ST308 - Lent term
Bayesian Inference

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Credible Intervals - Priors - Multiparameter models

Outline

Topics covered: Symmetric Credible Intervals, Highest Region Sets, Prior Elicitation, Jeffreys prior, joint and marginal posteriors

Outline

Interval Estimation problem

- Collect the data $x = (x_1, \ldots, x_n)$ from an experiment.
- Assign likelihood (model).
- (In Bayesian Inference) Assign a prior to the unknown parameters θ.
- Based on the above decide on a 'good' interval/set of values for θ.

Confidence vs Credible Intervals

Confidence Interval with level 1-α

If the experiment was repeated many times, the interval would contain the true value of θ in $100(1-\alpha)\%$ of them.

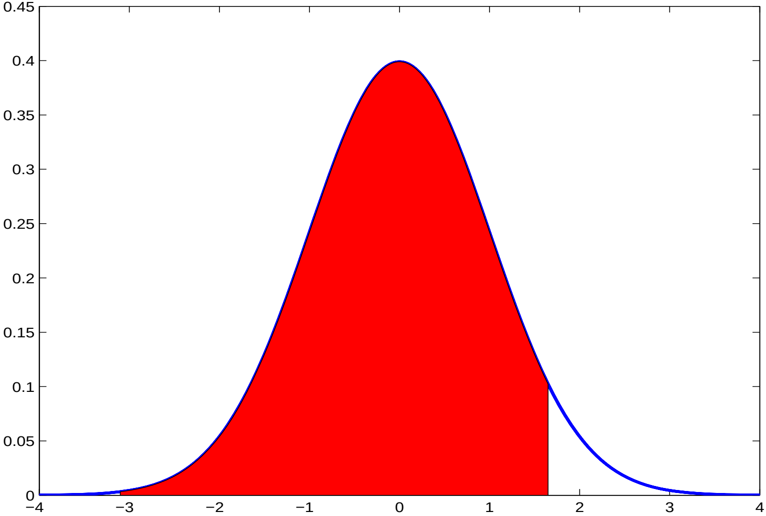
Credible Intervals of 1-α

The parameter θ is in the credible interval with probability $1-\alpha$.

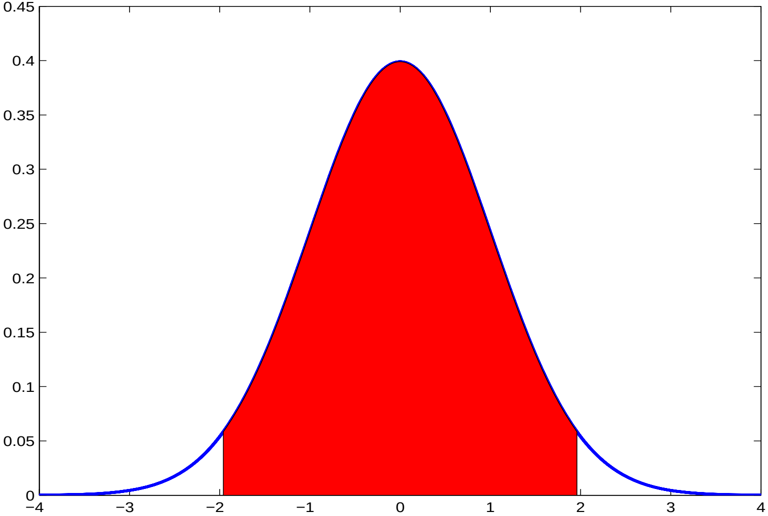
A $100(1-\alpha)\%$ credible set C_α for θ satisfies

