Math Camp Lesson 3

Calculus

UW–Madison Political Science

August 22 & 23, 2018

Overview

Calculus evaluates the behavior of functions:

- Increasing/decreasing
- Rate of change
- Change in the rate of change
- Area of the region they define

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Concepts from calculus underlie a wide variety of mathematics, particularly in the applied math that we use in political science

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Calculating the probability density in regions of continuous distributions.

Solving for the choice that maximizes a decision maker's utility.

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More Derivatives
- Integrals
- Applications

Limits

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The limit of a function characterizes its behavior given a certain input, or as an input value changes.

Limits (cont'd)

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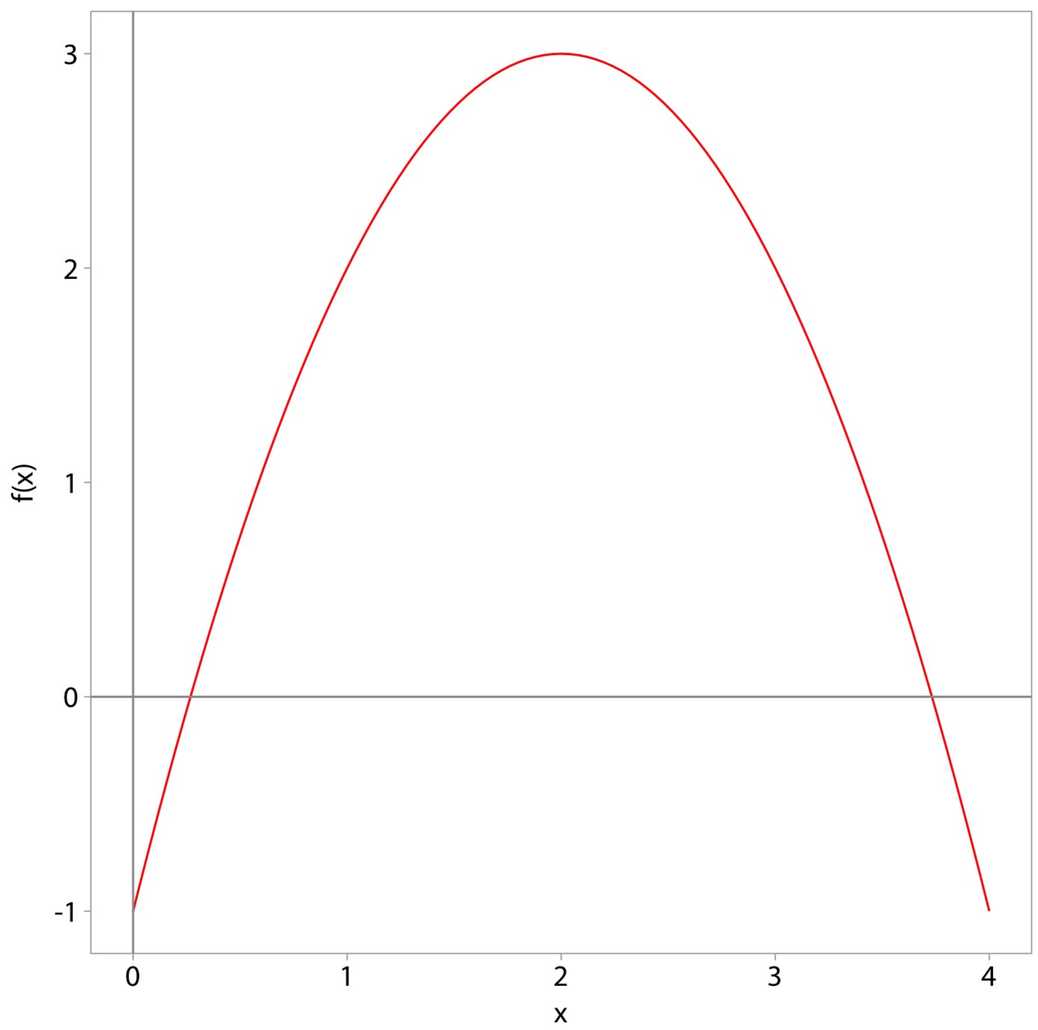
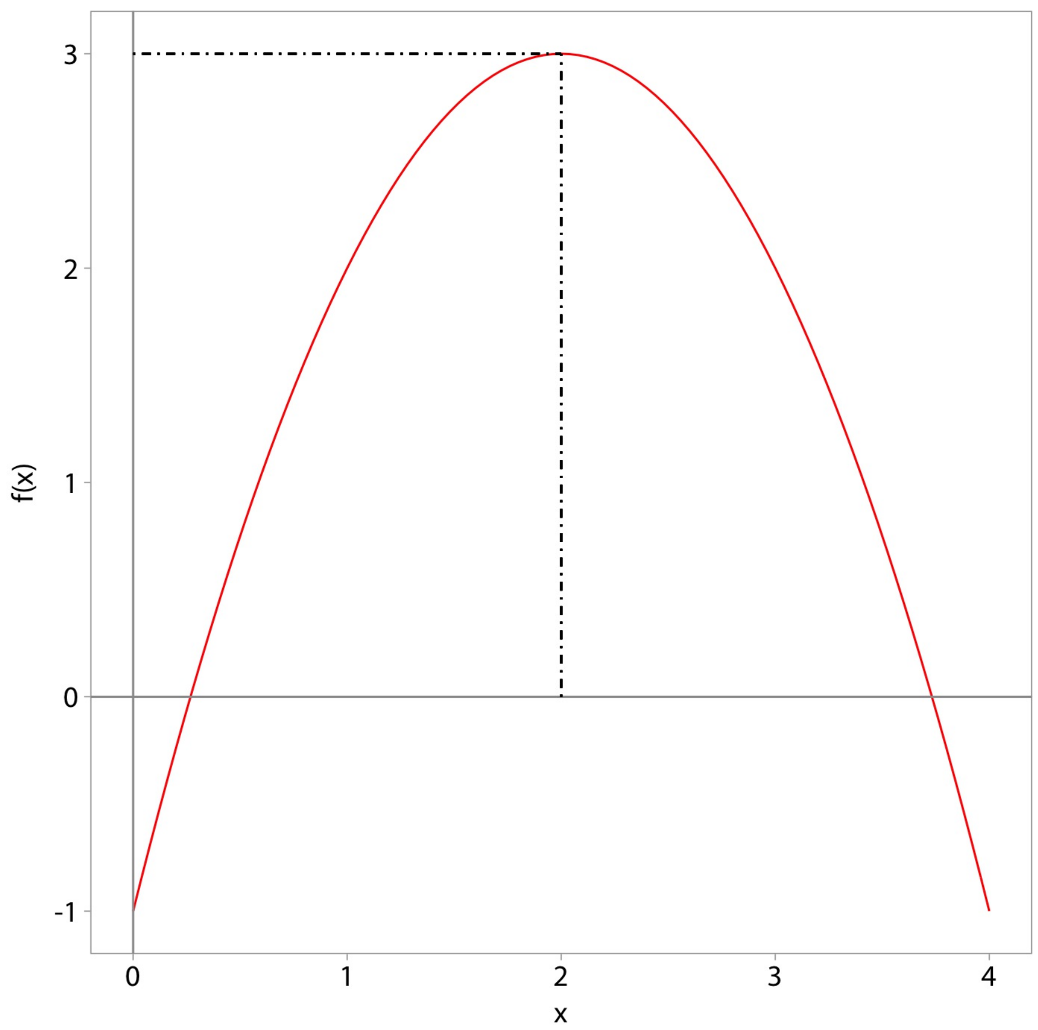


Let's consider the simple function,
$f(x) = y = 3 - (x-2)^2$, plotted to the left.
What is the limit of $f(x)$ as x approaches 2?

Limits (cont'd)



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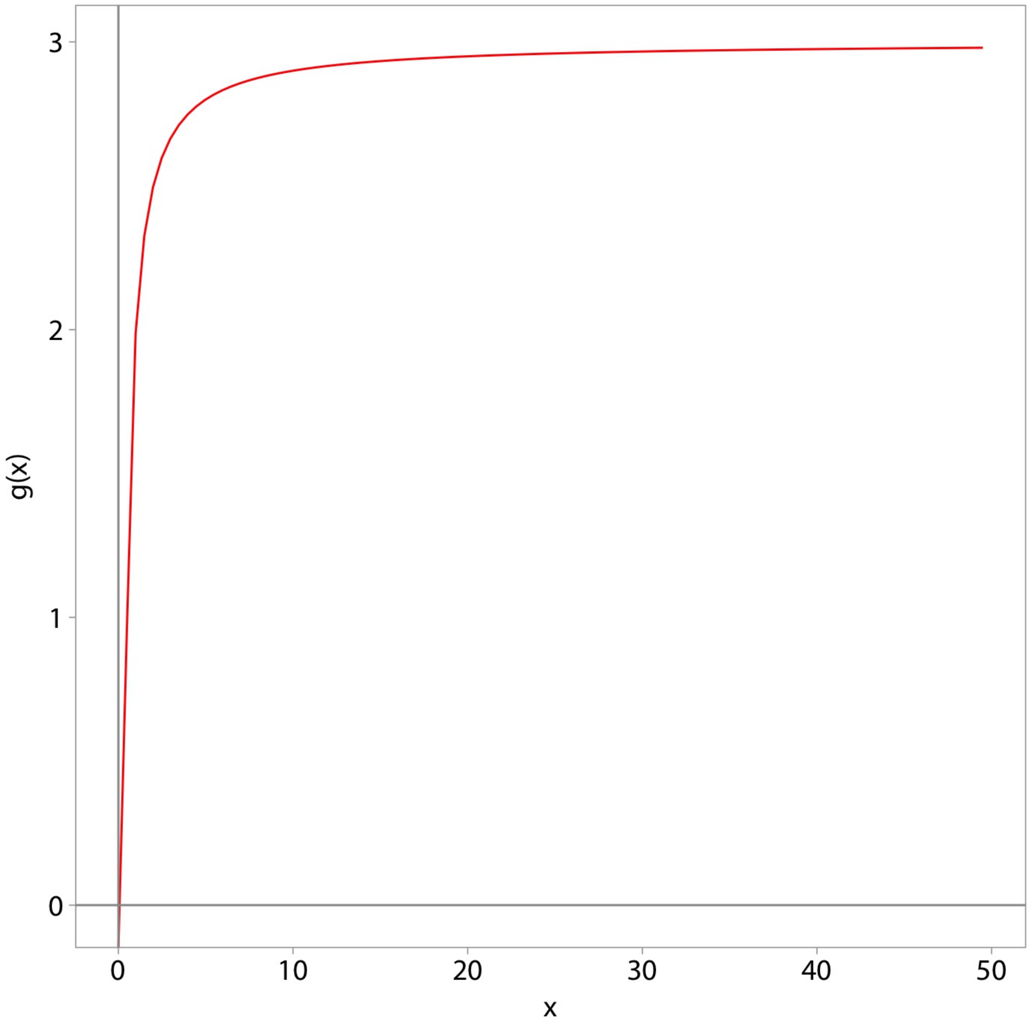
As x approaches 2, $f(x)$ or y approaches $f(2) = 3$.

Limits (cont'd)



Let's consider a less simple function, $g(x) = y = 3 - \frac{1}{x}$, plotted to the left. What is the limit of $g(x)$ as x approaches ∞?

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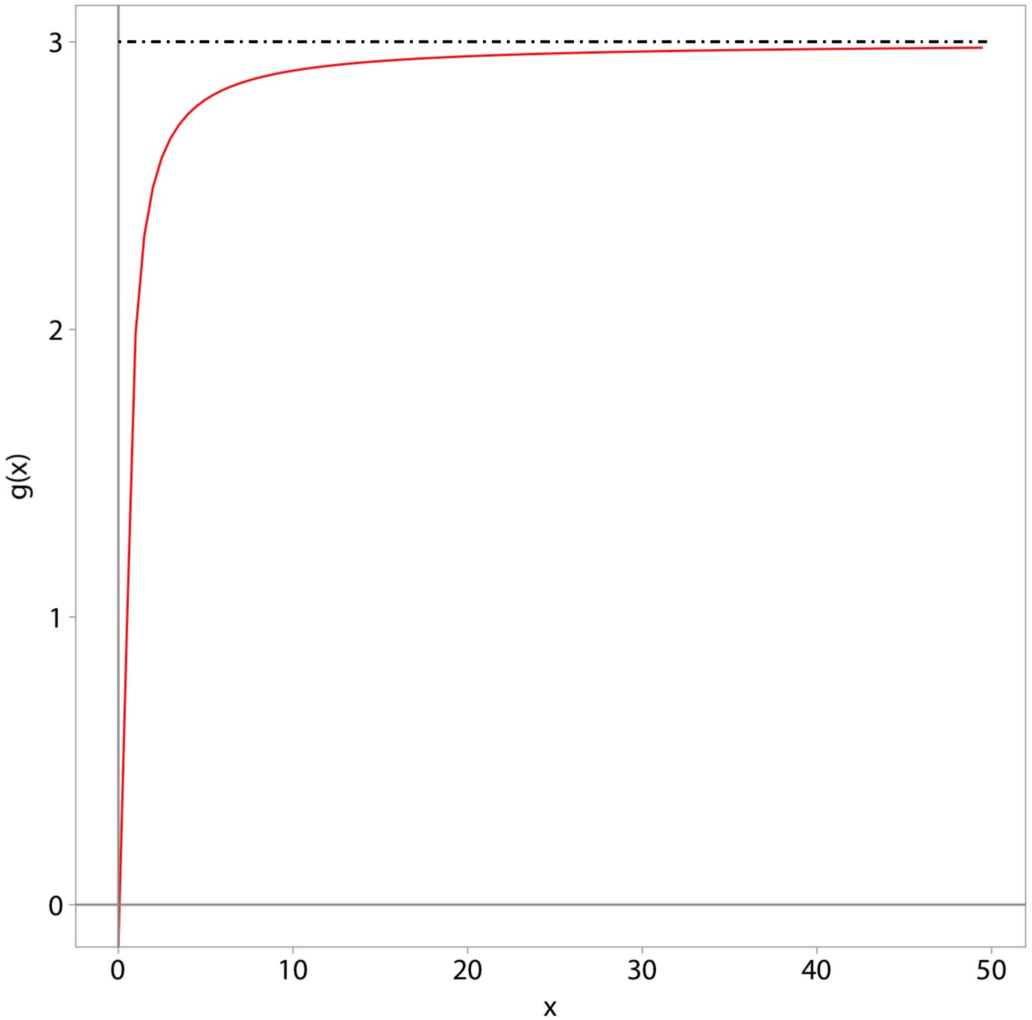

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As x approaches ∞, $g(x)$ approaches 3. Why?

