

# Day 3: Calculus 2

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August 21, 2019

# Today

- Today
  - ▶ Partial derivative
  - ▶ Integral
    - ★ Definition
    - ★ Calculation rules
    - ★ Calculation techniques

# Partial Derivative

- What to do when there are more than one input variables?
- **Partial derivative** of a function with more than one input variable is defined as the derivative with respect to one of those variables with others held constant.
- Formally, the partial derivative of  $f(x_1, x_2, \dots, x_n)$  with respect to  $x_i$  is defined as

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_n) = \lim_{h \rightarrow 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{h}$$

- As in the case of derivative of univariate functions, we can take higher-order derivatives.

# Partial Derivative: Example

- Let  $f(x, y) = 3x^2y + 2y^3$ . Then,

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= (3x^2y)' + (2y^3)' \\ &= 3y \cdot (x^2)' = 6xy\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= (3x^2y)' + (2y^3)' \\ &= 3x^2 \cdot (y)' + 2 \cdot (y^3)' = 3x^2 + 6y^2\end{aligned}$$

## Partial Derivative: Example (cont.)

- We can take second-order derivative as

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} f(x, y) \right) \\ &= (6xy)' = 6y \cdot (x)' = 6y\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} f(x, y) \right) \\ &= (3x^2)' + (6y^2)' = 6 \cdot (y^2)' = 12y\end{aligned}$$

## Partial Derivative: Example (cont.)

- We can also take second-order **mixed derivative** as

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x, y) &= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} f(x, y) \right) \\ &= (3x^2)' + (6y^2)' = 3 \cdot (x^2) = 6x\end{aligned}$$

- The order of differentiation does not matter:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x, y) &= \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} f(x, y) \right) \\ &= (6xy)' = 6x \cdot (y)' = 6x\end{aligned}$$

# Partial Derivative: Exercises

- Let  $f(x, y) = 3x^3y^2 - 3xy^2 - \sqrt{y} + x$ . Calculate the following partial derivatives.
  1.  $\frac{\partial f}{\partial x}(x, y)$
  2.  $\frac{\partial f}{\partial y}(x, y)$
  3.  $\frac{\partial^2 f}{\partial x^2}(x, y)$
  4.  $\frac{\partial^2 f}{\partial y^2}(x, y)$
  5.  $\frac{\partial^2 f}{\partial x \partial y}(x, y)$

# Partial Derivative: Application

- Same as the univariate case, partial derivative stand for the **rate of change** of a function.
- Application examples
  - ▶ **Marginal effects**
    - ★ How much does the value of  $y$  change due to one-unit change in  $x$ ?
    - ★ e.g.,  $y = \alpha + \beta_1 x + \beta_2 z + \beta_3 xz$
  - ▶ **Multivariate optimization**

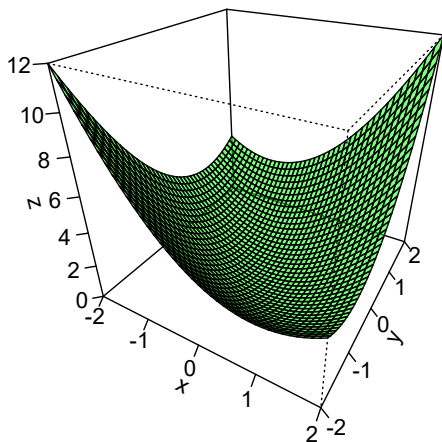


# Partial Derivative: Visual Explanation

- For a point on a graph of a function with more than one input, there are an infinite number of tangent lines.
- Then how do we determine the rate of change of the function?
- Partial derivative of  $y = f(x_1, x_2, \dots, x_n)$  with regard to  $x_i$  computes the rate of change by finding the slope of a tangent line parallel to the  $x_i y$ -plane.
- Let's take a look at an example of a function with two independent variables.

