Day 3: Calculus 2

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Today

- Today
  - Partial derivative
  - Integral
    - Definition
    - Calculation rules
    - Calculation techniques

Partial Derivative

- What to do when there are more than one input variables?

- **Partial derivative** of a function with more than one input variable is defined as the derivative with respect to one of those variables with others held constant.

- Formally, the partial derivative of $f(x_1, x_2, \cdots, x_n)$ with respect to x_i is defined as

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \cdots, x_n) = \lim_{h \to 0} \frac{f(x_1, x_2, \cdots, x_i + h, \cdots, x_n) - f(x_1, x_2, \cdots, x_i, \cdots, x_n)}{h}$$

- As in the case of derivative of univariate functions, we can take higher-order derivatives.

Partial Derivative: Example

- Let $f(x,y) = 3x^2y + 2y^3$. Then,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= (3x^2y)' + (2y^3)' \\ &= 3y \cdot (x^2)' = 6xy \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) &= (3x^2y)' + (2y^3)' \\ &= 3x^2 \cdot (y)' + 2 \cdot (y^3)' = 3x^2 + 6y^2 \end{aligned}$$

Partial Derivative: Example (cont.)

- We can take second-order derivative as

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x, y)\right) \\ &= (6xy)' = 6y \cdot (x)' = 6y\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x, y)\right) \\ &= (3x^2)' + (6y^2)' = 6 \cdot (y^2)' = 12y\end{aligned}$$

Partial Derivative: Example (cont.)

- We can also take second-order **mixed derivative** as

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial y} f(x,y)\Big) \\ &= (3x^2)' + (6y^2)' = 3 \cdot (x^2) = 6x\end{aligned}$$

- The order of differentiation does not matter:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial y}\Big(\frac{\partial}{\partial x} f(x,y)\Big) \\ &= (6xy)' = 6x \cdot (y)' = 6x\end{aligned}$$

Partial Derivative: Exercises

- Let $f(x, y) = 3x^3y^2 - 3xy^2 - \sqrt{y} + x$. Calculate the following partial derivatives.
  1. $\frac{\partial f}{\partial x}(x, y)$
  2. $\frac{\partial f}{\partial y}(x, y)$
  3. $\frac{\partial^2 f}{\partial x^2}(x, y)$
  4. $\frac{\partial^2 f}{\partial y^2}(x, y)$
  5. $\frac{\partial^2 f}{\partial x \partial y}(x, y)$

Partial Derivative: Application

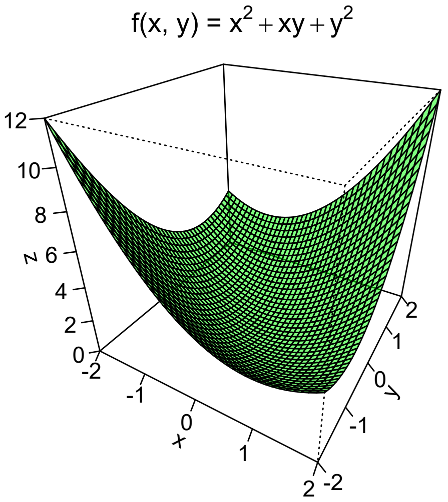
- Same as the univariate case, partial derivative stand for the **rate of change** of a function.

- Application examples
  - **Marginal effects**
    - How much does the value of y change due to one-unit change in x?
    - e.g., $y = \alpha + \beta_1 x + \beta_2 z + \beta_3 xz$
  - **Multivariate optimization**

Partial Derivative: Visual Explanation

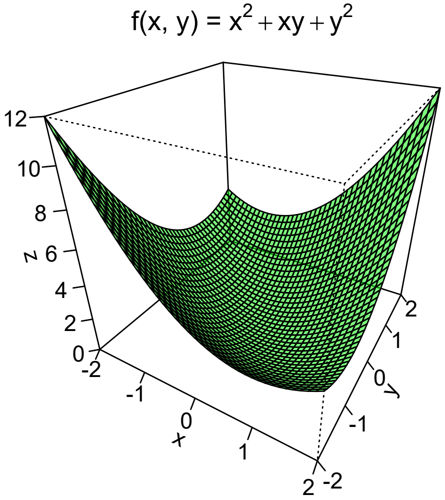
- For a point on a graph of a function with more than one input, there are an infinite number of tangent lines.
- Then how do we determine the rate of change of the function?
- Partial derivative of $y = f(x_1, x_2, \cdots, x_n)$ with regard to x_i computes the rate of change by finding the slope of a tangent line parallel to the $x_i y$-plane.
- Let's take a look at an example of a function with two independent variables.

