

Math Camp Lesson 2

Vectors and Matrices (Linear Algebra)

UW–Madison Political Science

August 21, 2018

Introductions

Linear Algebra

Review/Overview

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Algebra is a fundamental basis for more advanced mathematical manipulation:

- Use to derive statistical estimators, and to understand their properties and the assumptions necessary to apply them.
- Use to evaluate the optimal choices of strategic actors.

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If no range is indicated ($\sum x_i$), this implies all observations are included.

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$$\begin{aligned}\sum_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) + (x_2^2 + 3) + (x_3^2 + 3) \\ &= (3^2 + 3) + (4^2 + 3) + (1^2 + 3) \\ &= 35\end{aligned}$$

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$$\begin{aligned}\prod_{i=1}^3 (x_i^2 + 3) &= (x_1^2 + 3) \times (x_2^2 + 3) \times (x_3^2 + 3) \\ &= (3^2 + 3) \times (4^2 + 3) \times (1^2 + 3) \\ &= 912\end{aligned}$$

Summations and Products

Given these data: $x_1 = 3$, $x_2 = 4$, $x_3 = 1$, and $x_4 = 0$; and $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, and $y_4 = 4$.
Find these quantities:

- $\sum x_i + \sum y_i$
- $\sum (x_i + y_i)$
- $\prod x_i + \prod y_i$
- $\prod (x_i \times y_i)$

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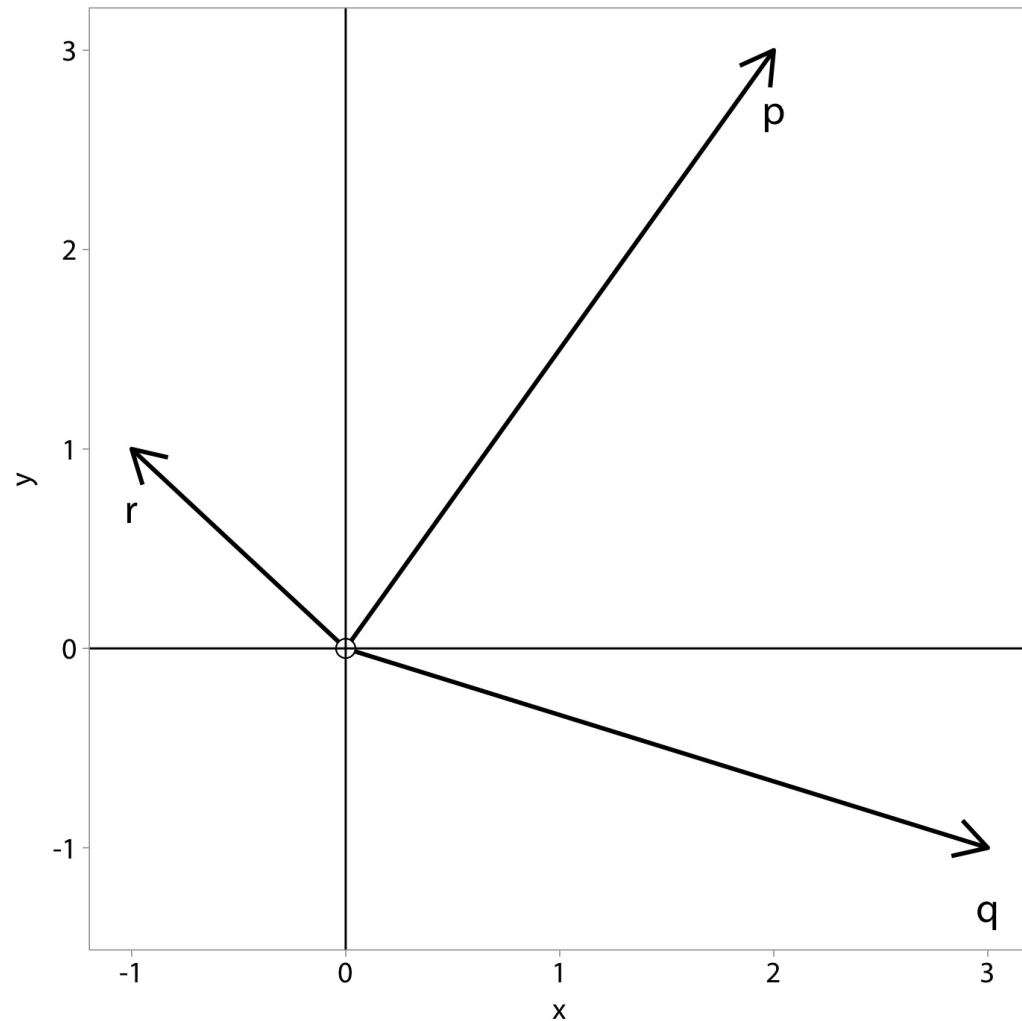
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or a **column** vectors like

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Ordered simply means that $[v_1, v_2, v_3, v_4] \neq [v_4, v_3, v_2, v_1]$

Vectors in Space



Vectors can be thought of as lines from the origin in k -dimensional space (where k is the number of vector elements) going to a point with the coordinates of the elements of the vector.

