applications of dynamic programming it is not possible to express the value function by a formula in the state variables and the best that we can do is to use a computer to calculate the value function at a number of possible values of the state variable using Bellman's equation.

But our present problem is a particularly simple one. Looking at (11) and (12) suggests to us the guess,

$$V_{T-n}(w) = \frac{n(n+1)}{2}\log(R) + (n+1)\log\left(\frac{w}{n+1}\right) \tag{13}$$

[Remember $1+2+\cdots+n = n(n+1)/2$]

How do we check that our guess is right? We will use the principle of mathematical induction. By comparing to (11) we see that (13) is correct for $n=1$. Suppose that the equation is true for $n=k$. What then would be $V_{T-(k+1)}$? We once again write down the Bellman equation

$$\begin{aligned}
V_{T-(k+1)} &= \max_c[\log(c) + V_{T-k}(R(w-c))] \\
&= \max_c\left[\log(c) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{R(w-c)}{k+1}\right)\right] \\
&\text{[assuming (13)]}
\end{aligned} \tag{14}$$

The first-order condition is:

$$\begin{aligned}
\frac{1}{c} + (k+1)\left(\frac{k+1}{R(w-c)}\right)\left(\frac{-R}{k+1}\right) &= 0 \\
(k+1)\frac{1}{(w-c)} &= \frac{1}{c} \\
c &= w/(k+2)
\end{aligned} \tag{15}$$

Substituting this into (14) we have

$$\begin{aligned}
V_{T-(k+1)} &= \log(c) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{R(w-c)}{k+1}\right) \\
&\text{substituting (15),} \\
&= \log\left(\frac{w}{k+2}\right) + \frac{k(k+1)}{2}\log(R) + (k+1)\log\left(\frac{Rw}{k+2}\right) \\
&= \log\left(\frac{w}{k+2}\right) + \frac{k(k+1)}{2}\log(R) + (k+1)\log(R) + (k+1)\log\left(\frac{w}{k+2}\right) \\
&= \frac{(k+2)(k+1)}{2}\log(R) + (k+2)\log\left(\frac{w}{k+2}\right)
\end{aligned} \tag{16}$$

But this is the same as (13) for $n = k+1$. We therefore conclude that if (13) is true for $n = k$ it is also true for $n = k+1$. We have already checked that (13) is true for $n = 1$. Hence we conclude by the principle of mathematical induction that the value function for our dynamic programming problem is given by (13) for $n = 1, \dots, T$.

Also, now that we have verified that (13) is indeed the value function of the problem, (15) gives the policy function, i.e.

$$g_{T-n}(w) = w/(n+1) \tag{17}$$

4. THE EULER EQUATION

As we discussed in the last section, for most dynamic programming problems it is not possible to compute the value and policy functions in terms of simple formulae. The best we can do is to calculate numerical values. But even if we cannot find an exact formula for the solution to our optimisation problem, it may still be possible to get some qualitative information about the problem by studying the consequences of the Bellman equation. That is the subject of this section.

Let's recall the Bellman equation,

$$V_t(w_t) = \max_{c_t}[u(c_t) + V_{t+1}(R(w_t - c_t))$$

The first order condition for this maximisation problem is:

$$u'(c_t) = RV'_{t+1}(R(w_t - c_t)) \tag{18}$$

By itself (18) does not seem very useful unless we know $V_{t+1}(\cdot)$ and can calculate its derivative. But there is a trick that we can use to eliminate this unknown derivative from (18).[3]

Let $c_t^*(w_t)$ be the optimal consumption in period t when period t wealth is w_t. From (18) we already know that,

$$u'[c_t^*(w_t)] = RV'_{t+1}[R(w_t - c_t^*(w_t))] = RV'_{t+1}(w_{t+1}) \tag{19}$$

[3]The 'trick' is a particular case of a general result known as the envelope theorem. See section M.L of Mas-Colell, Whinston and Green or some mathematical methods book for more detail.

But from the definition of the value function

$$V_t(w_t) = u[c_t^*(w_t)] + V_{t+1}[R(w_t - c_t^*(w_t))]$$

Differentiating with respect to w_t we have,

$$\begin{aligned} V_t'(w_t) &= u'[c_t^*(w)]c_t^{*\prime}(w_t) + V_{t+1}'[R(w_t - c_t^*(w_t))][R(1 - c_t^{*\prime}(w_t))] \\ &= c_t^{*\prime}(w_t)[u'(\cdot)) - RV_{t+1}'(\cdot)] + RV_{t+1}'[R(w_t - c_t^*(w_t))] \end{aligned}$$

From (19) the first terms equals 0, so,

$$V_t'(w_t) = RV_{t+1}'(w_{t+1})$$

Using (19)

$$V_t'(w_t) = u'(c_t)$$

The equation above was derived for arbitrary t. So it is equally good for $t+1$, i.e.

$$V_{t+1}'(w_{t+1}) = u'(c_{t+1})$$

Substituting this in (19) we have,

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{R} = \delta \tag{20}$$

This condition is known as the Euler[4] equation for our dynamic programming problem. We can alternatively derive it by starting out with an optimal consumption plan, increasing consumption in period t by a small amount Δc and reducing consumption in period $t+1$ by $R\Delta c$ so that wealth at the end of the period $t+1$ is once again the same as what it would have been under the optimal plan. The first-order change in utility from this deviation is

$$\Delta u = u'(c_t)[\Delta c] - u'(c_{t+1})[R\Delta c]$$

Now for the original plan to have been optimal Δu must be 0 since if $\Delta u > 0$ the deviation considered above increases total utility whereas if $\Delta u < 0$ then the opposite of the deviation considered above increases total utility. But $\Delta u = 0$ implies

$$u'(c_t) - Ru'(c_{t+1}) = 0$$

which is again our Euler equation (20).

The Euler equation also follows from the first-order conditions (6) of the Lagrange-multiplier approach, showing that we have come full circle.

[4]Pronounced "oiler". Named after a eighteenth-century mathematician who was among the earliest to study dynamic optimisation problems.

The Euler equation tells us how consumption should grow or decline. It does not tell us the level of the consumption. But we can characterise the entire consumption path if we keep track of the path of wealth implied by the path of consumption and impose, in addition to the Euler equation, the conditions

$$w_0 = \overline{w_0}$$

which comes to us as a given data and

$$w_T = 0$$

which comes to us from our no bequest, no terminal borrowing, monotonic utility assumptions about the terminal period.

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