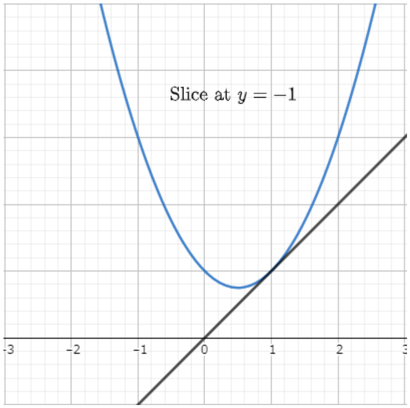
Partial Derivative: Visual Explanation (cont.)



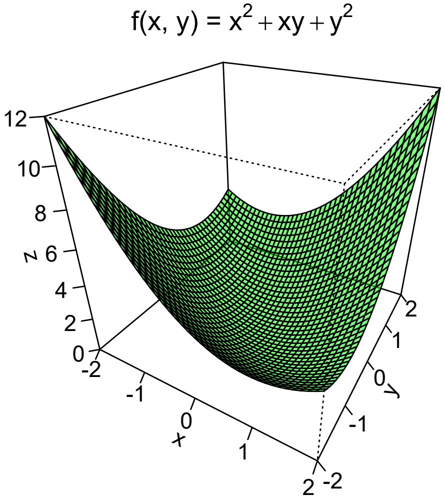
- Let
  $z = f(x, y) = x^2 + xy + y^2$
- $\frac{\partial f}{\partial x}(x, y) = 2x + y$ describes how slope of the tangent lines paralell to the xz-plane changes with values of x and y
- It is same as
  - first slicing the graph at a specific value of $y = y_0$
  - then finding the slope of a tangent line on the sliced plane

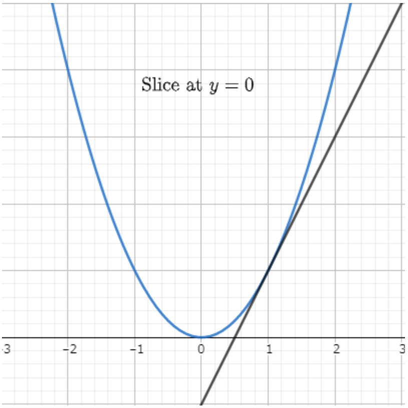
Partial Derivative: Visual Explanation (cont.)



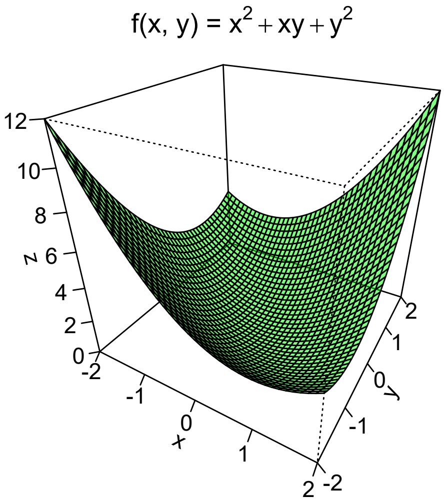


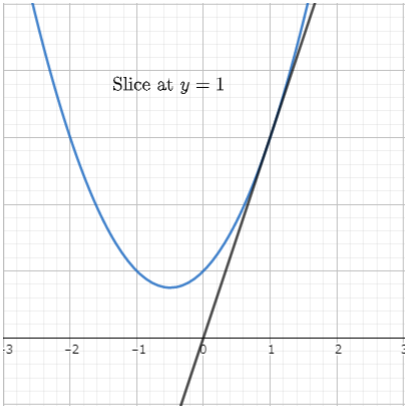
Partial Derivative: Visual Explanation (cont.)





Partial Derivative: Visual Explanation (cont.)