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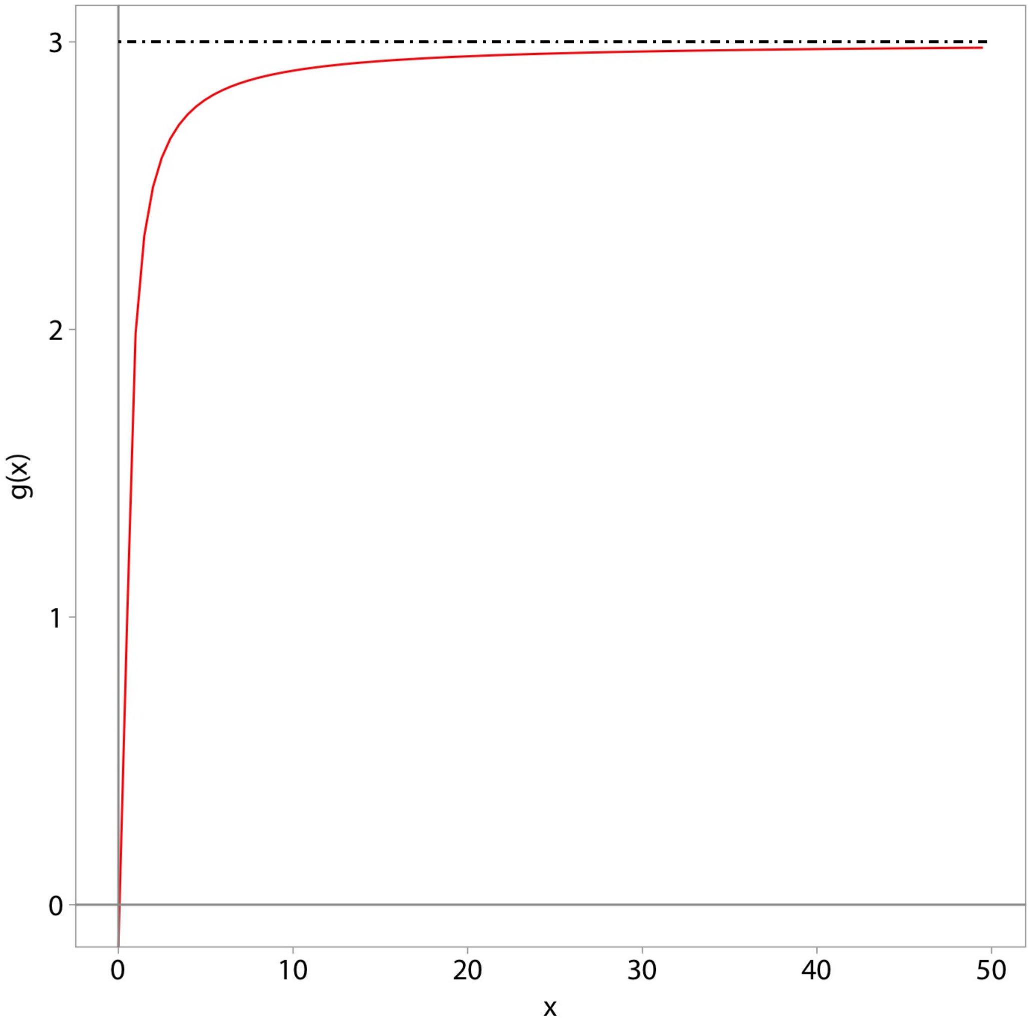

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As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \dots\right)$$

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Many times, you will often see this expression written as $\lim_{x \to c^-} f(x) = L$ or $\lim_{x \to c^+} f(x) = L$. A negative sign $(-)$ implies "As x approaches c from the left" and a positive sign $(+)$ implies "As x approaches c from the right"

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If you can, evaluate the function at the limit.

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For functions that are well-behaved (continuous), the limit as x approaches a finite point is generally the value of the function at that point (if it exists)

Examples of Limits

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$$\begin{aligned}
\lim_{x \to 2} x^2 - 3x + 1 &= \lim_{x \to 2} x^2 - 2x + 1 \\
&= \lim_{x \to 2} x^2 - 2 \lim_{x \to 2} x + \lim_{x \to 2} 1 \\
&= 2^2 - 2(2) + 1 \\
&= 1
\end{aligned}$$

Examples of Limits (cont'd)

Now, let's consider $\lim_{x \to \infty} \dfrac{4x^4 + 7x^2 + 8}{3x^4}$

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$$\lim_{x\to\infty} \frac{4x^4 + 7x^2 + 8}{3x^4} = \lim_{x\to\infty} \frac{4x^4 + 7x^2 + 8}{3x^4}$$

$$= \lim_{x\to\infty} \frac{4x^4}{3x^4} + \lim_{x\to\infty} \frac{7x^2}{3x^4} + \lim_{x\to\infty} \frac{8}{3x^4}$$

$$= \lim_{x\to\infty} \frac{4}{3} + \lim_{x\to\infty} \frac{7}{3x^2} + \lim_{x\to\infty} \frac{8}{3x^4}$$

$$= \frac{4}{3} + 0 + 0$$

$$= \frac{4}{3}$$

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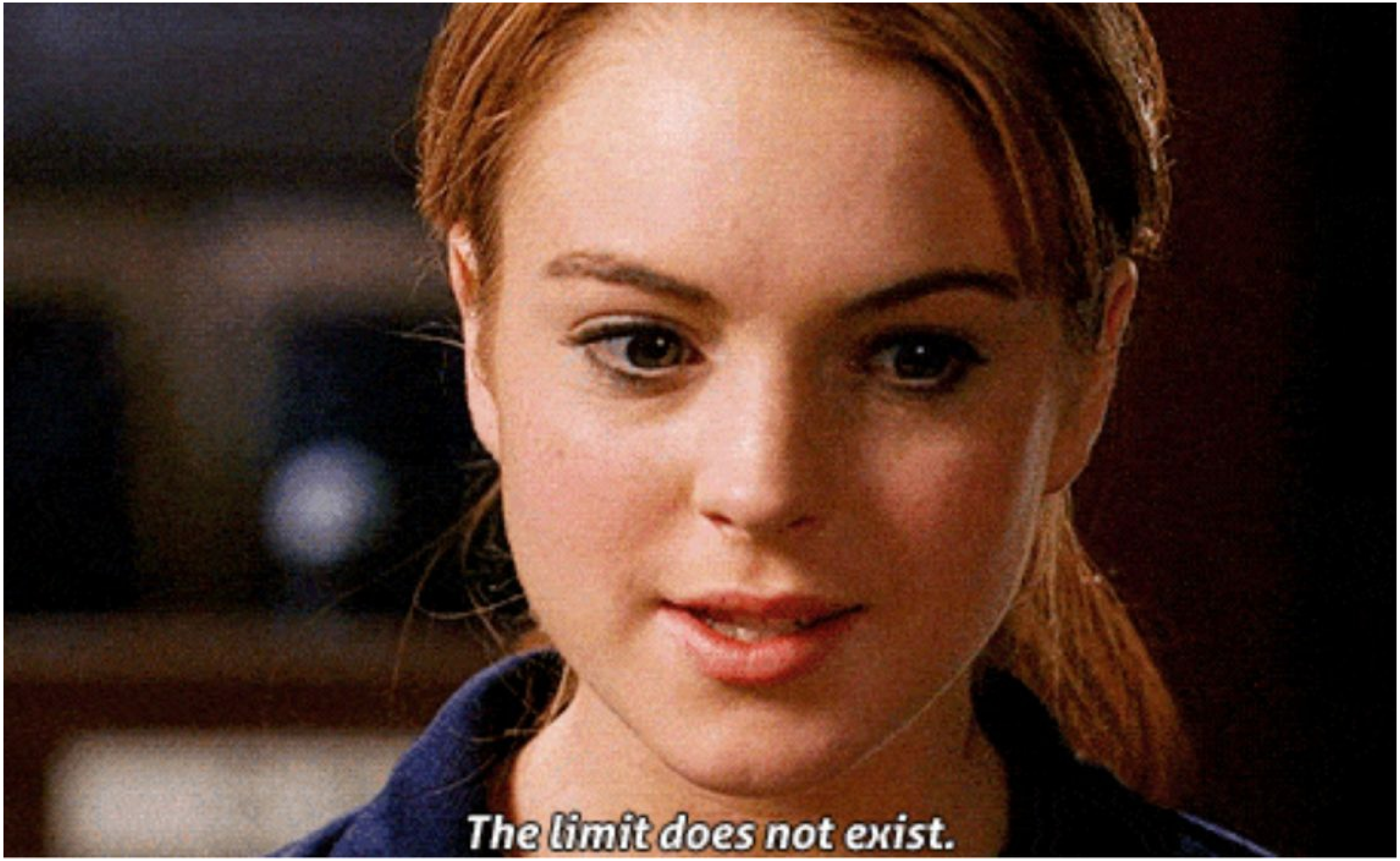
$$= \frac{4}{3} + \frac{7}{0^2} + \frac{8}{0^4}$$


The limit does not exist.

Just Kidding

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The limit does exist but the limit is ∞.

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$$\lim_{x\to 0} \frac{4x^4 + 7x^2 + 8}{3x^4} = \frac{4}{3} + \lim_{x\to 0} \frac{7}{3x^2} + \lim_{x\to 0} \frac{8}{3x^4}$$

$$= \frac{4}{3} + \text{Huge number} + \text{Huge number}$$

$$= \infty$$

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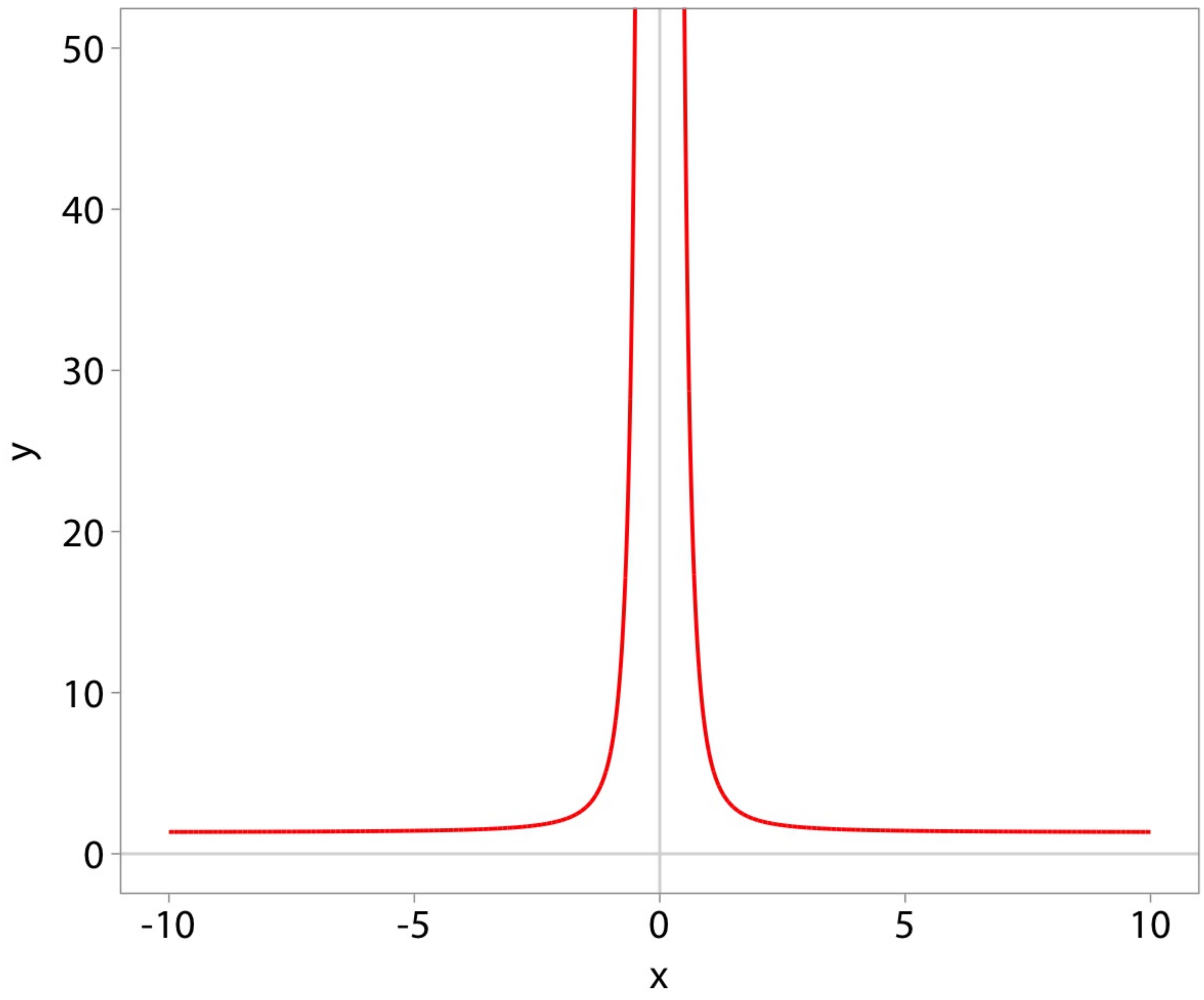
$$= \frac{4}{3} + \text{Huge number} + \text{Huge number}$$

$$= \infty$$

As x approaches 0, the function retains some positive value in the numerator while denominator positively approach 0. This means that you are dividing by a smaller and smaller fraction, which means the entire statement is getting larger and approaches ∞.



50
40
30
20
10
0
y
-10
-5
0
5
10
x

Derivatives

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The derivative of a function is its rate of change in the output as the value of its input changes.

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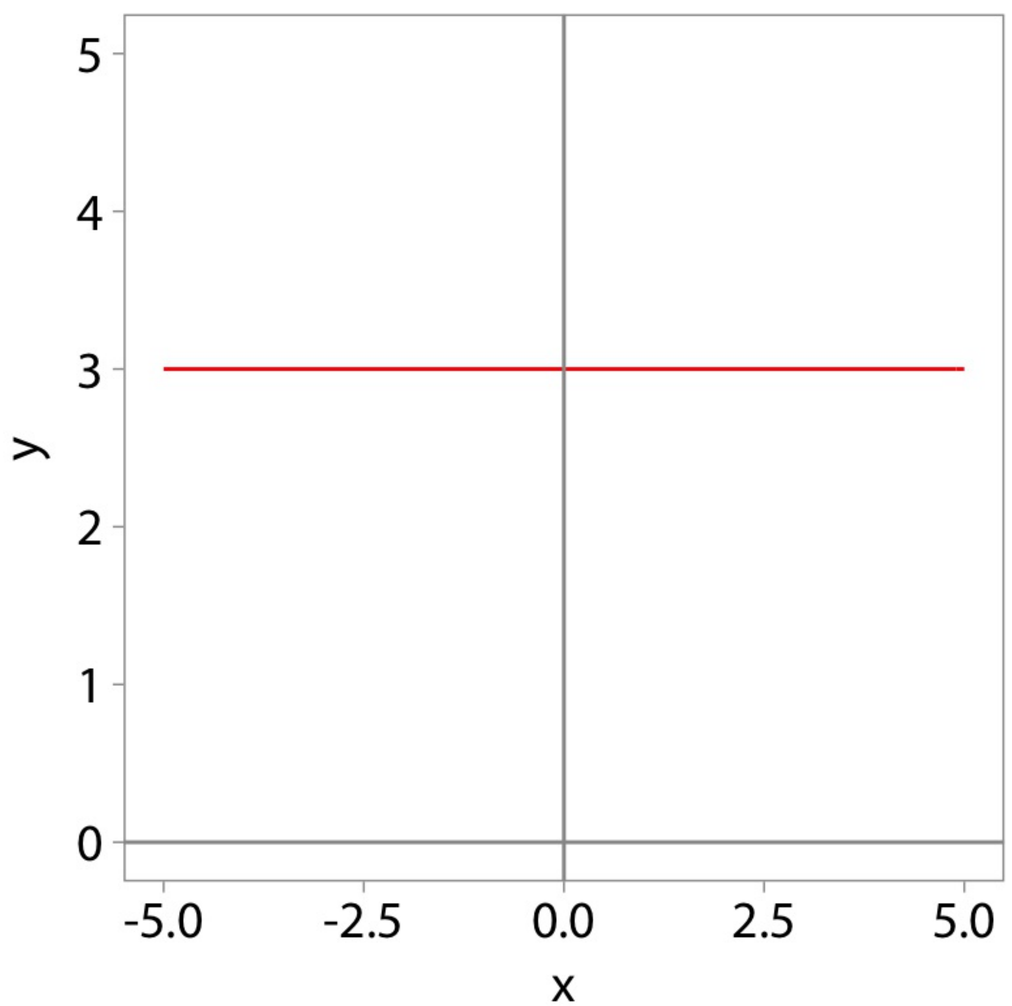
The instantaneous slope of the line at any given point.

The slope of a function is how much the output changes as a result of changes in the input. Using Δ to signify 'change', this is $\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ or ''rise-over-run''.