$$P(\theta \in C_\alpha | x) = 1 - \alpha$$

A 95% credible set



A shorter 95% credible set



Highest region sets

Highest (posterior density) region sets

Let $H_\gamma = \{\theta : \pi(\theta|x) > \gamma\}$. If we set γ such that

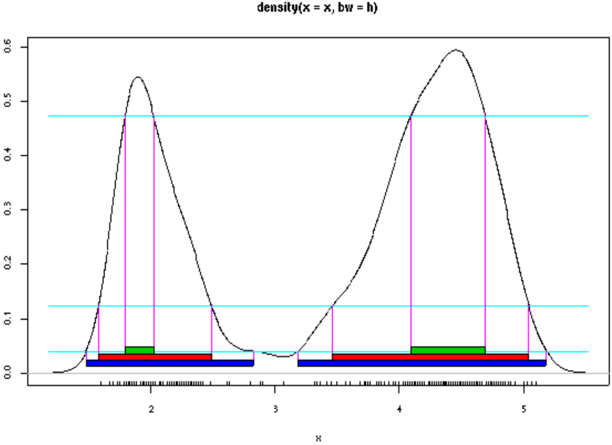
$$P(\theta \in H_\gamma | x) = 1 - \alpha,$$

we get the $100(1-\alpha)\%$ highest region credible set.

Notes:

- Highest region sets are the shortest. Also intuitive, as we want to report the most plausible values of θ.
- Not always easy to calculate highest region sets. In most cases symmetric sets just report the $\frac{\alpha}{2}$, $1 - \frac{\alpha}{2}$ posterior percentiles.

Highest posterior density region sets



Example: Normal(θ, σ^2) - σ^2 known

Let $x = (x_1, \ldots, x_n)$ be a random sample from the $N(\theta, \sigma^2)$, with σ^2 known and the prior for θ being a $N(\mu, \tau^2)$. The posterior for θ is

$$N\left(\frac{\bar{x}\tau^2 + \mu\frac{\sigma^2}{n}}{(\frac{\sigma^2}{n} + \tau^2)}, \frac{\frac{\sigma^2}{n}\tau^2}{(\frac{\sigma^2}{n} + \tau^2)}\right).$$

The symmetric and highest region $100(1-\alpha)\%$ credible set is then

$$\frac{\bar{x}\tau^2 + \mu\frac{\sigma^2}{n}}{(\frac{\sigma^2}{n} + \tau^2)} \pm \mathcal{Z}_{\alpha/2}\sqrt{\frac{\frac{\sigma^2}{n}\tau^2}{(\frac{\sigma^2}{n} + \tau^2)}}.$$

where $\mathcal{Z}_{\alpha/2}$ is the $\frac{\alpha}{2}$ percentile of a $N(0, 1)$.

For $\pi(\theta) \propto 1$, we get $\bar{x} \pm \mathcal{Z}_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}$. Same as in the frequentist case but with different interpretation.

Numerical example: IQ scores

A student uses Bayesian inference in modelling his IQ score $x|\theta$ which is $\mathrm{N}(\theta, 80)$. His prior is $\mathrm{N}(110, 120)$. After a score of $x = 98$ the posterior becomes $\mathrm{N}(102.8, 48)$.
The frequentist 95% confidence interval is

$$\left[98 - 1.96\sqrt{80},\ 98 + 1.96\sqrt{80}\right] = [80.5,\ 115.5]$$

The HPD (and symmetric) 95% credible interval is

$$\left[102.8 - 1.96\sqrt{48},\ 102.8 + 1.96\sqrt{48}\right] = [89.2,\ 116.4]$$

which is shorter as it includes prior information.

Outline

Prior distributions

- The choice of the prior distribution may be very influential.
- In the presence of information, the prior should reflect it appropriately.
- If no information is available a vague, low-informative prior should be chosen.
- Bayes and Laplace used uniform priors - Not always possible/sensible.

Conjugate priors

Posterior and prior have the same distribution. Only the parameters are updated. Used primarily for convenience.