Multivariate Optimization

- How do we find local minima/maxima for functions with several inputs?
- Let's again use an example of a function $f(x, y)$ with two independent variables.
- At a local minimum/maximum, slope of the tangent lines on the xz-plane and the yz-place must be 0!
- Therefore, we need to solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{cases}$$

Multivariate Optimization (cont.)

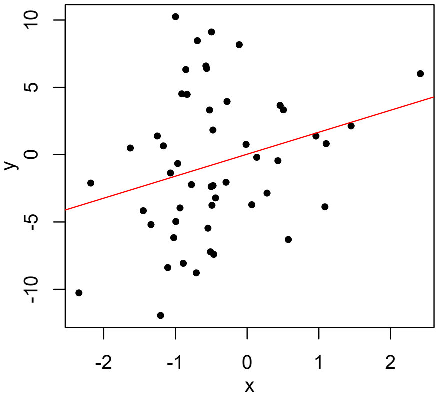
- By extending the logic, to find local minima/maxima of a function $f(x_1, x_2, \cdots, x_n)$, we need to solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial x_1}(x_1, x_2, \cdots, x_n) = 0 \\ \frac{\partial f}{\partial x_2}(x_1, x_2, \cdots, x_n) = 0 \\ \quad\vdots \\ \frac{\partial f}{\partial x_i}(x_1, x_2, \cdots, x_n) = 0 \\ \quad\vdots \\ \frac{\partial f}{\partial x_n}(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

Multivariate Optimization (cont.)

- Second order condition?
  - Distinguishing local maxima/minima is much harder in a multivariate case.
  - We'll briefly touch on this issue tomorrow.

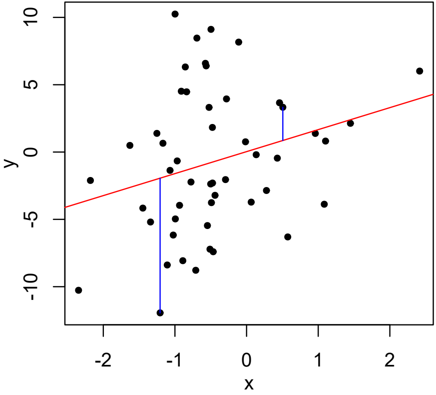
Multivariate Optimization: Example



- We often want to know the relationship between one variable (x) and another (y).
  - e.g., GRE score and grades in graduate school
  - Notation
    - Total number of observations/points: n
    - Denote each observation/point as $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

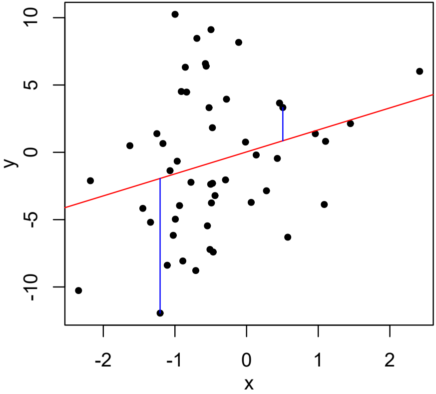
- Let's consider drawing a line that best fits the data.
  - $y = a + bx$

Multivariate Optimization: Example (cont.)



- **Ordinary Least Square (OLS)**: find a line which minimizes the sum of squared residuals
- What is residual?
  - Distance between the line and each point.
- Residual for point (x_i, y_i):
  - A point on a line at $x = x_i$: $(x_i, a + bx_i)$
  - Therefore, residual for point (x_i, y_i) is $y_i - (a + bx_i)$

Multivariate Optimization: Example (cont.)



- Residual for observation i

$$y_i - (a + bx_i)$$

- Squared residual for observation i

$$\{y_i - (a + bx_i)\}^2$$

- Sum of squared residuals

$$\sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2$$

Multivariate Optimization: Example (cont.)

- Let's find the values of a and b which minimize the sum of squared residuals.

$$\operatorname*{argmin}_{a,b} \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2$$

- Following the discussion so far, we need to solve the system of equations

$$\begin{cases} \frac{\partial}{\partial a} \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2 = 0 \\ \frac{\partial}{\partial b} \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2 = 0 \end{cases}$$

- NB: Here we treat a and b as variables and x_is and y_is and as constant!

Multivariate Optimization: Example (cont.)