# Partial Derivative: Visual Explanation (cont.)

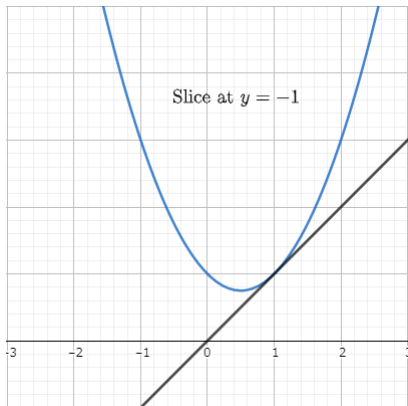
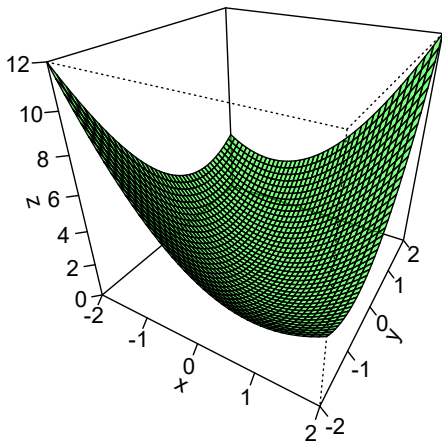
$$f(x, y) = x^2 + xy + y^2$$



- Let  $z = f(x, y) = x^2 + xy + y^2$
- $\frac{\partial f}{\partial x}(x, y) = 2x + y$  describes how slope of the tangent lines parallel to the  $xz$ -plane changes with values of  $x$  and  $y$
- It is same as
  - ▶ first slicing the graph at a specific value of  $y = y_0$
  - ▶ then finding the slope of a tangent line on the sliced plane

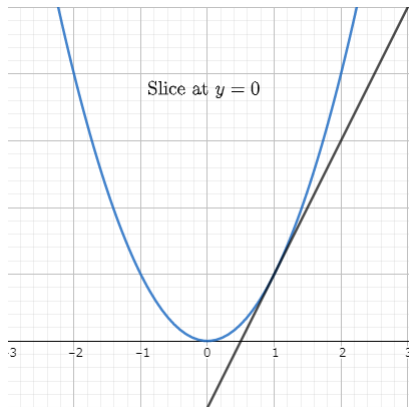
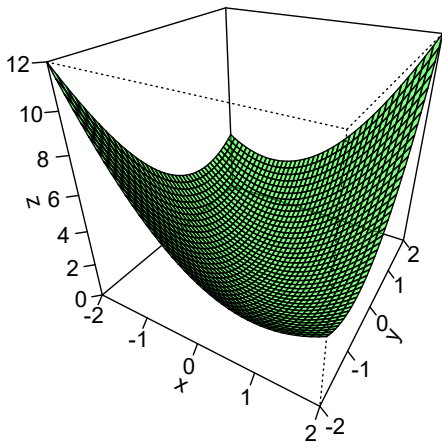
# Partial Derivative: Visual Explanation (cont.)

$$f(x, y) = x^2 + xy + y^2$$



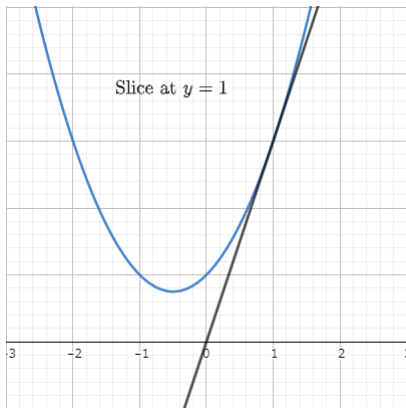
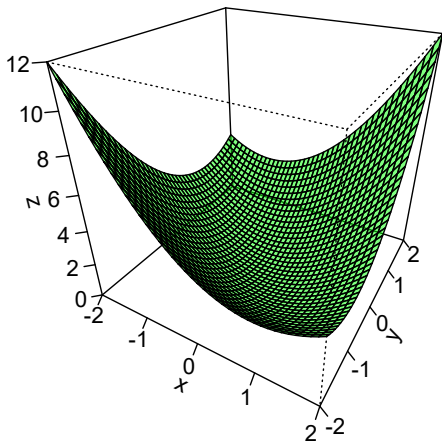
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# Partial Derivative: Visual Explanation (cont.)

$$f(x, y) = x^2 + xy + y^2$$



# Multivariate Optimization

- How do we find local minima/maxima for functions with several inputs?
- Let's again use an example of a function  $f(x, y)$  with two independent variables.
- At a local minimum/maximum, slope of the tangent lines on the  $xz$ -plane and the  $yz$ -plane must be 0!
- Therefore, we need to solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{cases}$$

# Multivariate Optimization (cont.)

- By extending the logic, to find local minima/maxima of a function  $f(x_1, x_2, \dots, x_n)$ , we need to solve the system of equations

