Problem Set 4

August 22, 2019

Please use `R` to do questions 2 - 4.

1. Given the following matrices, perform the calculations below.

$$\boldsymbol{A} = \begin{pmatrix} 5 & 1 & 2 \\ 6 & 2 & 3 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 3 & 4 & 5 \\ -2 & -3 & 6 \end{pmatrix},\ \boldsymbol{C} = \begin{pmatrix} 1 & 2 \\ -5 & 3 \\ -3 & 1 \end{pmatrix},\ \boldsymbol{D} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

   (a) $\boldsymbol{A} - \boldsymbol{B}$
   (b) $\boldsymbol{A} + 5\boldsymbol{B}$
   (c) $\boldsymbol{B}' - \boldsymbol{C}$
   (d) $\boldsymbol{D}\boldsymbol{A}$
   (e) $\boldsymbol{B}\boldsymbol{A}'$

2. Given the following matrices, perform the calculations below.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & -2 & 1 & 1 \\ 3 & 1 & 2 & 0 \end{pmatrix},\ \boldsymbol{C} = \begin{pmatrix} 2 & 1 & -3 & 9 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

   (a) $\boldsymbol{A} - (\boldsymbol{B} + \boldsymbol{C})$
   (b) $(\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C})'$
   (c) $\boldsymbol{B}\boldsymbol{C}'$
   (d) $\boldsymbol{B}'\boldsymbol{C}$

3. Find the (i) determinants and (ii) inverses of the following matrices.

   (a)

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2 & -4 \\ -1 & -5 & 1 \\ 3 & 2 & 3 \end{pmatrix}$$

(b)

$$\boldsymbol{B} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

4. Solve the following systems of equations.

   (a)

$$\begin{cases} 2x + 3y - z = -3 \\ -x - y + 2z = 3 \\ x + z = 0 \end{cases}$$

   (b)

$$\begin{cases} x - y + z = 3 \\ -x + y + z = -1 \\ x + y - z = -1 \end{cases}$$

5. Let $\boldsymbol{y} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}'$ be the vector of dependent variable, $\boldsymbol{X} = \begin{pmatrix} 1 & x_{11} & x_{12} \dots & x_{1k} \\ 1 & x_{21} & x_{22} \dots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} \dots & x_{nk} \end{pmatrix}$ be the matrix of independent variables, and $\boldsymbol{b} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_k \end{pmatrix}'$ be the vector of regression coefficients, and $\boldsymbol{e} = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \end{pmatrix}'$ be the vector of residuals. Then, as we covered in class, the system of regression equations

$$\begin{cases} y_1 = b_0 + b_1 x_{11} + \cdots + b_k x_{1k} + e_1 \\ y_2 = b_0 + b_1 x_{21} + \cdots + b_k x_{2k} + e_2 \\ \quad \vdots \\ y_n = b_0 + b_1 x_{n1} + \cdots + b_k x_{nk} + e_n \end{cases}$$

   can be concisely written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{b} + \boldsymbol{e}$$

   Now, let's find the coefficients $\boldsymbol{b}$ that best fit the data.

   (a) Check that the sum of squared residuals $\sum_{i=1}^{n} e_i^2$ is equal to the dot product of vector $\boldsymbol{e}$ with itself (i.e., $\boldsymbol{e}'\boldsymbol{e}$).

   (b) Take the derivative of the sum of squared residuals with respect to $\boldsymbol{b}$ and represent the results with $\boldsymbol{y}$, $\boldsymbol{X}$, and $\boldsymbol{b}$.

   (c) Set the derivative computed in (b) to $\boldsymbol{0}$ and solve the system of equations for $\boldsymbol{b}$.