Slope



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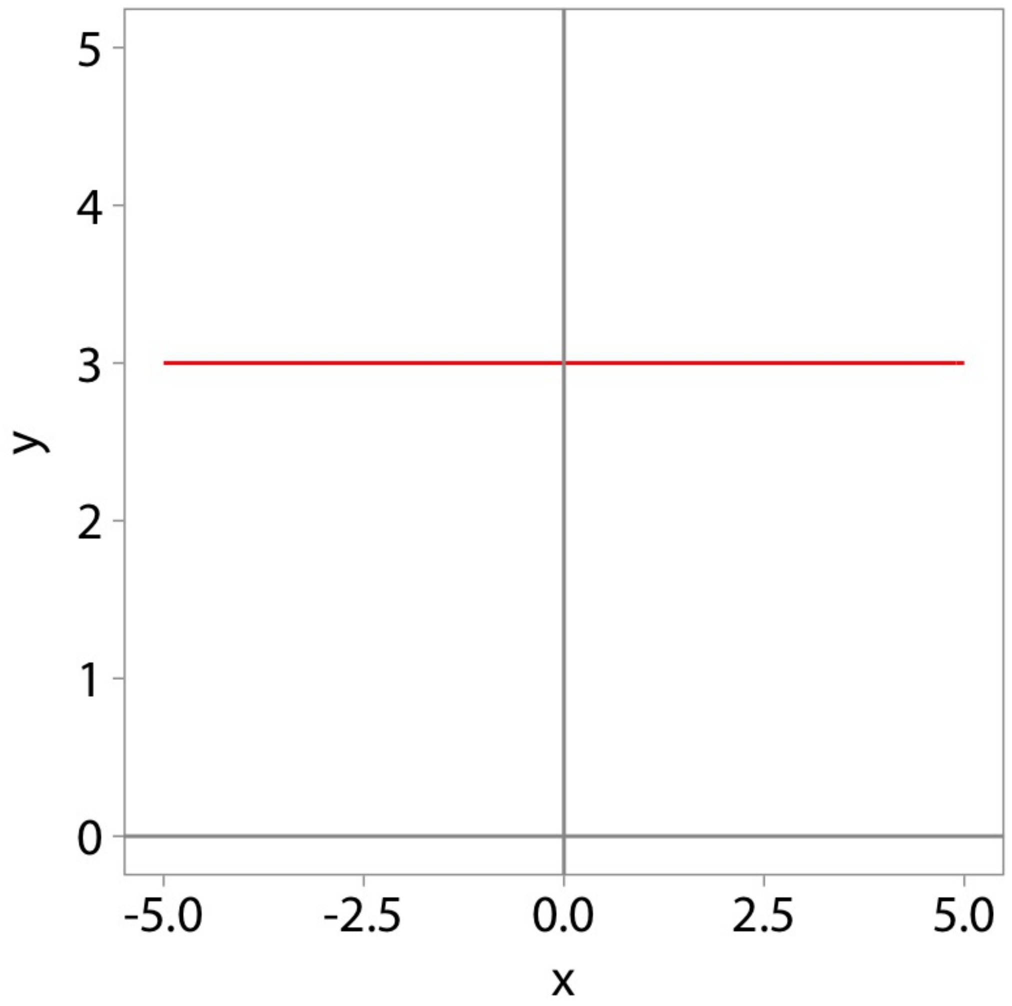
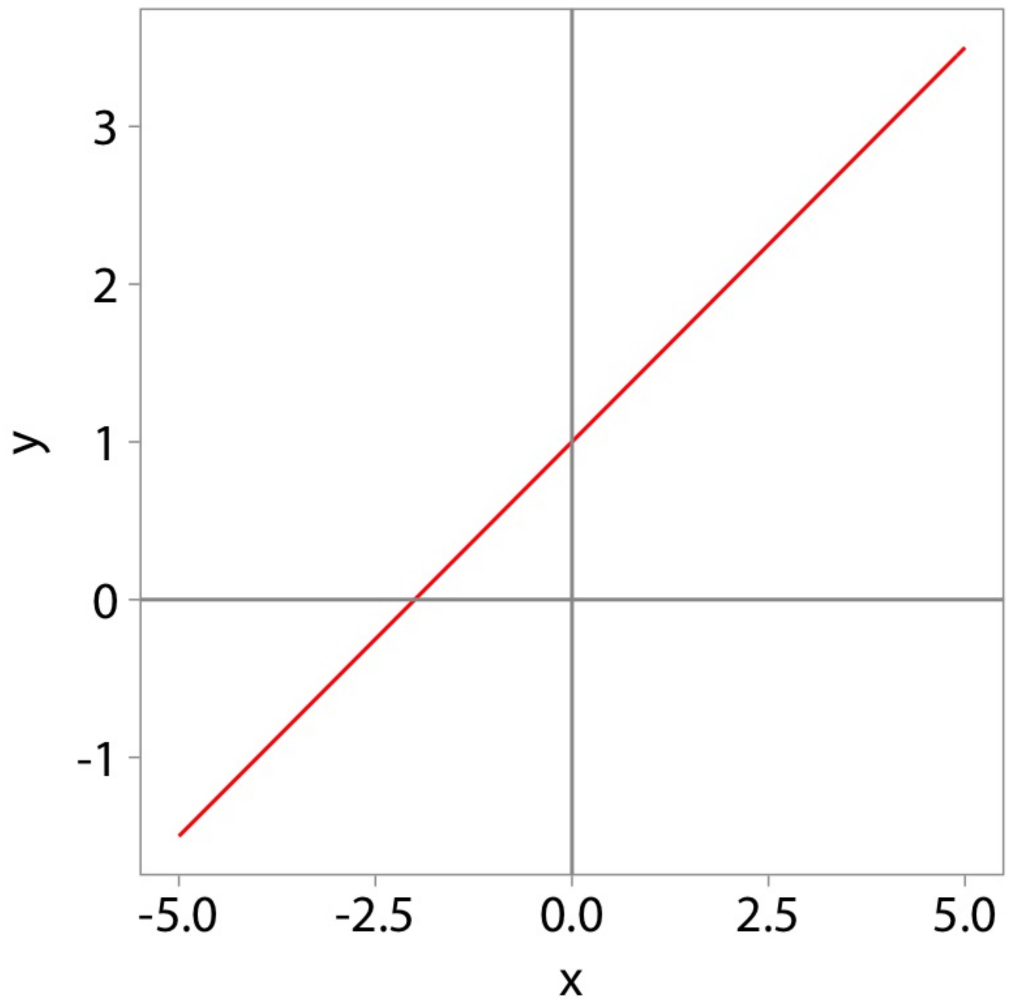
Slope



Let's consider the function, $y = 3$, plotted to the left. What is its ''slope''?

Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slope



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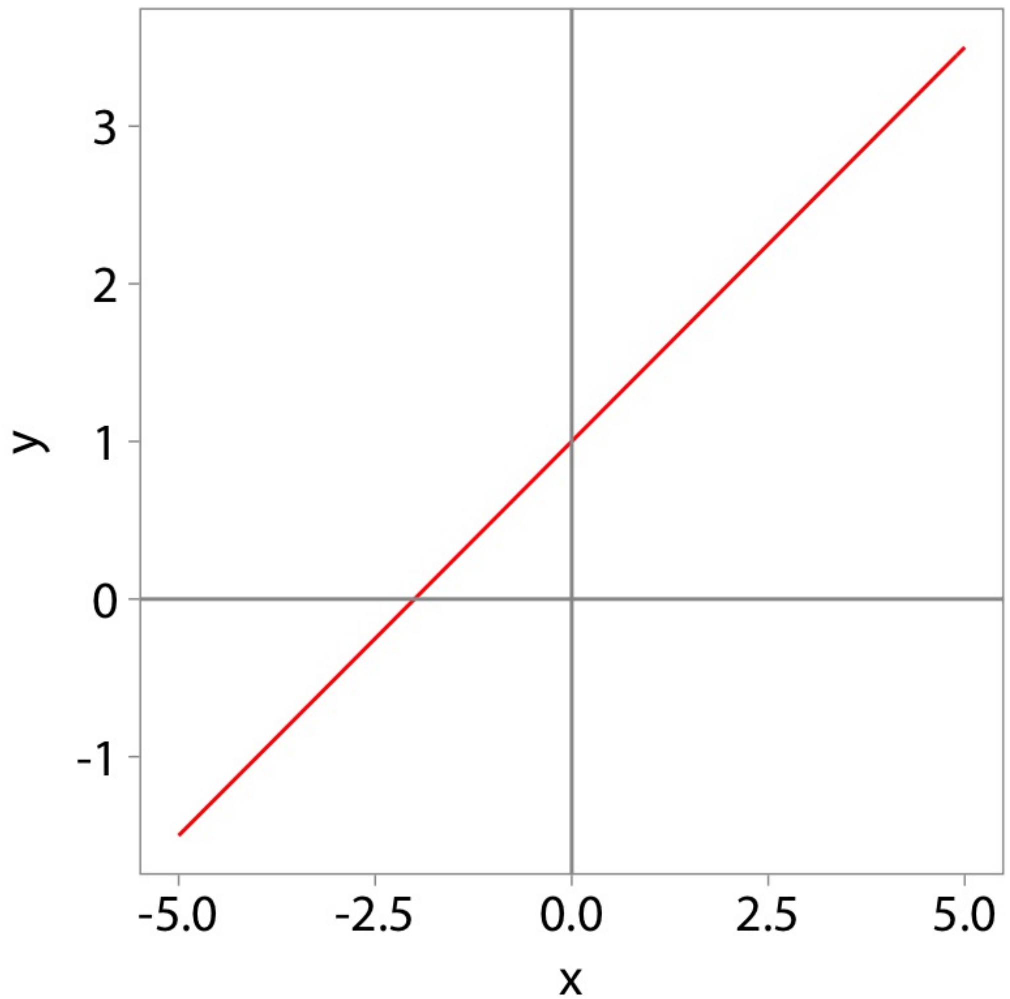
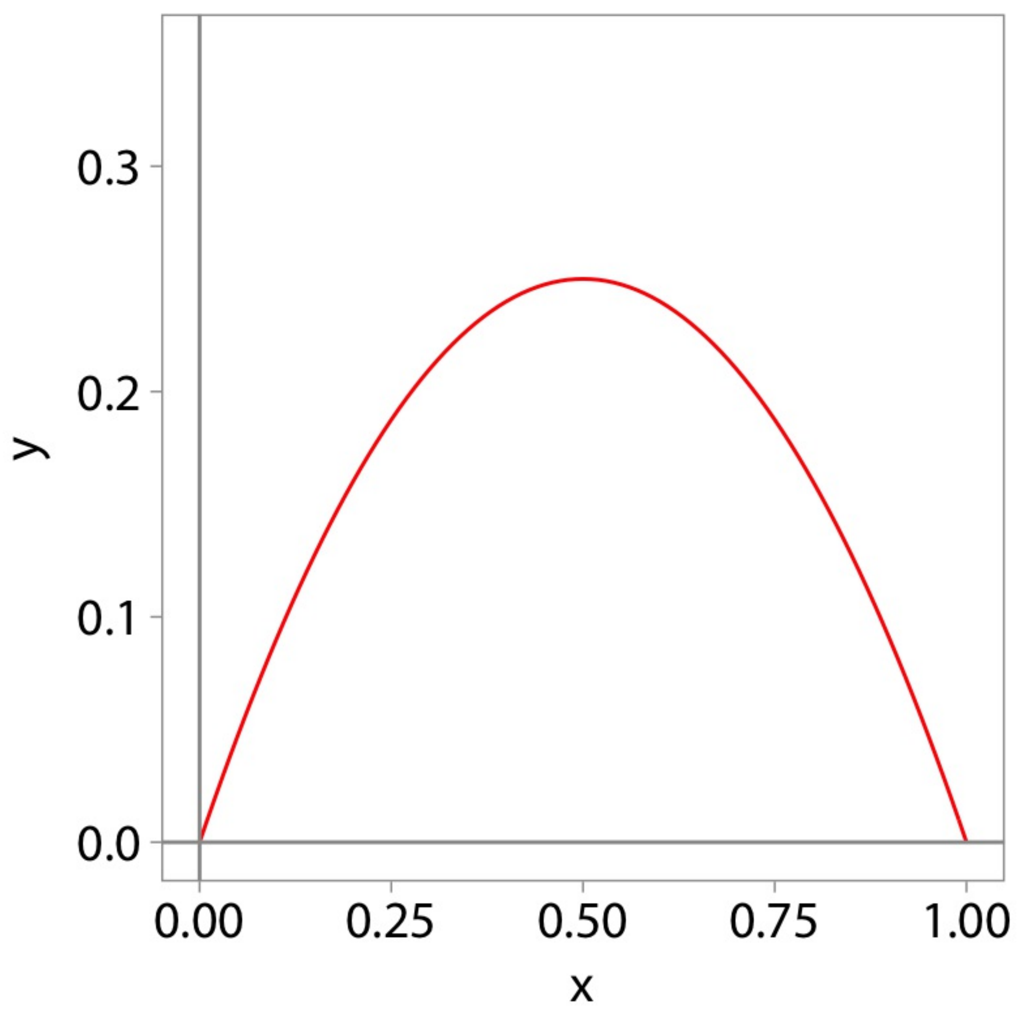
Slope



Let's consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

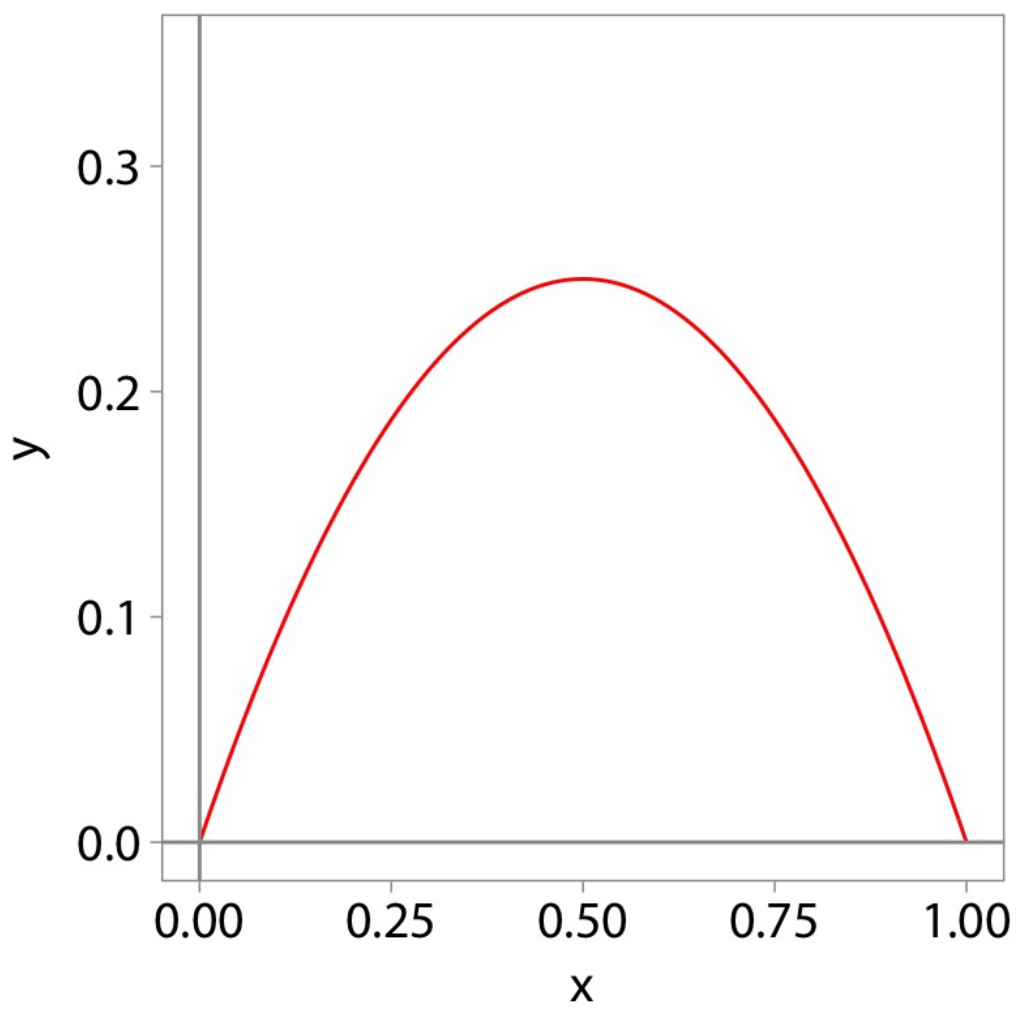
Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$. [Recall: $y = mx + b$ from Day 1]

Slope



Let's consider even more complicated function, $y = x - x^2$, plotted to the left. What is its "slope"?

Slope



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Its slope or $\frac{\Delta f(x)}{\Delta x}$ =???

Derivatives as a Limit

Approximate the slope at a certain location by picking a point nearby on the line and finding the slope of the straight line connecting them.