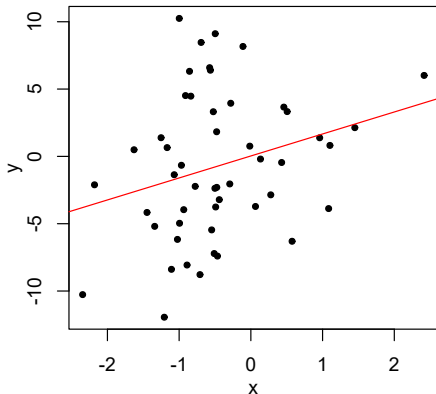
$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, x_2, \dots, x_n) = 0 \\ \frac{\partial f}{\partial x_2}(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

# Multivariate Optimization (cont.)

- Second order condition?
  - ▶ Distinguishing local maxima/minima is much harder in a multivariate case.
  - ▶ We'll briefly touch on this issue tomorrow.

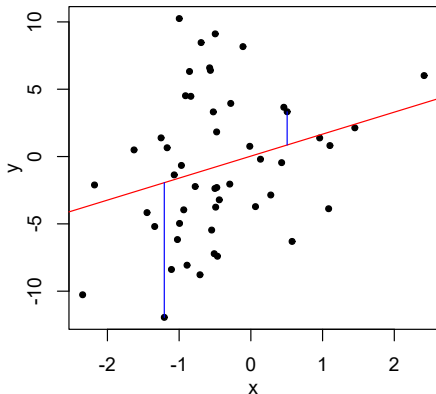


# Multivariate Optimization: Example



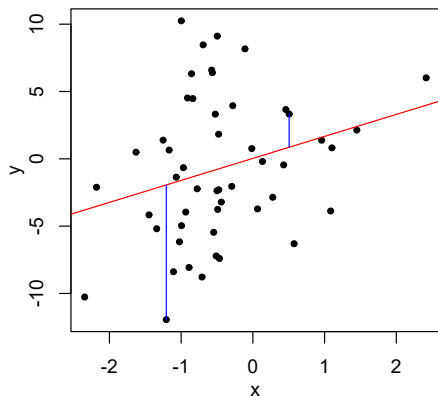
- We often want to know the relationship between one variable ( $x$ ) and another ( $y$ ).
  - ▶ e.g., GRE score and grades in graduate school
  - ▶ Notation
    - ★ Total number of observations/points:  $n$
    - ★ Denote each observation/point as  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$
- Let's consider drawing a line that best fits the data.
  - ▶  $y = a + bx$

# Multivariate Optimization: Example (cont.)



- **Ordinary Least Square (OLS):** find a line which minimizes the sum of squared residuals
- What is residual?
  - ▶ Distance between the line and each point.
- Residual for point  $(x_i, y_i)$ :
  - ▶ A point on a line at  $x = x_i$ :  $(x_i, a + bx_i)$
  - ▶ Therefore, residual for point  $(x_i, y_i)$  is  $y_i - (a + bx_i)$

# Multivariate Optimization: Example (cont.)



- Residual for observation  $i$

$$y_i - (a + bx_i)$$

- Squared residual for observation  $i$

$$\{y_i - (a + bx_i)\}^2$$

- Sum of squared residuals

$$\sum_{i=1}^n \{y_i - (a + bx_i)\}^2$$

# Multivariate Optimization: Example (cont.)

- Let's find the values of  $a$  and  $b$  which minimize the sum of squared residuals.

$$\operatorname{argmin}_{a,b} \sum_{i=1}^n \{y_i - (a + bx_i)\}^2$$

- Following the discussion so far, we need to solve the system of equations

$$\begin{cases} \frac{\partial}{\partial a} \sum_{i=1}^n \{y_i - (a + bx_i)\}^2 = 0 \\ \frac{\partial}{\partial b} \sum_{i=1}^n \{y_i - (a + bx_i)\}^2 = 0 \end{cases}$$

- NB: Here we treat  $a$  and  $b$  as variables and  $x_i$ s and  $y_i$ s as constants!

