Math Camp: Lesson 1

Basics, Notation, Pre-Calculus

UW–Madison Political Science

August 20, 2018

Welcome

First…

Numbers

Math: not just numbers

Math is a general framework for manipulating various concepts, one of which is "numbers"

Types of numbers:

- Integers (whole numbers, including negative): $\mathbb{Z}$
- Real numbers (the continuous number line): $\mathbb{R}$
- Positive and negative real numbers: $\mathbb{R}^+$ and $\mathbb{R}^-$
- Real numbers in n dimensions: $\mathbb{R}^n$
- Complex numbers: $\mathbb{C}$
  - $1 + 2i$
  - where $i = \sqrt{-1}$

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  - What happens if $z = 0$?
  - If $z = 1$?

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∞: Infinity. Not really a number but a boundless quantity

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We can generalize these statements by using variables

- $1 + \mathrm{x}$
- $3 - \mathrm{y}$
- $5 \times \mathrm{a}$
- $7 \div \mathrm{m}$

Variable: a symbol to represent an entity that could take different values

With me so far?

Equations

Statements about equality and inequality

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Inequalities show whether LHS or RHS is greater

- Usually we use them to find the conditions under which a statement holds

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Remember to flip an inequality if you multiply by a negative number!

Data

Data

The information we record about what we study

- Cases: The units being studied (rows)
- Variables: Characteristics that describe units (columns)
- Values: Specific realization of a variable (cells)

Establishment	Location	Coffee	Vibe	Notable Flaw
Aldo's	Campus	7	good	no plugs
Ancora	Capitol	8	great	hours
Barriques	Capitol	6	good	bathroom key
Colectivo	State St.	7	fair	spotty wifi
Fair Trade	State St.	8	good	expensive, tables
Michelangelo's	Capitol	5	meh	bathroom key
Steep & Brew	Bascom Hill	0	fair	bad coffee (espresso OK)

Classifying data

Quantitative vs. Qualitative analysis

- Broad, sometimes contentious, arguably artificial divides in the study of politics
- Quantitative: larger n, statistical description and inference
- Qualitative: smaller n, rich description, non-statistical inference

Quantitative vs. Qualitative variables

- Quantitative: countable, numeric (age, number of toes)
- Qualitative: not countable but descriptive (gender, party preference)
- Unlike qualitative research, qualitative variables can still be organized in a data table

I advise not getting too hung up these classification systems. Use them only until you can lose them

Discrete vs. Continuous

Discrete

- Variables take specific values from a finite set of possible values
- Could be categories, could be discrete numbers
- Level of the atmosphere, number of parties, country of origin

Dichotomous

- Special type of discrete variable, two possible values
- 0 or 1, yes or no, war or peace, win or lose, voted or not

Continuous

- Variables take values from a continuous number line
- Could be a bounded number line
- GDP, vote share, percentage of turnout, unemployment rate

Related: the "levels of measurement"

Nominal / Categorical

- Unordered categories
- e.g. party affiliation, gender, country of origin, occupation

Ordinal

- Ordered or ranked outcomes
- Could be categories, but numbers are possible (e.g. rankings)
- No fixed "distance" between levels
- e.g. highest level of educational attainment, levels of democratization, issue prioritization

Related: the "levels of measurement"

Interval

- Ordered values with fixed distance between levels
- But no true zero point
- Issue scales, day of the year, Likert scales (debatable)

Ratio:

- Ordered, fixed intervals, and true zero
- Vote percentage, turnout, minutes in line to vote

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  - Interval: least-squares regression
  - Ratio: least-squares, count/rate models, duration/survival models

Do you buy that?

Sets

(useful for "speaking math" about data)

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Sets could contain individual numbers, but they could contain other sorts of entities

- vectors, matrices
- functions, probability distributions (which are...?)

We need only some set notation to help us work with data

What's in a set?

A set could contain individual elements, written as

$$A = \{1,2,3,4,5,6,7,8,9,10\}$$

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- Is it the case that $A = B$?
- $C = (0,11)$ (parentheses indicate that endpoints are not included)
- Does $B = C$?

Set notation

$\cup$: the union of two sets

- elements that are members of either set
- if $A = \{1,2\}$ and $B = \{2,3\}$, then $A \cup B = \{1,2,3\}$

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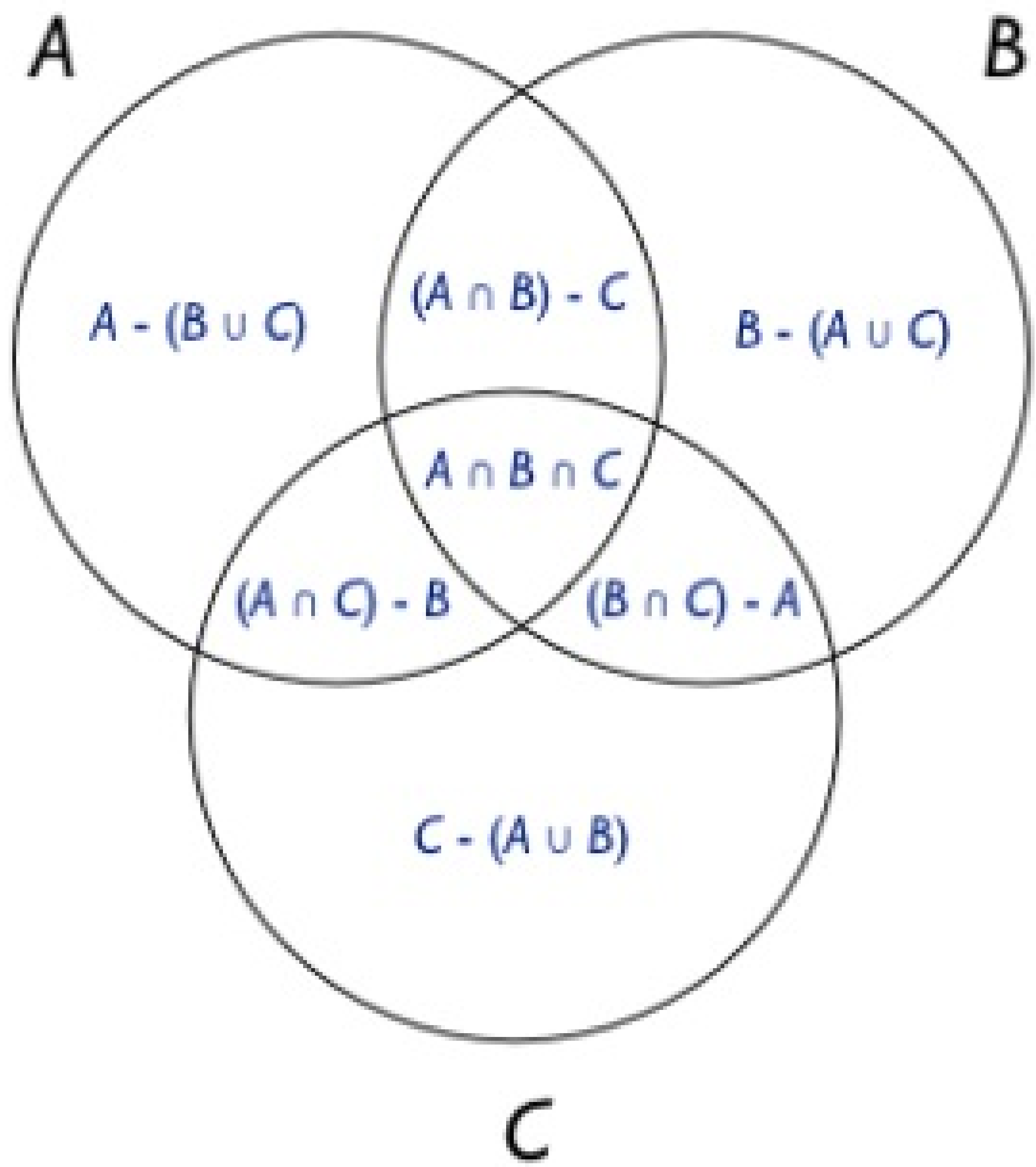
$\emptyset$: the empty set (null set)



https://bosker.wordpress.com/2013/07/10/venn-diagram-partitioning/

Symbols and Set Notation

A handful of symbols are commonly used when we represent data mathematically.

Symbol	Meaning
$>$	greater than
$\geq$	greater than or equal to
$<$	(less than)
$\leq$	less than or equal to
$\approx$	approximately equal to $(x \approx y)$
$\equiv$	equivalent to (for establishing identities)
$\propto$	proportional to $(4x \propto x)$

Symbols and Set Notation

Symbol	Meaning
$\neg$	not
/	not (if through a symbol)
$\mid$	given that $(A \mid B)$
$\in$	is an element of a set $(x_i \in \mathbf{x})$
$\rightarrow$	implies $(A \rightarrow B)$
$\leftrightarrow$	if and only if $([x = y] \leftrightarrow [y = x])$

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"Proper subset"

- $A \subset B$
- A isn't a proper subset of A
- proper subsets can't be equivalent to their superset

Indexing

Variables can be sets, and they can take different values for different individuals in a dataset. It is convenient to index individual observations using a subscript (typically i).

Student	Math Courses in College
1	3
2	0
3	1
4	4

If x represents the number of math courses, x_i refers to the ith observation in x

- $x_1 = 3$
- $x_2 = 0$
- $x_3 = ?$
- $x_4 = ?$

Set practice

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$A = \{10, 20, 30\}$

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- is $(A \cup C) \subset C$?

Set notation in the wild

Almost verbatim from a paper about congressional votes ("roll call votes")*

- The data consist of n legislators voting on m different roll calls [bills].
- Each roll call $j = 1, \ldots, m$ presents legislators $i = 1, \ldots, n$ with a choice between a 'Yea' position ζ_j and a 'Nay' position Ψ_j, locations in $\mathbb{R}$
- Let $y_{ij} = 1$ if legislator i votes Yea on the jth roll call and $y_{ij} = 0$ otherwise.

Legislator (i)	Bill (j)	Vote (y)
1	1	0
1	2	1
$\vdots$	$\vdots$	$\vdots$
n	m	y_{nm}

* - Clinton, Jackman, and Rivers. "The Statistical Analysis of Roll Call Data." APSR 2004

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Analogies include:

- algorithms, machines, black box, routinized process

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Other symbols also are fine, e.g.

- $\Phi(\cdot)$
- $\Gamma(\cdot)$
- $B(\cdot)$
- $\Lambda(\cdot)$

Operators

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Several common operators serve as the building blocks of mathematical functions

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Order of operations:

- operations within parentheses
- Exponents
- Multiplication and division (left to right)
- Addition and subtraction (left to right)

Function examples

x	y	z	$f(x,y) = x - y$	$g(z) = 2z - 1$	$h(x,y,z) = \dfrac{x+y}{z}$
5	0	5			
2	5	8			
0	3	9			
3	2	0			
8	4	2			
1	2	4			

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5	0	5	5	9	1
2	5	8	-3	15	.875
0	3	9	-3	17	.333
3	2	0	1	-1	undefined
8	4	2	4	3	6
1	2	4	-1	7	.75

Nested functions

Given that all functions do is map an input to an output, we can nest functions

- Imagine we perform one function $f(\cdot)$
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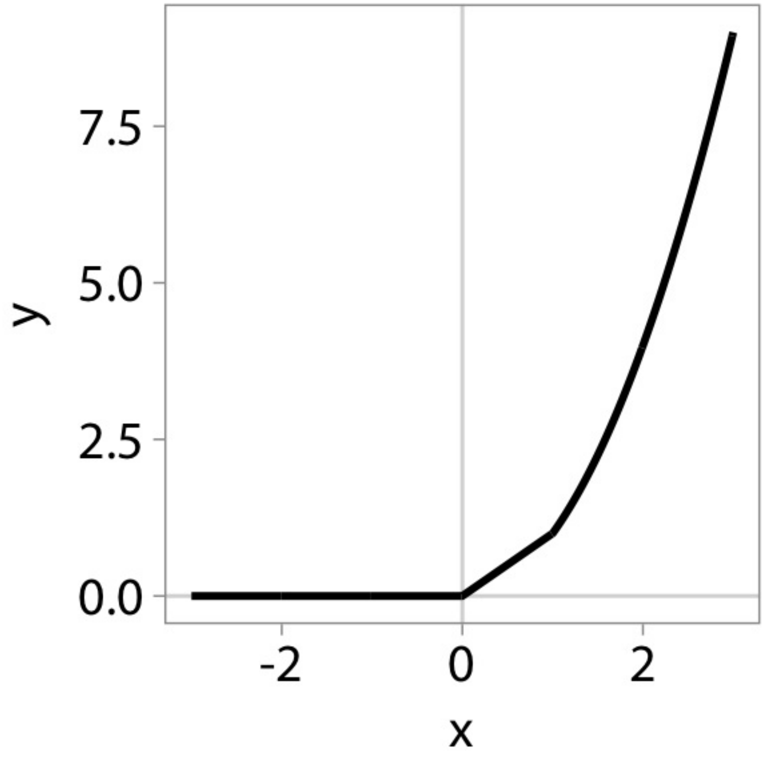
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \in [0, 1] \\ x^2 & \text{if } x > 1 \end{cases}$$