- Let's compute the partial derivatives.

$$
\begin{aligned}
& \frac{\partial}{\partial a} \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2 \\
= \; & \frac{\partial}{\partial a} \Big(\{y_1 - (a + bx_1)\}^2 + \{y_2 - (a + bx_2)\}^2 \cdots + \{y_n - (a + bx_n)\}^2 \Big) \\
= \; & \frac{\partial}{\partial a} \{y_1 - (a + bx_1)\}^2 + \frac{\partial}{\partial a} \{y_2 - (a + bx_2)\}^2 + \cdots + \frac{\partial}{\partial a} \{y_n - (a + bx_n)\}^2 \\
= \; & -2 \{y_1 - (a + bx_1)\} - 2 \{y_2 - (a + bx_2)\} - \cdots - 2 \{y_n - (a + bx_n)\} \\
= \; & -2 \Big(\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \cdots + \{y_n - (a + bx_n)\} \Big) \\
= \; & -2 \sum_{i=1}^{n} \{y_i - (a + bx_i)\}
\end{aligned}
$$

Multivariate Optimization: Example (cont.)

and

$$
\begin{aligned}
& \frac{\partial}{\partial b} \sum_{i=1}^{n} \{y_i - (a + bx_i)\}^2 \\
= \; & \frac{\partial}{\partial b} \big(\{y_1 - (a + bx_1)\}^2 + \{y_2 - (a + bx_2)\}^2 \cdots + \{y_n - (a + bx_n)\}^2 \big) \\
= \; & \frac{\partial}{\partial b} \{y_1 - (a + bx_1)\}^2 + \frac{\partial}{\partial b} \{y_2 - (a + bx_2)\}^2 + \cdots + \frac{\partial}{\partial b} \{y_n - (a + bx_n)\}^2 \\
= \; & -2x_1 \{y_1 - (a + bx_1)\} - 2x_2 \{y_2 - (a + bx_2)\} - \cdots - 2x_n \{y_n - (a + bx_n)\} \\
= \; & -2\big(x_1 \{y_1 - (a + bx_1)\} + x_2 \{y_2 - (a + bx_2)\} + \cdots + x_n \{y_n - (a + bx_n)\}\big) \\
= \; & -2 \sum_{i=1}^{n} x_i \{y_i - (a + bx_i)\}
\end{aligned}
$$

Multivariate Optimization: Example (cont.)

- The system of equations

$$\begin{cases} -2\sum_{i=1}^{n}\{y_i-(a+bx_i)\}=0 \\ -2\sum_{i=1}^{n}x_i\{y_i-(a+bx_i)\}=0 \end{cases}$$

  is called the **normal equation**.

- Rest is to solve the system of equations above!
  - which you are asked to do in GOVT 701...

Integral

- What is an integral?
    1. **Antiderivative**
        - $\star$ inverse of a derivative
    2. **Area under the curve**

- **Fundamental theorem of calculus** connects these two explanations.

Indefinite Integral

- **Antiderivative** or **indefinite integral** of a function $f(x)$, often denoted as $F(x)$, is a function whose derivative is equal to the original function $f(x)$. Formally,

$$F'(x) = f(x)$$

- Indefinite integral of $f(x)$ is also written as

$$\int f(x)dx,$$

which is equal to $F(x)$.

Indefinite Integral (cont.)

- Example: The function $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, since $F'(x) = f(x)$.

- However, this is not the only antiderivative.
  - $F(x) = x^3 + 1$
  - $F(x) = x^3 + 100$
  - $F(x) = x^3 - 23$
  - $F(x) = x^3 - 1108564$
  - $\cdots$

- Since the derivative of a constant is 0, there are infinite number of antiderivatives!

Indefinite Integral (cont.)

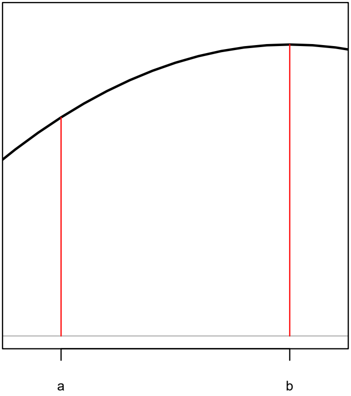
- To represent this situation, we add an arbitrary constant, C, known as the *constant of integration*.