# Multivariate Optimization: Example (cont.)

- Let's compute the partial derivatives.

$$\begin{aligned}& \frac{\partial}{\partial a} \sum_{i=1}^n \{y_i - (a + bx_i)\}^2 \\&= \frac{\partial}{\partial a} (\{y_1 - (a + bx_1)\}^2 + \{y_2 - (a + bx_2)\}^2 \cdots + \{y_n - (a + bx_n)\}^2) \\&= \frac{\partial}{\partial a} \{y_1 - (a + bx_1)\}^2 + \frac{\partial}{\partial a} \{y_2 - (a + bx_2)\}^2 + \cdots + \frac{\partial}{\partial a} \{y_n - (a + bx_n)\}^2 \\&= -2\{y_1 - (a + bx_1)\} - 2\{y_2 - (a + bx_2)\} - \cdots - 2\{y_n - (a + bx_n)\} \\&= -2(\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \cdots + \{y_n - (a + bx_n)\}) \\&= -2 \sum_{i=1}^n \{y_i - (a + bx_i)\}\end{aligned}$$

# Multivariate Optimization: Example (cont.)

and

$$\begin{aligned}& \frac{\partial}{\partial b} \sum_{i=1}^n \{y_i - (a + bx_i)\}^2 \\&= \frac{\partial}{\partial b} (\{y_1 - (a + bx_1)\}^2 + \{y_2 - (a + bx_2)\}^2 + \cdots + \{y_n - (a + bx_n)\}^2) \\&= \frac{\partial}{\partial b} \{y_1 - (a + bx_1)\}^2 + \frac{\partial}{\partial b} \{y_2 - (a + bx_2)\}^2 + \cdots + \frac{\partial}{\partial b} \{y_n - (a + bx_n)\}^2 \\&= -2x_1 \{y_1 - (a + bx_1)\} - 2x_2 \{y_2 - (a + bx_2)\} - \cdots - 2x_n \{y_n - (a + bx_n)\} \\&= -2(x_1 \{y_1 - (a + bx_1)\} + x_2 \{y_2 - (a + bx_2)\} + \cdots + x_n \{y_n - (a + bx_n)\}) \\&= -2 \sum_{i=1}^n x_i \{y_i - (a + bx_i)\}\end{aligned}$$

# Multivariate Optimization: Example (cont.)

- The system of equations

$$\begin{cases} -2 \sum_{i=1}^n \{y_i - (a + bx_i)\} = 0 \\ -2 \sum_{i=1}^n x_i \{y_i - (a + bx_i)\} = 0 \end{cases}$$

is called the **normal equation**.

- Rest is to solve the system of equations above!
  - ▶ which you are asked to do in GOVT 701...

# Integral

- What is an integral?
  1. **Antiderivative**
    - ★ inverse of a derivative
  2. **Area under the curve**
- **Fundamental theorem of calculus** connects these two explanations.



# Indefinite Integral

- **Antiderivative** or **indefinite integral** of a function  $f(x)$ , often denoted as  $F(x)$ , is a function whose derivative is equal to the original function  $f(x)$ . Formally,

$$F'(x) = f(x)$$

- Indefinite integral of  $f(x)$  is also written as

$$\int f(x)dx,$$

which is equal to  $F(x)$ .

# Indefinite Integral (cont.)

- Example: The function  $F(x) = x^3$  is an antiderivative of  $f(x) = 3x^2$ , since  $F'(x) = f(x)$ .
- However, this is not the only antiderivative.
  - ▶  $F(x) = x^3 + 1$
  - ▶  $F(x) = x^3 + 100$
  - ▶  $F(x) = x^3 - 23$
  - ▶  $F(x) = x^3 - 1108564$
  - ▶ ...
- Since the derivative of a constant is 0, there are infinite number of antiderivatives!

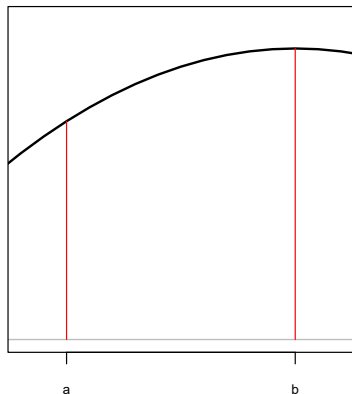
# Indefinite Integral (cont.)

- To represent this situation, we add an arbitrary constant,  $C$ , known as the *constant of integration*.
- Example: antiderivative of  $f(x) = 3x^2$  is

$$F(x) = x^3 + C$$

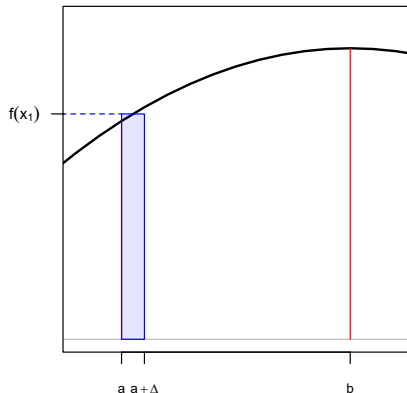
- ▶ All values of  $F(x)$  can be obtained by changing the values of  $C$ .