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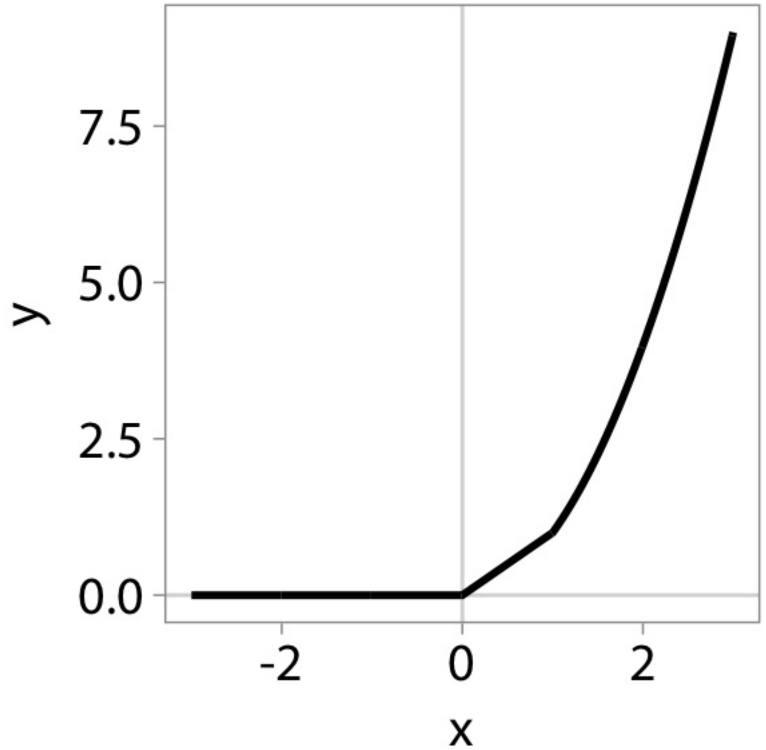
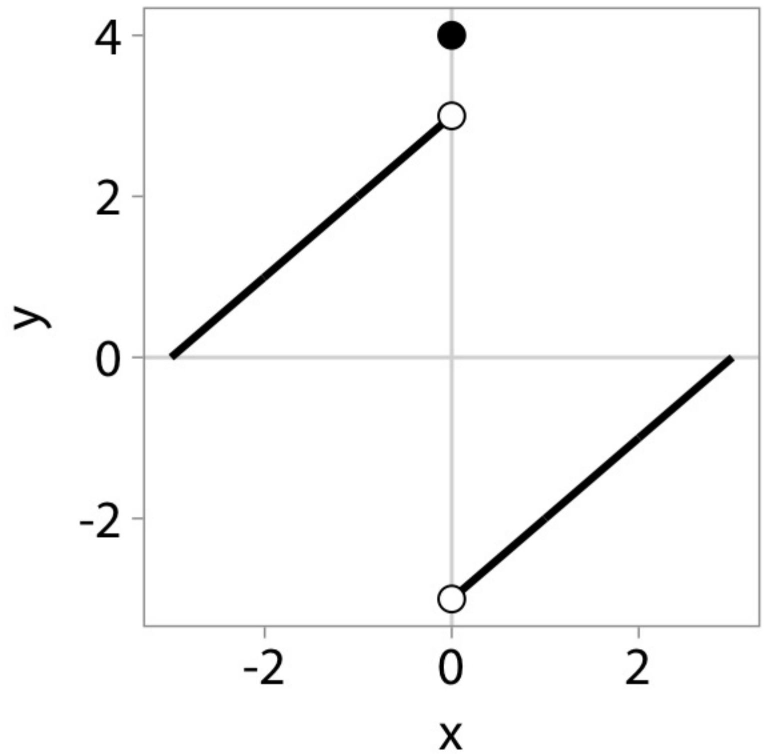
$$g(x) = \begin{cases} x + 3 & \text{if } x \in (-\infty, 0) \\ 4 & \text{if } x = 0 \\ x - 3 & \text{if } x \in (0, \infty) \end{cases}$$





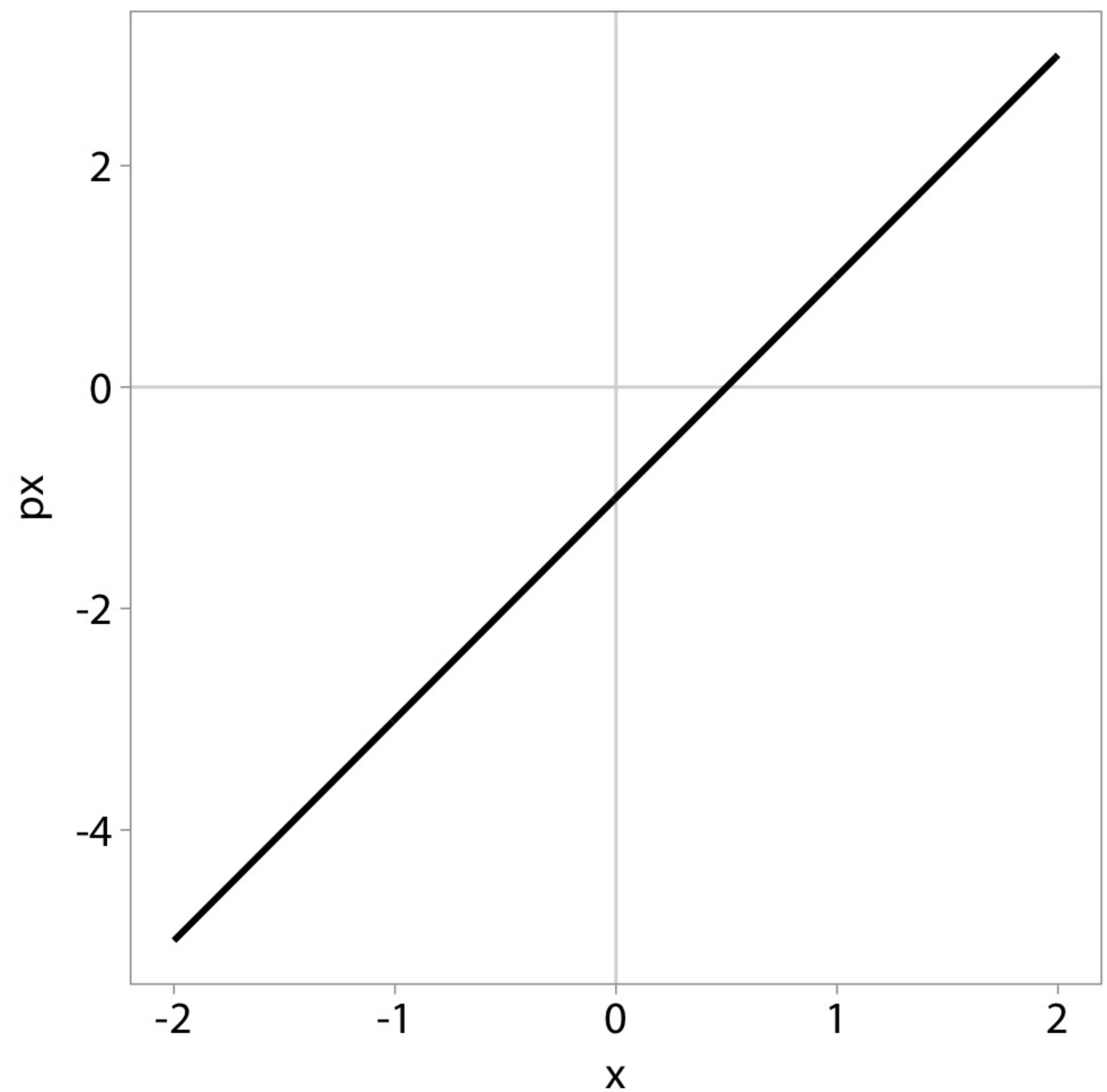
Function practice

Practice

Sketch graphs of the following functions:

- $p(x) = 2x - 1$, on the interval $[-2, 2]$
- $q(x) = x * -1^x$, for integers $\{0, 1, 2, 3, 4, 5\}$
- $r(x) = 2x^2 - 3x + 4$, on the interval $(0, 4)$

$p(x) = 2x - 1$, on the interval $[-2, 2]$



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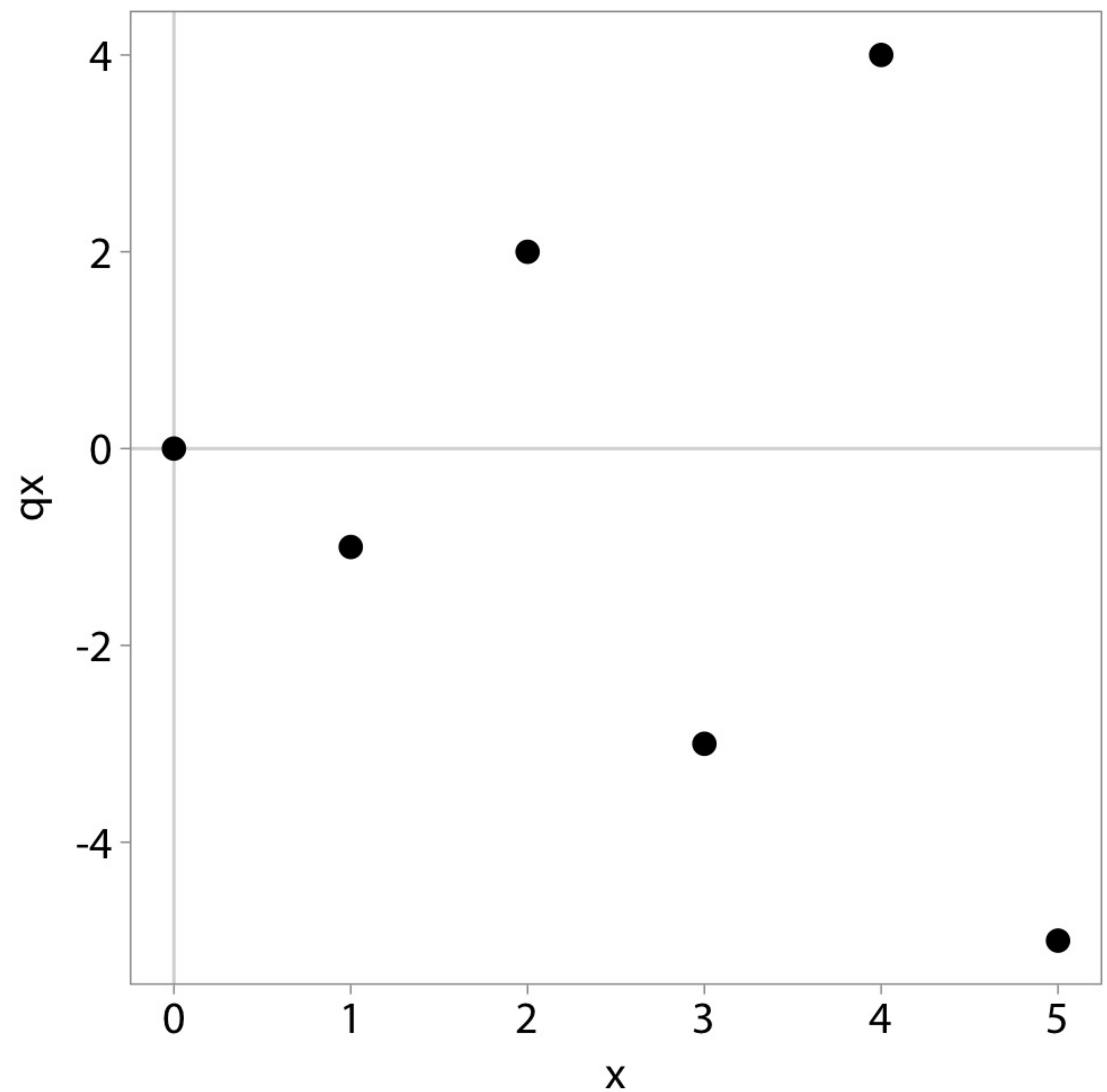