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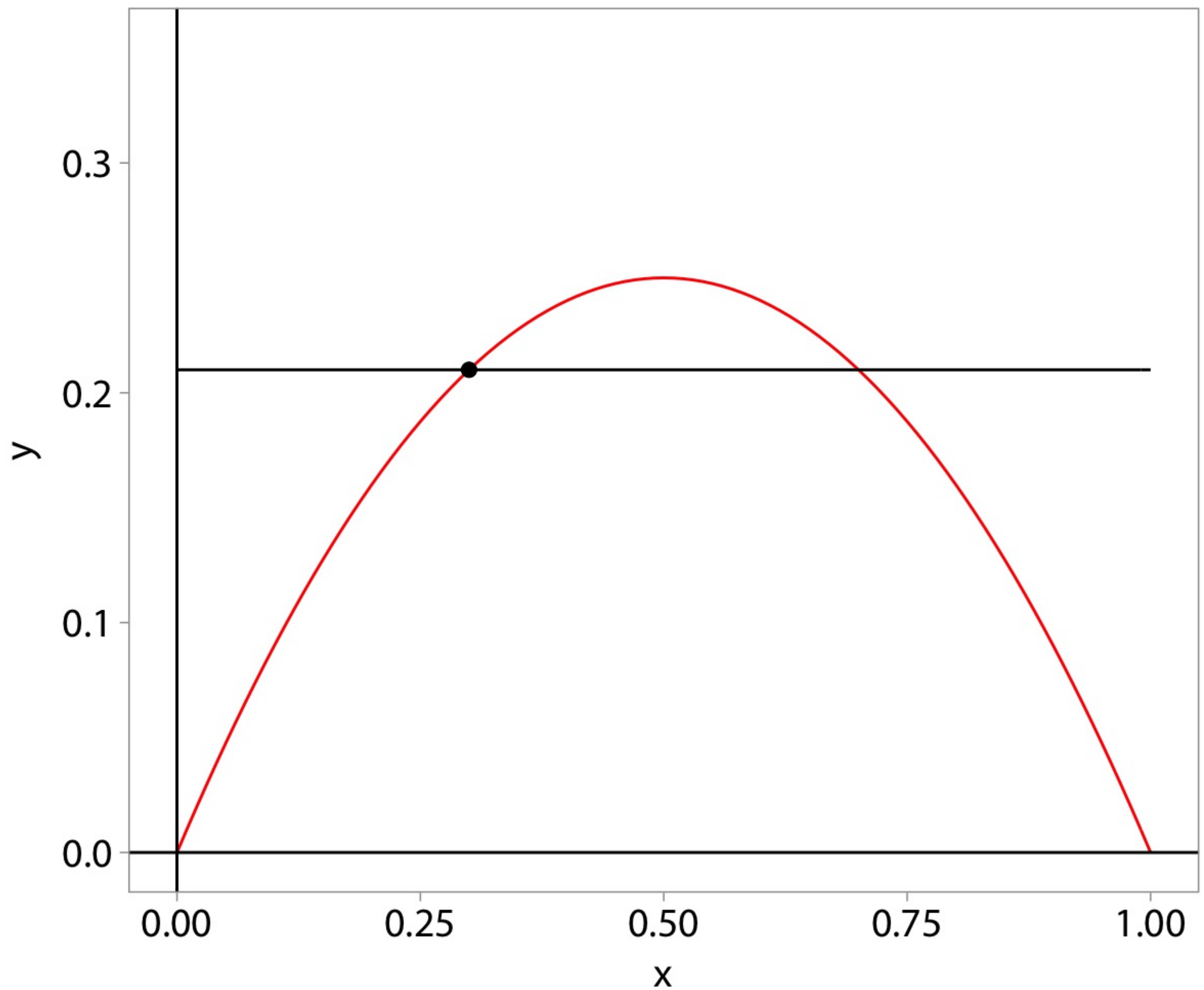
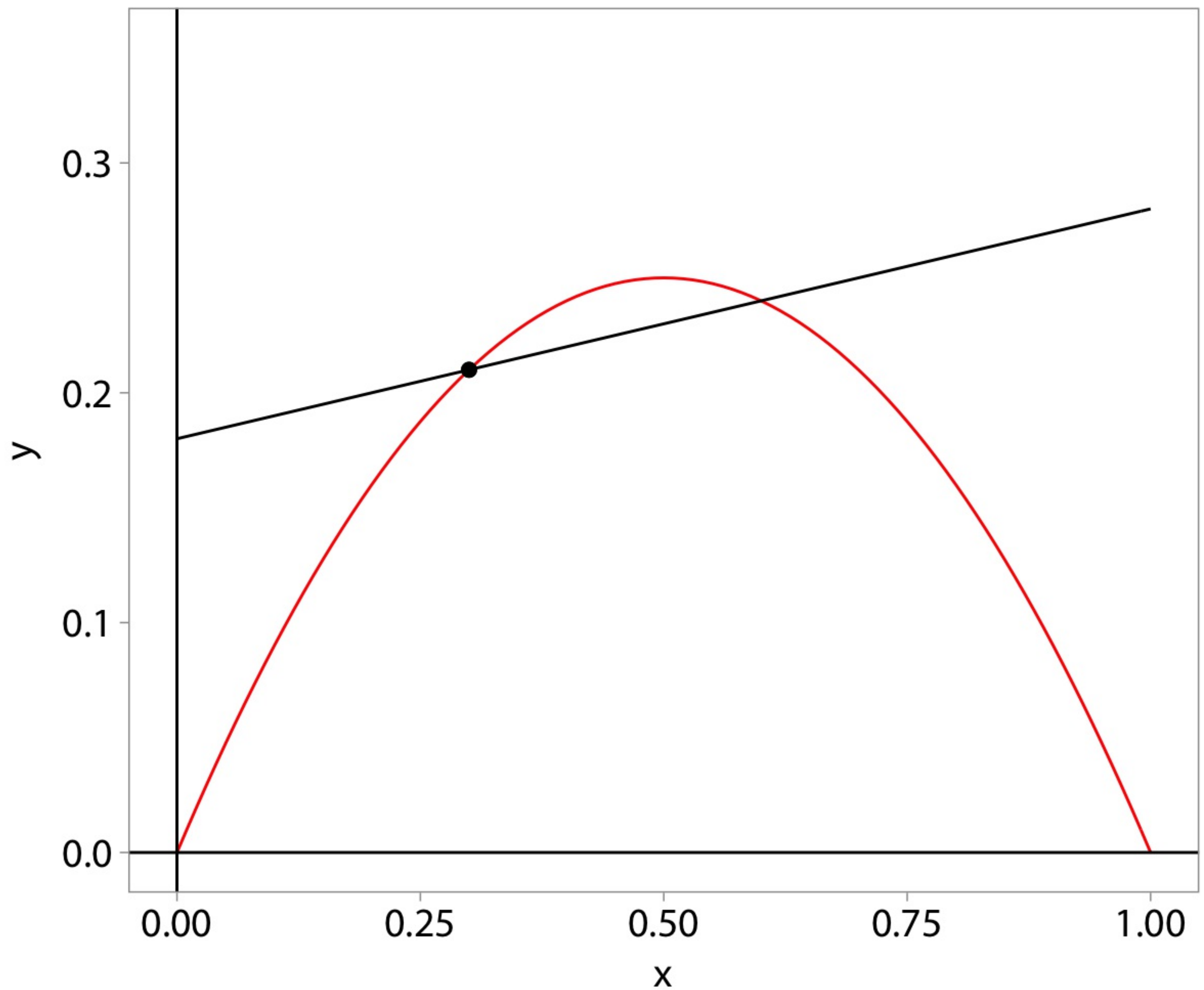
Let's consider a few examples of this



0.3
0.2
0.1
0.0
y
0.00
0.25
0.50
0.75
1.00
x



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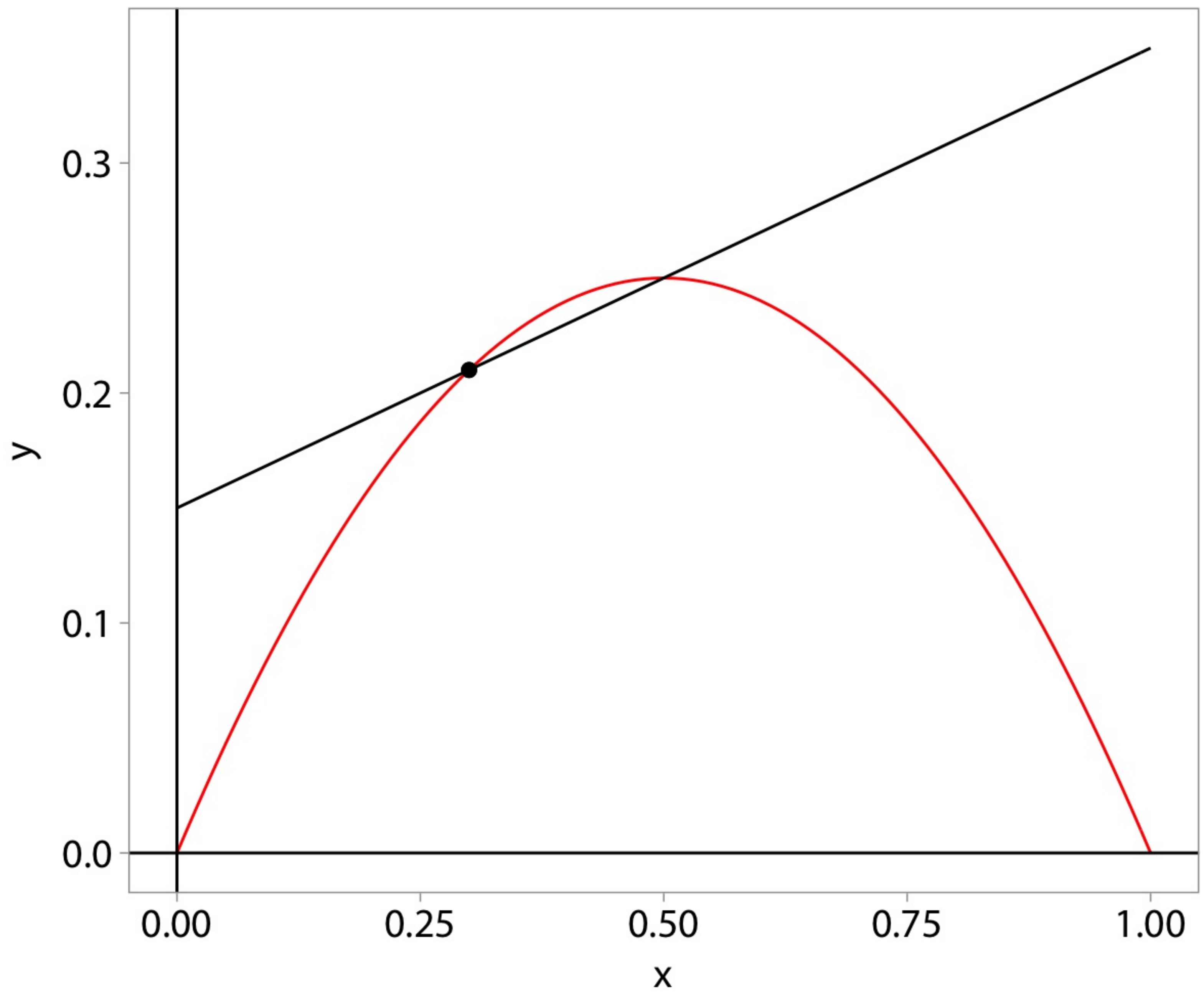

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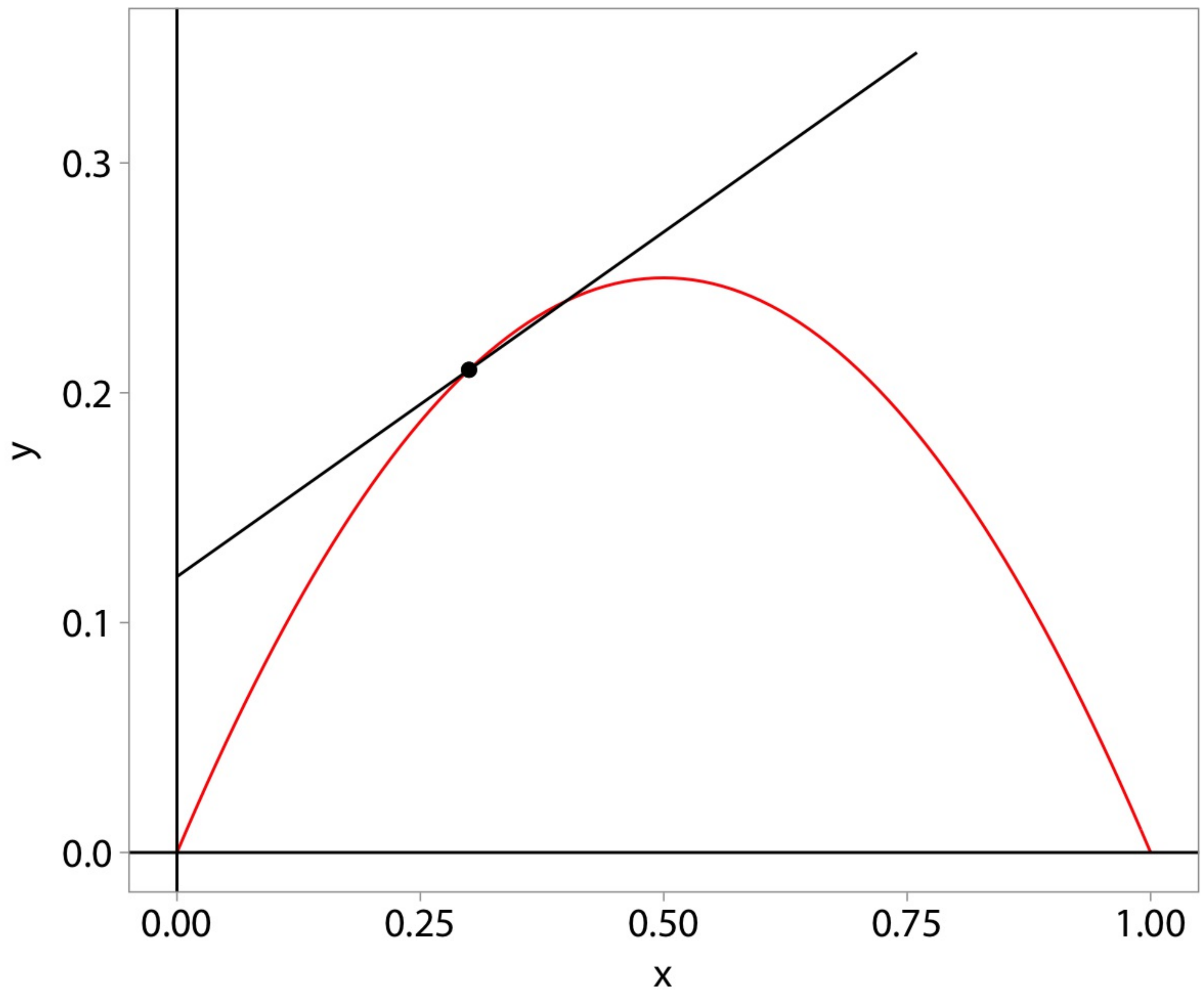