# Definite Integral



- Suppose we want to find the area under the curve defined by a function  $f(x)$  and some interval  $x \in [a, b]$ .
- Let's approximate the area using rectangles.
  - ▶ First partition the interval into  $n$  regions.
  - ▶ Let the length of each subinterval be  $\Delta x$
  - ▶ Denote the mid-points of subintervals as  $x_1, x_2, \dots, x_n$

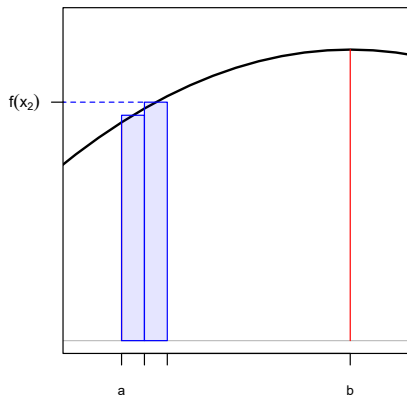
# Definite Integral (cont.)



- Area of first rectangle:  
 $f(x_1)\Delta x$
- Area of second rectangle:  
 $f(x_2)\Delta x$
- ...
- Add area of all the rectangles  
(**Riemann sum**):

$$\sum_{i=1}^n f(x_i)\Delta x$$

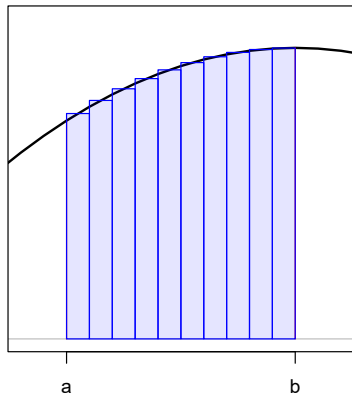
# Definite Integral (cont.)



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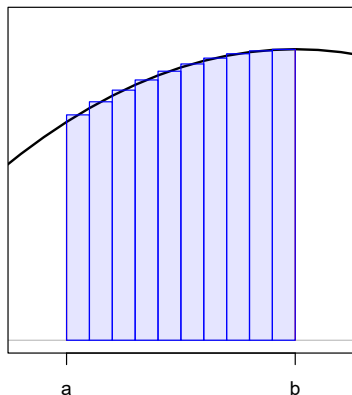
# Definite Integral (cont.)



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# Definite Integral (cont.)

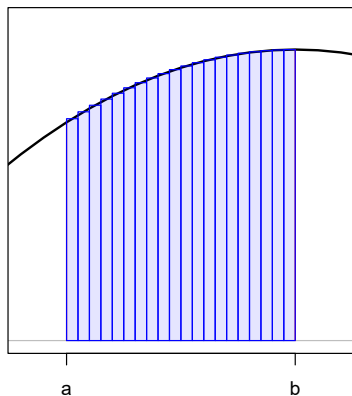


- By making  $\Delta x$  smaller and smaller (i.e., partitioning the interval  $[a, b]$  into smaller regions)...
- We can approximate the area closer and closer!
- Formally stated, we can find the area under the curve by taking the limit of the Riemann sum as  $\Delta x \rightarrow 0$ :

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x$$



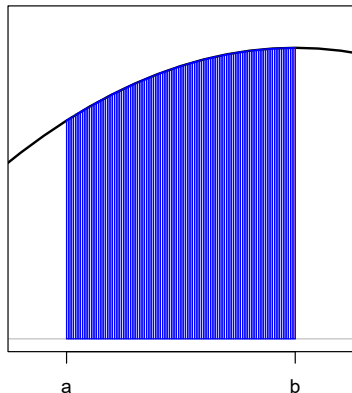
# Definite Integral (cont.)



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# Definite Integral (cont.)

- **Riemann integral/Definite integral** of function  $f(x)$  from  $a$  to  $b$  is defined as the limit of Riemann sum of  $f(x)$  on that interval as  $\Delta x \rightarrow 0$  and denoted as

$$\int_a^b f(x)dx = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i)\Delta x$$

# Fundamental Theorem of Calculus

- (First) **Fundamental Theorem of Calculus:** Let  $f(x)$  be a function on an interval  $[a, b]$ , and  $F(x)$  be a function defined on  $[a, b]$  by

$$F(x) = \int_a^x f(x)dx$$

Then,

$$F'(x) = f(x)$$

for all  $x \in (a, b)$ .