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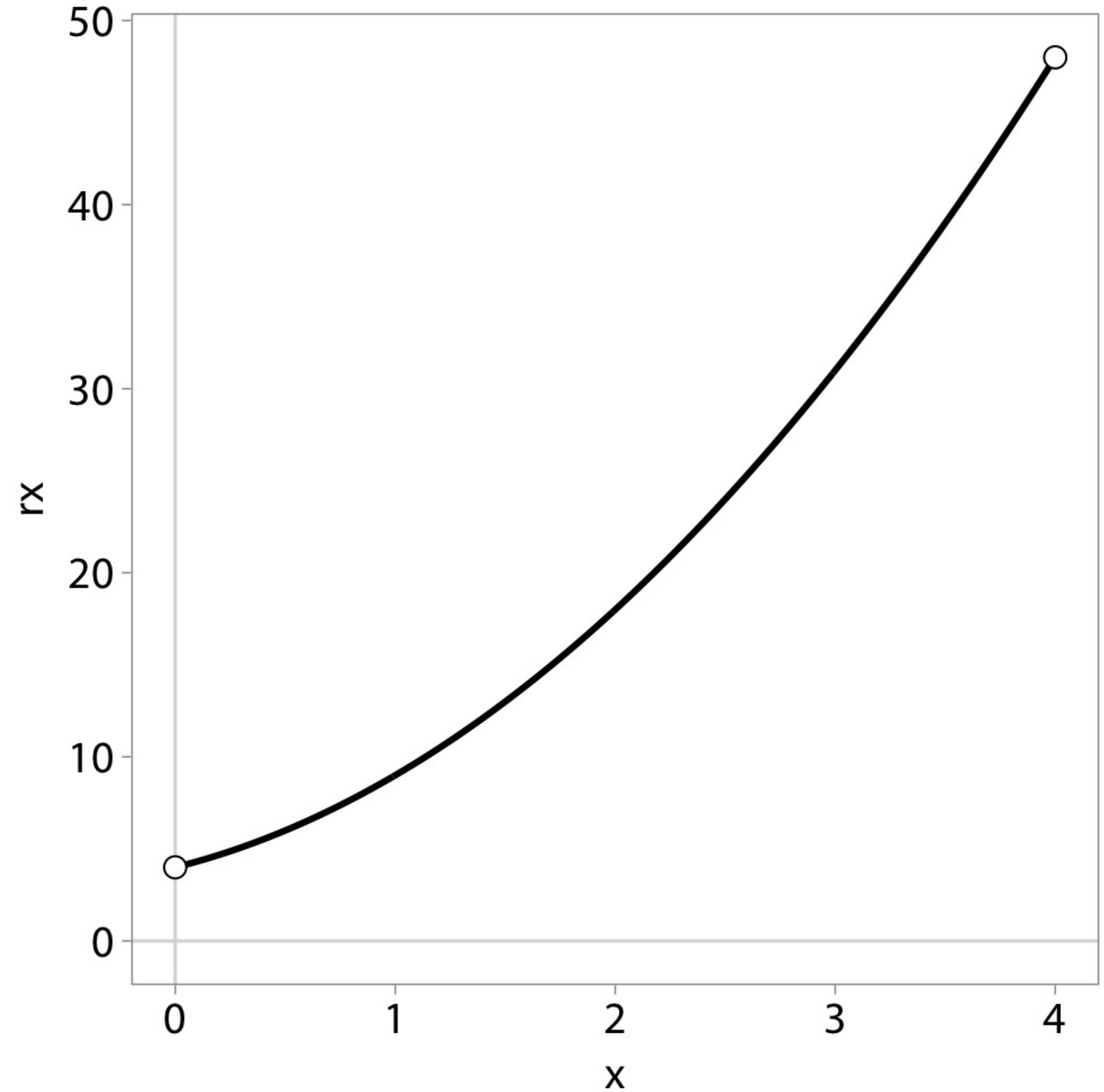


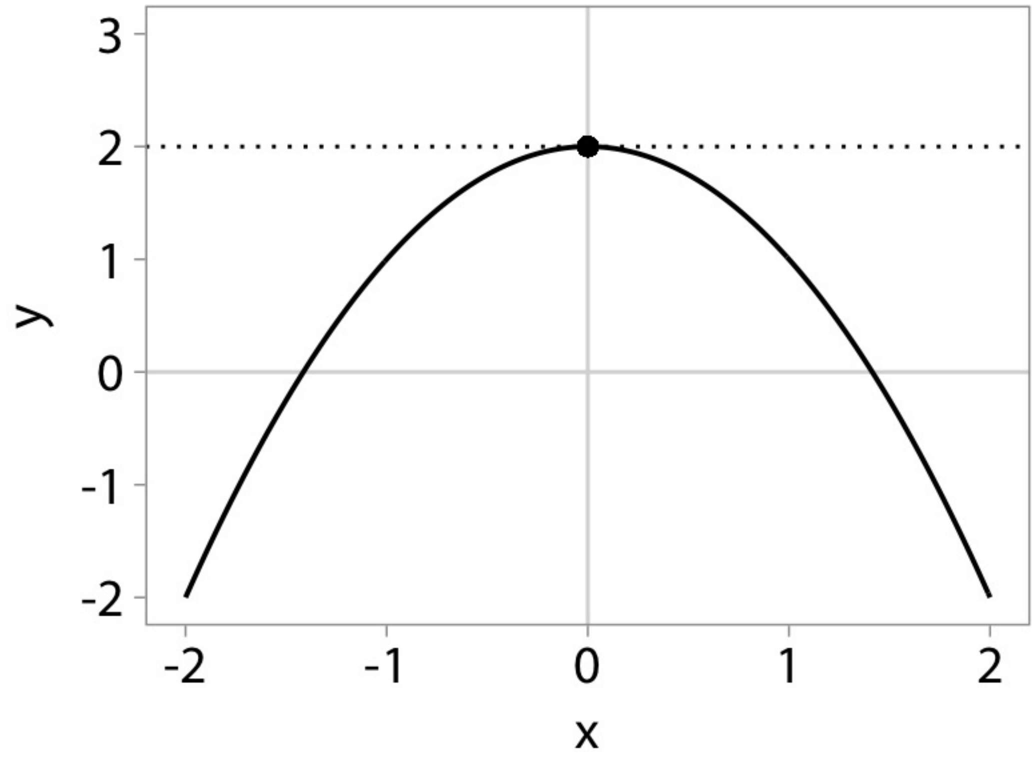
Important functions, routines, and properties

Limits

Limits will help us formally define concepts in this lecture.

A limit describes a function's behavior at a given input: $\lim_{x \to 0} \left(2 - x^2\right) = 2$

...or as the input value changes: $\lim_{x \to \infty} \left(2 - x^2\right) = -\infty$

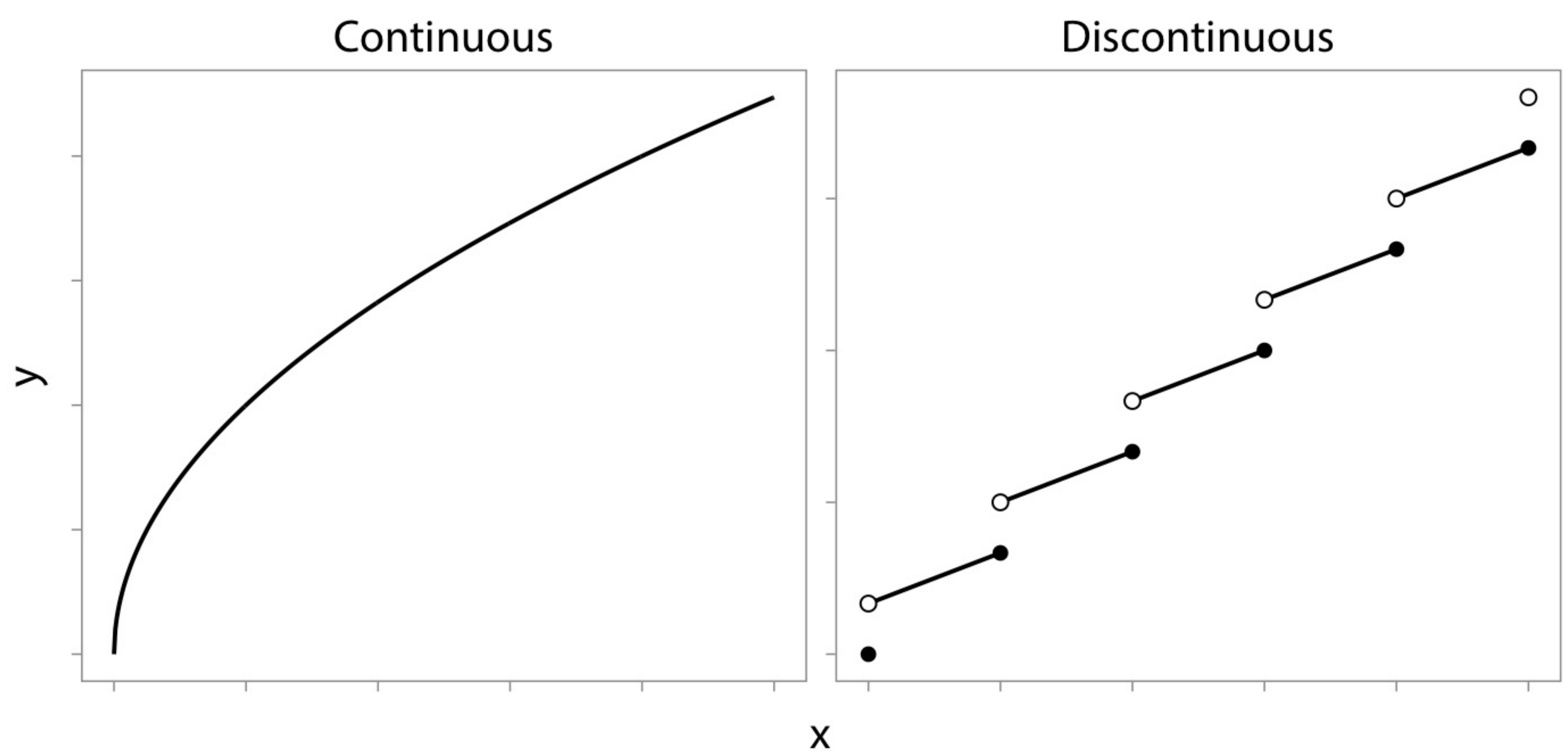


Continuity

A function is continuous if it has no gaps or jumps.

$$\lim_{x \to a} f(x) = f(a)$$

Meaning... small changes in input produce small changes in output



In the wild

In the wild: Continuity and Discontinuity

Why we care about continuity

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- More on Wednesday

Continuity and Discontinuity

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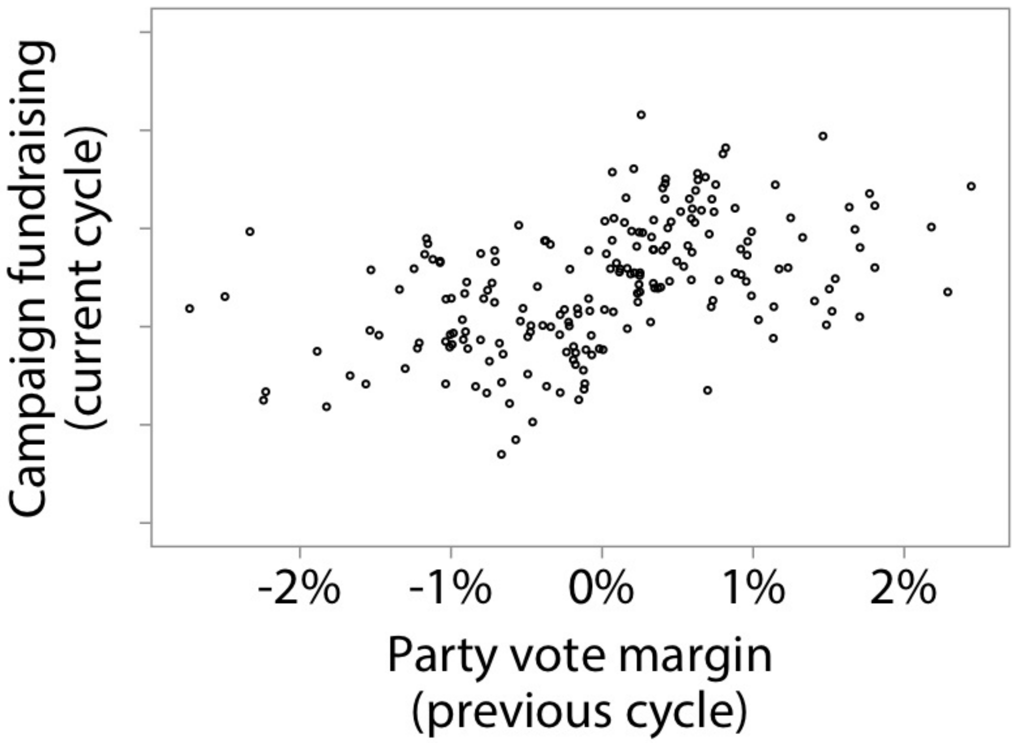

