Day 2: Calculus 1

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Today

- Calculus 1
    - Limit
    - Derivative
        - Definition
        - Calculation rules
    - Unconstrained optimization
    - (if time permits) Taylor series expansion/approximation

Limit

- The **limit** of function $f(x)$ is the value that $f(x)$ approaches as x approaches to some value (say a).

- Notation: if $f(x)$ approaches to b when x approaches a, we write

$$\lim_{x \to a} f(x) = b$$

or

$$f(x) \to b \text{ as } x \to a$$

Limit (cont.)

- How about just plug in the number!?

- We need the concept limit because...
  - We often want to think about the behavior of functions when it approaches to infinity/negative infinity
  - We may want to evaluate the value of $f(x)$ where it is not defined
    - ⋆ e.g., what happens to $\log(x)$ at $x = 0$?
  - Functions can be discontinuous

Limit (cont.)

- Example: Let's compute

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2}$$

  - Answer: Since

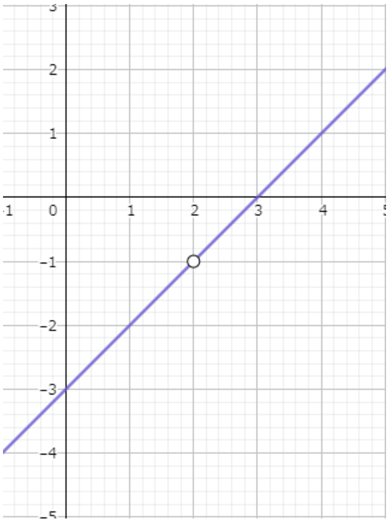
$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x-2)(x-3)}{x-2} = x - 3 \ (x \neq 2),$$

    we can see that

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x - 2} = -1$$

    even if the function is not defined at $x = 2$.

Limit (cont.)

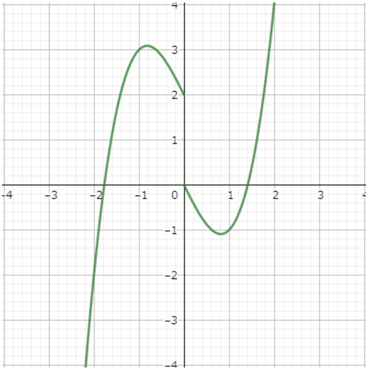


- Behavior of $f(x) = \frac{x^2-5x+6}{x-2}$ around $x = 2$

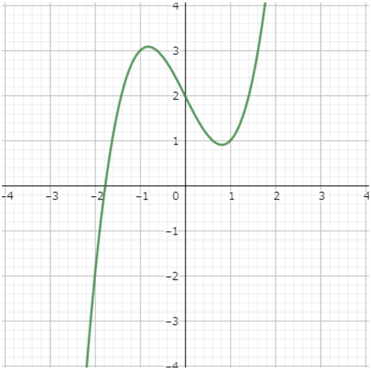
x	$f(x)$
1.99	1.01
1.999	-1.001
1.9999	-1.0001
1.99999	-1.00001
1.999999	-1.000001
2	Not defined
2.000001	-0.999999
2.00001	-0.99999
2.0001	-0.9999
2.001	-0.999
2.01	-0.99

Limit: Continuity

$f(x)$ is not continuous



$f(x)$ is continuous



Limit: Continuity (cont.)

- One-sided limit
  - **Right-sided limit** is the limit when x approaches a from above (from the right)
  - **Left-sided limit** is the limit when x approaches a from below (from the left)

- Notation: We denote the right-sided limit as

$$\lim_{x \to a^+} f(x) \text{ or } \lim_{x \downarrow a} f(x)$$

and left-sided limit as

$$\lim_{x \to a^-} f(x) \text{ or } \lim_{x \uparrow a} f(x)$$

Limit: Continuity (cont.)

- Function $f(x)$ is continuous at a if

$$\lim_{x \downarrow a} f(x) = \lim_{x \uparrow a} f(x)$$

and

$$\lim_{x \to a} f(x) = f(a)$$

  - If $\lim_{x \downarrow a} f(x) \neq \lim_{x \uparrow a} f(x)$, $\lim_{x \to a} f(x)$ is not defined.