- The definite integral of  $f(x)$  is one of its antiderivatives!

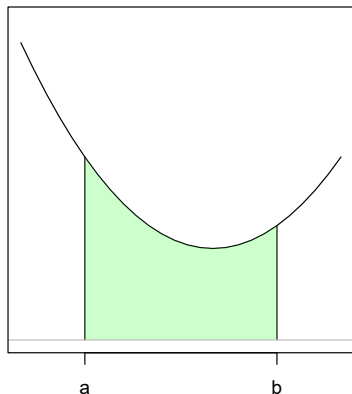
# Fundamental Theorem of Calculus (cont.)

- (Second) **Fundamental Theorem of Calculus:** Let  $f(x)$  be a function on an interval  $[a, b]$ , and  $F(x)$  be an antiderivative of  $f(x)$  on  $[a, b]$ . Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

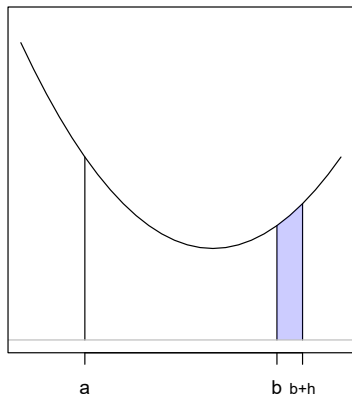
- Therefore, we can calculate the definite integral (i.e., area under the curve) from  $b$  and  $a$  by subtracting the indefinite integral evaluated at  $a$  from the indefinite integral evaluated at  $b$ !

# Fundamental Theorem of Calculus (cont.)



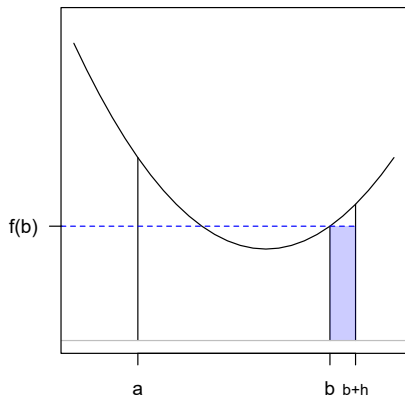
- But why?
- Let  $S(b)$  be a function representing the area under the curve defined by the function  $f(x)$  and interval  $x \in [a, b]$ .

# Fundamental Theorem of Calculus (cont.)



- Let's consider how much the area changes if we move rightward from  $b$  by  $h$ .
- The area shown in blue can be computed as  $S(b+h) - S(b)$

# Fundamental Theorem of Calculus (cont.)

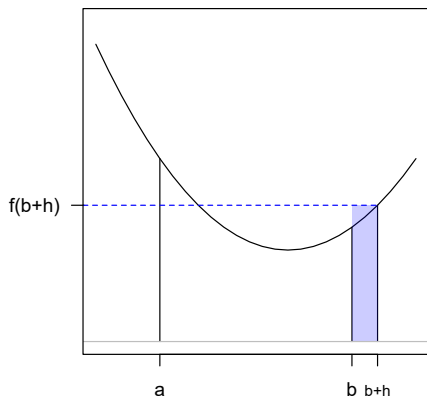


- Here let's assume that  $f(x)$  is increasing around  $b$ .
- Then, as we can see from the figure,  $S(b+h) - S(b)$  is larger than  $f(b)h$  and smaller than  $f(b+h)h$ . Dividing each by  $h$ , we get

$$f(b) < \frac{S(b+h) - S(b)}{h} < f(b+h)$$



# Fundamental Theorem of Calculus (cont.)



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$$f(b) < \frac{S(b+h) - S(b)}{h} < f(b+h)$$

# Fundamental Theorem of Calculus (cont.)

- Take the limit of each as  $h \rightarrow 0$ .

$$\lim_{h \rightarrow 0} f(b) < \lim_{h \rightarrow 0} \frac{S(b+h) - S(b)}{h} < \lim_{h \rightarrow 0} f(b+h)$$

- Since  $\lim_{h \rightarrow 0} f(b) = \lim_{h \rightarrow 0} f(b+h) = f(b)$ ,

$$\lim_{h \rightarrow 0} \frac{S(b+h) - S(b)}{h} = f(b)$$

- From the definition of derivative,

$$S'(b) = f(b)$$

# Fundamental Theorem of Calculus (cont.)

- Find the antiderivative of both:

$$S(b) = F(b) + C$$

- Since  $S(a) = 0$ ,  $C = -F(a)$ . Therefore,

$$S(b) = F(b) - F(a)$$

- We can compute the area under the curve by calculating the differences in indefinite integrals!