- Example: antiderivative of $f(x) = 3x^2$ is

$$F(x) = x^3 + C$$

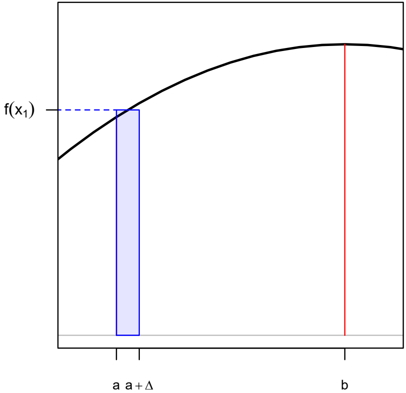
  - All values of $F(x)$ can be obtained by changing the values of C.

Definite Integral



- Suppose we want to find the area under the curve defined by a function $f(x)$ and some interval $x \in [a, b]$.
- Let's approximate the area using rectangles.
  - First partition the interval into n regions.
  - Let the length of each subinterval be Δx
  - Denote the mid-points of subintervals as $x_1, x_2, \cdots, x_n$

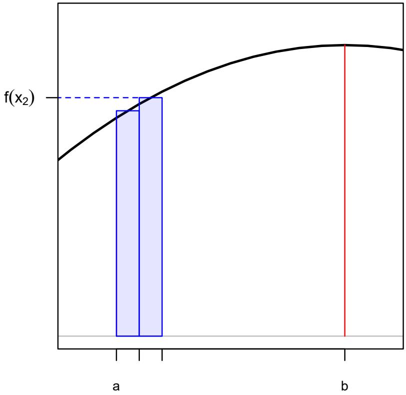
Definite Integral (cont.)



- Area of first rectangle: $f(x_1)\Delta x$
- Area of second rectangle: $f(x_2)\Delta x$
- $\cdots$
- Add area of all the rectangles (**Riemann sum**):

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Definite Integral (cont.)



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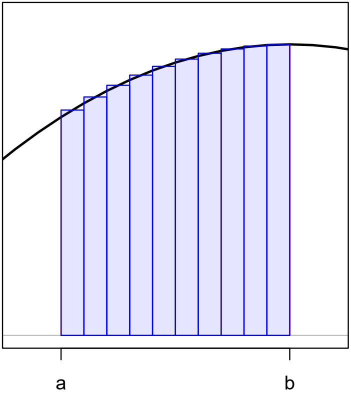

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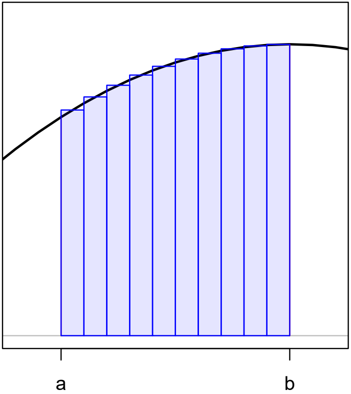
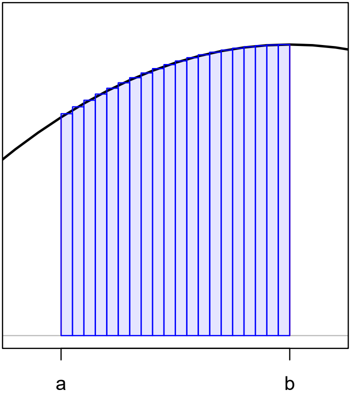
Definite Integral (cont.)



- By making Δx smaller and smaller (i.e., partitioning the interval $[a, b]$ into smaller regions)...
- We can approximate the area closer and closer!
- Formally stated, we can find the area under the curve by taking the limit of the Riemann sum as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

Definite Integral (cont.)



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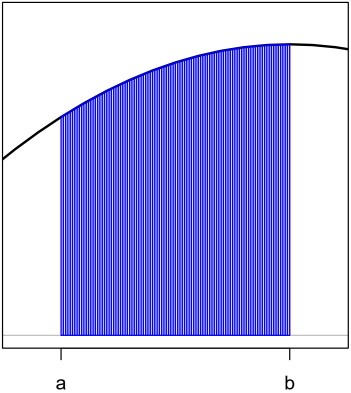
Definite Integral (cont.)



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Definite Integral (cont.)