$$\mathbf{p} = [2, 5]$$

$$\mathbf{q} = [4, -1]$$

$$\mathbf{r} = [-1, 1]$$

Vector Operations: Addition and Subtraction

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$$\mathbf{u} + \mathbf{v} = \mathbf{w}$$

$$[1, 2, 3, 4] + [4, 8, 12, 16] = \mathbf{w}$$

$$[1 + 4, 2 + 8, 3 + 12, 4 + 16] = \mathbf{w}$$

$$[5, 10, 15, 20] = \mathbf{w}$$

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$$\mathbf{u} - \mathbf{v} = \mathbf{w}$$

$$[1, 2, 3, 4] - [4, 8, 12, 16] = \mathbf{w}$$

$$[1 - 4, 2 - 8, 3 - 12, 4 - 16] = \mathbf{w}$$

$$[-3, -6, -9, -12] = \mathbf{w}$$

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It is important to note that conformability does not matter for scalar multiplication and division.

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$$\mathbf{x} \cdot \mathbf{y} = [x_1 \times y_1 + x_2 \times y_2, \dots + x_{k-1} \times y_{k-1} + x_k \times y_k] = \sum_{i=1}^k x_i y_i$$

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The dot product will start with two vectors and result in a scalar.

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Commutative property: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Associative Property: $s(\mathbf{u} \cdot \mathbf{v}) = s(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot s(\mathbf{v})$

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Distributive Property: $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

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$$\mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Vectors

Given vectors $\mathbf{x} = [1, 2, 0, 4]$ and $\mathbf{y} = [5, 3, 2, 3]$, find:

- \mathbf{x}^T
- $\mathbf{x} + \mathbf{y}$
- $\mathbf{x} \cdot \mathbf{y}$

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Like vectors, each value is referred to as an element. When referring to elements of a matrix, we will not bold the vector and add a subscript to denote their position. For example, $x_{1,2}$ refers to the element in the first row, second column of \mathbf{X} .

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This switches the dimensions (here, from 2×3 to 3×2).

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$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}$$

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Such that, e.g. $a_{2,1} + b_{2,1} = c_{2,1}$

Matrix Operations: Matrix Addition and Subtraction

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+4 & 3+6 \\ 4+8 & 5+10 & 6+12 \end{bmatrix} \\ = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

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or

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-4 & 3-6 \\ 4-8 & 5-10 & 6-12 \end{bmatrix} \\ = \begin{bmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{bmatrix}$$

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$$4 \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \times 1 & 4 \times 2 & 4 \times 3 \\ 4 \times 4 & 4 \times 5 & 4 \times 6 \end{bmatrix} \\ = \begin{bmatrix} 4 & 8 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$

Matrix Multiplication

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Such that, e.g. $c_{1,1} = a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + a_{1,3}b_{3,1}$

Let's do one more example to clarify. The product of a 1×2 matrix and a 2×1 matrix is a 1×1 matrix, or a scalar!

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What do c and d equal?

$$c = 3 \times 5 + 4 \times 7 = 43$$

$$d = 3 \times 6 + 4 \times 8 = 50$$

Matrix Multiplication

Therefore,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

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Unlike with scalars, order matters. Reversing the order may result in a different product, or may not even be possible depending on the dimensions of the matrices.

Properties of Matrix Multiplication

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- Additive Distributive Property: $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$
- Identity Property: $\mathbf{XI} = \mathbf{IX} = \mathbf{X}$

Matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

Given the matrices above, calculate

- $\mathbf{A} + \mathbf{B}$
- \mathbf{C}^T
- \mathbf{DB}
- \mathbf{CD}^T

Matrix Inversion

The operation most closely analogous to division for matrices is inversion. The inverse of a matrix (denoted with the superscript $^{-1}$) is the matrix that, when multiplied by the original, produces the identity matrix:

$$\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$$

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Matrix inversion is only possible with [some square matrices](#). If a square matrix is not invertible it is called a singular or non-invertible matrix.

Matrix Inversion (cont'd)

A handy shortcut to find the inverse of a 2×2 matrix, calculate the **determinant** (product of the main diagonal minus the product of the off diagonal) and adjust the elements as such:

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If the determinant is zero, there is no inverse. Calculating inverses of larger (square) matrices is more complicated.

Matrix Inversion

Consider:

$$\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \mathbf{X}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

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Demonstrate that ($\mathbf{X}^{-1} \mathbf{X} = \mathbf{I}$)

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Consider a system of equations:

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Let's call of these matrices and vector.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

Solving Systems of Equations (cont'd)

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$$\mathbf{A}^{-1} = \frac{1}{2 \times 4 - 6 \times 3} \begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

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Now we can solve for \mathbf{v} .

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Let's verify our results

$$2(-10) + 6(5) = 10$$

$$3(-10) + 4(5) = -10$$

Where is linear algebra in political
science?

Regression

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$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \alpha + \beta \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

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or ...

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Matrix-form regression

Supposing that we refer to the coefficient vector $\beta = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$

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It so happens that when we do the matrix calculus to solve for β ...

$$\beta = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

Point being...

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These basic principles apply to all regression modeling

Point being...

These basic principles apply to all regression modeling
and tons of political science boils down to regression modeling

End Day 2