# Rules of Integration

- Common rules of integration

1. **Linearity:**  $\int \{\alpha f(x) + \beta g(x)\} dx = \alpha \int f(x) dx + \beta \int g(x) dx$

2. **Reverse power rule:**  $\int x^n dx = \frac{1}{n+1} x^{n+1} + C \quad (n \neq -1)$

3. Exception to 2. when  $n = -1$ :  $\int \frac{1}{x} dx = \log |x| + C$

4. **Exponential rule:**  $\int a^x dx = \frac{a^x}{\log a} + C$

5. Special case of 4.:  $\int e^x dx = e^x + C$

# Rules of Integration (cont.)

- To calculate the definite integral  $\int_a^b f(x)dx$ :
  1. Find the indefinite integral  $F(x)$
  2. Calculate  $F(b) - F(a)$ 
    - ▶ We often write the second step as  $\int_a^b f(x)dx = F(x)\Big|_a^b$
- Common rules for definite integral
  1.  $\int_a^a f(x)dx = 0$
  2.  $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  3.  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$  ( $c \in [a, b]$ )
  4.  $\int_a^b \{\alpha f(x) + \beta g(x)\} dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

# Rules of Integration: Example

- Find the antiderivative of  $f(x) = 3x^3 - 4x^2 + x + 6$

► Answer:

$$\begin{aligned}\int f(x)dx &= \int (3x^3 - 4x^2 + x + 6)dx \\&= 3 \int x^3 dx - 4 \int x^2 dx + \int x dx + 6 \int dx \\&= \frac{3}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 6x + C\end{aligned}$$

# Rules of Integration: Example (cont.)

- Calculate the definite integral  $\int_{-1}^4 (3x^2 + 2x - 1)dx$

► Answer:

$$\begin{aligned}& \int_{-1}^4 (3x^2 + 2x - 1)dx \\&= 3 \int_{-1}^4 x^2 dx + 2 \int_{-1}^4 x dx - \int_{-1}^4 dx \\&= 3 \cdot \frac{1}{3} x^3 \Big|_{-1}^4 + 2 \cdot \frac{1}{2} x^2 \Big|_{-1}^4 - x \Big|_{-1}^4 \\&= \{64 - (-1)\} + (16 - 1) - \{4 - (-1)\} \\&= 65 + 15 - 5 = 75.\end{aligned}$$

# Rules of Integration: Exercises

1. Calculate the following integrals.

1.1  $\int (4x^5 + 2x^2 + 5)dx$

1.2  $\int (\sqrt{x} + 2x^3)dx$

1.3  $\int (3x^{\frac{3}{2}} + 4^x)dx$

1.4  $\int_{-2}^2 (3x^2 + x)dx$

1.5  $\int_2^4 (3x^2 + x + 5)dx$

1.6  $\int_1^{16} (5x^{\frac{3}{2}} - 2x^{-\frac{5}{4}})dx$

2. Find the area surrounded by  $f(x) = x^2 - 3x + 3$  and  $g(x) = -x^2 + 5x + 13$ .



# Integration Techniques

1. Integration by parts
2. Integration by substitution
3. L'Hopital's rule

# Integration by Parts

- If the integrand  $h(x)$  can be represented as the product of two functions,  $f(x)$  and  $g'(x)$ , then

$$\int h(x)dx = \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- In the case of definite integral:

$$\int_a^b h(x)dx = \int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

- This formula can easily be derived from the product rule of derivative calculation.

# Integration by Parts: Example

- Question: find the indefinite integral of  $\log(x)$ .

► Answer: As  $\log(x) = \log(x) \cdot (x)'$ ,

$$\begin{aligned}\int \log(x) dx &= \int \log(x) \cdot (x)' dx \\ &= x \log(x) - \int \frac{1}{x} \cdot x dx \\ &= x \log(x) - \int dx \\ &= x \log(x) - x + C.\end{aligned}$$

# Integration by Substitution

- If we can find a function  $t(x)$  and write the integrand  $f(x)$  as

$$f(x) = f(t) \frac{dt}{dx},$$

we can apply the integration by substitution formula

$$\int f(x) dx = \int f(t) \frac{dt}{dx} dx = \int f(t) dt$$

- In the case of definite integral: if  $t$  moves from  $\alpha$  to  $\beta$  when  $x$  moves from  $a$  to  $b$ , then

$$\int_a^b f(x) dx = \int_a^b f(t) \frac{dt}{dx} dx = \int_\alpha^\beta f(t) dt$$

- ▶ Note that the interval of integration can change due to the variable transformation.