- **Riemann integral/Definite integral** of function $f(x)$ from a to b is defined as the limit of Riemann sum of $f(x)$ on that interval as $\Delta x \to 0$ and denoted as

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$

Fundamental Theorem of Calculus

- (First) **Fundamental Theorem of Calculus**: Let $f(x)$ be a function on an interval $[a, b]$, and $F(x)$ be a function defined on $[a, b]$ by

  $$F(x) = \int_a^x f(x)dx$$

  Then,

  $$F'(x) = f(x)$$

  for all $x \in (a, b)$.

- The definite integral of $f(x)$ is one of its antiderivatives!

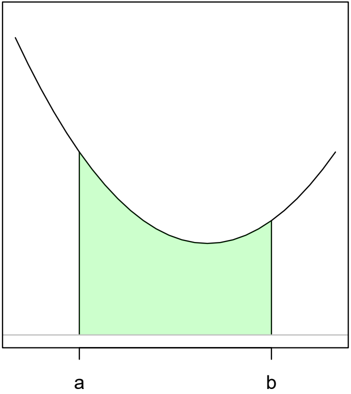
Fundamental Theorem of Calculus (cont.)

- (Second) **Fundamental Theorem of Calculus**: Let $f(x)$ be a function on an interval $[a, b]$, and $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

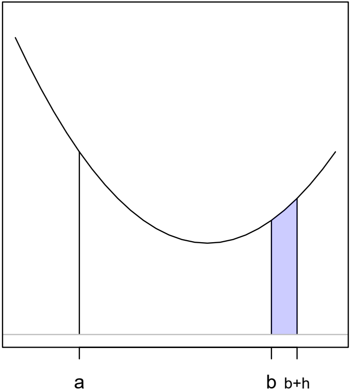
- Therefore, we can calcuate the definite integral (i.e., area under the curve) from b and a by subtracting the indefinite integral evaluated at a from the indefinite integral evaluated at b!

Fundamental Theorem of Calculus (cont.)



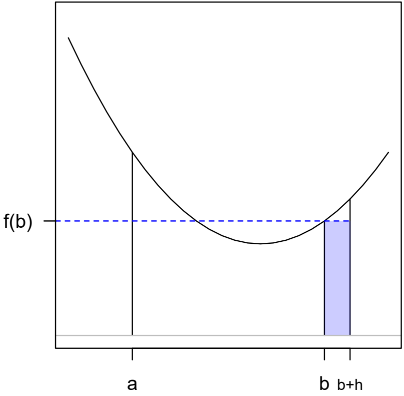
- But why?
- Let $S(b)$ be a function representing the area under the curve defined by the function $f(x)$ and interval $x \in [a, b]$.

Fundamental Theorem of Calculus (cont.)



- Let's consider how much the area changes if we move rightward from b by h.
- The area shown in blue can computed as $S(b+h) - S(b)$

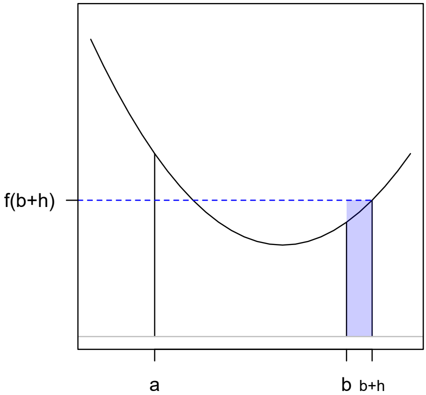
Fundamental Theorem of Calculus (cont.)



- Here let's assume that $f(x)$ is increasing around b.
- Then, as we can see from the figure, $S(b+h) - S(b)$ is larger than $f(b)h$ and smaller than $f(b+h)h$. Dividing each by h, we get

$$f(b) < \frac{S(b+h) - S(b)}{h} < f(b+h)$$

Fundamental Theorem of Calculus (cont.)



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Fundamental Theorem of Calculus (cont.)

- Take the limit of each as $h \to 0$.

$$\lim_{h \to 0} f(b) < \lim_{h \to 0} \frac{S(b+h) - S(b)}{h} < \lim_{h \to 0} f(b+h)$$

- Since $\lim_{h \to 0} f(b) = \lim_{h \to 0} f(b+h) = f(b)$,

$$\lim_{h \to 0} \frac{S(b+h) - S(b)}{h} = f(b)$$

- From the definition of derivative,

$$S'(b) = f(b)$$

Fundamental Theorem of Calculus (cont.)

