# Math Camp Lesson 2

## Vectors and Matrices (Linear Algebra)

UW–Madison Political Science

August 21, 2018

# Introductions

# Linear Algebra

# Review/Overview

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Algebra is a fundamental basis for more advanced mathematical manipulation:

- Use to derive statistical estimators, and to understand their properties and the assumptions necessary to apply them.
- Use to evaluate the optimal choices of strategic actors.

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If no range is indicated $\left(\sum x_i\right)$, this implies all observations are included.

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Suppose we have data: $x_1 = 3$, $x_2 = 4$, $x_3 = 1$, $x_4 = 0$, and $x_5 = 2$

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$$\begin{aligned}\sum_{i=1}^{3} (x_i^2 + 3) &= (x_1^2 + 3) + (x_2^2 + 3) + (x_3^2 + 3) \\ &= (3^2 + 3) + (4^2 + 3) + (1^2 + 3) \\ &= 35\end{aligned}$$

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$$\begin{aligned}\prod_{i=1}^{3} (x_i^2 + 3) &= (x_1^2 + 3) \times (x_2^2 + 3) \times (x_3^2 + 3) \\ &= (3^2 + 3) \times (4^2 + 3) \times (1^2 + 3) \\ &= 912\end{aligned}$$

# Summations and Products

Given these data: $x_1 = 3$, $x_2 = 4$, $x_3 = 1$, and $x_4 = 0$; and $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, and $y_4 = 4$.
Find these quantities:

- $\sum x_i + \sum y_i$
- $\sum (x_i + y_i)$
- $\prod x_i + \prod y_i$
- $\prod (x_i \times y_i)$

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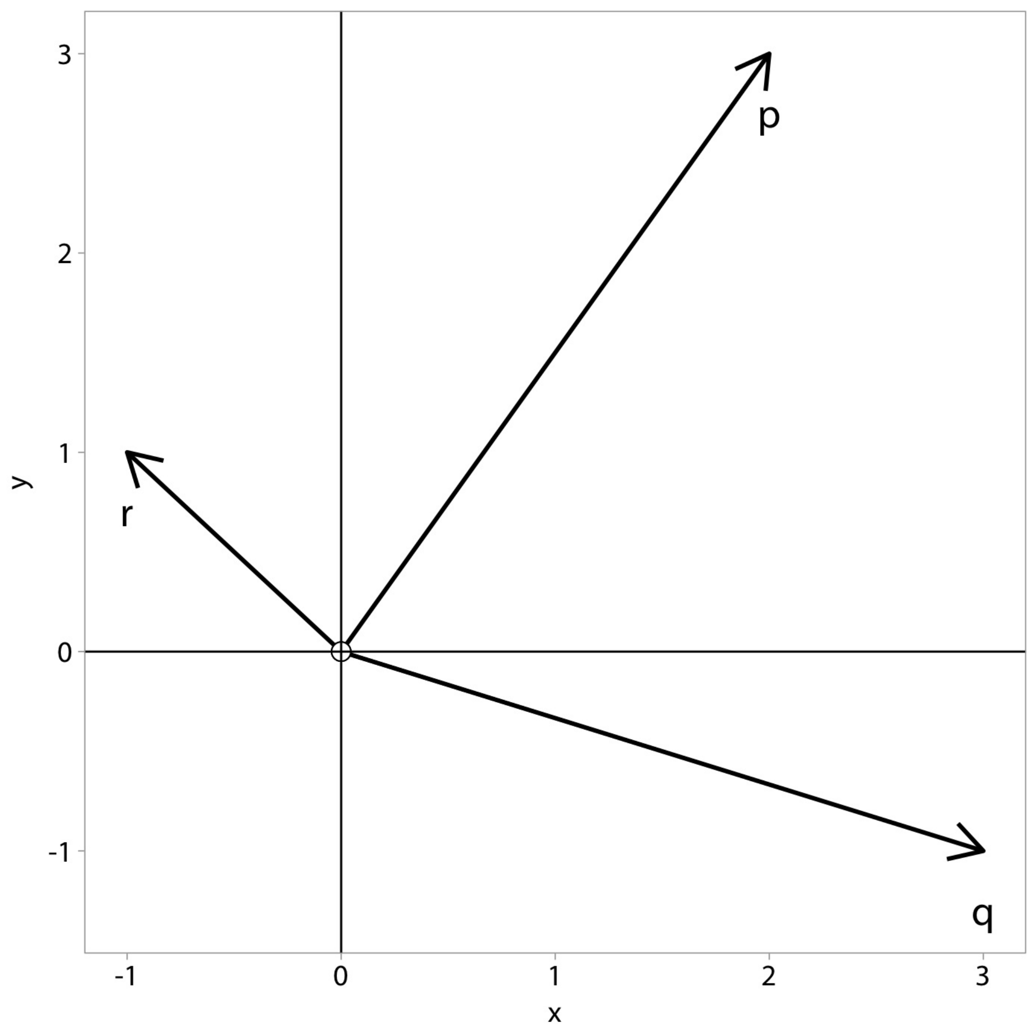
Ordered simply means that $[v_1, v_2, v_3, v_4] \neq [v_4, v_3, v_2, v_1]$

# Vectors in Space




Vectors can be thought of as lines from the origin in k-dimensional space (where k is the number of vector elements) going to a point with the coordinates of the elements of the vector.

