

Math Camp Lesson 2

Vectors and Matrices (Linear Algebra)

UW–Madison Political Science

August 21, 2018

Introductions

Linear Algebra

Review/Overview

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"Solving for unknown values"

Algebra is a fundamental basis for more advanced mathematical manipulation:

- Use to derive statistical estimators, and to understand their properties and the assumptions necessary to apply them.
- Use to evaluate the optimal choices of strategic actors.

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If no range is indicated ($\sum x_i$), this implies all observations are included.

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$$\begin{aligned}\sum_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) + (x_2^2 + 3) + (x_3^2 + 3) \\ &= (3^2 + 3) + (4^2 + 3) + (1^2 + 3) \\ &= 35\end{aligned}$$

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$$\begin{aligned}\prod_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) \times (x_2^2 + 3) \times (x_3^2 + 3) \\ &= (3^2 + 3) \times (4^2 + 3) \times (1^2 + 3) \\ &= 912\end{aligned}$$

Summations and Products

Given these data: $x_1 = 3$, $x_2 = 4$, $x_3 = 1$, and $x_4 = 0$; and $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, and $y_4 = 4$.

Find these quantities:

- $\sum x_i + \sum y_i$
- $\sum (x_i + y_i)$
- $\prod x_i + \prod y_i$
- $\prod (x_i \times y_i)$

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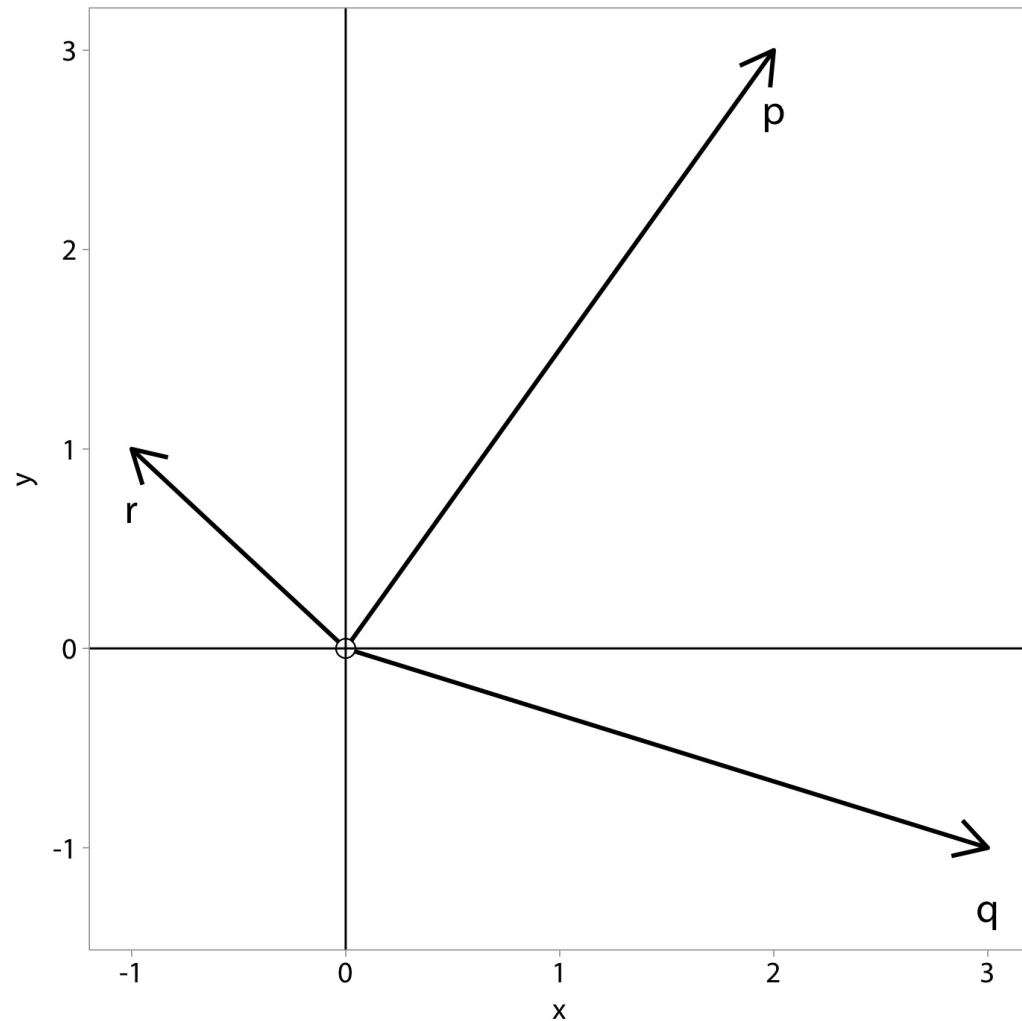
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or a **column** vectors like

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Ordered simply means that $[v_1, v_2, v_3, v_4] \neq [v_4, v_3, v_2, v_1]$

Vectors in Space



Vectors can be thought of as lines from the origin in k -dimensional space (where k is the number of vector elements) going to a point with the coordinates of the elements of the vector.

$$\mathbf{p} = [2, 3]$$

$$\mathbf{q} = [3, -1]$$

$$\mathbf{r} = [-1, 1]$$

Vector Operations: Addition and Subtraction

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$$\mathbf{u} + \mathbf{v} = \mathbf{w}$$

$$[1, 2, 3, 4] + [4, 8, 12, 16] = \mathbf{w}$$

$$[1 + 4, 2 + 8, 3 + 12, 4 + 16] = \mathbf{w}$$

$$[5, 10, 15, 20] = \mathbf{w}$$

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$$\mathbf{u} - \mathbf{v} = \mathbf{w}$$

$$[1, 2, 3, 4] - [4, 8, 12, 16] = \mathbf{w}$$

$$[1 - 4, 2 - 8, 3 - 12, 4 - 16] = \mathbf{w}$$

$$[-3, -6, -9, -12] = \mathbf{w}$$

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It is important to note that conformability does not matter for scalar multiplication and division.