Limit: Properties

- Properties of limit operators
  1. $\lim_{x\to a}\{\alpha f(x)+\beta g(x)\}=\alpha\lim_{x\to a}f(x)+\beta\lim_{x\to a}g(x)$
  2. $\lim_{x\to a}f(x)g(x)=\lim_{x\to a}f(x)\lim_{x\to a}g(x)$
  3. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$ $(\lim_{x\to a}g(x)\neq 0)$

- Property 1. is called the **linearity**, and operators with this property is called the linear operators.
  - e.g., $\sum$

Limit: Calculating a Limit of a Function

- Tips
  - Simplify the function as much as possible before computing the limit
  - Graphing the function is often helpful

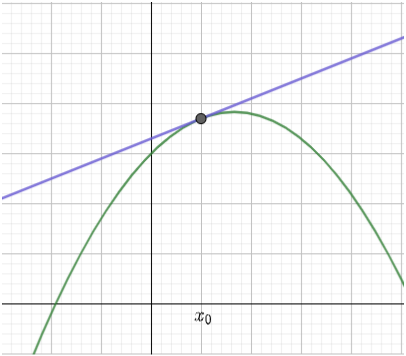
Limit: Exercises

- For the following functions at the specified values, please answer (i) whether they are defined, (ii) whether the limit is defined, and (iii) if the limit is defined, its value.
  1. x^2 at $x = 3$
  2. $3x^2 + 5x - 9$ at $x = 2$
  3. $|3x - 2|$ at $x = \frac{2}{3}$
  4. $\frac{x^2-4x-5}{x+1}$ at $x = -1$

Derivative: Introduction

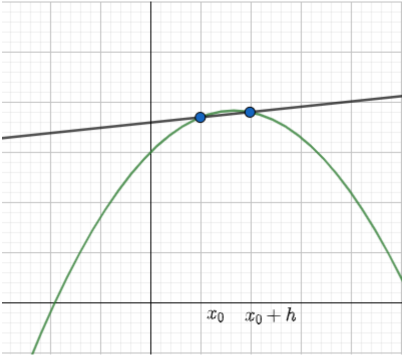
- We want to know how a function $f(x)$ is curved.
    - Is it increasing/decreasing? How fast?
        - What is **the rate of change**?
    - At which point does it start to increase/decrease?
- Let's examine the behavior of $f(x)$ at $x = x_0$.

Derivative: Introduction (cont.)



- How do we know that $f(x)$ is increasing/decreasing at $x = x_0$?
- Let's take a look at the slope of the tangent line.
- Tangent line: a line that just "touches" the curve at $x = x_0$.
- But how do we calculate the slope of the tangent line?

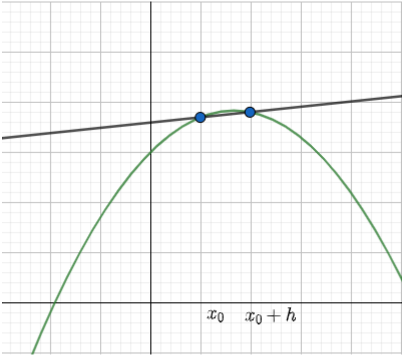
Derivative: Introduction (cont.)



- Think about approximating the tangent by a line connecting the two points on $f(x)$, $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$
- Slope of the line:

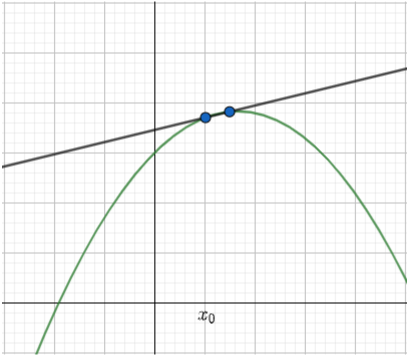
$$
\begin{aligned}
& \frac{f(x_0 + h) - f(x_0)}{(x + h) - x} \\
= \; & \frac{f(x_0 + h) - f(x_0)}{h}
\end{aligned}
$$

Derivative: Introduction (cont.)



- Now let's make the increment h smaller and smaller...
- The line mathces the tangent line!

Derivative: Introduction (cont.)



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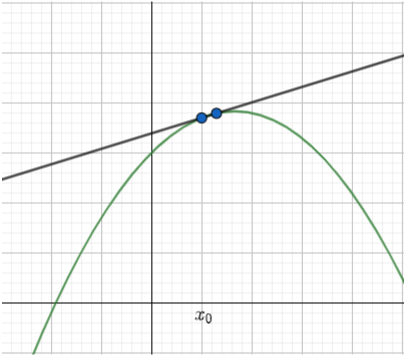