$\mathbf{p} = [2, 3]$

$\mathbf{q} = [3, -1]$

$\mathbf{r} = [-1, 1]$

# Vector Operations: Addition and Subtraction

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$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{w} \\ [1,2,3,4] + [4,8,12,16] &= \mathbf{w} \\ [1+4, 2+8, 3+12, 4+16] &= \mathbf{w} \\ [5,10,15,20] &= \mathbf{w} \end{aligned}$$

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$$\begin{aligned} \mathbf{u} - \mathbf{v} &= \mathbf{w} \\ [1,2,3,4] - [4,8,12,16] &= \mathbf{w} \\ [1-4, 2-8, 3-12, 4-16] &= \mathbf{w} \\ [-3,-6,-9,-12] &= \mathbf{w} \end{aligned}$$

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It is important to note that conformability does not matter for scalar multiplication and division.

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$$\mathbf{x} \cdot \mathbf{y} = [x_1 \times y_1 + x_2 \times y_2, \cdots + x_{k-1} \times y_{k-1} + x_k \times y_k] = \sum_{i=1}^{k} x_i y_i$$

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The dot product will start with two vectors and result in a scalar.

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Associative Property: $\mathrm{s}(\mathbf{u} \cdot \mathbf{v}) = \mathrm{s}(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathrm{s}(\mathbf{v})$

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Distributive Property: $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

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$$\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

# Vectors

Given vectors $\mathbf{x} = [1, 2, 0, 4]$ and $\mathbf{y} = [5, 3, 2, 3]$, find:

- $\mathbf{x}^{\mathrm{T}}$
- $\mathbf{x} + \mathbf{y}$
- $\mathbf{x} \cdot \mathbf{y}$

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Like vectors, each value is referred to as an element. When referring to elements of a matrix, we will not bold the vector and add a subscript to denote their position. For example, $x_{1,2}$ refers to the element in the first row, second column of $\mathbf{X}$.

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- A very general square matrix form is the symmetric matrix. This is a matrix that is symmetric across the diagonal from the upper left-hand corner to the lower right-hand corner (also called the main diagonal) .

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

This switches the dimensions (here, from $2 \times 3$ to $3 \times 2$).

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$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

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Such that, e.g. $a_{2,1} + b_{2,1} = c_{2,1}$

# Matrix Operations: Matrix Addition and Substraction

Consider:

$$
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ 4+8 & 5+10 & 6+12 \end{bmatrix}
$$

$$
= \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}
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$$= \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-4 & 3-6 \\ 4-8 & 5-10 & 6-12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

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To multiply (or divide) a matrix by a scalar, simply multiply (or divide) each element of the matrix by the scalar.

For example,

$$\begin{aligned} 4 \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} &= \begin{bmatrix} 4 \times 1 & 4 \times 2 & 4 \times 3 \\ 4 \times 4 & 4 \times 5 & 4 \times 6 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix} \end{aligned}$$

# Matrix Multiplication

To multiply a matrix by another matrix, each element of the result is found by taking the corresponding row of the first matrix, turning it sideways, multiplying each element by the corresponding column in the second matrix, and summing the result, or the dot product!

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Such that, e.g. $c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}$

Let's do one more example to clarify. The product of a $1 \times 2$ matrix and a $2 \times 1$ matrix is a $1 \times 1$ matrix, or a scalar!

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$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$

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What do $c$ and $d$ equal?

$$c = 3 \times 5 + 4 \times 7 = 43$$
$$d = 3 \times 6 + 4 \times 8 = 50$$

# Matrix Multiplication

Therefore,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

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Unlike with scalars, order matters. Reversing the order may result in a different product, or may not even be possible depending on the dimensions of the matrices.

# Properties of Matrix Multiplication

Like vector multiplication, matrix multiplication has some special properties consider for conformable matrices.

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- Additive Distributive Property: $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$
- Identity Property: $\mathbf{XI} = \mathbf{IX} = \mathbf{X}$

# Matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

Given the matrices above, calculate

- $\mathbf{A} + \mathbf{B}$
- $\mathbf{C}^{\mathrm{T}}$
- $\mathbf{DB}$
- $\mathbf{CD}^{\mathrm{T}}$

# Matrix Inversion

The operation most closely analogous to division for matrices is inversion. The inverse of a matrix (denoted with the superscript $^{-1}$) is the matrix that, when multiplied by the original, produces the identity matrix:

$$\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$$

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Matrix inversion is only possible with some square matrices. If a square matrix is not invertible it is called a singular or non-invertible matrix.

# Matrix Inversion (cont'd)

A handy shortcut to find the inverse of a $2 \times 2$ matrix, calculate the determinant (product of the main diagonal minus the product of the off diagonal) and adjust the elements as such:

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$$\begin{aligned} \mathbf{X}^{-1} &= \frac{1}{\det(\mathbf{X})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \end{aligned}$$

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If the determinant is zero, there is no inverse. Calculating inverses of larger (square) matrices is more complicated.

# Matrix Inversion

Consider:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

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Demonstrate that $\left(\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}\right)$

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Let's call of these matrices and vector.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

# Solving Systems of Equations (cont'd)

Now, we have expression: $\mathbf{A}\mathbf{v} = \mathbf{B}$ and we want to solve for $\mathbf{v}$.

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$$\mathbf{A}^{-1} = \frac{1}{2 \times 4 - 6 \times 3} \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

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Let's verify our results

$$2(-10) + 6(5) = 10$$
$$3(-10) + 4(5) = -10$$

# Where is linear algebra in political science?

More like, where isn't it?

# Regression

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In matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \alpha + \beta \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

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or ...

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

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$$\mathbf{y} = \mathbf{X}\beta + \varepsilon$$

It so happens that when we do the matrix calculus to solve for $\beta$ ...

$$\beta = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$$

# Point being…

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and tons of political science boils down to regression modeling

End Day 2