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$$\mathbf{x} \cdot \mathbf{y} = [x_1 \times y_1 + x_2 \times y_2, \dots + x_{k-1} \times y_{k-1} + x_k \times y_k] = \sum_{i=1}^k x_i y_i$$

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The dot product will start with two vectors and result in a scalar.

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Commutative property: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Associative Property: $s(\mathbf{u} \cdot \mathbf{v}) = s(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot s(\mathbf{v})$

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Distributive Property: $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

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$$\mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Vectors

Given vectors $\mathbf{x} = [1, 2, 0, 4]$ and $\mathbf{y} = [5, 3, 2, 3]$, find:

- \mathbf{x}^T
- $\mathbf{x} + \mathbf{y}$
- $\mathbf{x} \cdot \mathbf{y}$

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Like vectors, each value is referred to as an element. When referring to elements of a matrix, we will not bold the vector and add a subscript to denote their position. For example, $x_{1,2}$ refers to the element in the first row, second column of \mathbf{X} .

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This switches the dimensions (here, from 2×3 to 3×2).

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Such that, e.g. $a_{2,1} + b_{2,1} = c_{2,1}$

Matrix Operations: Matrix Addition and Subtraction

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ 4+8 & 5+10 & 6+12 \end{bmatrix} \\ = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

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or

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-4 & 3-6 \\ 4-8 & 5-10 & 6-12 \end{bmatrix} \\ = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

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$$4 \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \times 1 & 4 \times 2 & 4 \times 3 \\ 4 \times 4 & 4 \times 5 & 4 \times 6 \end{bmatrix} \\ = \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$

Matrix Multiplication

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Such that, e.g. $c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}$

Let's do one more example to clarify. The product of a 1×2 matrix and a 2×1 matrix is a 1×1 matrix, or a scalar!

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What do c and d equal?

$$c = 3 \times 5 + 4 \times 7 = 43$$

$$d = 3 \times 6 + 4 \times 8 = 50$$

Matrix Multiplication

Therefore,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

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Unlike with scalars, order matters. Reversing the order may result in a different product, or may not even be possible depending on the dimensions of the matrices.

Properties of Matrix Multiplication

Like vector multiplication, matrix multiplication has some special properties consider for conformable matrices.

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- Additive Distributive Property: $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$
- Identity Property: $\mathbf{XI} = \mathbf{IX} = \mathbf{X}$

Matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

Given the matrices above, calculate

- $\mathbf{A} + \mathbf{B}$
- \mathbf{C}^T
- \mathbf{DB}
- \mathbf{CD}^T

Matrix Inversion

The operation most closely analogous to division for matrices is inversion. The inverse of a matrix (denoted with the superscript $^{-1}$) is the matrix that, when multiplied by the original, produces the identity matrix:

$$\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$$

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Matrix inversion is only possible with [some square matrices](#). If a square matrix is not invertible it is called a singular or non-invertible matrix.

Matrix Inversion (cont'd)

A handy shortcut to find the inverse of a 2×2 matrix, calculate the **determinant** (product of the main diagonal minus the product of the off diagonal) and adjust the elements as such:

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$$\begin{aligned} \mathbf{X}^{-1} &= \frac{1}{\det(\mathbf{X})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \end{aligned}$$

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If the determinant is zero, there is no inverse. Calculating inverses of larger (square) matrices is more complicated.

Matrix Inversion

Consider:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

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Demonstrate that ($\mathbf{X}^{-1} \mathbf{X} = \mathbf{I}$)

Solving Systems of Equations

Consider a system of equations:

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Let's call of these matrices and vector.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

Solving Systems of Equations (cont'd)

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$$\mathbf{A}^{-1} = \frac{1}{2 \times 4 - 6 \times 3} \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

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Now we can solve for \mathbf{v} .

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Let's verify our results

$$2(-10) + 6(5) = 10$$

$$3(-10) + 4(5) = -10$$

Where is linear algebra in political
science?

Regression

We want to estimate the linear relationship between an independent variable \mathbf{x} and a dependent variable \mathbf{y} . How does \mathbf{x} affect \mathbf{y} ?

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In matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \alpha + \beta \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

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or ...

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Matrix-form regression

Supposing that we refer to the coefficient vector $\beta = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$

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It so happens that when we do the matrix calculus to solve for β ...

$$\beta = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

Point being...

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These basic principles apply to all regression modeling

Point being...

These basic principles apply to all regression modeling
and tons of political science boils down to regression modeling

End Day 2