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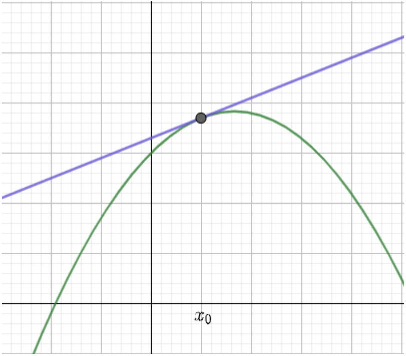


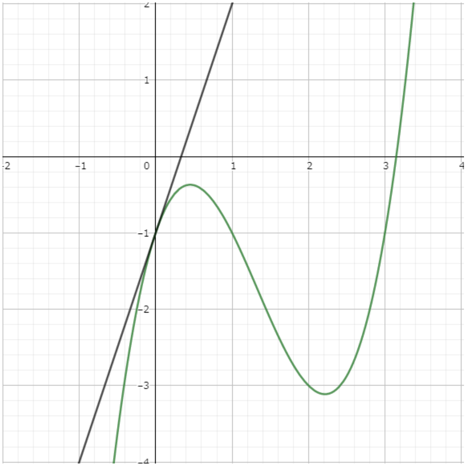
- Now let's make the increment h smaller and smaller...
- The line mathces the tangent line!

Derivative: Introduction (cont.)

- We call the slope of the tangent line as **derivative**.
- Using the limit notation we introduced earlier, the derivative of $f(x)$ at a point $x = x_0$ is denoted as

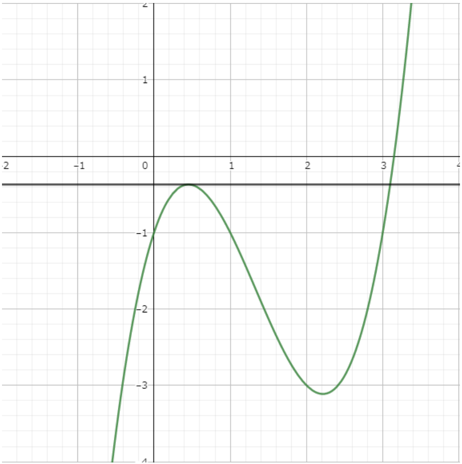
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Derivative: Introduction (cont.)



- The slope of the tangent line is different at different values of x.
- We can also think about derivative as a function of x.

Derivative: Introduction (cont.)



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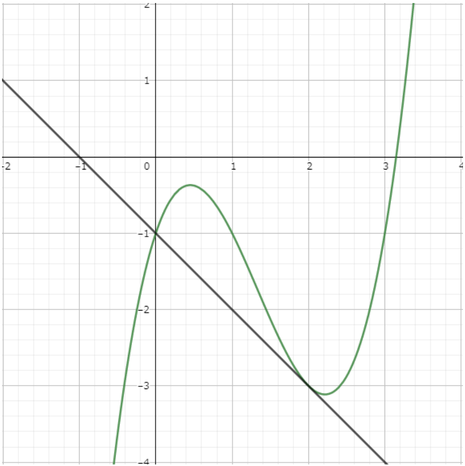

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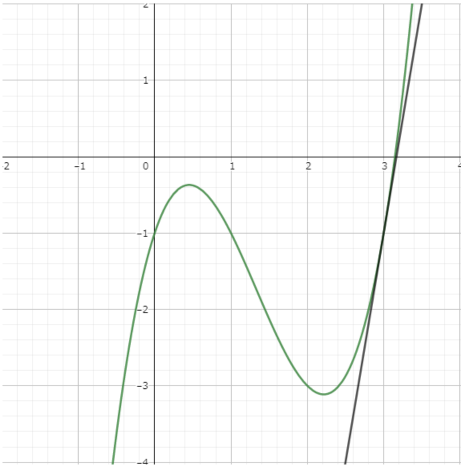


- The slope of the tangent line is different at different values of x.
- We can also think about derivative as a function of x.

Derivative: Introduction (cont.)

- **Definition**: the derivative of $f(x)$ with regard to x is defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - By plugging in concrete numbers, we can get the slope of the curve (tangent) at specific points.
  - We also say "to take the derivative" as "**differentiate**"

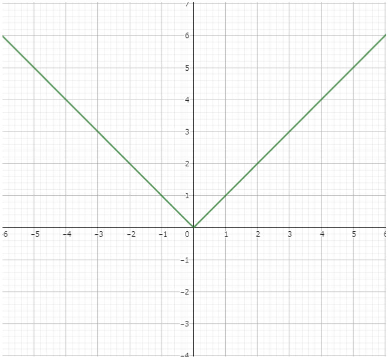
- Notation: the derivative of $y = f(x)$ with regard to x is denoted as

$$\frac{dy}{dx} \text{ or } \frac{df(x)}{dx}.$$

If the variable we take the derivative is obvious, we also write as

$$f'(x)$$

Derivative: Introduction (cont.)



