ST308 Bayesian Inference
Week 7

Mapping to Examples in the Slides:

Exercise 1: Aims to test understanding of the definitions in slides 18, 20, 21 and 22. The notation in this exercise may be complicated. Feel free to skip it and move to the next one, the main purpose is to complement the slides.

Exercise 2: See example in slides 23 and 24.

Exercises

1. (Optional) Verify that each step of a Gibbs Sampler can be viewed as a Metropolis Hastings algorithm with acceptance probability of 1. For each step you can consider the proposal that draws θ_i from $\pi(\theta_i|y,\theta_{-1})$ and keeps θ_{-i} to its current value, hence $q(\theta^{(*)}|\theta^{(t)}) = \pi(\theta_i^{(*)}|\theta_{-i},y)$.

   Answer: In each update of each $\theta^{(t)}$ we will propose $\theta_i^{(*)}$ from $\pi(\theta_i|\theta_{-i}^{(t)},x)$ setting $\theta^{(*)} = (\theta_i^*,\theta_{-1}^{(t)})$, whereas $\theta^{(t)} = (\theta_i^{(t)},\theta_{-1}^{(t)})$.

   Note that then $q(\theta^{(*)}|\theta^{(t)}) = \pi(\theta_i^{(*)}|\theta_{-i},x)$. We will therefore accept with probability 1:

$$\alpha(\theta^{(t)},\theta^{(*)}) = \min\left(1,\ \frac{\pi(\theta^{(*)}|x)\pi(\theta_i^{(t)}|\theta_{-i}^{(t)},x)}{\pi(\theta^{(t)}|x)\pi(\theta_i^{(*)}|\theta_{-i}^{(t)},x)}\right)$$
$$= \min\left(1,\ \frac{\pi(\theta_i^{(*)}|\theta_{-i}^{(t)},x)\pi(\theta_{-i}^{(t)}|x)\pi(\theta_i^{(t)}|\theta_{-i}^{(t)},x)}{\pi(\theta_i^{(t)}|\theta_{-i}^{(t)},x)\pi(\theta_{-i}^{(t)}|x)\pi(\theta_i^{(*)}|\theta_{-i}^{(t)},x)}\right) = 1$$

2. Let $y = (y_1,\dots,y_n)$ be a r.s. from a $\mathrm{N}(\mu,v)$ where v is distributed according to an $\mathrm{IGamma}(\frac{\nu}{2},\frac{\nu}{2}\sigma^2)$. The parameter ν is assumed to be known. Finalise the model with an improper prior on μ,σ^2, $\pi(\mu,\sigma^2)\propto 1$.

   (a) Write down the posterior up to proportionality.

   Answer: The posterior can be written as

$$\begin{aligned}\pi(v,\mu,\sigma^2|y) &\propto f(y_1,\dots,y_n|v,\mu)f(v|\sigma^2)\\ &\propto \prod_{i=1}^n\left\{v^{-1/2}\exp\left(-\frac{(y_i-\mu)^2}{2v}\right)\right\}\frac{\left(\frac{\nu}{2}\sigma^2\right)^{\nu/2}}{\Gamma(\nu/2)}v^{-\nu/2-1}\exp\left(-\frac{\frac{\nu}{2}\sigma^2}{v}\right)\\ &\propto v^{-\frac{n+\nu}{2}-1}\exp\left(-\frac{\sum_{i=1}^n(y_i-\mu)^2}{2v}\right)(\sigma^2)^{\nu/2}\exp\left(-\frac{\frac{\nu}{2}\sigma^2}{v}\right)\end{aligned}$$

   (b) Specify the details needed to construct a Gibbs sampler to draw from the posterior of μ, v and σ^2.

Answer: The full conditional posterior for v is

$$\begin{aligned}
\pi(v|\mu,\sigma^2,y) &\propto v^{-\frac{n+\nu}{2}-1}\exp\left(-\frac{\frac{\sum_{i=1}^n(y_i-\mu)^2}{2}}{v}\right)\exp\left(-\frac{\frac{\nu}{2}\sigma^2}{v}\right)\\
&= v^{-\frac{n+\nu}{2}-1}\exp\left(-\frac{\frac{\nu\sigma^2+\sum_{i=1}^n(y_i-\mu)^2}{2}}{v}\right)\\
&\overset{\mathcal{D}}{=} \mathrm{IGamma}\left(\frac{n+\nu}{2},\frac{\nu\sigma^2+\sum_{i=1}^n(y_i-\mu)^2}{2}\right),
\end{aligned}$$

whereas for μ is

$$\begin{aligned}
\pi(\mu|v,\sigma^2,y) &\propto \exp\left(-\sum_{i=1}^n\frac{(\mu^2-2\mu x_i+y_i^2)}{2v}\right)\propto\exp\left(-\frac{n\mu^2-2\mu\sum_{i=1}^n y_i}{2v}\right)\\
&= \exp\left(-\frac{\mu^2-2\mu\bar{x}}{2v/n}\right)\overset{\mathcal{D}}{=}\ \mathrm{N}\left(\bar{y},\frac{v}{n}\right),
\end{aligned}$$

and for σ^2

$$\pi(\sigma^2|v,\mu,y)\propto(\sigma^2)^{\nu/2+1-1}\exp\left(-\frac{\nu}{2v}\sigma^2\right)\overset{\mathcal{D}}{=}\ \mathrm{Gamma}\left(\nu/2+1,\frac{\nu}{2v}\right),$$

A Gibbs Sampler initiates μ, v and σ^2 and then draws from the three conditional posterior distributions in turn at each iteration.

(c) Generate 200 numbers from the model above with $\mu=0$, $\sigma^2=1$, $v=1$ and $\nu=20$, and set these numbers as $y=(y_1,\ldots,y_{200})$. Write a R script to run 10,000 iterations of the Gibbs sampler derived in the previous part based on the data you generated.

(d) Provide posterior summaries and traceplots for μ, v and σ^2.

(e) Repeat with RStan and compare the results with the previous part.

Answer: For parts (c), (d) and (e), see the R markdown flie 'exercise2cde.Rmd'