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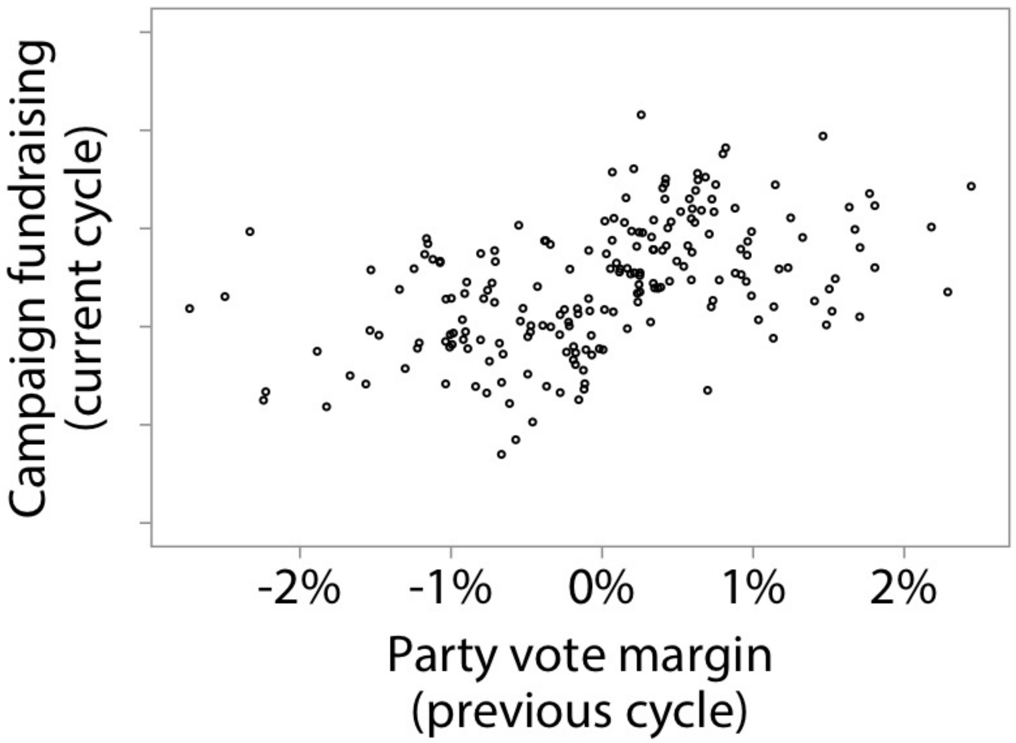
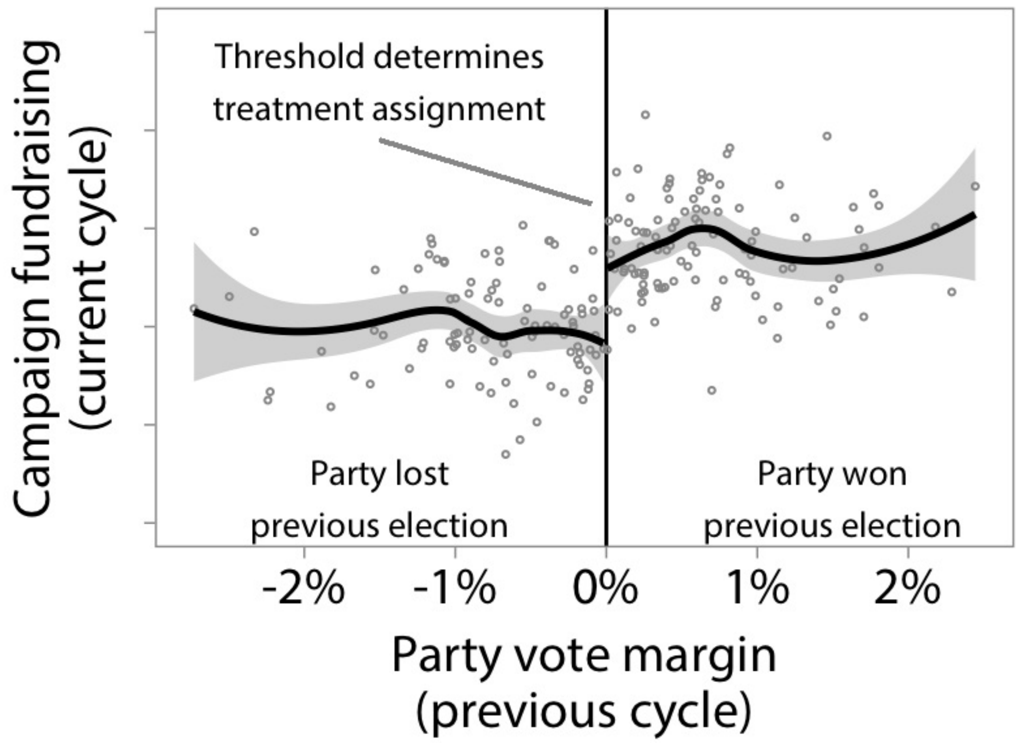
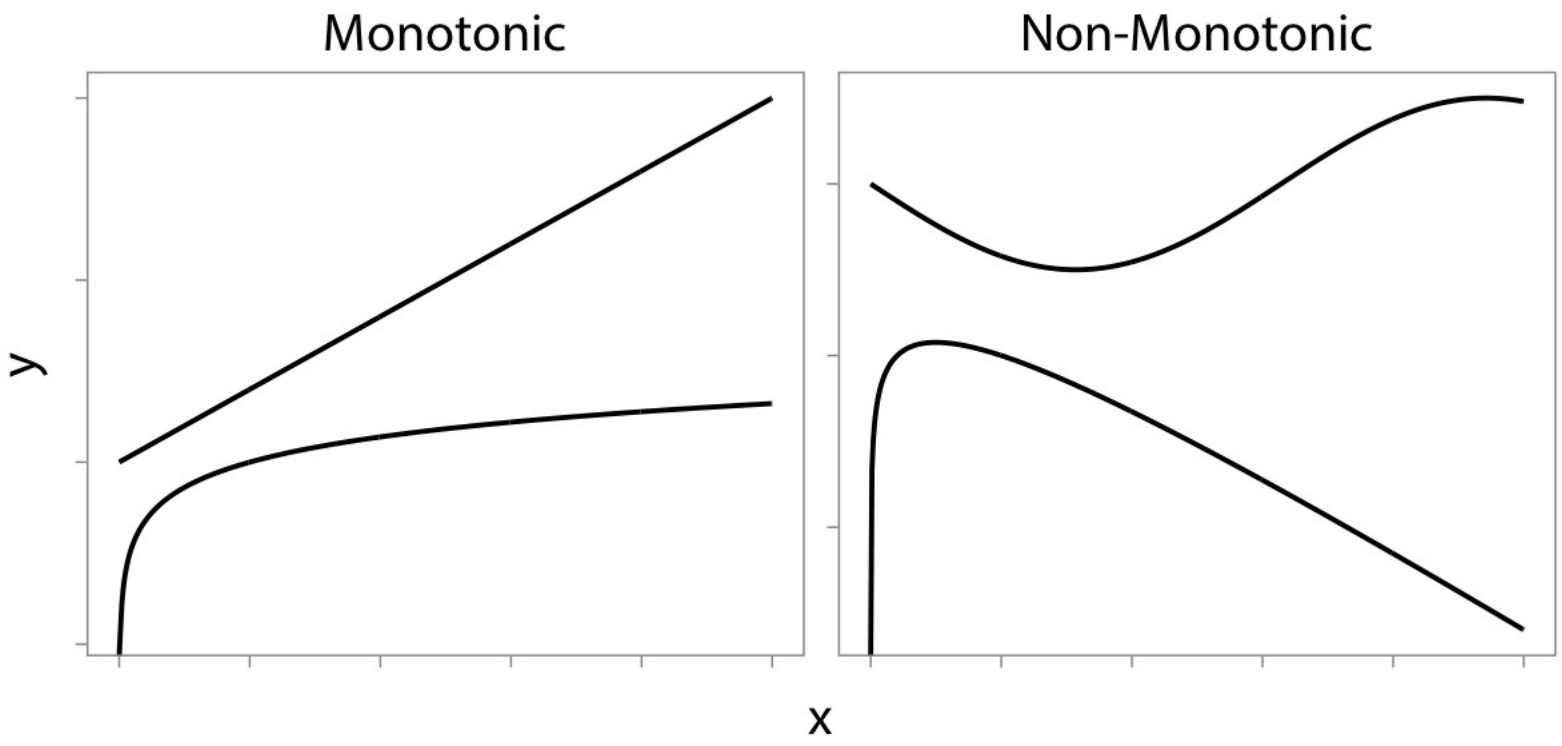




Monotonicity

A function is monotonic if it always increases (monotonically increasing) or always decreases (monotonically decreasing)

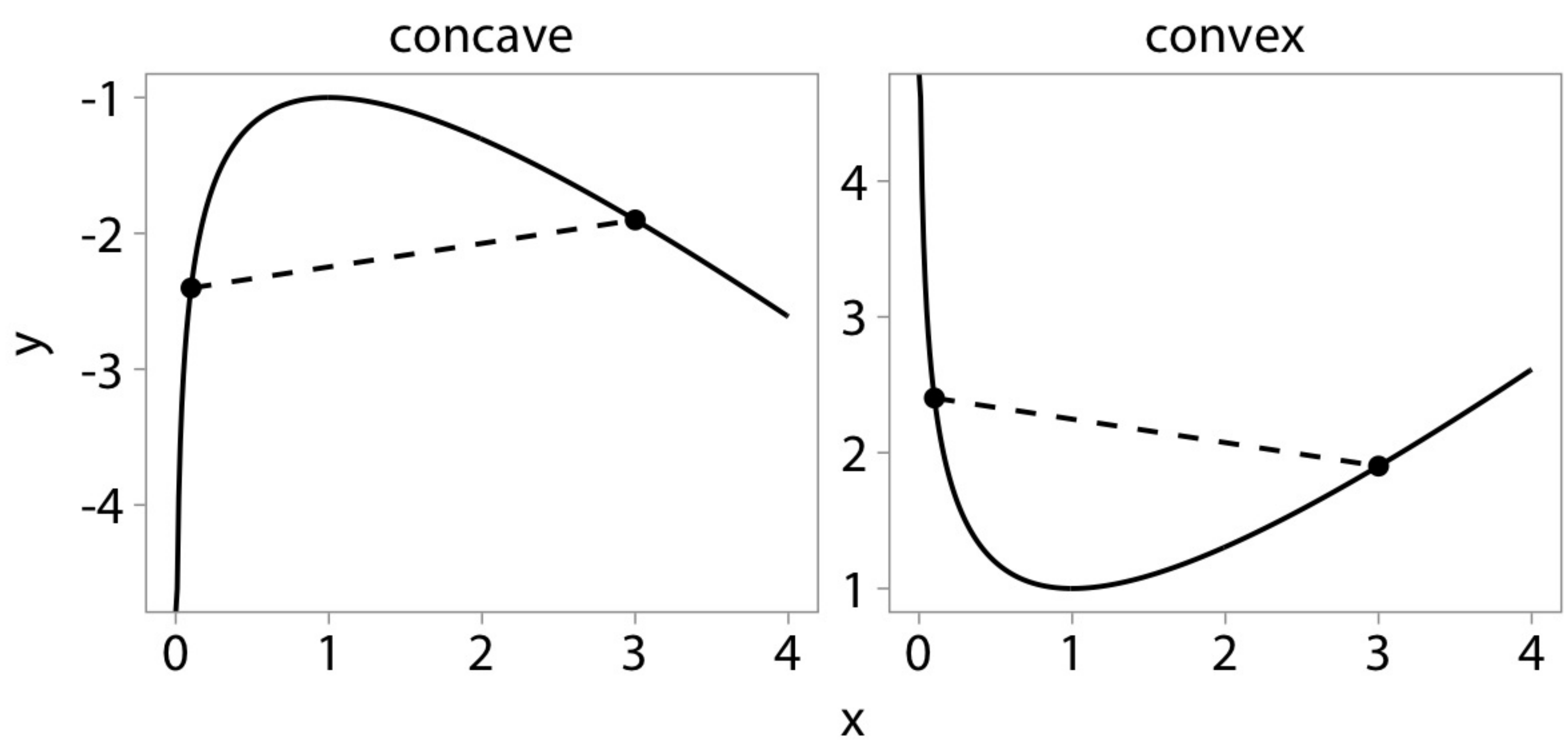
- increasing: for any $x_1 > x_2$, then $f(x_1) > f(x_2)$
- decreasing: for any $x_1 < x_2$, then $f(x_1) < f(x_2)$



Concavity and Convexity

Imagine you draw a line between two points along a function. A function (or segment of a function) is concave if this line is below the function, and convex if the line is above the function.

- concave: $\frac{f(x_1)+f(x_2)}{2} < f\left(\frac{x_1+x_2}{2}\right)$
- convex: $\frac{f(x_1)+f(x_2)}{2} > f\left(\frac{x_1+x_2}{2}\right)$

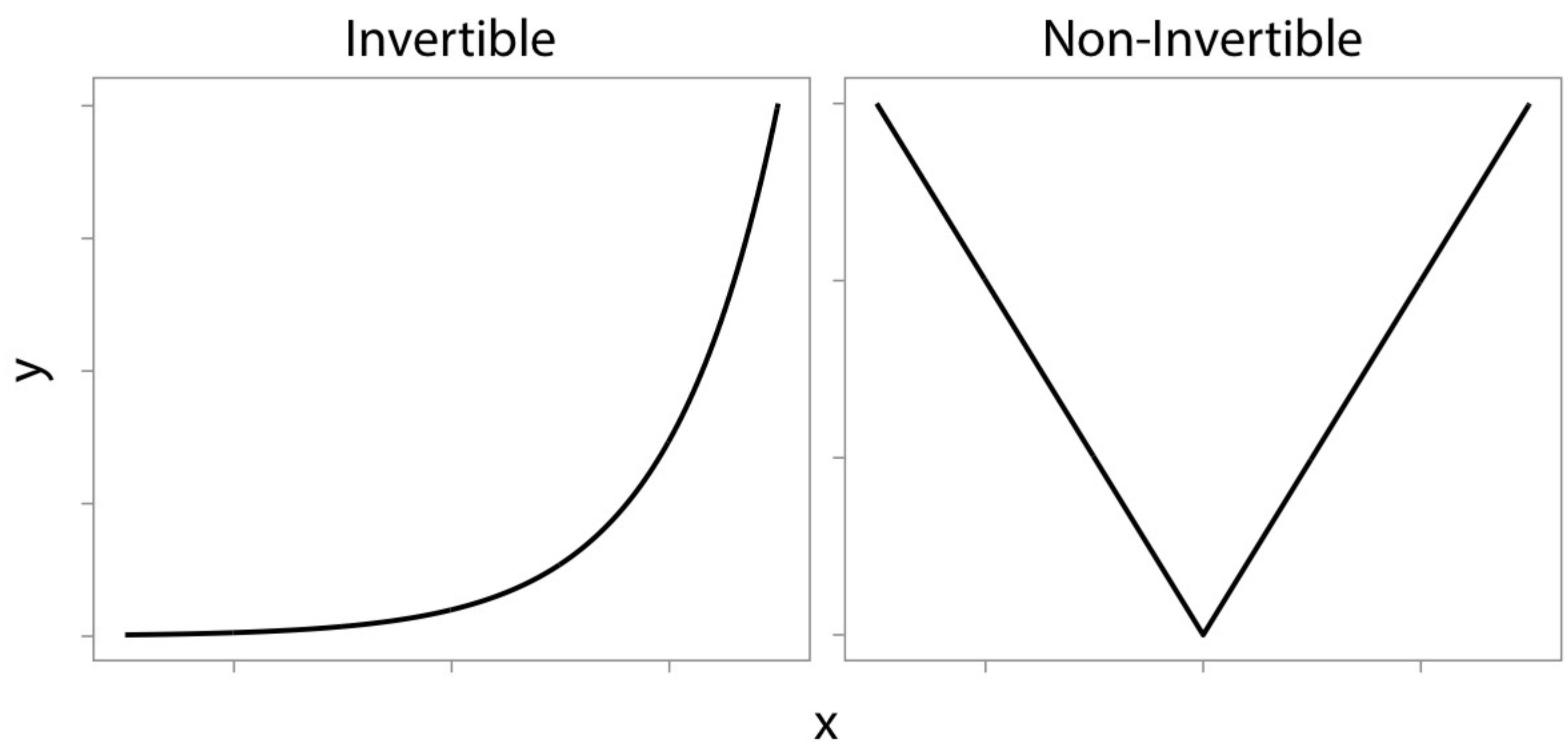


Invertible functions

A function maps an input to an output. A function is invertible if there exists a reverse function that maps the output back to the input.