- Not all the functions are differentiable.
  - Differentiable: the derivative $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists
  - e.g., $f(x) = |x|$
- Differentiability and continuity
  - Differentiable functions are always continuous
  - Continuity does not necessarily mean differentiability
    - ⋆ e.g., $f(x) = |x|$

Derivative: Exercises

1. Following the definition introduced earlier, calculate the derivative of $f(x) = x^2 + 3x$.

2. Define $f(x) = x^3$
   2.1 Following the defition introduced earlier, compute $f'(x)$.
   2.2 Calculate (a) $f'(2)$, (b) $f'(-1)$, and (c) $f'(4)$

Rules for Differentiation

- It is cumbersome to calculate the derivatives using formal definition!
- So we rely on rules of differentiation.
- Rules of Differentiation
  1. **Power Rule**: $(x^n)' = nx^{n-1}$
  2. **Summation Rule**: $\{\alpha f(x) + \beta g(x)\}' = \alpha f'(x) + \beta g'(x)$
  3. **Product Rule**: $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$
  4. **Quotient Rule**: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$

Rules for Differentiation (cont.)

- Example: Differentiate $f(x) = x^3 + x^2 - x + 8$
  - Answer:

$$
\begin{aligned}
f'(x) &= (x^3 + x^2 - x + 8)' \\
&= (x^3)' + (x^2)' + (-x)' + (8)' \\
&= 3x^2 + 2x - 1
\end{aligned}
$$

Rules for Differentiation (cont.)

- Example: Differentiate $f(x) = (x-2)(x^2+3x+1)$
  - Answer:

$$\begin{aligned}
f'(x) &= \left\{(x-2)(x^2+3x+1)\right\}' \\
&= (x-2)'(x^2+3x+1) + (x-2)(x^2+3x+1)' \\
&= (x^2+3x+1) + (x-2)(2x+3) \\
&= (x^2+3x+1) + (2x^2+3x-4x-6) \\
&= 3x^2+2x-5
\end{aligned}$$

Rules for Differentiation (cont.)

- Example: Differentiate $f(x) = \frac{2x+5}{3x^2}$
  - Answer:

$$
\begin{aligned}
f'(x) &= \left(\frac{2x+5}{3x^2}\right)' \\
&= \frac{(2x+5)' \cdot (3x^2) - (2x+5) \cdot (3x^2)'}{(3x^2)^2} \\
&= \frac{(2) \cdot (3x^2) - (2x+5) \cdot (6x)}{9x^4} \\
&= \frac{6x^2 - 12x^2 - 30x}{9x^4} \\
&= -\frac{2(x+5)}{3x^3}
\end{aligned}
$$

Rules for Differentiation: Exponents and Logs

- Rules related with exponential and log functions
  1. $(e^x)' = e^x$
  2. $(\log x)' = \frac{1}{x}$
  3. $(a^x)' = a^x \log x$

Rules for Differentiation: Exponents and Logs (cont.)

- Example: Differentiate $f(x) = 2x^2e^x$
  - Answer:

$$\begin{aligned} f'(x) &= (2x^2e^x)' \\ &= (2x^2)' \cdot (e^x) + (2x^2) \cdot (e^x)' \\ &= (4x) \cdot (e^x) + (2x^2) \cdot (e^x) \\ &= (2x^2 + 4x)e^x \\ &= 2x(x+2)e^x \end{aligned}$$

Rules for Differentiation: Chain Rule

- Chain rule: used to differentiate composite functions

- **Composite function** is a function whose input is the output of another function, denoted as

$$h(x) = f(g(x)) = (f \circ g)(x)$$

  - Note that the range of the inner function (i.e., $g(x)$) must be contained in the domain of the outer function (i.e., $f(x)$).
  - $(f \circ g)(x)$ and $(g \circ f)(x)$ are generally different.
  - e.g., Let $f(x) = x^2$ and $g(x) = \log x$. Then,

$$\begin{aligned} (f \circ g)(x) &= (\log x)^2 \\ (g \circ f)(x) &= \log x^2 \end{aligned}$$

Rules for Differentiation: Chain Rule (cont.)

- **Chain Rule**: Derivative of $h(x) = f(g(x))$ with respect to x is

$$h(x)' = f'(g(x)) \cdot g'(x)$$

  - Chain rule is also denoted as

$$\frac{dh(x)}{dx} = \frac{dh(x)}{dg(x)} \frac{dg(x)}{dx}$$

  - *The derivative of a composite function is the derivative of outer times the derivative of inner*

Rules for Differentiation: Chain Rule (cont.)

