ST308 Bayesian Inference

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Week 1: Exercises

1. Let $x = (x_1, \ldots, x_n)$ be a random sample from an Exponential(λ) distribution. Set the prior for λ to be a Gamma(α, β) and derive its posterior distribution.

   Answer: The likelihood can be written as

$$f(x|\lambda) = f(x_1, \ldots, x_n|\lambda) = \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

   The prior is set to a Gamma(α, β), so we can write

$$\pi(\lambda) \propto \lambda^{\alpha-1} \exp(-\beta\lambda)$$

   The posterior is then proportional to

$$\begin{aligned}\pi(\lambda|x) \propto f(x|\theta)\pi(\theta) &\propto \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right) \lambda^{\alpha-1} \exp(-\beta\lambda)\\ &= \lambda^{n+\alpha-1} \exp\left[-\lambda\left(\beta + \sum_{i=1}^{n} x_i\right)\right]\\ &\overset{\mathcal{D}}{=} \text{Gamma}\left(n+\alpha, \beta + \sum_{i=1}^{n} x_i\right)\end{aligned}$$

2. Let $x = (x_1, \ldots, x_n)$ be a random sample from a N(θ, σ^2) distribution with σ^2 known.

   (a) Show that the likelihood is proportional to

$$f(x|\theta) \propto \exp\left(-\frac{n(\bar{x}-\theta)^2 + (n-1)S^2}{2\sigma^2}\right).$$

   where $\bar{x}$ is the sample mean and S^2 is the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

Hence the likelihood simplifies to

$$f(x|\theta) \propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right)$$

(b) Set the prior for θ to be $\mathrm{N}(\mu, \tau^2)$ and derive its posterior distribution. (You can use the above result)

Answer:

2 (a) The joint density of the sample x is

$$\begin{aligned} f(x|\theta,\sigma^2) = f(x_1,\ldots,x_n|\theta,\sigma^2) &= \prod_{i=1}^{n}(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{(x_i-\theta)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}\right) \end{aligned}$$

Hence, it suffices to show that

$$\sum_{i=1}^{n}(x_i-\theta)^2 = n(\bar{x}-\theta)^2 + \sum_{i=1}^{n}(x_i-\bar{x})^2.$$

Note that

$$\sum_{i=1}^{n}(x_i-\theta)^2 = \sum_{i=1}^{n}(x_i^2 - 2\theta x_i + \theta^2) = \sum_{i=1}^{n} x_i^2 - 2\theta n\bar{x} + n\theta^2$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$$

Subtracting the second of these equations from the first yields

$$\sum_{i=1}^{n}(x_i-\theta)^2 - \sum_{i=1}^{n}(x_i-\bar{x})^2 = n\bar{x}^2 - 2\theta n\bar{x} + n\theta^2 = n(\bar{x}-\theta)^2$$

Since σ^2 is known, we are interested in $f(x|\theta)$ which is proportional to

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{n(\bar{x}-\theta)^2 + (n-1)S^2}{2\sigma^2}\right) \\ &\propto \exp\left(-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right)\exp\left(-\frac{(n-1)S^2}{2\sigma^2}\right) \\ &\propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right) \end{aligned}$$

2 (b) The prior for θ is set to be $\mathrm{N}(\mu, \tau^2)$. Hence we can write

$$\pi(\theta) \propto \exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right).$$

Using the result of part (i), the posterior is then proportional to

$$\begin{aligned}
\pi(\theta|x) \propto f(x|\theta)\pi(\theta) &\propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right) \\
&= \exp\left(-\frac{\theta^2+2\theta\bar{x}}{2\frac{\sigma^2}{n}}-\frac{\theta^2-2\theta\mu}{2\tau^2}\right)\exp\left(-\frac{\bar{x}^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{\mu^2}{2\tau^2}\right) \\
&\propto \exp\left(-\frac{\theta^2+2\theta\bar{x}}{2\frac{\sigma^2}{n}}-\frac{\theta^2-2\theta\mu}{2\tau^2}\right) \\
&= \exp\left(-\frac{\tau^2\theta^2-2\theta\bar{x}\tau^2+\frac{\sigma^2}{n}\theta^2-2\theta\mu\frac{\sigma^2}{n}}{2\frac{\sigma^2}{n}\tau^2}\right) \\
&= \exp\left(-\frac{(\frac{\sigma^2}{n}+\tau^2)\theta^2-2\theta(\bar{x}\tau^2+\mu\frac{\sigma^2}{n})}{2\frac{\sigma^2}{n}\tau^2}\right) \\
&= \exp\left(-\frac{\theta^2-2\theta\frac{\bar{x}\tau^2+\mu\frac{\sigma^2}{n}}{(\frac{\sigma^2}{n}+\tau^2)}}{2\frac{\frac{\sigma^2}{n}\tau^2}{(\frac{\sigma^2}{n}+\tau^2)}}\right) \overset{\mathcal{D}}{=} \mathrm{N}\left(\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\tau^2+\frac{\sigma^2}{n}}, \frac{\tau^2\frac{\sigma^2}{n}}{\tau^2+\frac{\sigma^2}{n}}\right)
\end{aligned}$$

3. Let $x = (x_1, \ldots, x_n)$ be a random sample from a $\mathrm{N}(0, \sigma^2)$ distribution. Set the prior for σ^2 to be $\mathrm{IGamma}(\alpha, \beta)$ and derive its posterior distribution.

Answer:

$$\begin{aligned}
f(x|\sigma^2) = f(x_1, \ldots, x_n|\sigma^2) &= \prod_{i=1}^{n}(2\pi)^{-1/2}(\sigma^2)^{-1/2}\exp\left(-\frac{x_i^2}{2\sigma^2}\right) \\
&\propto (\sigma^2)^{-n/2}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)
\end{aligned}$$

The prior is set to a $\mathrm{IGamma}(\alpha, \beta)$, so we can write

$$\pi(\sigma^2) \propto (\sigma^2)^{-\alpha-1}\exp\left(-\frac{\beta}{\sigma^2}\right)$$

The posterior is then proportional to

$$
\begin{aligned}
\pi(\sigma^2|x) &\propto f(x|\theta)\pi(\theta) \propto (\sigma^2)^{-n/2}\exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right)(\sigma^2)^{-\alpha-1}\exp\left(-\frac{\beta}{\sigma^2}\right) \\
&= (\sigma^2)^{-(n/2+\alpha)-1}\exp\left[-\frac{\beta+\frac{1}{2}\sum_{i=1}^n x_i^2}{\sigma^2}\right] \\
&\overset{\mathcal{D}}{=} \mathrm{IGamma}\left(n/2+\alpha, \beta+\frac{1}{2}\sum_{i=1}^n x_i^2\right)
\end{aligned}
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