Formally: if $y = f(x)$, then $f^{-1}(y) = x$

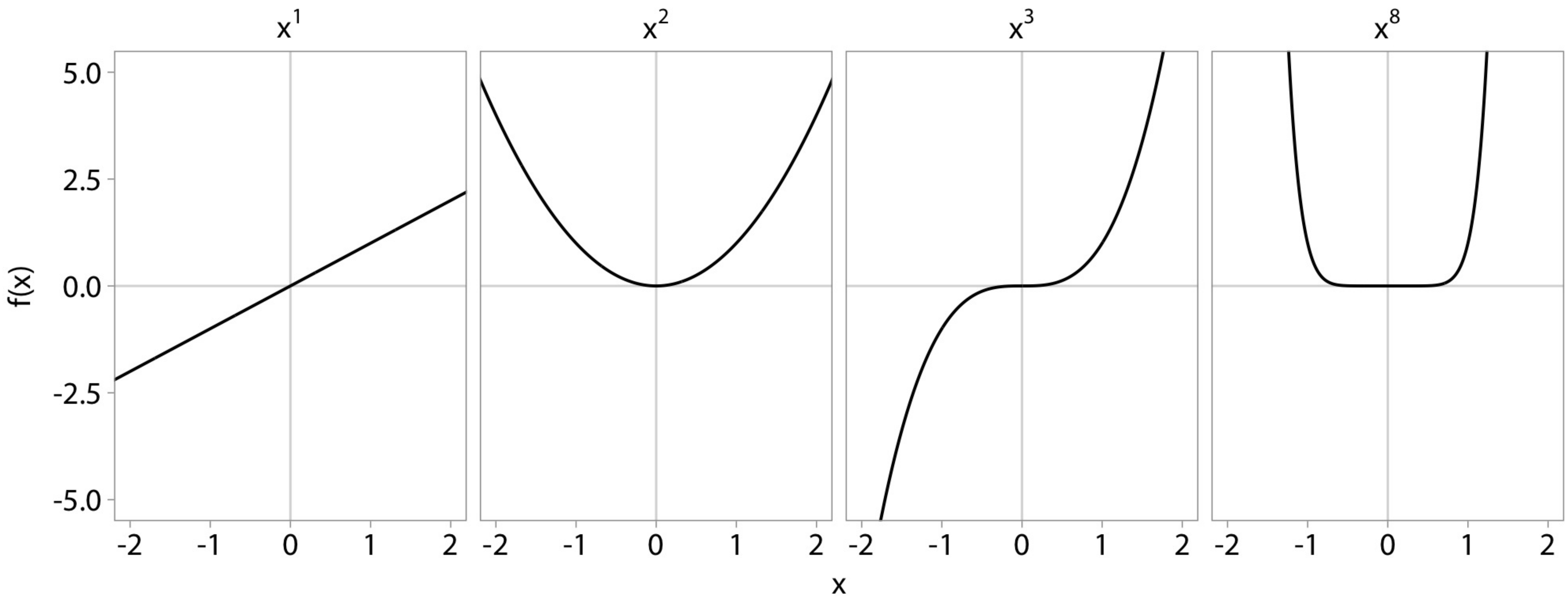
Also: $f^{-1}(f(x)) = x$



Exponents

The exponent operator multiplies a number by itself the number of times indicated in the exponent

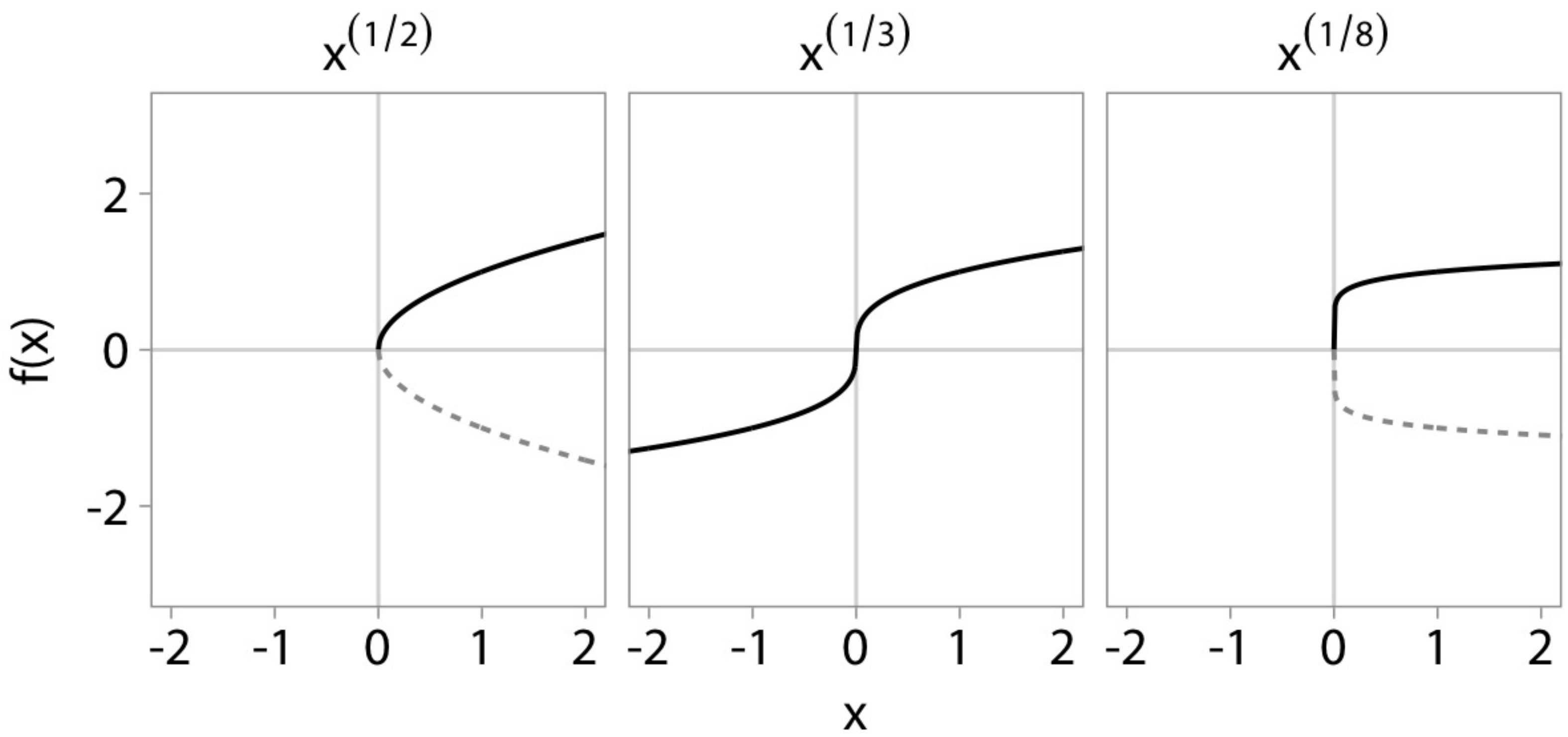
$$x^n = x * x * \ldots * x \quad \text{(n times)}$$



Roots

Root operators return the number that, when multiplied by itself the number of times indicated in the root, is equal to the input. When no number is given, that indicates the square root.

$$x = \sqrt[n]{x} * \sqrt[n]{x} \ldots \sqrt[n]{x} \quad \text{(n times)}$$



Exponents and Roots

All roots can be expressed as exponents (with the same properties)

$$\sqrt[n]{x} \equiv x^{\frac{1}{n}}$$

Important properties for exponents and roots

Zeroth power	$x^0 = 1$
Negative powers	$x^{-n} = \frac{1}{x^n}$
Inversion using exponents	$x^{-1} = \frac{1}{x}$
Distribution of powers (multiplication)	$(x * y)^n = x^n * y^n$
Distribution of powers (division)	$(\frac{x}{y})^n = \frac{x^n}{y^n}$
Product of powers	$x^n * x^m = x^{n+m}$
Nested powers	$(x^n)^m = x^{n*m}$

Two important continuous, monotonic, invertible functions: Exponentials and Logarithms

Exponentials and Logarithms

The logarithm (log) of some value y (with base b) is...

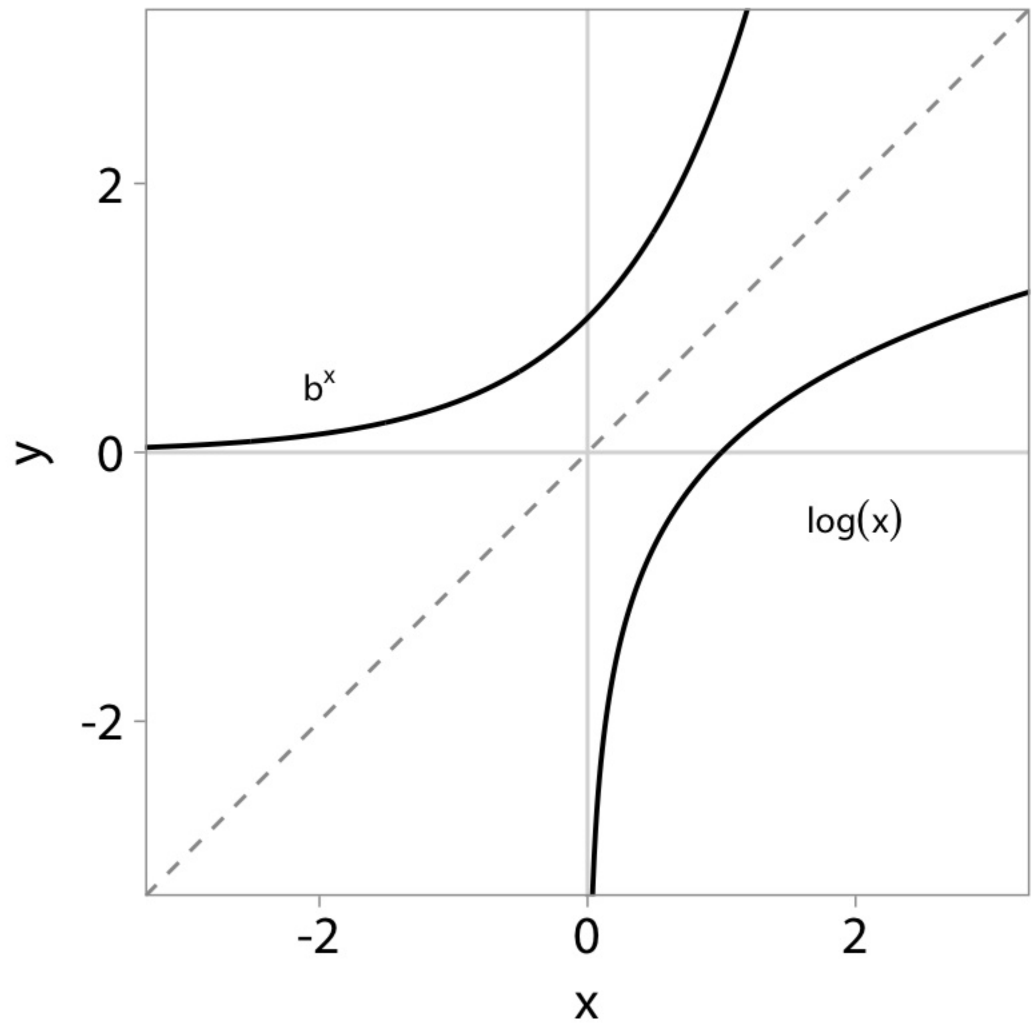
...the power to which that base would need to be raised to equal y

$$\text{If } b^x = y, \text{ then } \log_b(y) = x$$

Exponentials and logarithms are inverse functions. Logs "undo" exponentials, and exponentials "undo" logs.