- Example: Differentiate $h(x) = (\log x)^2$
  - Answer:

$$
\begin{aligned}
h'(x) &= \left\{(\log x)^2\right\}' \\
&= \frac{d(\log x)^2}{d\log x}\frac{d\log x}{dx} \\
&= 2\log x \cdot \frac{1}{x} \\
&= \frac{2\log x}{x}
\end{aligned}
$$

Rules for Differentiation: Chain Rule (cont.)

- Example: Differentiate $h(x) = \log x^2$
  - Answer:

$$\begin{aligned} h'(x) &= (\log x^2)' \\ &= \frac{d \log x^2}{dx^2} \frac{dx^2}{dx} \\ &= \frac{1}{x^2} \cdot (2x) \\ &= \frac{2}{x} \end{aligned}$$

Rules for Differentiation: Exercises

- Differentiate the following functions with respect to x.
  1. $4x^3 + \frac{1}{3}x^2 + 2x + 7$
  2. $(2x+3)(x^2-13)$
  3. $\frac{1}{\log x}$
  4. $(\sqrt{x}+3)(x^3-x^2+1)$
  5. $(2x+5)^3$
  6. $\exp(x^2)$
  7. $\frac{1}{1+\exp(-x)}$
  8. $(x+3)(x^2-3x+9)$
  9. $\sqrt{x^2+1}$
  10. $\log e^x$

Optimization

- One major application of differential calculus
- When does the function takes the largest/smallest value?
  - Many applications in formal modeling and statistics!
- "Unconstrained" optimization
  - Find the value of inputs which maximizes/minimizes a function.
  - Constrained optimization: Find the value of inputs which maximizes/minimizes a function under some constraints
    - ⋆ e.g., Find the value of (x, y) which minimizes $f(x, y)$ under the constraint that $x + y \leq 5$.

Optimization: Terms and Notations

- **Objective function**: a function we want to maximize/minimize
- The values of the variable x which maximize the function $f(x)$ is denoted as

$$\underset{x}{\operatorname{argmax}} f(x)$$

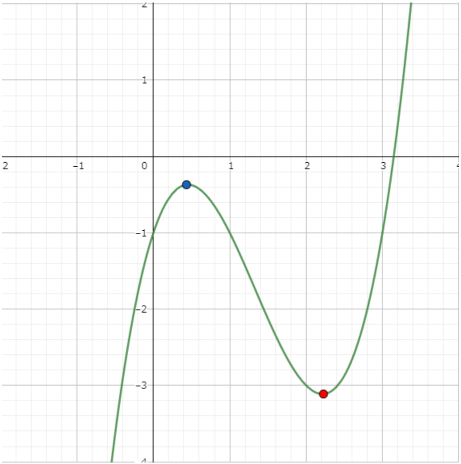
Similarly, the values of the variable x which minimize the function $f(x)$ is denoted as

$$\underset{x}{\operatorname{argmin}} f(x)$$

Optimization: Terms and Notations (cont.)

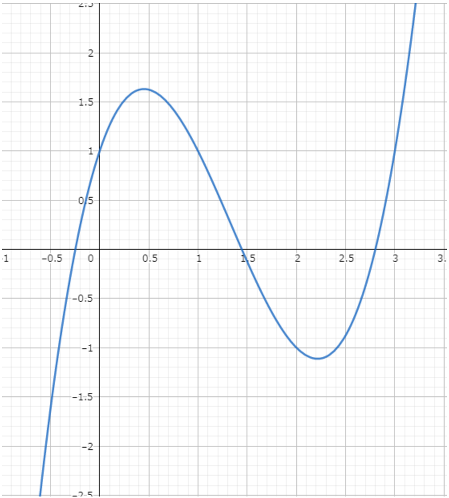
- **Extrema** of a function are any points where the value of a function is the largest (**maxima**) or smallest (**minima**).
- Global v. Local
  - A point $(x_0, f(x_0))$ is a **global maximum** if $f(x_0) \geq f(x)$ for all x in the domain.
  - A point $(x_0, f(x_0))$ is a **global minimum** if $f(x_0) \leq f(x)$ for all x in the domain.
  - A point $(x_0, f(x_0))$ is a **local maximum** if $f(x_0) \geq f(x)$ for all x within some open interval containing x_0.
  - A point $(x_0, f(x_0))$ is a **local minimum** if $f(x_0) \leq f(x)$ for all x within some open interval containing x_0.