# Integration by Substitution: Example

- Question: find the indefinite integral of  $f(x) = 18x^2\sqrt[3]{6x^3 - 7}$ 
  - ▶ Answer: Let  $t = 6x^3 - 7$ . Then,  $\frac{dt}{dx} = 18x^2$ . Therefore,

$$\begin{aligned}\int f(x)dx &= \int 18x^2\sqrt[3]{6x^3 - 7}dx \\&= \int \sqrt[3]{t} \frac{dt}{dx} dx \\&= \int t^{\frac{1}{3}} dt \\&= \frac{3}{4}t^{\frac{4}{3}} + C \\&= \frac{3}{4}(6x^3 - 7)^{\frac{4}{3}} + C\end{aligned}$$

# Integration by Substitution: Summary of Steps

1. Identify some part of  $f(x)$  which can be simplified by substituting in a single variable  $t$  (which is a function of  $x$ )
2. Compute  $\frac{dt}{dx}$ , and reexpress  $f(x)$  using  $t$  and  $\frac{dt}{dx}$
3. Solve the indefinite integral
4. (For indefinite integration) Substitute back in for  $x$
5. (For definite integration) Determine the new interval of integration  $[\alpha, \beta]$ , and evaluate the antiderivative at the boundary points.

# Integration by Substitution: Another Example

- Question: find the definite integral  $\int_0^1 x\sqrt{x^2+1}dx$
- ▶ Answer: Let  $t = x^2 + 1$ . Then,  $\frac{dt}{dx} = 2x$ . Also, when  $x$  moves from 0 to 1,  $t$  moves from 1 to 2. Therefore,

$$\begin{aligned} & \int_0^1 x\sqrt{x^2+1}dx \\ &= \int_0^1 \sqrt{t} \cdot \frac{1}{2} \frac{dt}{dx} dx \\ &= \frac{1}{2} \int_1^2 \sqrt{t} dt \\ &= \frac{1}{2} \cdot \frac{2}{3} t^{\frac{3}{2}} \Big|_1^2 \\ &= \frac{1}{3} (2\sqrt{2} - 1) \end{aligned}$$

# L'Hopital's Rule

- Let's calculate the definite integral  $\int_0^\infty xe^{-x}dx$ .

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= \int_0^\infty x(-e^{-x})'dx \\ &= -(xe^{-x}|_0^\infty) - \int_0^\infty (x)' \cdot (-e^{-x})dx \\ &= -(xe^{-x}|_0^\infty) + \int_0^\infty e^{-x}dx\end{aligned}$$

- How can we evaluate  $xe^{-x}$  at  $x \rightarrow \infty$ ?
  - ▶  $\lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty}??$
  - ▶ In such cases, L'Hopital's rule help you circumvent the problem.



# L'Hopital's Rule (cont.)

- **L'Hopital's rule:** Let functions  $f(x)$  and  $g(x)$  be differentiable at an open interval close to  $c$ . If  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$  or  $\pm\infty$ , and  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

- Applying the L'Hopital's rule, we can see that

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{x'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$$

# Integration Techniques: Exercises

- Calculate the following integrals.

1.  $\int \frac{1}{1+\exp(-x)} dx$

2.  $\int x \exp(x) dx$

3.  $\int \frac{2x}{5x^2-7} dx$

4.  $\int x^2(x^3 + 15)^{\frac{3}{2}} dx$

5.  $\int_0^1 x \exp(x^2) dx$

6.  $\int_1^2 x \log x dx$

7.  $\int_0^1 \log(1 + \sqrt{x}) dx$  (Hint: set  $t = 1 + \sqrt{x}$  and perform integration by substitution)

# Tomorrow

- Problem set 3  $\rightarrow$  review in the morning
- Tomorrow
  - ▶ Vector and matrix algebra
  - ▶ Vector/matrix notations for multivariate calculus
  - ▶ Geometric meanings of vector/matrix algebra
  - ▶ Moore & Siegel, Chapters 12, 13, 15.2.2, 15.2.4, & 16.1