Exponentials and Logarithms

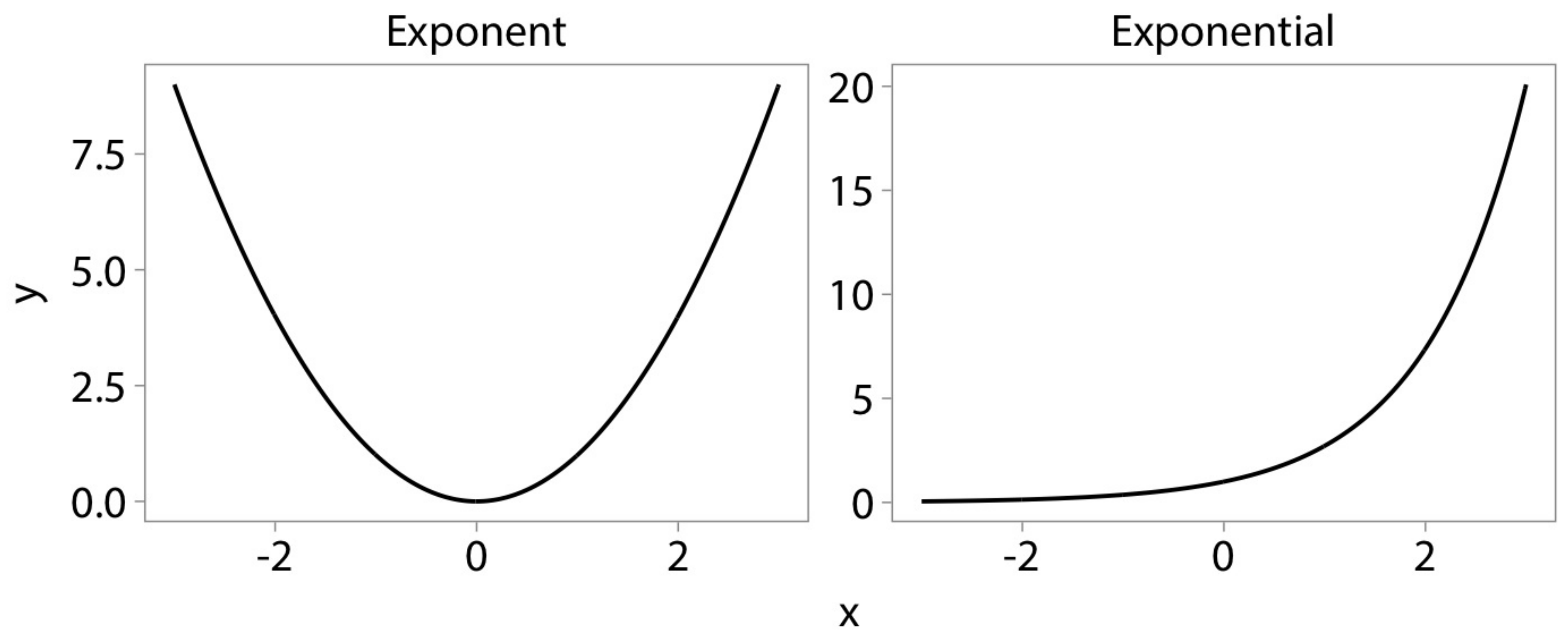
We can see this because exponentials and logs are reflections of each other over $y = x$ (one way to identify inverse functions)



Exponents ≠ Exponentials

Exponent: x^2 (x is the base)

Exponentials: 2^x (x is the power)



Better yet...

What happens when you exponentiate a parabola?

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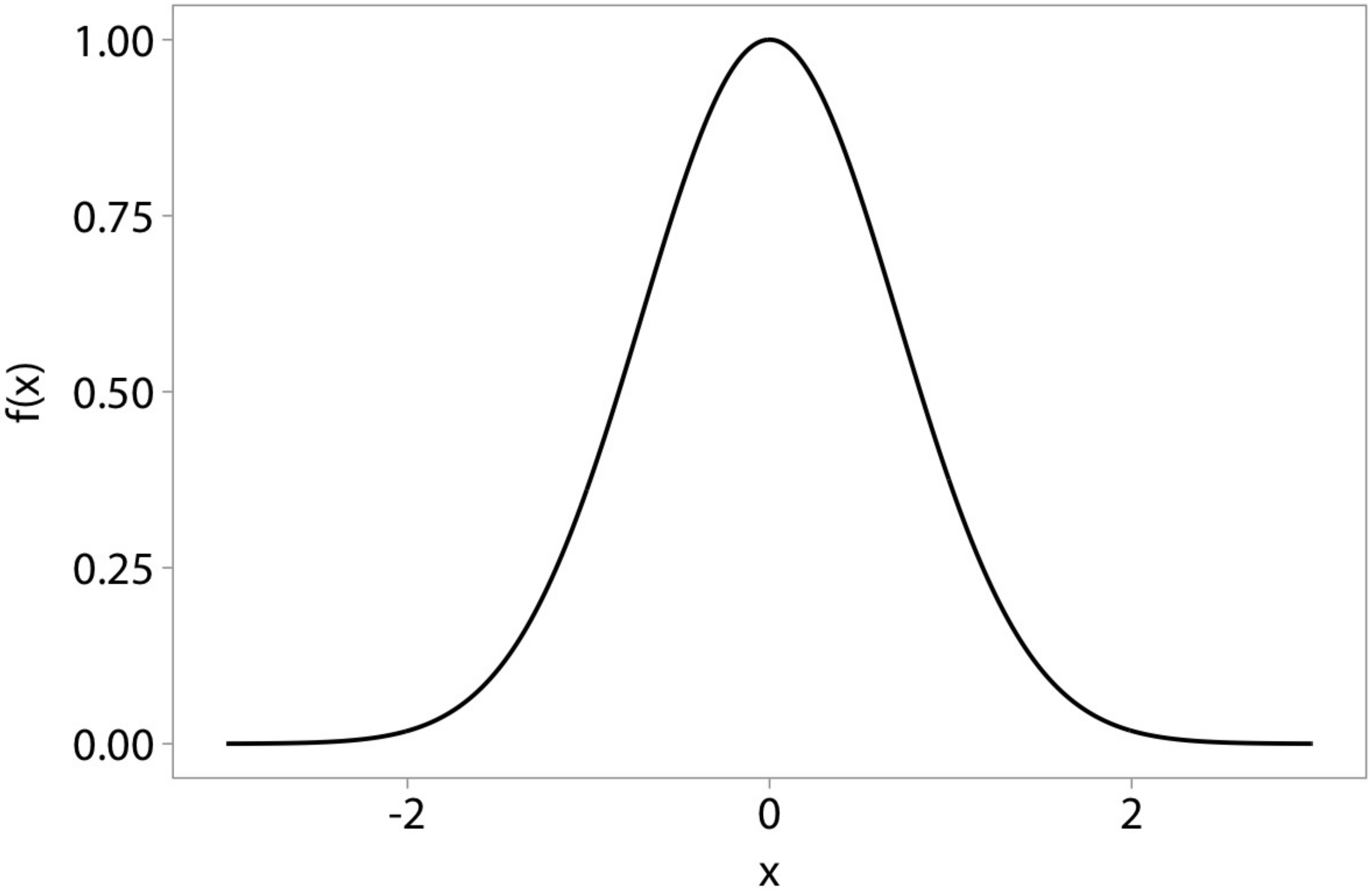


Rules for logarithms

The following apply to all logs, regardless of base

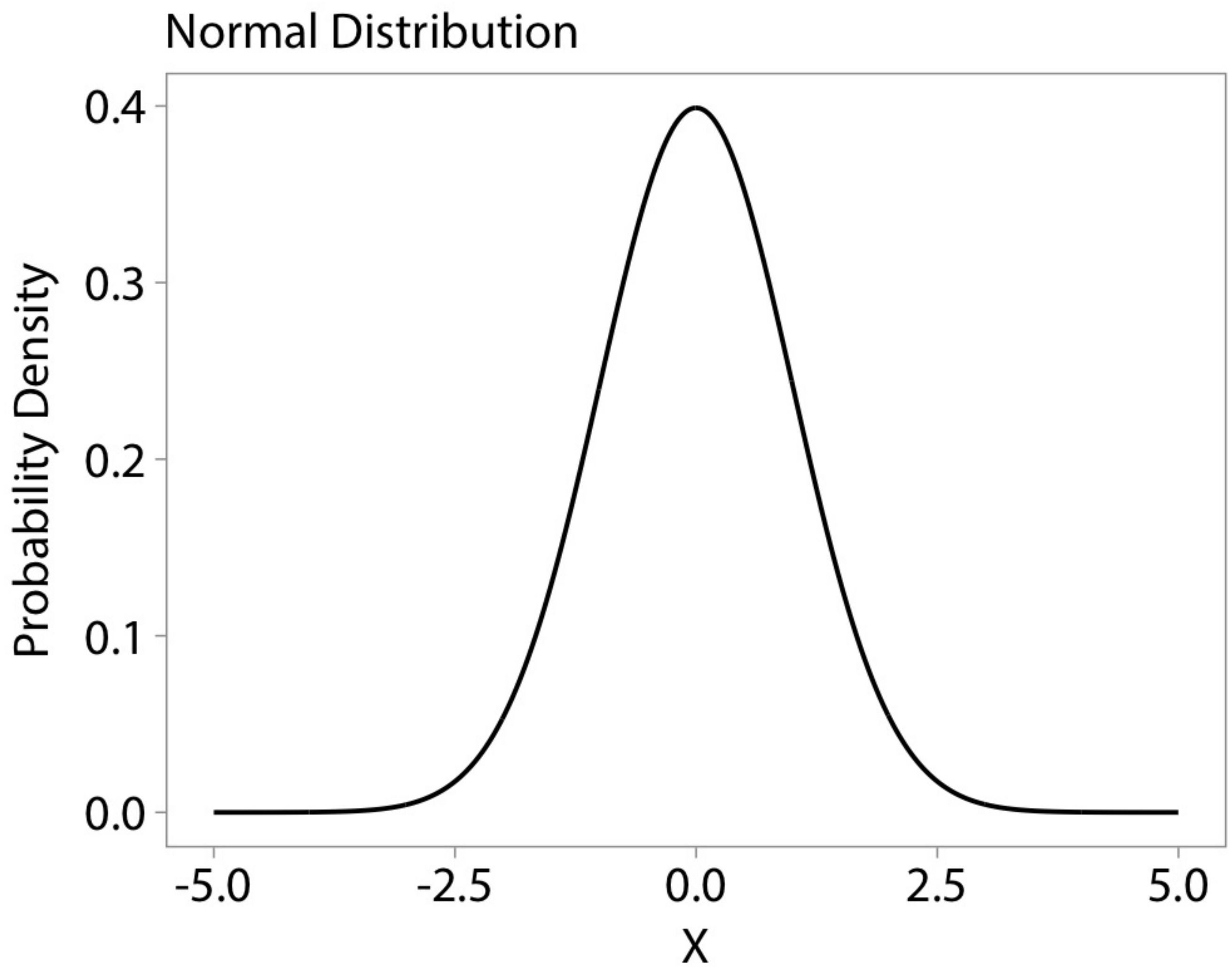
Log 1	$\log(1) = 0$
Log 0	undefined, approaches $-\infty$
Multiplication	$\log(x * y) = \log(x) + \log(y)$
Division	$\log(\frac{x}{y}) = \log(x) - \log(y)$
Exponentiation	$\log(x^a) = a * \log(x)$
Basis	$\log_b(b^x) = x$