Optimization: Terms and Notations (cont.)



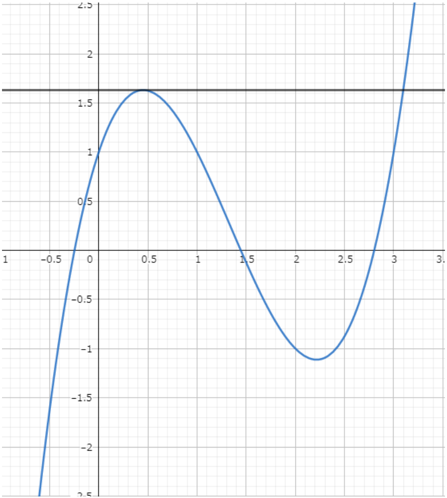
- The blue point is a local maximum, and the red is a local minimum.
- Whether they are also the global maximum/minimum depends on the domain of the function.

Optimization: First Order Condition



- Let's find the local maxima of a function $(x^3 - 4x^2 + 3x + 1)$ graphed on the left.
- Because the local maxima are the points where the curve stops sloping upward and it starts sloping downward...

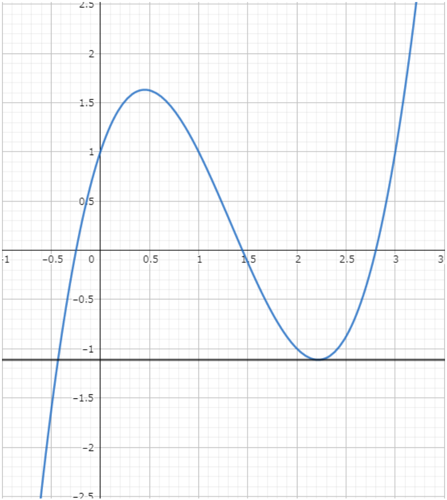
Optimization: First Order Condition (cont.)



- Local maxima are the points where the slope of the tangent line equals to zero!
- **First Order Condition**: To find (local) extrema of $f(x)$, we need to look at points $(x_0, f(x_0))$ where

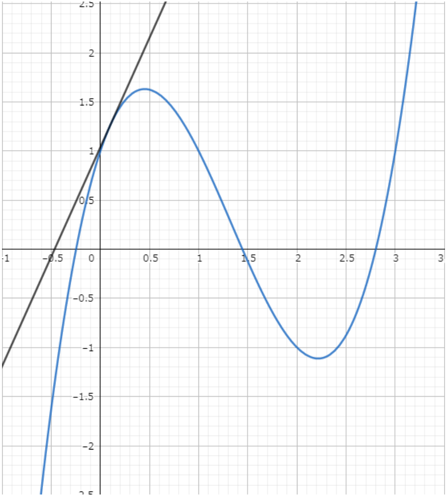
$$f'(x_0) = 0$$

Optimization: Second Order Condition



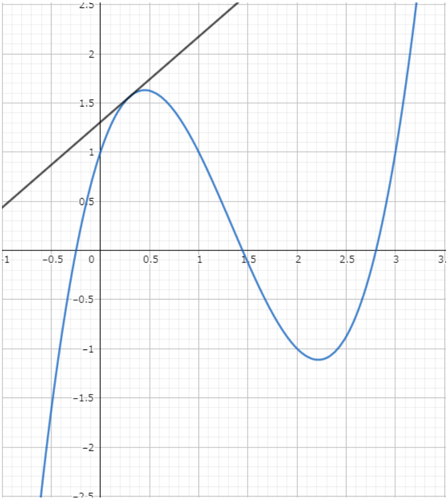
- First order condition cannot distinguish the maxima and minima, as the slope of the tangent line is also zero for the latter!
- Then how should we do?

Optimization: Second Order Condition (cont.)



- Let's take a look at the slope of the tangent line (= values of the derivatives) around the local maxima.
- The slope of the tangent line is **decreasing** as x gets larger!

Optimization: Second Order Condition (cont.)