- Find the antiderivative of both:

$$S(b) = F(b) + C$$

- Since $S(a) = 0$, $C = -F(a)$. Therefore,

$$S(b) = F(b) - F(a)$$

- We can compute the area under the curve by calculating the differences in indefinite integrals!

Rules of Integration

- Common rules of integration
  1. **Linearity**: $\int \{\alpha f(x) + \beta g(x)\} dx = \alpha \int f(x)dx + \beta \int g(x)dx$
  2. **Reverse power rule**: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C \ (n \neq -1)$
  3. Exception to 2. when $n = -1$: $\int \frac{1}{x} dx = \log |x| + C$
  4. **Exponential rule**: $\int a^x dx = \frac{a^x}{\log a} + C$
  5. Special case of 4.: $\int e^x dx = e^x + C$

Rules of Integration (cont.)

- To calculate the definite integral $\int_a^b f(x)dx$:
  1. Find the indefinite integral $F(x)$
  2. Calculate $F(b) - F(a)$

  - We often write the second step as $\int_a^b f(x)dx = F(x)\Big|_a^b$

- Common rules for definite integral
  1. $\int_a^a f(x)dx = 0$
  2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \ (c \in [a, b])$
  4. $\int_a^b \{\alpha f(x) + \beta g(x)\} dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

Rules of Integration: Example

- Find the antiderivative of $f(x) = 3x^3 - 4x^2 + x + 6$
  - Answer:

$$\begin{aligned} \int f(x)dx &= \int (3x^3 - 4x^2 + x + 6)dx \\ &= 3\int x^3 dx - 4\int x^2 dx + \int x dx + 6\int dx \\ &= \frac{3}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 6x + C \end{aligned}$$

Rules of Integration: Example (cont.)

- Calculate the definite integral $\int_{-1}^{4}(3x^2 + 2x - 1)dx$
  - Answer:

$$\begin{aligned}
& \int_{-1}^{4}(3x^2 + 2x - 1)dx \\
= \; & 3\int_{-1}^{4} x^2 dx + 2\int_{-1}^{4} x dx - \int_{-1}^{4} dx \\
= \; & 3 \cdot \frac{1}{3}x^3\Big|_{-1}^{4} + 2 \cdot \frac{1}{2}x^2\Big|_{-1}^{4} - x\Big|_{-1}^{4} \\
= \; & \{64 - (-1)\} + (16 - 1) - \{4 - (-1)\} \\
= \; & 65 + 15 - 5 = 75.
\end{aligned}$$

Rules of Integration: Exercises

1. Calculate the following integrals.
   1.1 $\int (4x^5 + 2x^2 + 5)dx$
   1.2 $\int (\sqrt{x} + 2x^3)dx$
   1.3 $\int (3x^{\frac{3}{2}} + 4^x)dx$
   1.4 $\int_{-2}^{2} (3x^2 + x)dx$
   1.5 $\int_{2}^{4} (3x^2 + x + 5)dx$
   1.6 $\int_{1}^{16} (5x^{\frac{3}{2}} - 2x^{-\frac{5}{4}})dx$
2. Find the area surrounded by $f(x) = x^2 - 3x + 3$ and $g(x) = -x^2 + 5x + 13$.

Integration Techniques

1. Integration by parts
2. Integration by substitution
3. L'Hopital's rule

Integration by Parts

- If the integrand $h(x)$ can be represented as the product of two functions, $f(x)$ and $g'(x)$, then

$$\int h(x)dx = \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- In the case of definite integral:

$$\int_a^b h(x)dx = \int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

- This formula can easily be derived from the product rule of derivative calculation.