- **Binomial-Beta:** The prior is a Beta(α, β), the data consist of the observation x and the posterior is a Beta$(\alpha + x, n - x + \beta)$.
- **Poisson-Gamma:** The prior is a Gamma(α, β), the data consist of the random sample $x = (x_1, \ldots, x_n)$ and the posterior is a Gamma$(\alpha + \sum x_i, n + \beta)$.
- **Normal-Normal:** The prior is a N(μ, τ^2), the data consist of the random sample $x = (x_1, \ldots, x_n)$ and the posterior is a

$$\mathrm{N}\left(\frac{\frac{\sigma^2}{n}\mu + \tau^2\bar{x}}{\tau^2 + \frac{\sigma^2}{n}}, \frac{\tau^2\frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}}\right).$$

Prior elicitation

- Information may be available prior to the experiment about moments, percentiles, probabilities of certain intervals etc.
- This information can be incorporated in the parameters of the prior distribution (prior elicitation).
- If information is not available a vague / low-informative prior should be chosen.

Example: In a Poisson likelihood experiment we know that the mean of λ is around 5 and the variance around 4.

A Gamma(α, β) may be chosen. Since $E(\lambda) = \frac{\alpha}{\beta} = 5$ and $\mathrm{Var}(\lambda) = \frac{\alpha}{\beta^2} = 4$, we get that $\alpha = 6.25$ and $\beta = 1.25$

Jeffreys' priors

Consider a sample $x = (x_1, \ldots, x_n)$ with likelihood $f(x|\theta)$. Fisher information for θ is defined as

$$I(\theta) = E_X \left[\left(\frac{\partial \log f(x|\theta)}{\partial \theta} \right)^2 \right] = -E_X \left(\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2} \right)$$

Jeffreys suggested the following prior

$$\pi(\theta) \propto det\,(I(\theta))^{1/2}$$

and for the single parameter case

$$\pi(\theta) \propto I(\theta)^{1/2}$$

Invariant to transformations

Jeffreys prior is invariant to transformations.

Lemma

Let $\phi = h(\theta)$ or else $\theta = g(\phi)$. Then if $\theta \sim \pi_\theta(\theta)$ the pdf of ϕ is

$$\pi_\phi(\phi) = \pi_\theta(g(\phi)) \left| \frac{\partial g(\phi)}{\partial \phi} \right| = \pi_\theta(\theta) \left| \frac{\partial \theta}{\partial \phi} \right|$$

Now take

$$\pi_\phi(\phi) = \pi_\theta(\theta) \left| \frac{\partial \theta}{\partial \phi} \right| \propto E_X \left[\left(\frac{\partial \log f(x|\theta)}{\partial \theta} \right)^2 \right]^{1/2} \left(\left| \frac{\partial \theta}{\partial \phi} \right|^2 \right)^{1/2}$$

$$= E_X \left[\left(\frac{\partial \log f(x|\theta)}{\partial \theta} \frac{\partial \theta}{\partial \phi} \right)^2 \right]^{1/2} = E_X \left[\left(\frac{\partial \log f(x|\phi)}{\partial \phi} \right)^2 \right]^{1/2} = I(\phi)^{1/2}$$

Example 1: Normal likelihood

Let $x = (x_1, \ldots, x_n)'$ be a random sample from $N(\theta, \sigma^2)$, σ^2 known.

$$\log f(x|\theta) = \log\left[\exp\left(-\frac{\sum_{i=1}^n (x_i-\theta)^2}{2\sigma^2}\right)\right] = -\frac{\sum_{i=1}^n (x_i-\theta)^2}{2\sigma^2},$$

$$\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{\sum_{i=1}^n (x_i-\theta)}{\sigma^2} = \frac{\sum_{i=1}^n x_i - n\theta}{\sigma^2} = \frac{n(\bar{x}-\theta)}{\sigma^2}.$$

$$\mathcal{I}(\theta|x) = -E_X\left(\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right) = -E_X\left(-\frac{n}{\sigma^2}\right) = \frac{n}{\sigma^2}$$

Hence the Jeffreys prior is $\pi(\theta) \propto 1$.