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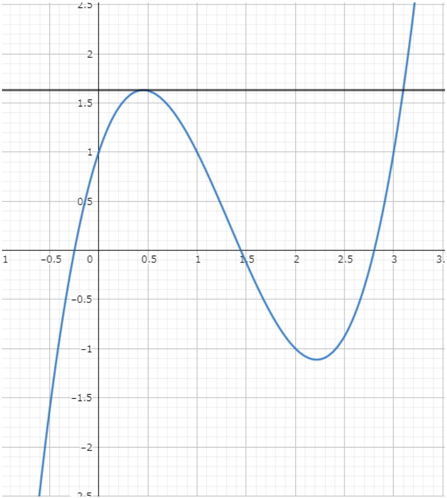

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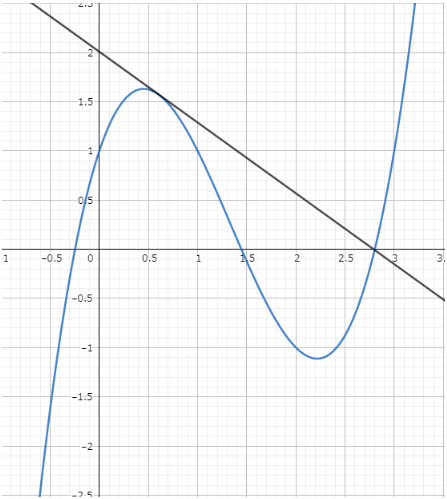

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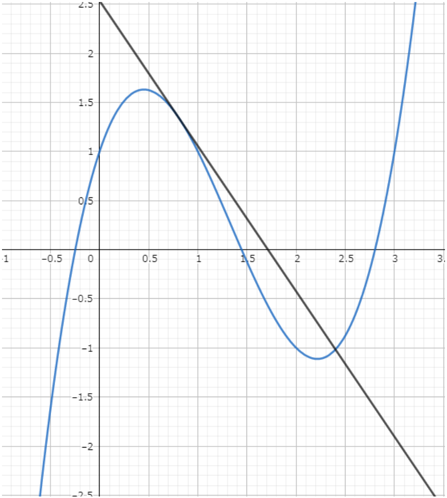

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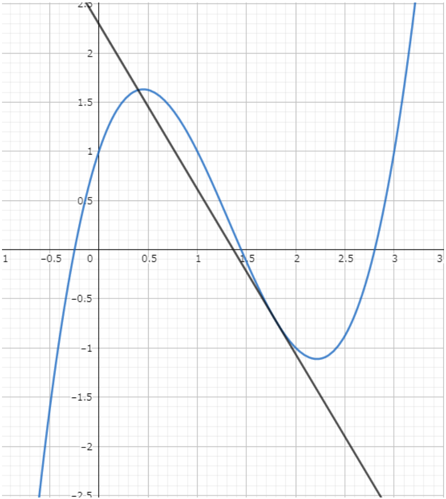
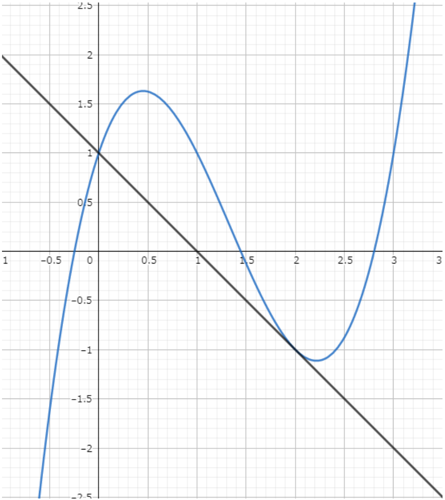


- Let's examine how the slope of the tangent line changes around the local minima.
- The slope of the tangent line is **increasing** as x gets larger!

Optimization: Second Order Condition (cont.)



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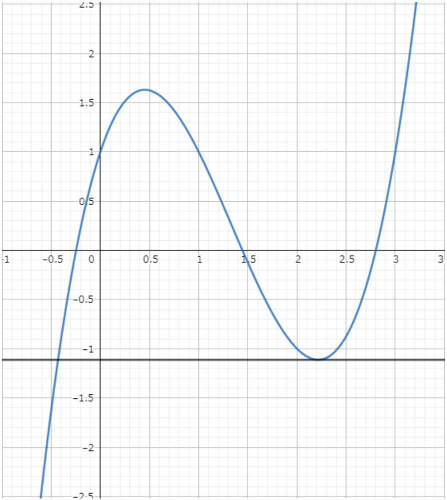

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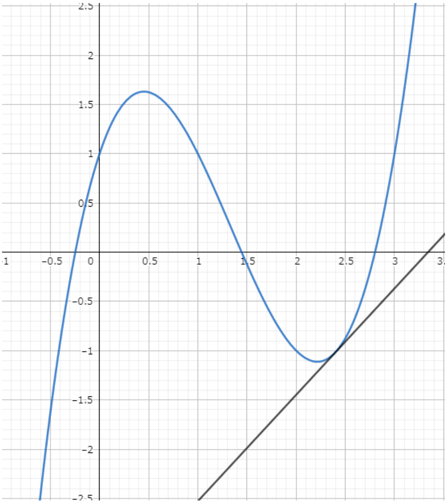