Integration by Parts: Example

- Question: find the indefinite integral of $\log(x)$.
  - Answer: As $\log(x) = \log(x) \cdot (x)'$,

$$
\begin{aligned}
\int \log(x)dx &= \int \log(x) \cdot (x)' dx \\
&= x\log(x) - \int \frac{1}{x} \cdot x dx \\
&= x\log(x) - \int dx \\
&= x\log(x) - x + C.
\end{aligned}
$$

Integration by Substitution

- If we can find a function $t(x)$ and write the integrand $f(x)$ as

$$f(x) = f(t)\frac{dt}{dx},$$

we can apply the integration by substitution formula

$$\int f(x)dx = \int f(t)\frac{dt}{dx}dx = \int f(t)dt$$

- In the case of definite integral: if t moves from α to β when x moves from a to b, then

$$\int_a^b f(x)dx = \int_a^b f(t)\frac{dt}{dx}dx = \int_\alpha^\beta f(t)dt$$

  - Note that the interval of integration can change due to the variable transformation.

Integration by Substitution: Example

- Question: find the indefinite integral of $f(x) = 18x^2\sqrt[3]{6x^3 - 7}$
  - Answer: Let $t = 6x^3 - 7$. Then, $\frac{dt}{dx} = 18x^2$. Therefore,

$$
\begin{aligned}
\int f(x)dx &= \int 18x^2\sqrt[3]{6x^3 - 7}dx \\
&= \int \sqrt[3]{t}\frac{dt}{dx}dx \\
&= \int t^{\frac{1}{3}}dt \\
&= \frac{3}{4}t^{\frac{4}{3}} + C \\
&= \frac{3}{4}(6x^3 - 7)^{\frac{4}{3}} + C
\end{aligned}
$$

Integration by Substitution: Summary of Steps

1. Identify some part of $f(x)$ which can be simplified by substituting in a single variable t (which is a function of x)
2. Compute $\frac{dt}{dx}$, and reexpress $f(x)$ using t and $\frac{dt}{dx}$
3. Solve the indefinite integral
4. (For indefinite integration) Substitute back in for x
5. (For definite integration) Determine the new interval of integration $[\alpha, \beta]$, and evaluate the antiderivative at the boundary points.

Integration by Substitution: Another Example

- Question: find the definite integral $\int_0^1 x\sqrt{x^2+1}dx$
  - Answer: Let $t = x^2 + 1$. Then, $\frac{dt}{dx} = 2x$. Also, when x moves from 0 to 1, t moves from 1 to 2. Therefore,

$$
\begin{aligned}
& \int_0^1 x\sqrt{x^2+1}dx \\
= \; & \int_0^1 \sqrt{t} \cdot \frac{1}{2}\frac{dt}{dx}dx \\
= \; & \frac{1}{2}\int_1^2 \sqrt{t}dt \\
= \; & \frac{1}{2} \cdot \frac{2}{3} t^{\frac{3}{2}}\Big|_1^2 \\
= \; & \frac{1}{3}(2\sqrt{2}-1)
\end{aligned}
$$

L'Hopital's Rule

- Let's calculate the definite integral $\int_0^\infty xe^{-x}dx$.

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= \int_0^\infty x(-e^{-x})'dx \\ &= -(xe^{-x}\big|_0^\infty) - \int_0^\infty (x)' \cdot (-e^{-x})dx \\ &= -(xe^{-x}\big|_0^\infty) + \int_0^\infty e^{-x}dx\end{aligned}$$

- How can we evaluate xe^{-x} at $x \to \infty$?
  - $\lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$??
  - In such cases, L'Hopital's rule help you circumvent the problem.

L'Hopital's Rule (cont.)

- **L'Hopital's rule**: Let functions $f(x)$ and $g(x)$ be differentiable at an open interval close to c. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

- Applying the L'Hopital's rule, we can see that

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Integration Techniques: Exercises

- Calculate the following integrals.
  1. $\int \frac{1}{1+\exp(-x)} dx$
  2. $\int x \exp(x) dx$
  3. $\int \frac{2x}{5x^2-7} dx$
  4. $\int x^2 (x^3 + 15)^{\frac{3}{2}} dx$
  5. $\int_0^1 x \exp(x^2) dx$
  6. $\int_1^2 x \log x dx$
  7. $\int_0^1 \log(1 + \sqrt{x}) dx$ (Hint: set $t = 1 + \sqrt{x}$ and perform integration by substitution)

Tomorrow

- Problem set 3 $\rightarrow$ review in the morning
- Tomorrow
  - Vector and matrix algebra
  - Vector/matrix notations for multivariate calculus
  - Geometric meanings of vector/matrix algebra
  - Moore & Siegel, Chapters 12, 13, 15.2.2, 15.2.4, & 16.1