(potentially) helpful video for understanding logs here

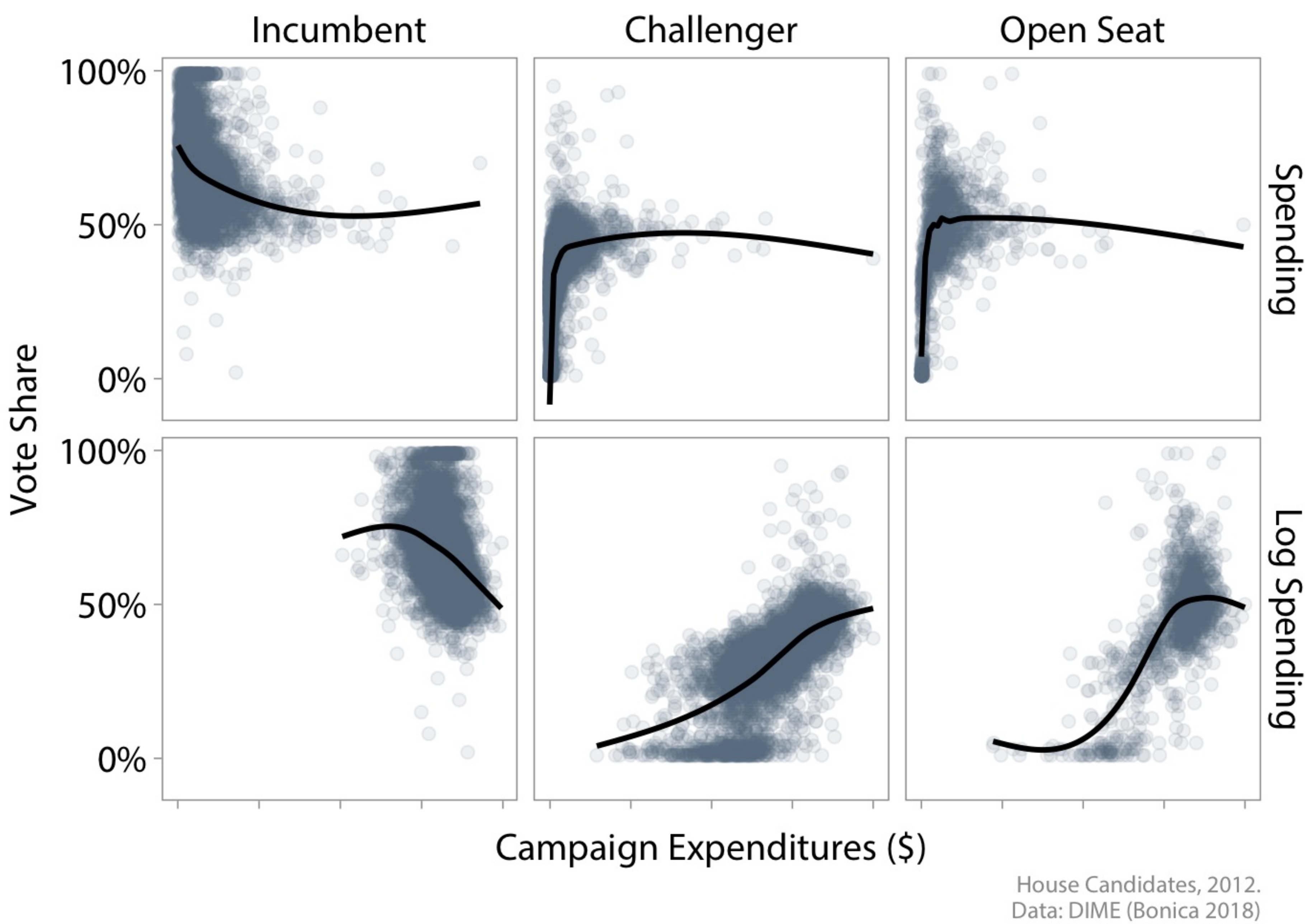
You WILL use logs and exponents

They are important for probability distributions

$$f(x \mid \mu, \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Common transformations in nonlinear models



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Probability mass function for a binomial outcome (n independent trials, y successes, success probability π): $\Pr(y = k \mid \pi) = \binom{n}{k}\pi^k(1-\pi)^{n-k}$

Plug in our data: $\Pr(y = 4 \mid \pi) = \binom{5}{4}\pi^4(1-\pi)^{5-4}$

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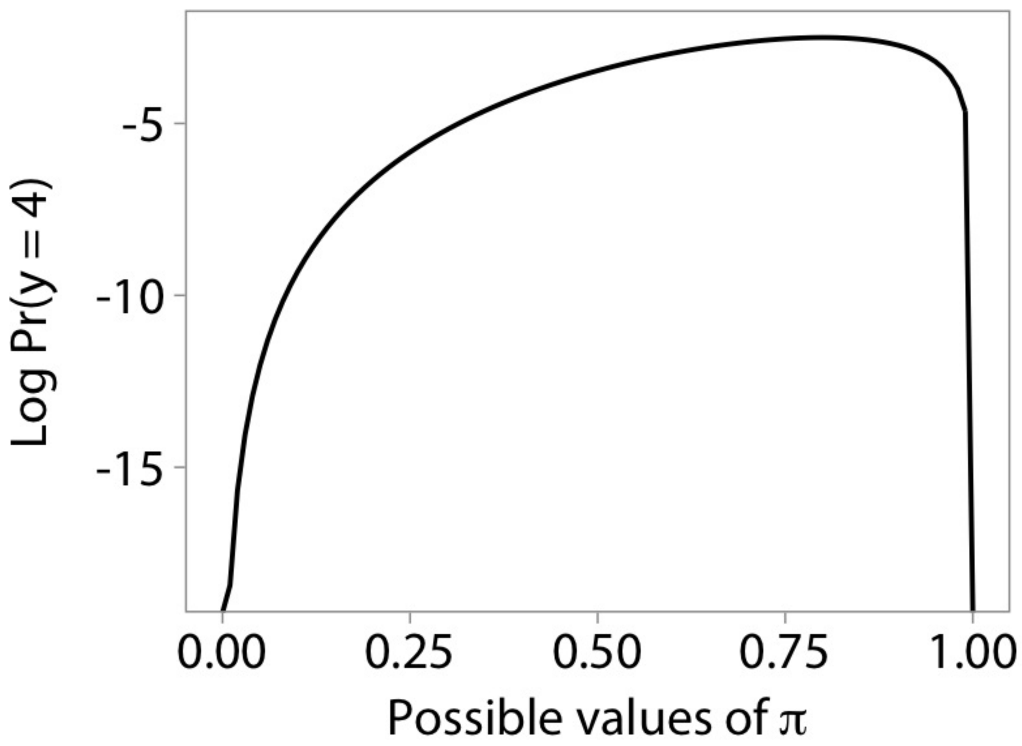
If we maximize this function with respect to π, we find the π value that gives us the greatest probability. That is, the most likely value of π that could give us these data.

This is how maximum likelihood works

Maximizing the log likelihood



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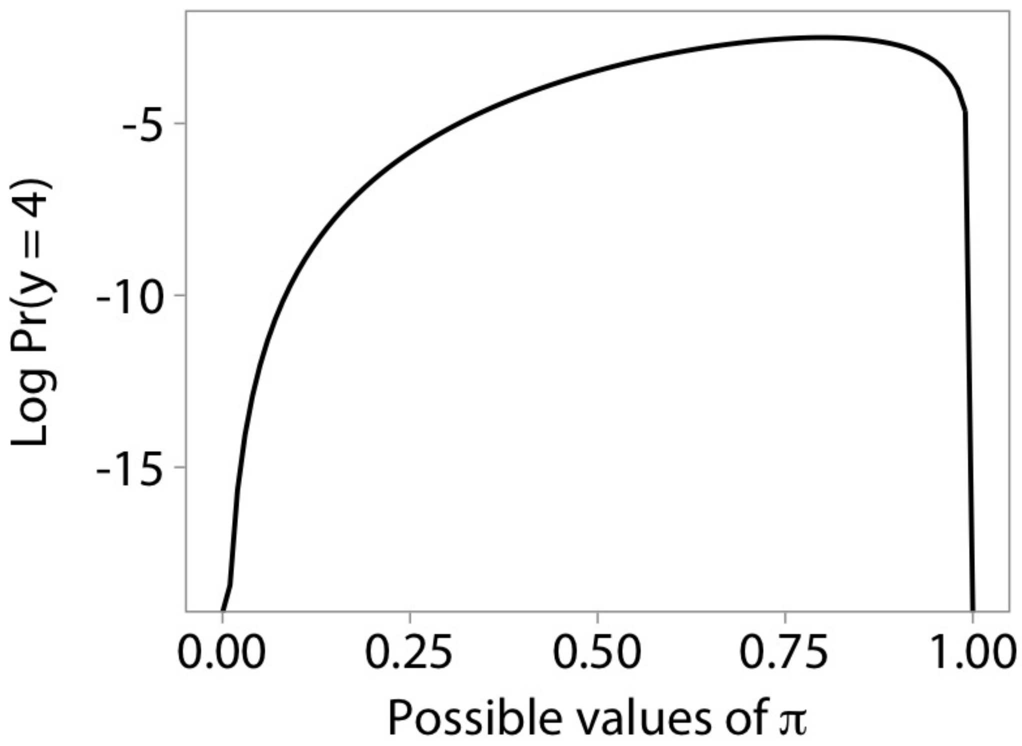
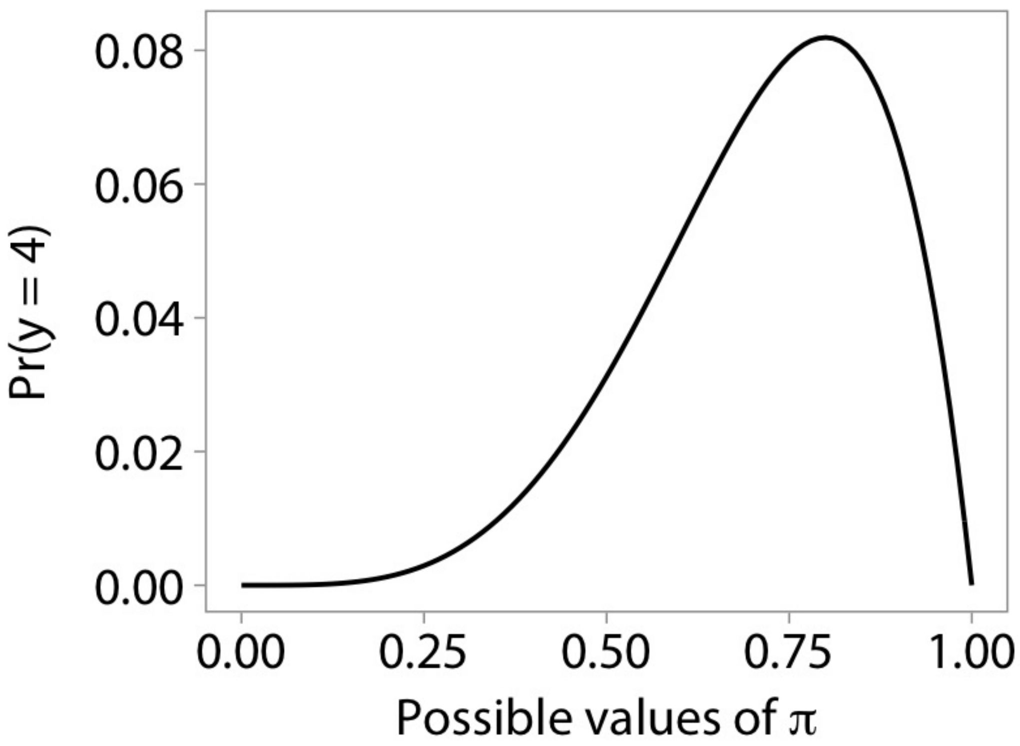


On the unlogged scale



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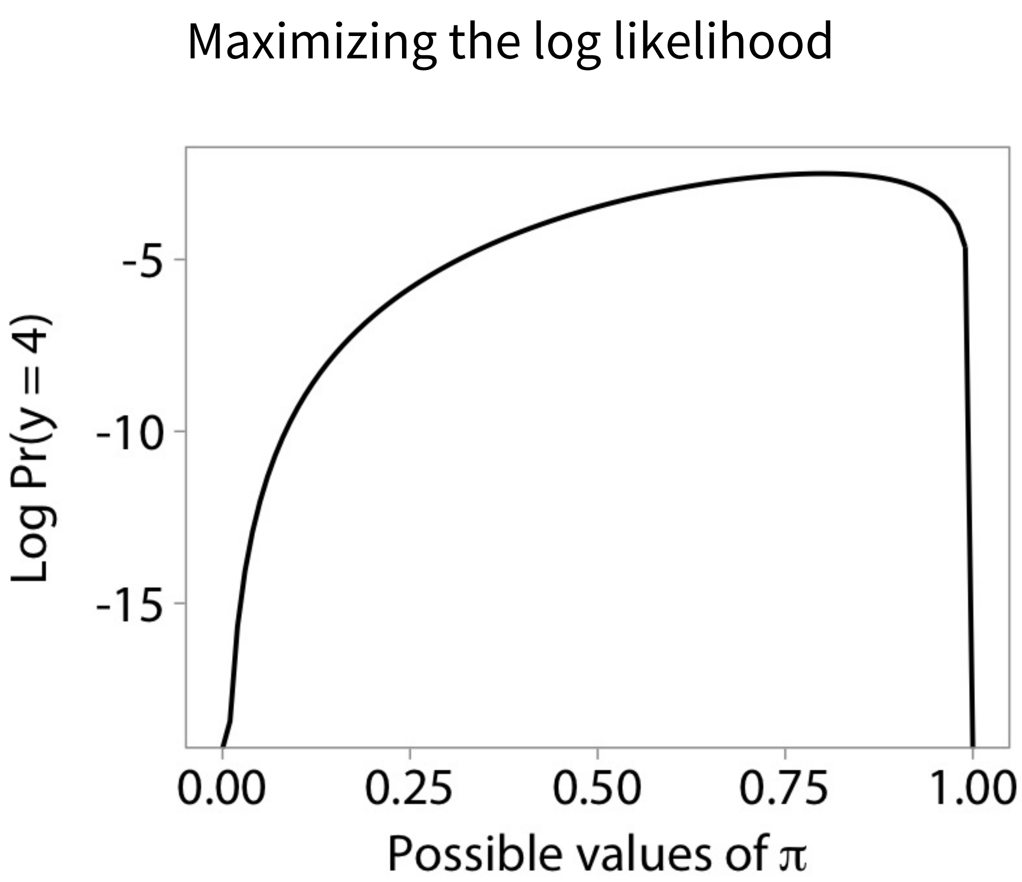