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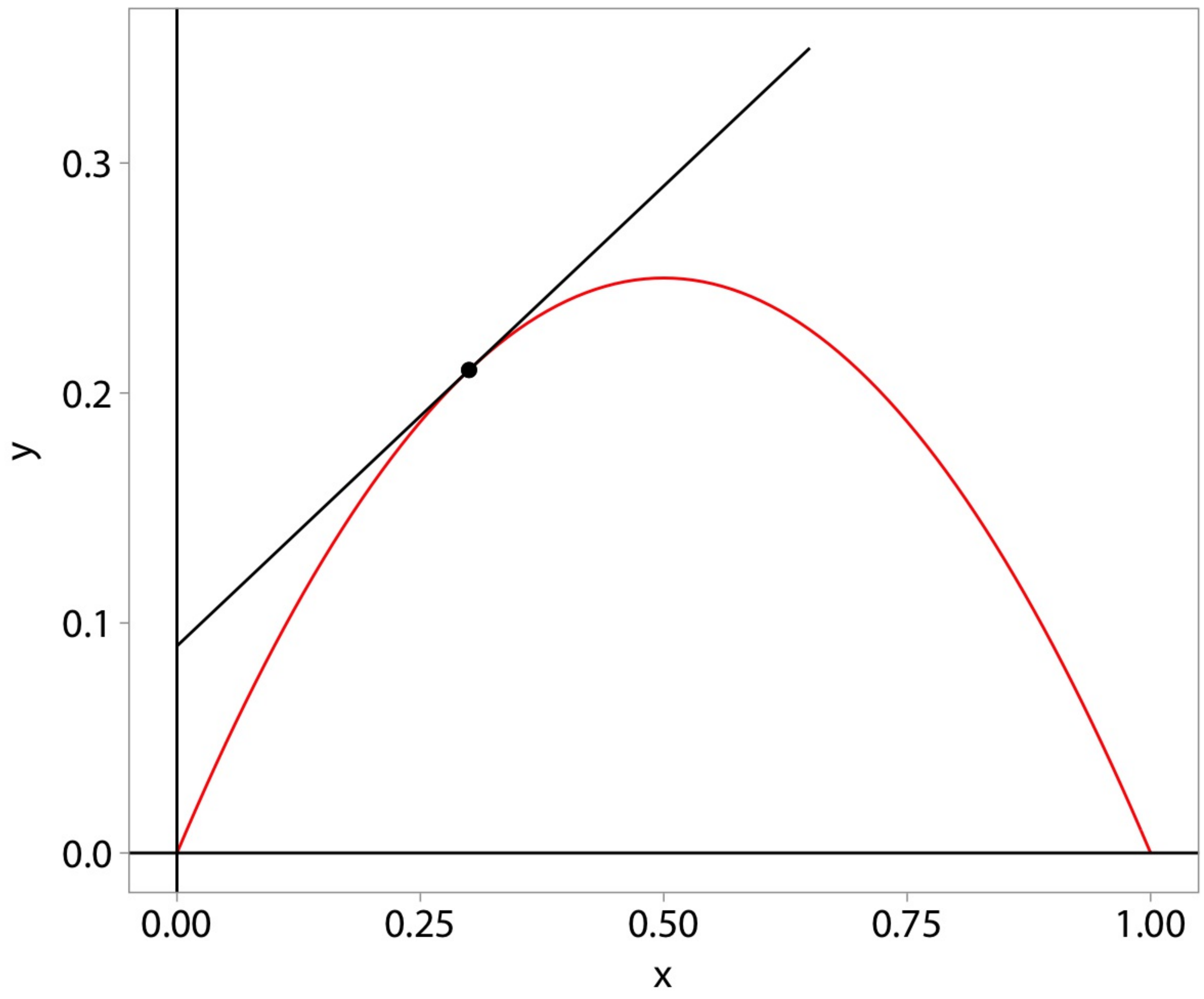
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So, as you reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point. Recall that $f(x) = x - x^2$ and consider a very small interval $\epsilon \ldots$

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$$\begin{aligned}\frac{\Delta f(x)}{\Delta x} &= \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - \epsilon} \\ &= \lim_{\epsilon \to 0} \frac{[(x+\epsilon) - (x+\epsilon)^2] - [(x) - (x)^2]}{(x+\epsilon) - x}\end{aligned}$$

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\end{aligned}
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$$= \lim_{\epsilon \to 0} \frac{\epsilon - 2\epsilon x - \epsilon^2}{\epsilon}$$

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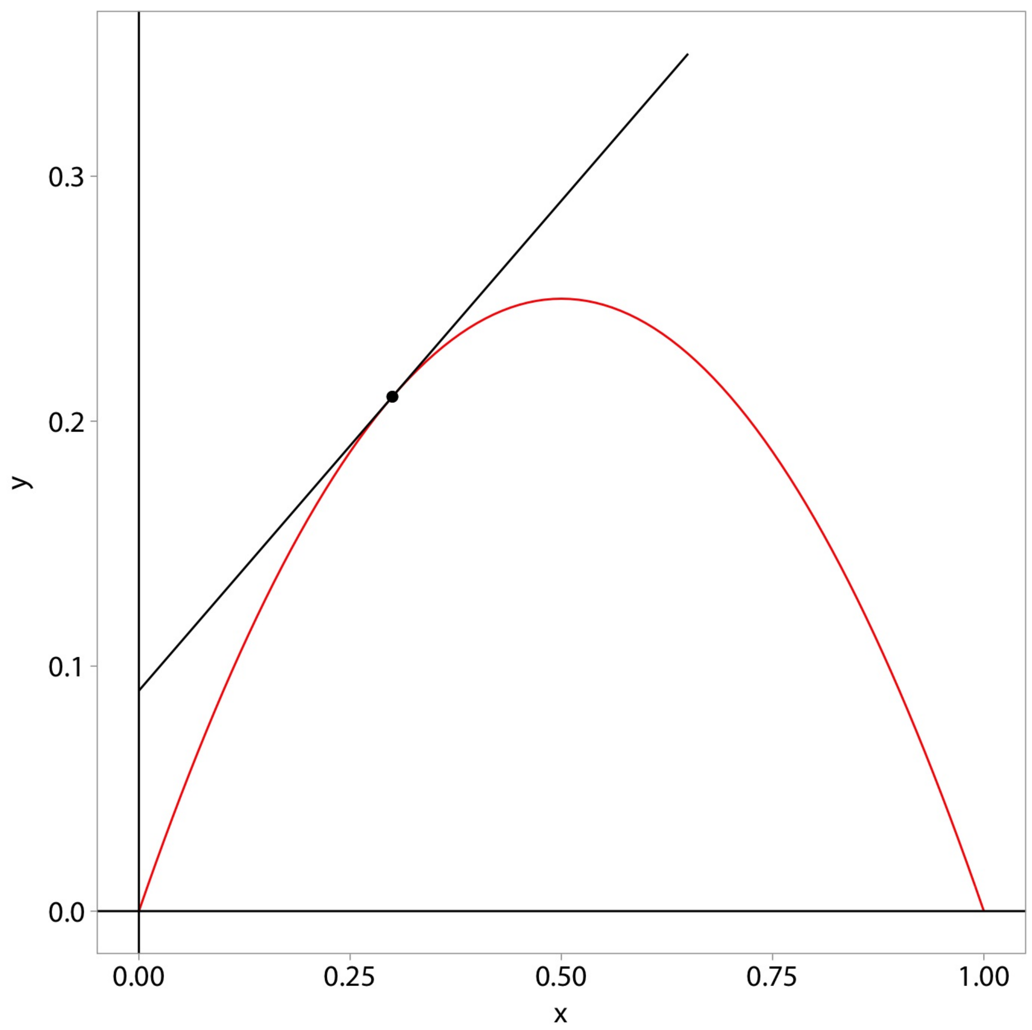
Derivative as a Limit



Using this formula, the slope of the curve at $x = .3$, or the point from the previous examples, is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.3) \\ &= 0.4 \end{aligned}$$

Alternatively, we could find the point at which the slope is exactly 0, or:

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Derivatives

Now, we can formally state that the derivative is equivalent to:

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Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a given slope

Derivative notation

Different notation systems for expressing derivatives:

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I'll be using $f'(x)$ and $f''(x)$ but you are free to choose whatever makes sense to you!

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps)
- The derivative exists (the limit of the slope is the same from the left as it is from the right) or no sharp corners.

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For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

Straightforward Derivatives

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$$
\begin{aligned}
f'(x) &= (3x)' \\
&= (3x^1)' \\
&= 1 \times 3x^{1-1} \\
&= 3x^0 = 3
\end{aligned}
$$

Straightforward Derivatives

Fortunately (for humans), it is not necessary to take limits with ϵ's to find derivatives every time. A few rules generate the derivatives of most functions. The first rule is the power rule. If $f(x) = cx^n$, then its derivative is $f'(x) = ncx^{n-1}$.

The simplest example of this is consider the line, $f(x) = 3x$. What's its derivative? It is 3 but why?

$$\begin{aligned} f'(x) &= (3x)' \\ &= (3x^1)' \\ &= 1 \times 3x^{1-1} \\ &= 3x^0 = 3 \end{aligned}$$

A special case of the power rule is that the derivative of a constant is zero.

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Let $h(x) = 7x^{\frac{1}{2}}$, then:

$$\begin{aligned} h'(x) &= \left(\frac{1}{2}\right)7x^{\frac{1}{2}-1} \\ &= \frac{7}{2}x^{-\frac{1}{2}} \end{aligned}$$

Straightforward Derivatives

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

$f(x) = \frac{1}{4}x^4$

$g(x) = \frac{2}{x^3}$ [Hint: What other ways can you express fractions?]

$h(x) = 4x^{\frac{5}{2}}$

$j(x) = \sqrt[3]{x}$ [Hint: What other ways can you express roots?]

Derivative of a Sum (or Difference)

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$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (4)2x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

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Let $f(x) = x$ and $g(x) = x^2$

$$\begin{aligned} f'(x) - g'(x) &= (x)' - (x^2)' \\ &= (1)x^{1-1} - (2)x^{2-1} \\ &= 1 - 2x \end{aligned}$$

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$$\begin{aligned} h'(x) &= (5x^5 - 10x^3 + 6x^2 - 3)' \\ &= (5x^5)' - (10x^3)' + (6x^2)' - (3)' \\ &= 5 \times 5x^{5-1} - 3 \times 10x^{3-1} + 2 \times 6x^{2-1} - 0 \\ &= 25x^4 - 30x^2 + 12x \end{aligned}$$

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What is the rate of change when x is equal to one?

$$\begin{aligned} h'(1) &= 25x^4 - 30x^2 + 12x \\ h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\ h'(1) &= 7 \end{aligned}$$

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&= (x^2 + 1)'(x^3 - 4x) + (x^2 + 1)(x^3 - 4x)' \\
&= (2x)(x^3 - 4x) + (x^2 + 1)(3x^2 - 4) \\
&= 2x^4 - 8x^2 + 3x^4 - 4x^2 + 3x^2 - 4 \\
&= 5x^4 - 9x^2 - 4
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This is easy to check by multiplying out the polynomial: $(x^2 + 1)(x^3 - 4x) = x^5 - 3x^3 - 4x$.
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We can also use the product rule to find the quotient rule. $\left(\frac{f(x)}{g(x)} = f(x)g(x)^{-1}\right)$

Derivative of a Quotient (cont'd)

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$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2+1}{x^3-4x}\right)' \\
&= \frac{(x^2+1)'(x^3-4x)-(x^2+1)(x^3-4x)'}{(x^3-4x)^2} \\
&= \frac{(2x)(3x^2-4)-(x^2+1)(3x^2-4)}{(x^3-4x)^2} \\
&= \frac{6x^3-8x-(3x^4-4x^2+3x^2-4)}{(x^3-4x)^2} \\
&= \frac{-3x^4+6x^3+x^2-8x+4}{(x^3-4x)^2}
\end{aligned}$$

Derivative of Products and Quotients

Find the derivative of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

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This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivative of Nested Functions (cont'd)

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$:

Derivative of Nested Functions (cont'd)

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$$\begin{aligned} h(x)' = (f(g(x)))' &= (6(3x^2 + 2)^4)' \\ &= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\ &= 24(x^2 - 5x + 20)^3(6x) \\ &= 144x(x^2 - 5x + 20)^3 \end{aligned}$$

Derivative of Nested Functions (cont'd)

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $k(x) = g(f(x))$:

Derivative of Nested Functions (cont'd)

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $k(x) = g(f(x))$:

$$\begin{aligned} k(x)' = (g(f(x)))' &= (3(6x^4)^2 + 2)' \\ &= (3(6x^4)^2)' + (2)' \\ &= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\ &= (2)3(6x^4)^{2-1}(24x^{4-1}) \\ &= (2)3(6x^4)(24x^3) \\ &= 864x^7 \end{aligned}$$

Derivative of Nested Functions (cont'd)

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

$(3x - 1)^4$

$2(x^4 + x^3) + 7$

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$$\begin{aligned}(\log_{\mathrm{e}}(\mathrm{x}))' = (\ln(\mathrm{x}))' &= \frac{1}{\ln(\mathrm{e})\mathrm{x}} \\ &= \frac{1}{\mathrm{x}}\end{aligned}$$

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$$\begin{aligned} g'(x) = (\ln(3x^2 + 4))' &= \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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$$\begin{aligned}(e^x)' &= \ln(e)e^x \\ &= 1 \times e^x \\ &= e^x\end{aligned}$$

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$$\begin{aligned} g'(x) = (2^{3x})' &= \ln(2) \times 2^{3x} \times (3x)' \\ &= 3\ln(2) \times 2^{3x} \end{aligned}$$

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Let $h(x) = 4e^x$,

$$h'(x) = (4e^x)' = 4e^x$$

End Day 3

Agenda

1) Second Derivatives

2) Partial Derivatives

3) Integrals

4) Optimization

Second Derivatives

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$$f'(x) = 15x^2 + 16x + 2$$

$$f''(x) = 30x + 16$$

Higher order (third, fourth, etc) derivatives also exist, but are rarely relevant.

Second Derivatives

Find the first and second derivative of the expressions below:

$$f(x) = 16x^3 - 3x^2 + 6$$

$$g(x) = x - x^2$$

$$h(x) = 4x^{-1} + 5x^{\frac{7}{2}}$$

Multivariate Functions & Partial Derivatives

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Consider $f(x, y, z) = 4x^2y^4 + 2xz^3 + 8y^2z^4 + 8x + 7y + 3z + 2$:

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Consider $f(x,y,z) = 4x^2y^4 + 2xz^3 + 8y^2z^4 + 8x + 7y + 3z + 2$:

$$\frac{\partial[f(x,y,z)]}{\partial x} = f_x(x,y,z) = 8xy^4 + 2z^3 + 8$$

$$\frac{\partial[f(x,y,z)]}{\partial y} = f_y(x,y,z) = 16x^2y^3 + 16yz^4 + 7$$

$$\frac{\partial[f(x,y,z)]}{\partial z} = f_z(x,y,z) = 6xz^2 + 32y^2z^3 + 3$$

Partial Derivatives

Find the partial derivatives of the expression below with respect to each variable

$8p^2q + 4pq - 7pq^2 + 18$

Partial Higher-Order Derivatives

It is possible to combine second-order (and higher) derivatives with partial derivatives. For example:

Consider $f(x,y) = 3x^3y^2$ and let's we wanted to find $\frac{\partial^2}{\partial x \partial y} f(x,y)$:

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}(3x^3y^2) &= \frac{\partial}{\partial y}((3)3x^{3-1}y^2) \\
&= \frac{\partial}{\partial y}(9x^2y^2) \\
&= (2)9x^2y^{2-1} \\
&= 18x^2y
\end{aligned}$$

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$$
\begin{aligned}
\frac{\partial^2}{\partial x \partial y}(3x^3y^2) &= \frac{\partial}{\partial y}((3)3x^{3-1}y^2) \\
&= \frac{\partial}{\partial y}(9x^2y^2) \\
&= (2)9x^2y^{2-1} \\
&= 18x^2y
\end{aligned}
$$

Pay attention to the denominator to give you guidance about what operations to perform. Here, we are taking the second derivative of the entire function, but are differentiating once with respect to x and once with respect to y overall.

If instead we were given $\frac{\partial^3}{\partial x^2 \partial y}$ we would differentiate 3 times overall, twice with respect to x and once with respect to y.