The posterior will be

$$\pi(\theta|x) \propto \exp\left(-\frac{\sum_{i=1}^n (x_i-\theta)^2}{2\sigma^2}\right) \propto \exp\left(-\frac{\theta^2 - 2\theta\bar{x}}{2\frac{\sigma^2}{n}}\right) \overset{\mathcal{D}}{=} N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

Example 2: Poisson likelihood

Let $x = (x_1, \ldots, x_n)'$ be a random sample from Poisson(λ).

$$\log f(x|\lambda) = \log\left[\exp(-n\lambda)\lambda^{\sum x_i}\right] = -n\lambda + \log(\lambda)\sum_{i=1}^{n} x_i,$$

$$\frac{\partial}{\partial\lambda}\log f(x|\lambda) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n.$$

$$\mathcal{I}(\lambda|x) = -E_X\left(\frac{\partial^2}{\partial\lambda^2}\log f(x|\lambda)\right) = -E_X\left(-\frac{\sum_{i=1}^{n} x_i}{\lambda^2}\right) = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

Jeffreys prior: $\pi(\lambda) \propto 1/\sqrt{\lambda}$. Posterior: Gamma$(\frac{1}{2} + \sum x_i, n)$.

Pros and Cons

- Jeffreys' prior provides a popular for 'objective' Bayesian inference.
- Usually improper, therefore the propriety of the posterior should also be checked.
- Requires Fisher information that can be very hard to compute, particularly in high dimensions.
- Overall, the problem of prior specification in higher dimensions is quite challenging.

Outline

Multiparameter models

We may have more than one parameters, say $\theta = (\theta_1, \theta_2)$. As before assign a prior $\pi(\theta_1, \theta_2)$ and obtain the posterior

$$\pi(\theta_1, \theta_2 | x) = \frac{f(x|\theta_1, \theta_2)\pi(\theta_1, \theta_2)}{\int \int f(x|\theta_1, \theta_2)\pi(\theta_1, \theta_2) d\theta_1 d\theta_2}$$

If interest mainly lies in θ_1, the marginal posterior of θ_1 may be used by averaging over θ_2.

$$\pi(\theta_1 | x) = \int \pi(\theta_1, \theta_2 | x) d\theta_2$$

Example 3: Normal likelihood for μ and σ^2

Let $x = (x_1, \ldots, x_n)$ be a random sample from a $N(\theta, \sigma^2)$ with both μ and σ^2 unknown.

Likelihood: The likelihood is (s^2 denotes the sample variance)

$$f(x|\theta) \propto (\sigma^2)^{-n/2} \exp\left(-\frac{(n-1)s^2 + n(\bar{x}-\theta)^2}{2\sigma^2}\right)$$

Prior: Consider the improper prior is $\pi(\theta, \sigma^2) \propto 1/\sigma^2$

Example 3: Normal likelihood for μ and σ^2 (cont'd)

Posterior: factorised as $\pi(\theta|x,\sigma^2)\pi(\sigma^2|x)$

$$
\begin{aligned}
\pi(\theta,\sigma^2|x) &\propto (\sigma^2)^{-n/2-1}\exp\left(-\frac{(n-1)s^2+n(\bar{x}-\theta)^2}{2\sigma^2}\right)\\
&= (\sigma^2)^{-n/2-1}\exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)\exp\left(-\frac{(\bar{x}-\theta)^2}{2\frac{\sigma^2}{n}}\right)\\
&= \sqrt{2\pi}(\frac{\sigma^2}{n})^{1/2}(\sigma^2)^{-n/2-1}\exp\left(-\frac{\frac{(n-1)s^2}{2}}{\sigma^2}\right)\times \mathrm{N}\left(\bar{x},\frac{\sigma^2}{n}\right)\\
&= \mathrm{IGamma}\left(\frac{n-1}{2},\frac{(n-1)s^2}{2}\right)\times \mathrm{N}\left(\bar{x},\frac{\sigma^2}{n}\right)
\end{aligned}
$$

(**Note:** The distributions above represent their densities)

Notes on multiparameter models

- Substantial increase in calculations. Inference is feasible only for a few models.
- This can be resolved using MCMC or approximate methods.
- Specification of priors may still be an issue.

Reading

J.O. Berger Sections 3.3, 4.2 and 4.3.2