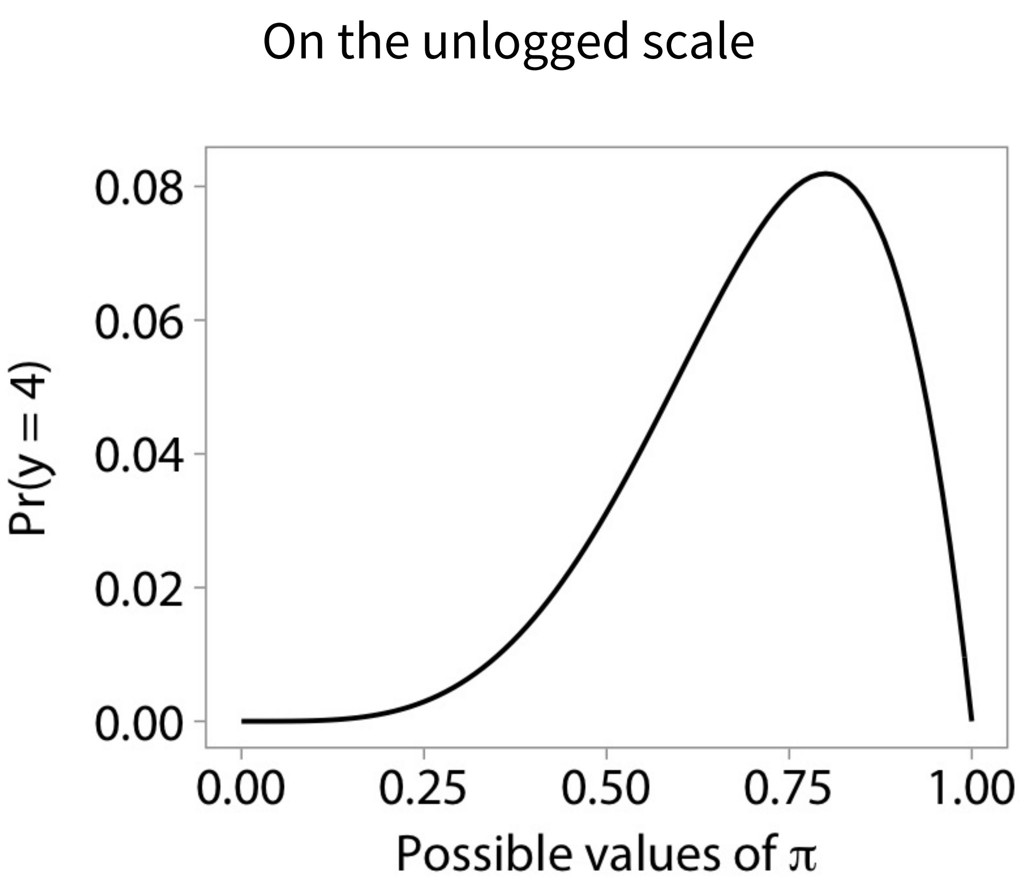

Point being: we use logs in MLE

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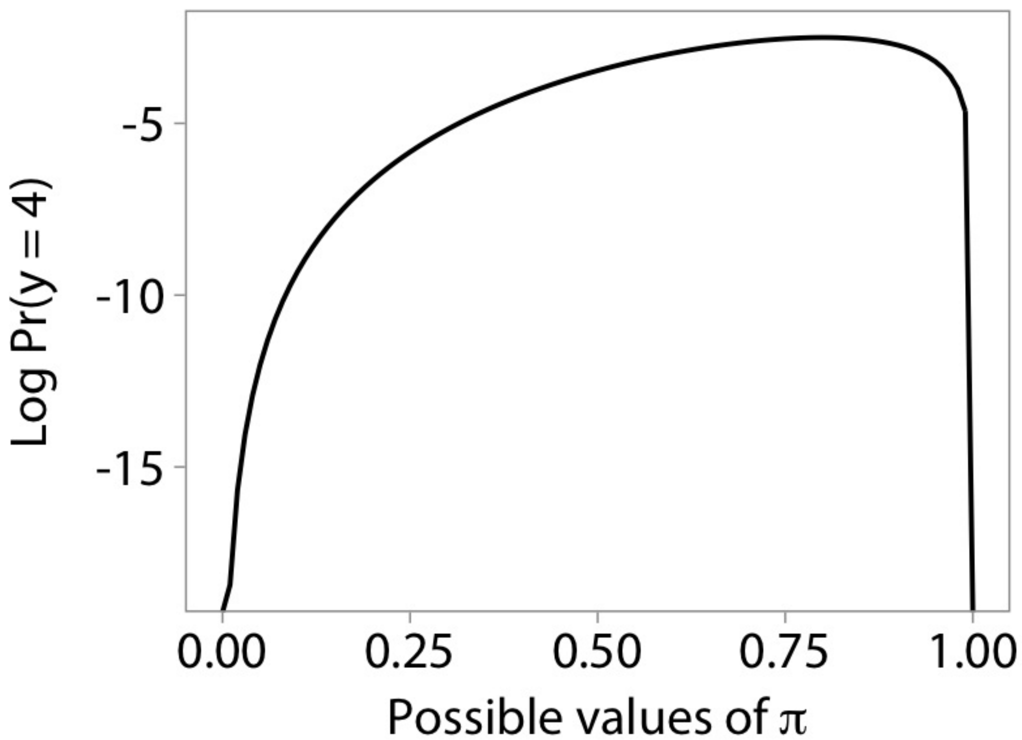

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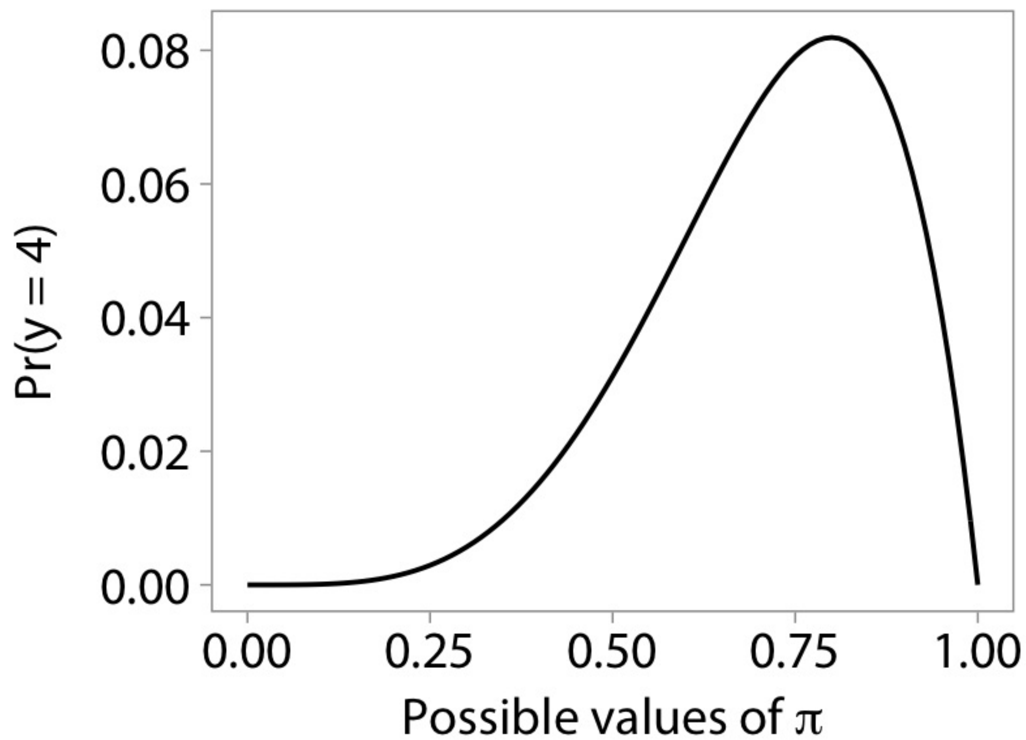

Point being: we use logs in MLE

(Note what the log transformation does to the y-axis)

Logs get easier with experience, which you will have

Base e

Although many early examples with logs use some arbitrary base (like base 10, the "common log"), most applications use base e ($\log_{\mathrm{e}}$, the "natural log", LN)

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We will emphasize the value of base e when we talk about probability (and the concept of odds)

Some practice with logs and exponents

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If $f(x) = \log_{10}(x)$, what is $f(1000)$?

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If $h(x) = \log_2\left(x^5\right)$, what is $h(4)$?

- $$\begin{aligned} \log_2\left(4^5\right) &= 5\log_2(4) \\ &= 5 * 2 \\ &= 10 \end{aligned}$$

Solve:

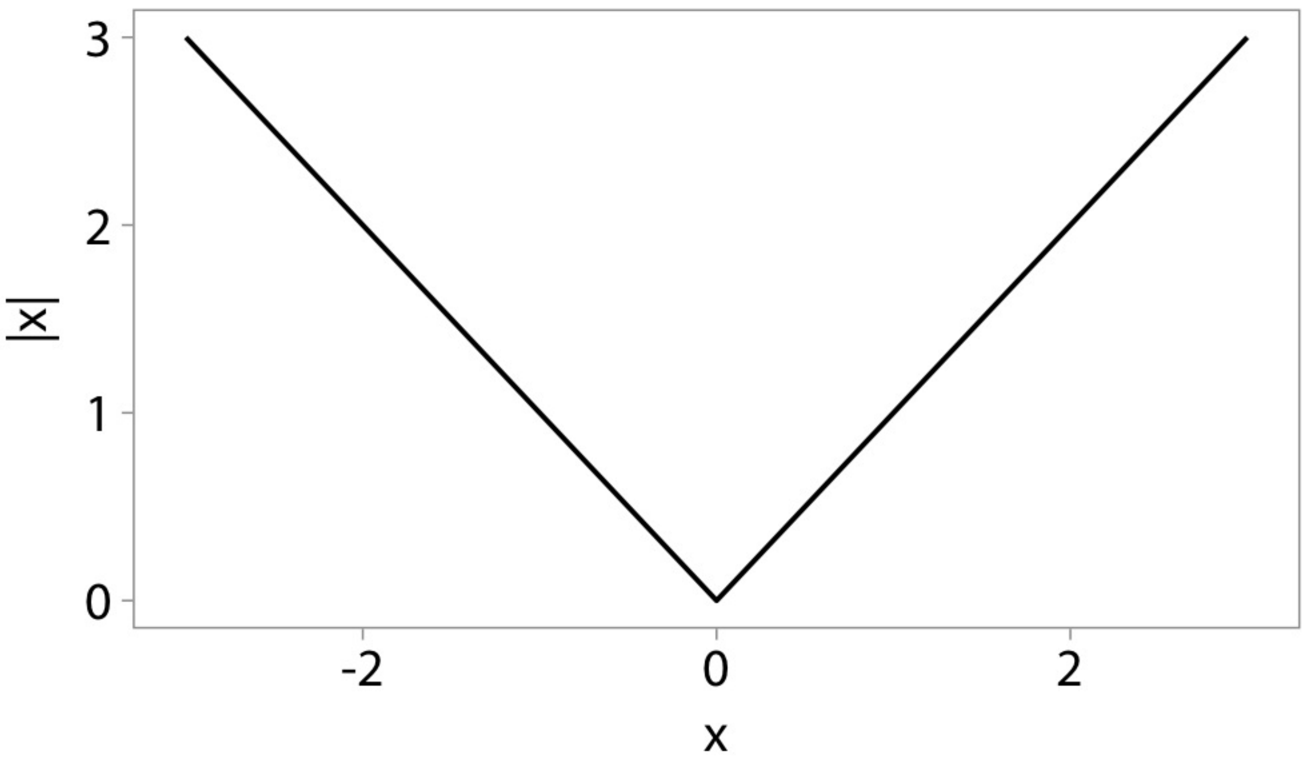
- $\log_2\left(4^3\right)$
- $\ln\left(\frac{x}{y} * q^4 * e\right)$

Now for something slightly easier

Absolute value

The absolute value operator returns the positive representation of a number

$$|x| = \begin{cases} x & \text{if } x \text{ is positive} \\ -x & \text{if } x \text{ is negative} \end{cases}$$



Multiplying polynomials

Polynomial: an expression with variables and coefficients, using only addition, subtraction, multiplication, and non-negative integer exponents

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Multiply them by distributing: every element of each polynomial must be multiplied by every element of the other polynomial, then group terms by the powers of the variables.

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$$ax\left(bx^2 + x\right) = abx^2 + acx$$

Multiplying polynomials: FOIL

First, Outside, Inside, Last: for polynomials that each have two terms

$$(ax + b) \cdot (cx + d)$$

How do we FOIL?

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- Last: $b \cdot d$

What do we get?

$$acx^2 + adx + bcx + bd$$

Longer polynomials

They work the same way, just keep track of all the terms. (FOIL is the same as distributing)

$$(2x^4 + 5x^3) \cdot (8x^2 + x + 3) = ?$$

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$$= (2x^4 \cdot 8x^2) + (2x^4 \cdot x) + (2x^4 \cdot 3) + (5x^3 \cdot 8x^2) + (5x^3 \cdot x) + (5x^3 \cdot 3)$$

$$= 16x^6 + 2x^5 + 6x^4 + 40x^5 + 5x^4 + 15x^3$$

$$= 16x^6 + 42x^5 + 11x^4 + 15x^3$$

Polynomial practice

Find the products:

- $(x^2 + 3) \cdot (x - 2)$
- $(3p + 4q) \cdot (p - 2q)$

Factorials

The factorial operator (denoted with an exclamation mark !) returns the product of an integer with all lesser integers.

$$x! = x \cdot (x - 1) \cdot (x - 2) \cdot \ldots \cdot 2 \cdot 1$$

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These come in handy when we do probabilities (combinations and permutations)

Let's call it a day

Homework is online

https://www.github.com/mikedecr/math-camp-2018