Partial Higher-Order Derivatives

Consider again $f(x,y) = 3x^3y^2$. Find:

- $\frac{\partial^3}{\partial x^2 \partial y}$
- $\frac{\partial^3}{\partial x \partial y^2}$

Integrals

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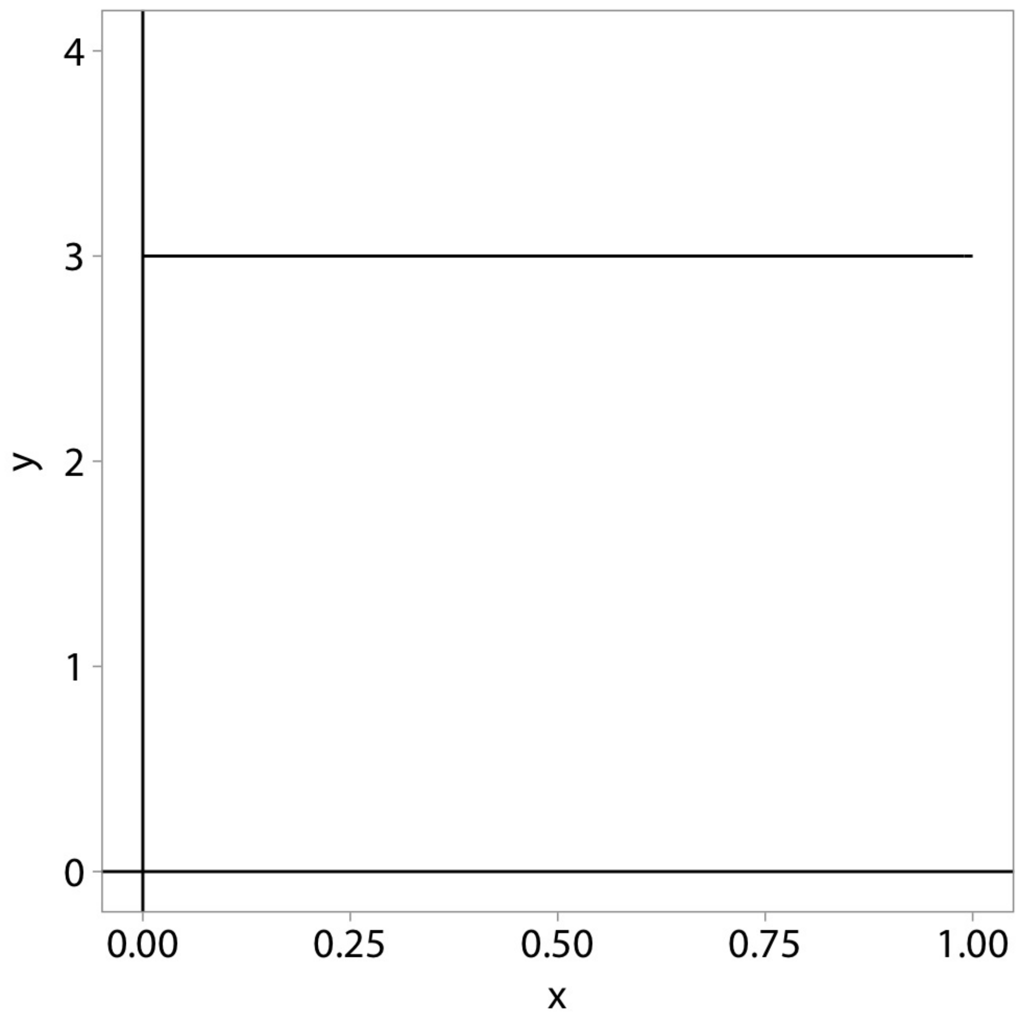
Either the total area as the function extends to infinity in either direction, or the area between two points.

Areas



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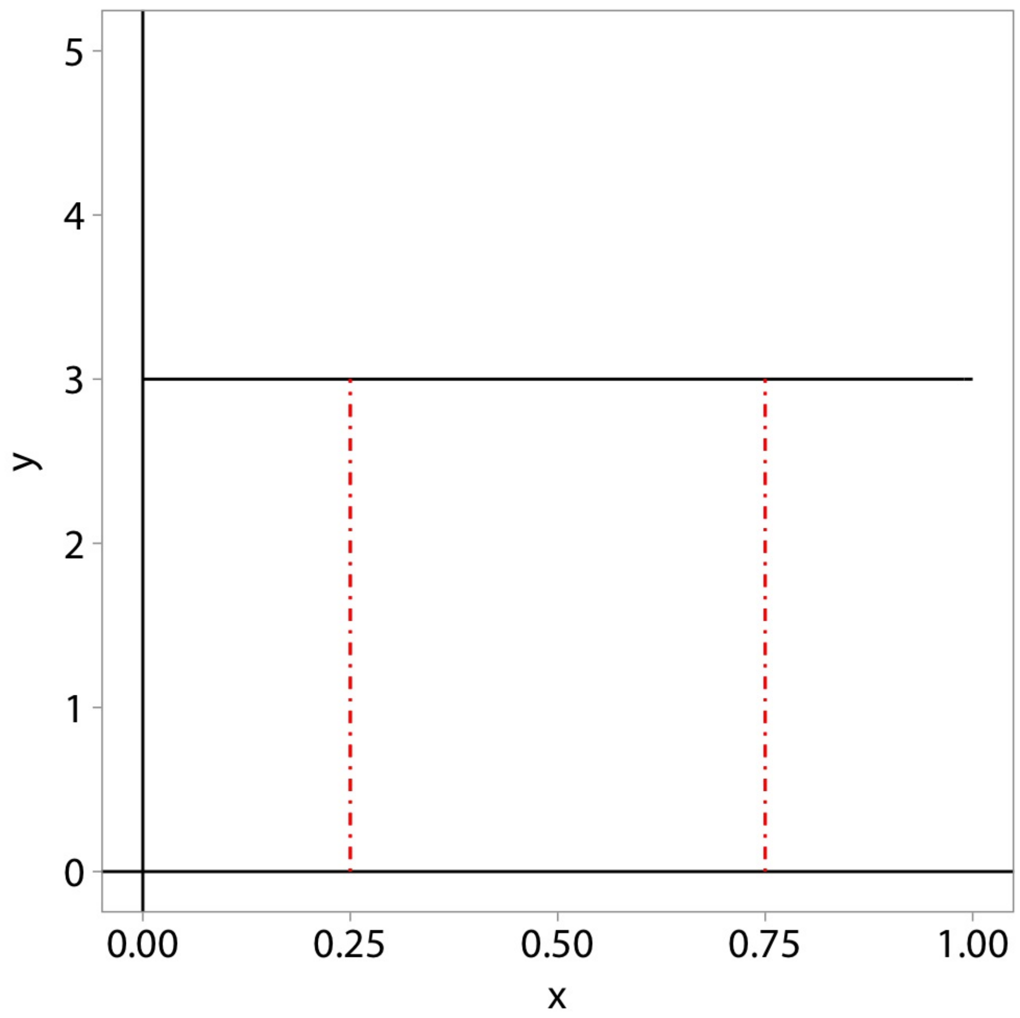
Areas



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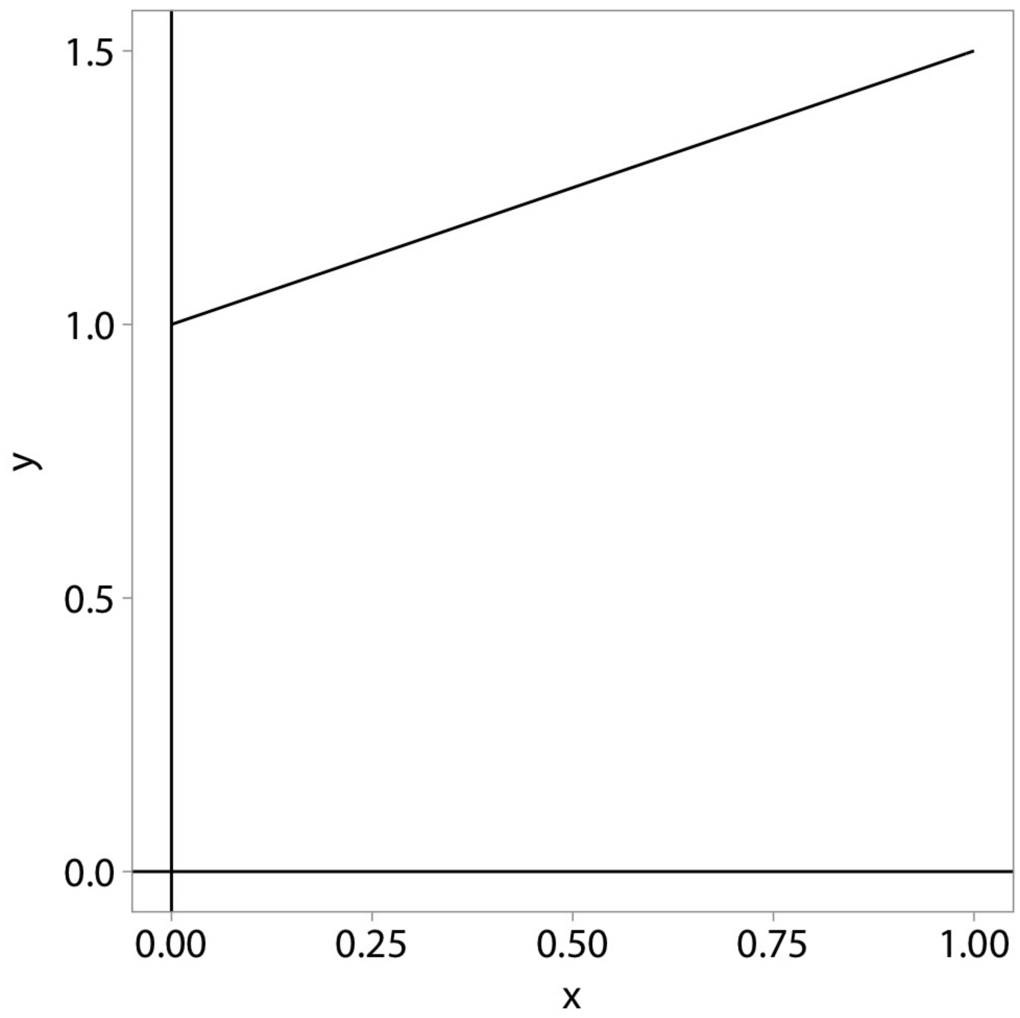
Given that this is a rectangle, the area between the function and the x-axis is
$\text{area} = (0.75 - 0.25) \times 3 = 1.5$

Areas



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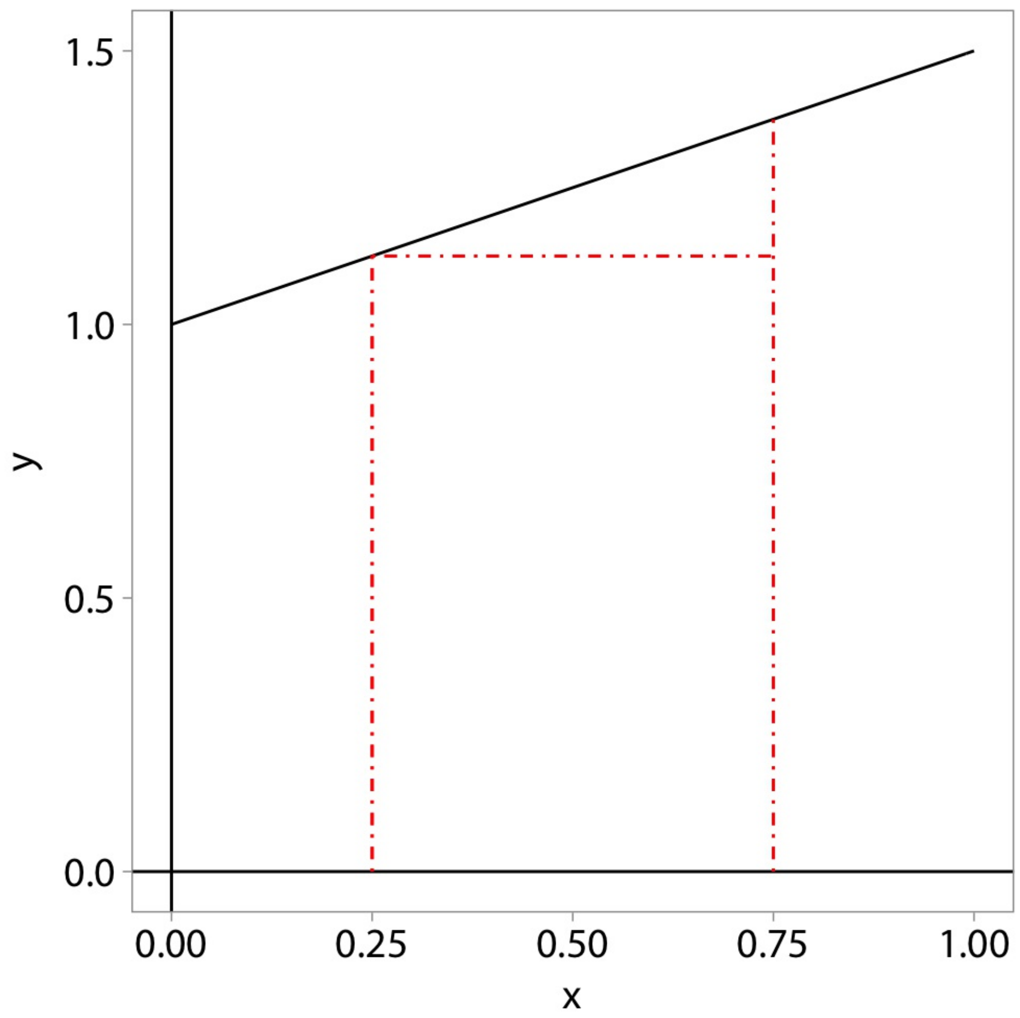

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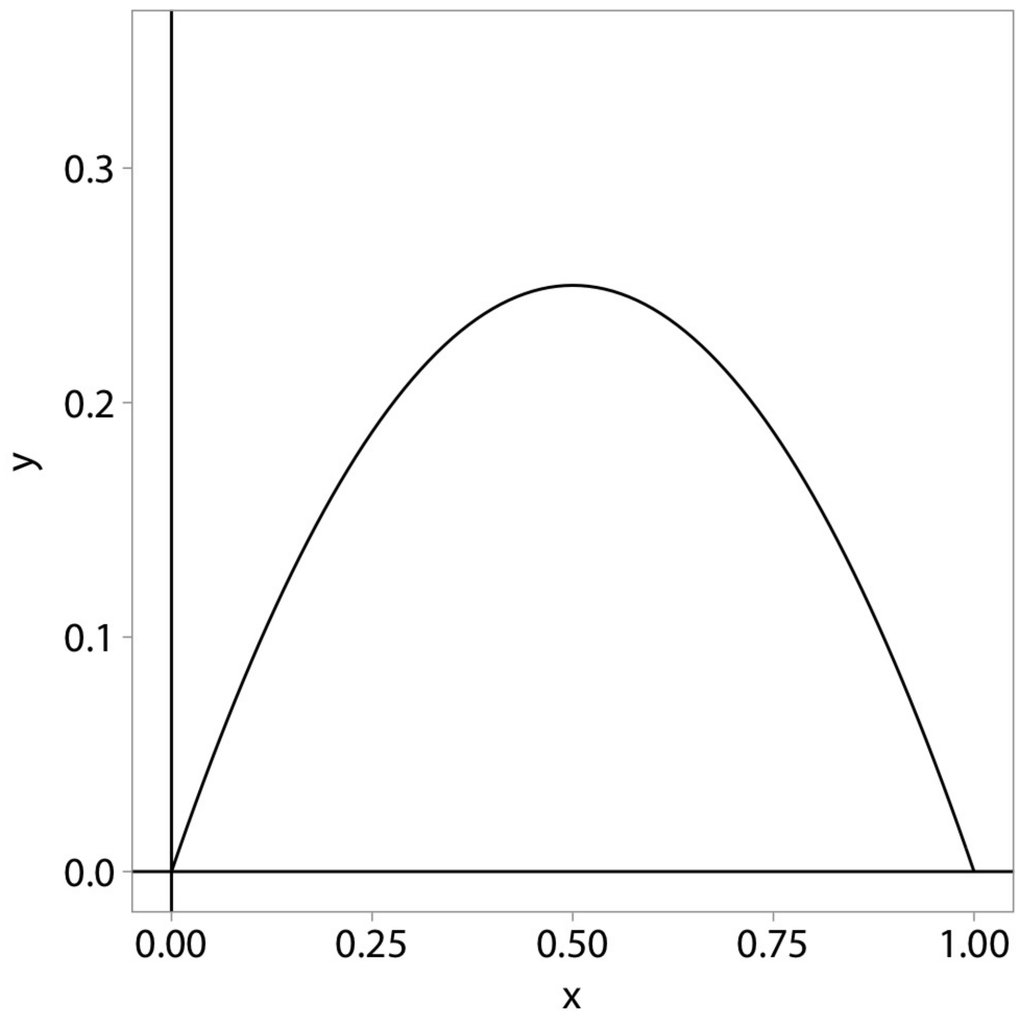
We can find the area between the function and the x-axis by taking advantage of the fact we can draw a triangle and rectangle and sum their areas. $\text{area}_{\text{tri}} = 0.0625$ and $\text{area}_{\text{rect}} = 0.5625$. Thus, the total area is 0.625.

Area



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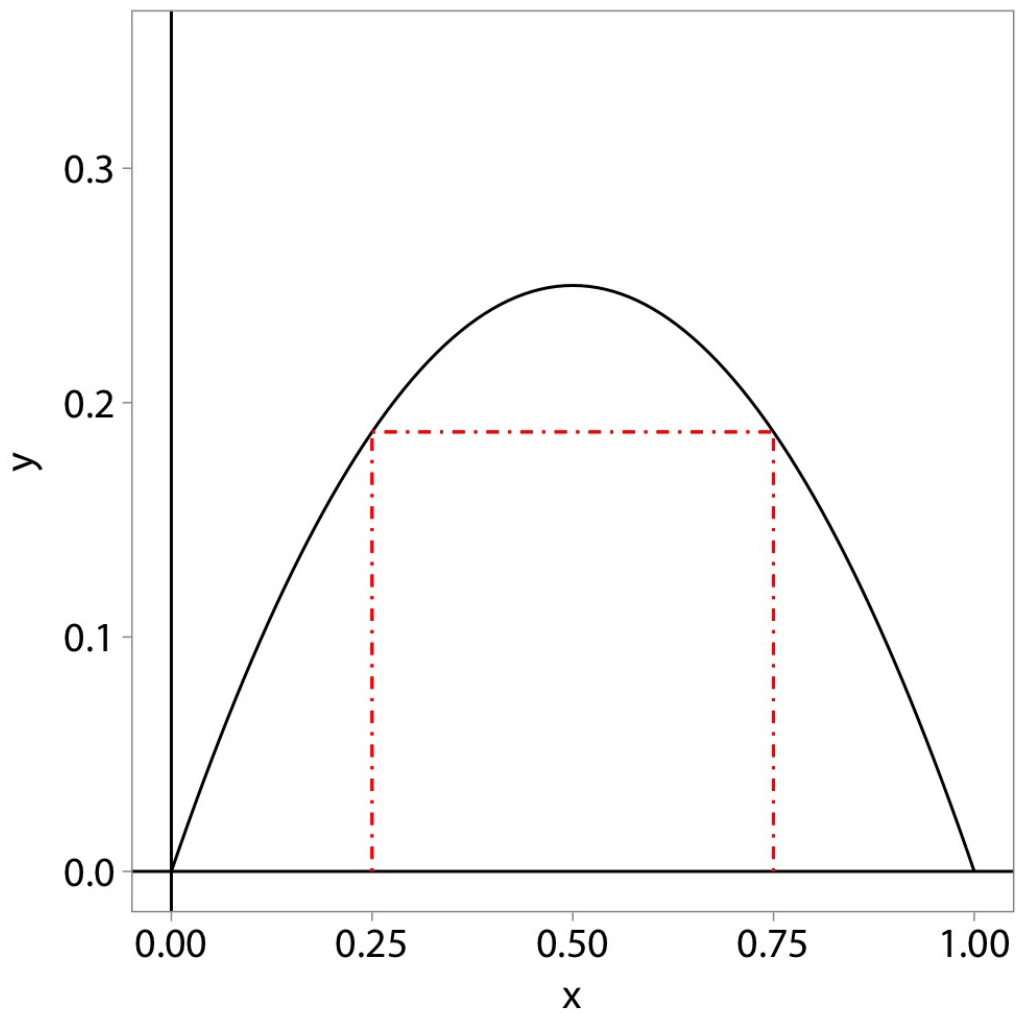

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The area under curve $x = 0.25$ and $x = 0.75$ is $???$

Integral as the Limit of a Sum

Imagine dividing the region into intervals and drawing a rectangle to capture the area for each interval, with height equal to the value of the function at the left (or right) of the interval, then summing the area of those rectangles.

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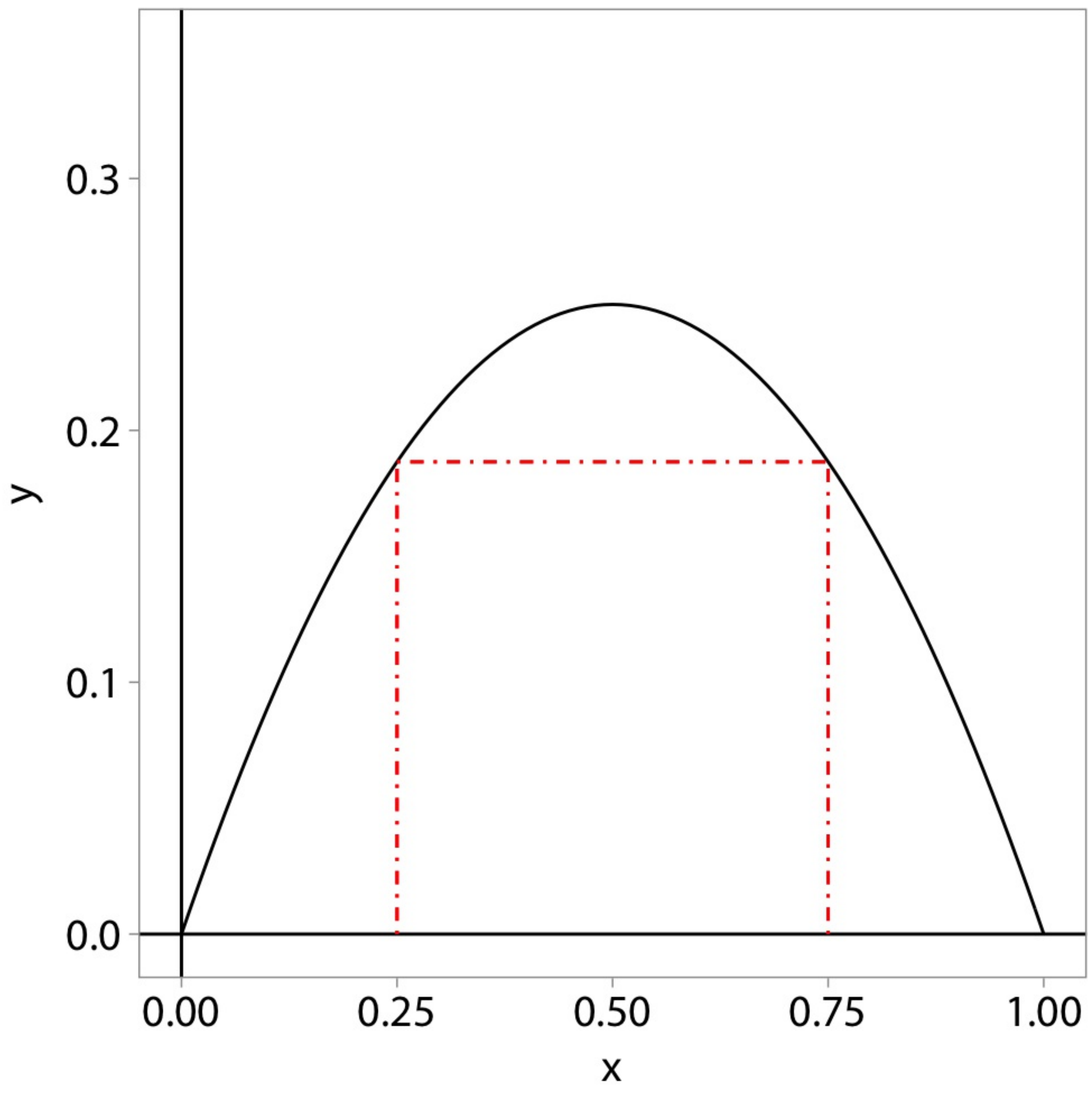
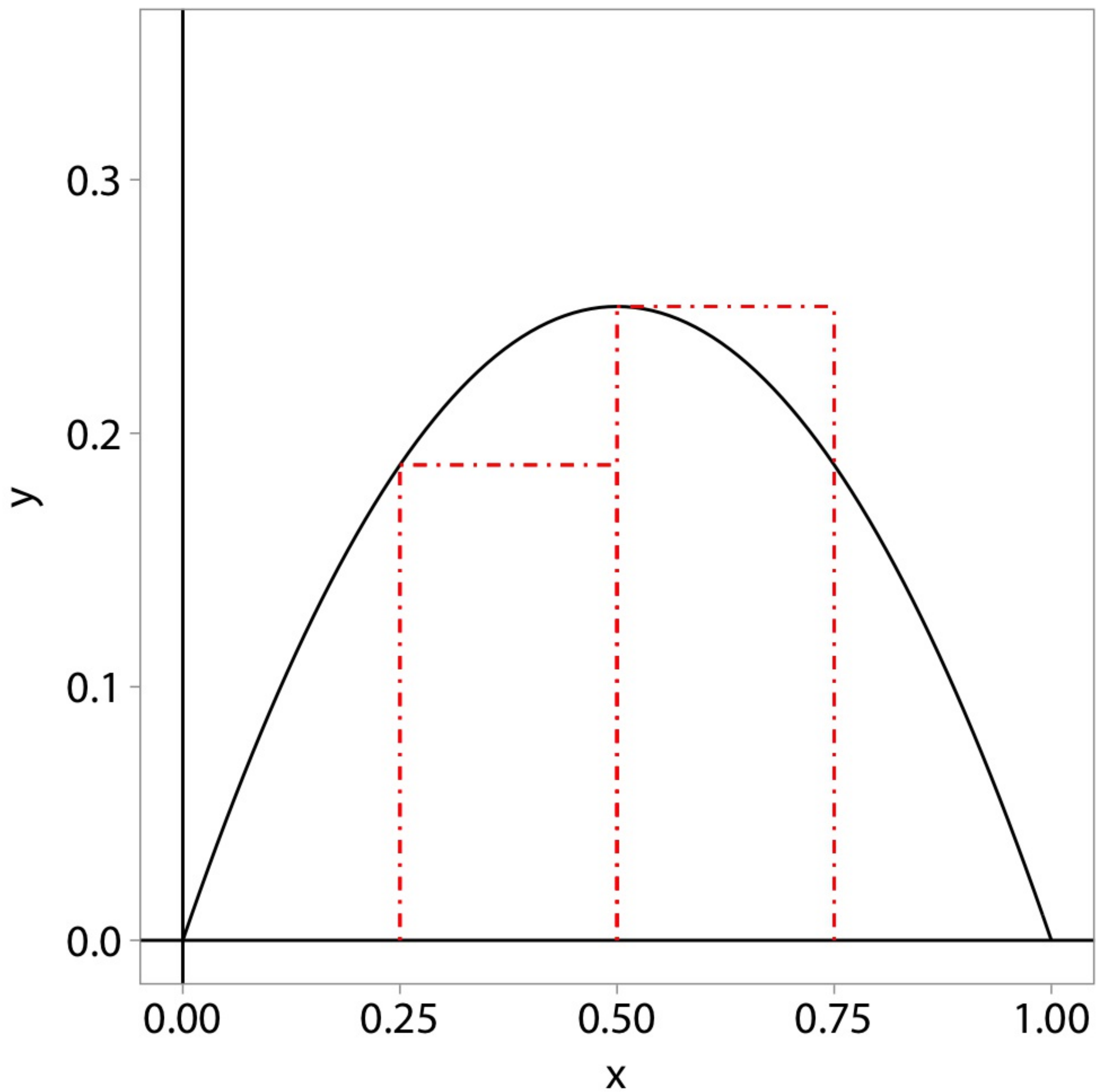
Let's see what happens as we add rectangles


0.3
0.2
0.1
0.0
y
0.00
0.25
0.50
0.75
1.00
x



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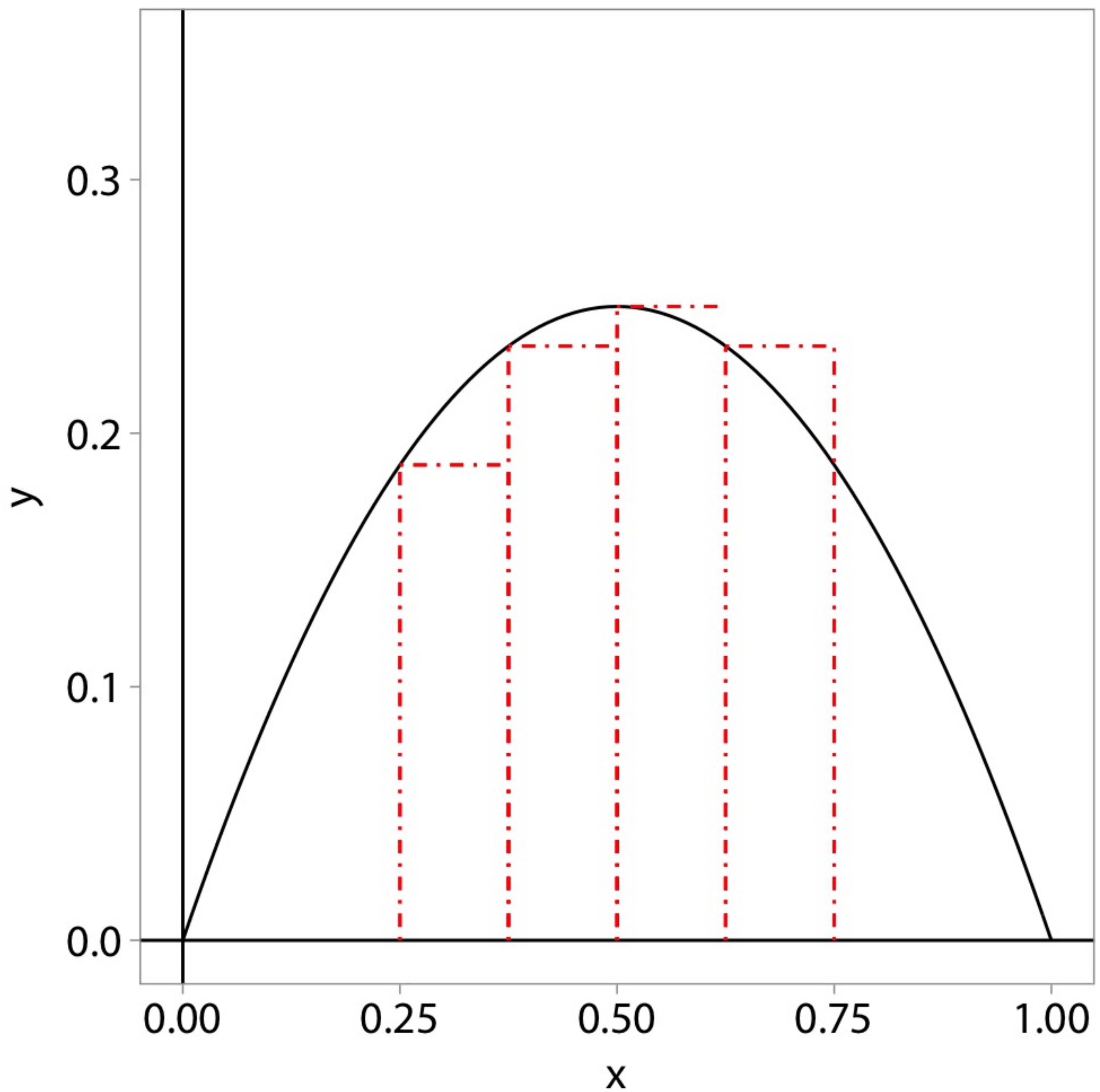
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Integral as the Limit of a Sum

Imagine dividing the region into intervals and drawing a rectangle to capture the area for each interval, with height equal to the value of the function at the left (or right) of the interval, then summing the area of those rectangles.

Approximation improves as the intervals become smaller.

Integrals

As you reduce the width of rectangles to zero, the summed areas of the rectangles converges to the area under the curve—including more and more of the area inside and less and less of the area outside.

$$\int_a^b f(x)dx = \lim_{h \to 0} \sum_{i=1}^{H} f(x_i)h_i$$

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Using this approach, we can find:

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- a general equation for the area between any two points

However, it is mathematically difficult to solve these using this approach.

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Essentially, this “unwinds” the derivative operation, or applies it backwards.

The Fundamental Theorem of Calculus

The fundamental theorem of calculus relates the derivative and the integral.

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\big|_a^b$$

Indefinite Integrals

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This expression uses exactly the same antiderivative as the definite integral but there is no subtraction and there's an arbitrary constant C added (since that'd disappear when taking the derivative).

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$$\begin{aligned}
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&= \frac{1}{1+1} x^{1+1} - \left(\frac{1}{2+1} x^{2+1} \right) + C \\
&= \frac{1}{2} x^2 - \frac{1}{3} x^3 + C
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$$\begin{aligned}\int_{.2}^{.75} x - x^2 dx &= \frac{1}{2} x^2 - \frac{1}{3} x^3 |_{.25}^{.75} \\ &= \left(\frac{1}{2} (.75)^2 - \frac{1}{3} (.75)^3 \right) - \left(\frac{1}{2} (.25)^2 - \frac{1}{3} (.25)^3 \right) \\ &= \frac{11}{96}\end{aligned}$$

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\int 9x^2 + 10x + 4dx &= 9\int x^2dx + 10\int xdx + \int 4dx \\
&= 9\left(\frac{1}{3}\right)x^3 + 10\left(\frac{1}{2}\right)x^2 + 4x + C \\
&= 3x^3 + 5x^2 + 4x + C
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Now consider the area of the curve specifically between 2 and 5:

$$\begin{aligned}\int_2^5 9x^2 + 10x + 4dx &= 3x^3 + 5x^2 + 4x|_2^5 \\ &= (3(5)^3 + 5(5)^2 + 4(5)) - (3(2)^3 + 5(2)^2 + 4(2)) \\ &= 468\end{aligned}$$

Straightforward Integrals (cont'd)

Find the indefinite integral of the function below, and calculate the area under the curve between 0 and 1:

$$\int(2x^3 - 3x^2 + 7x + 4)dx$$

Advanced Integrals

There are a number of techniques for computing the integrals of more complicated functions.

- Integration by Substitution
- Integration by Parts

These are beyond the scope of what we have time to cover here and, for the most part, beyond the scope of what you will need to do by hand in political science.

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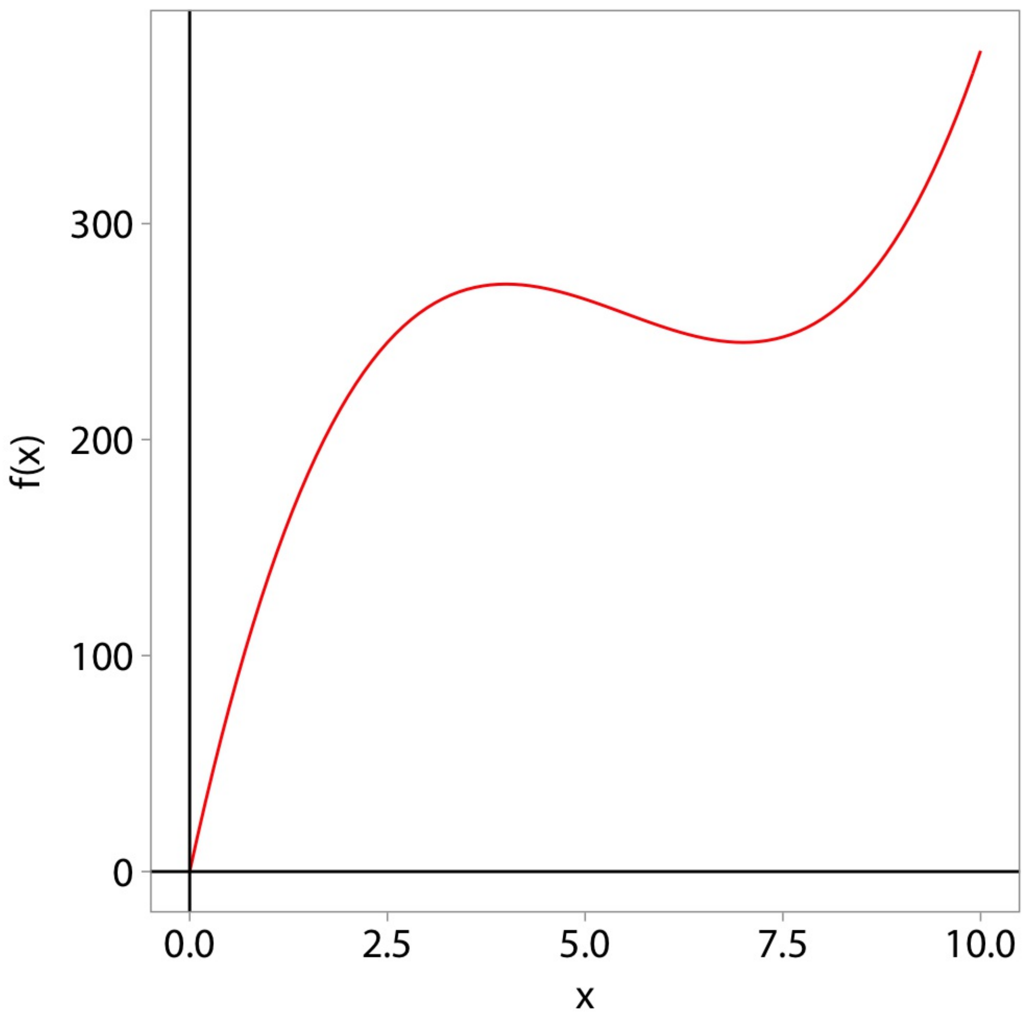
What does this look like in practical terms?

Optimization (cont'd)



Suppose we want to find all the maxima and minima for the function,
$f(x) = 2x^3 - 33x^2 + 168x$ when
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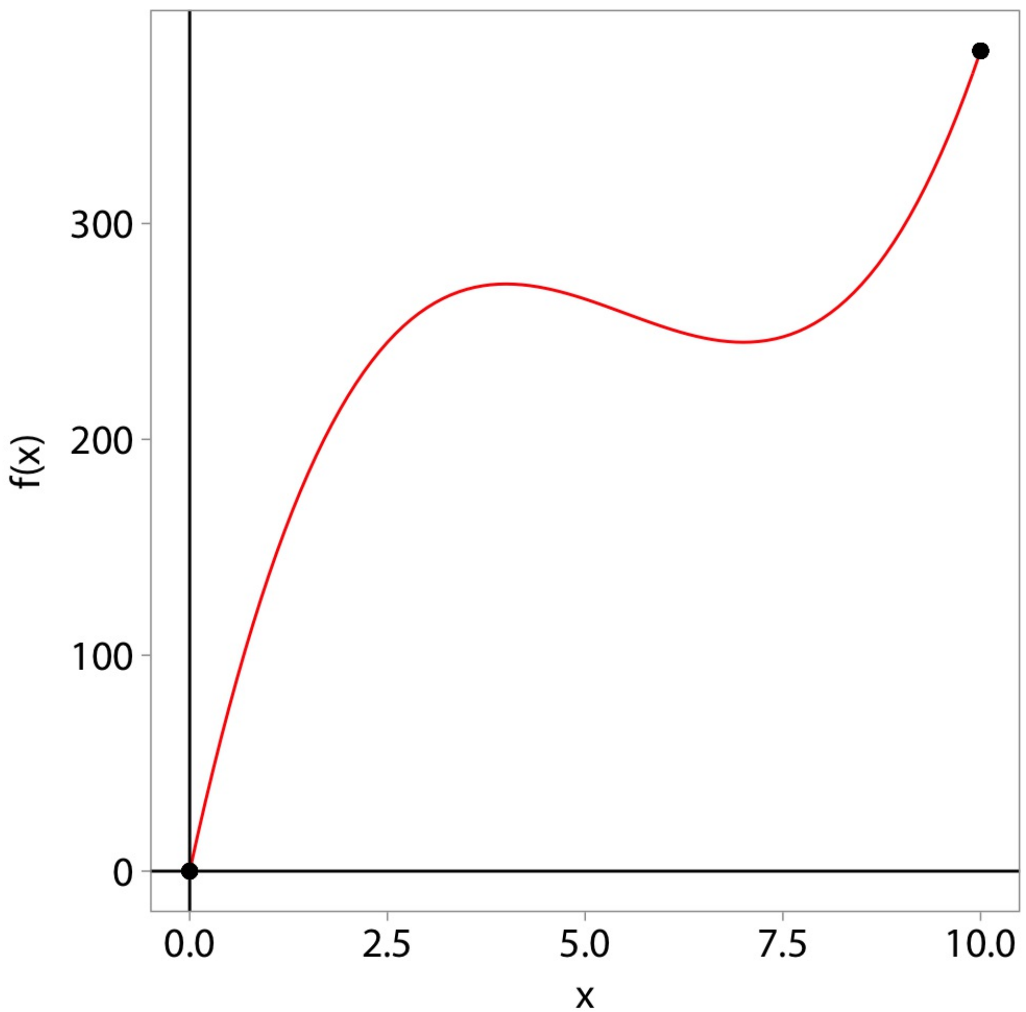
Optimization (cont'd)



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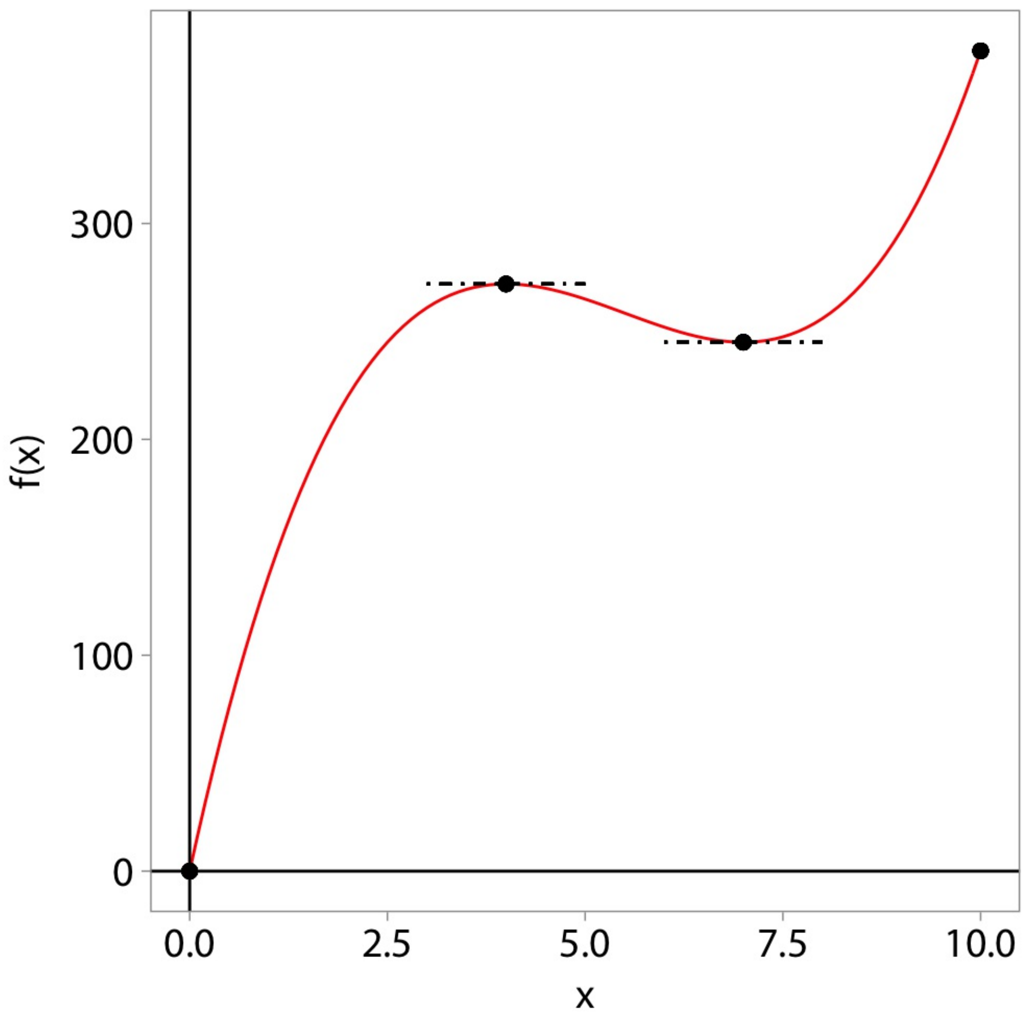

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The absolute maximum occurs at $x = 10$, and the absolute minimum occurs at $x = 0$ or the endpoints but there also appear to be a local maximum and a local minimum in between them.

To determine the precise location of these local maxima and minima, note that at these points, the slope of the line is flat. This means the derivative, which captures the slope of the tangent line is 0.

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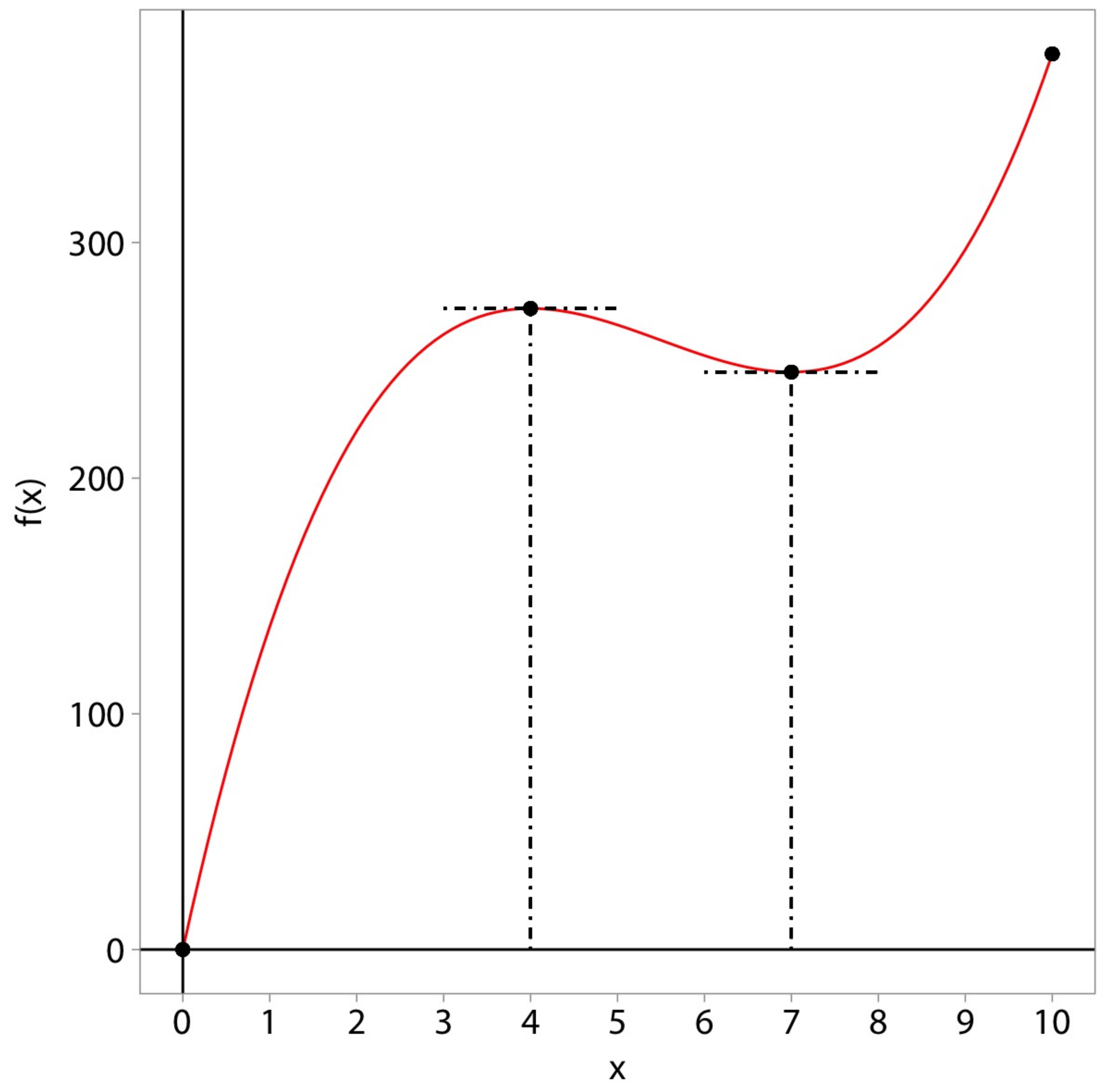
$$x = \frac{11 \pm \sqrt{(-11)^2 - 4(1)(28)}}{2(1)}$$
$$x = \frac{11 \pm 3}{2}$$
$$x = 4 \text{ or } 7$$



300
200
100
0
f(x)
0
1
2
3
4
5
6
7
8
9
10
x

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$$f'(x) = 6x^2 - 66x + 168$$
$$f''(x) = 12x - 66$$

when we evaluate the second derivatives at the local minimum and maximum, the results are $f''(4) = -18$ and $f''(7) = 18$.

Optimization (cont'd)

Find the local minimum and local maximum of the function below, and check mathematically which is the minimum and which is the maximum:

$$x^3 - x^2 + 1$$

End Day 4