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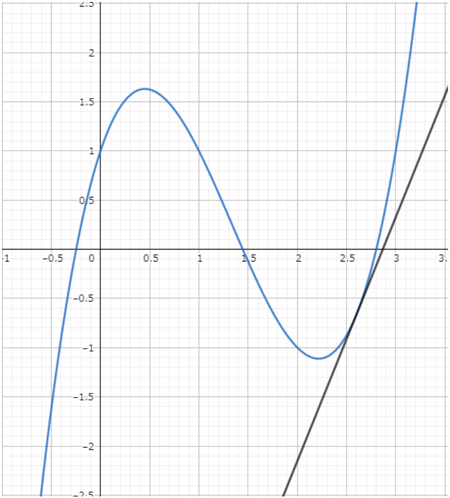
Optimization: Second Order Condition (cont.)



- Let's examine how the slope of the tangent line changes around the local minima.
- The slope of the tangent line is **increasing** as x gets larger!

Optimization: Second Order Condition (cont.)

- How can we describe the change in the slope of the tangent line?

- Because derivative describes the rate of change of a function ...
  $\rightarrow$ how about differentiate the function once more?

- **Second derivative** of $f(x)$ is the derivative of $f'(x)$, which is often denoted as

$$f''(x) \text{ or } \frac{d^2 f(x)}{dx^2}$$

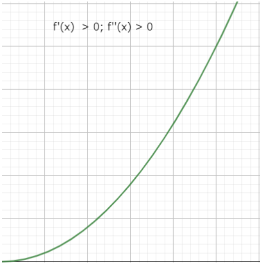
- By extention, we can also think of n-th derivative of $f(x)$, denoted as $f^{(n)}(x)$ or $\frac{d^n f(x)}{dx^n}$, by taking the derivative n times.
  - which describes the rate of change of rate of change of ...

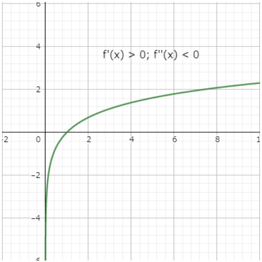
Optimization: Second Order Condition (cont.)

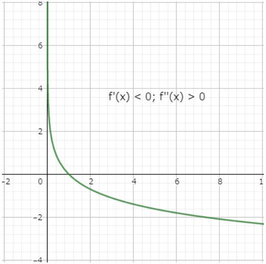
- Convex v. Concave
  - Function $f(x)$ is **convex** on an interval if $f''(x) > 0$ for all x in that interval.
    - ⋆ If $f(x)$ is convex, $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$ $(\alpha \in [0,1])$ for all x_1, x_2 in that interval.
  - Function $f(x)$ is **concave** on an interval if $f''(x) < 0$ for all x in that interval.
    - ⋆ If $f(x)$ is concave, $f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$ $(\alpha \in [0,1])$ for all x_1, x_2 in that interval.

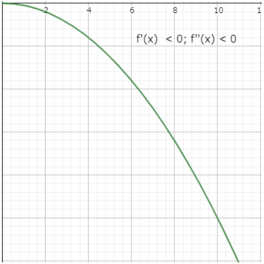
Optimization: Second Order Condition (cont.)

- Examples: convex (left column), concave (right column)









Optimization: Second Order Condition (cont.)

- **Second order condition**: when $f'(x_0) = 0$,
  - if $f''(x_0) < 0$ (i.e., $f(x)$ is concave in an interval containing x_0), the point $(x_0, f(x_0))$ is a **local maximum**
  - if $f''(x_0) > 0$ (i.e., $f(x)$ is convex in an interval containing x_0), the point $(x_0, f(x_0))$ is a **local minimum**

- When $f''(x_0) = 0$?
  - e.g., $f(x) = x^3$
  - Need to look at higher order derivatives
  - (You rarely encounter these cases. No worries!)

- To see whether x_0 is also a global maximum/minimum,
  - if the domain is bounded, check the values of $f(x)$ at the boundaries
  - examine the signs of $f'(x)$ (and $f''(x)$) $\rightarrow$ detect the shape of the curve in the entire domain

Optimization: Example

- Find all the extrema (local and global) of $f(x) = x^3 - x^2$ $(x \in [-1, 1])$, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain.
  - Answer: First calculate $f'(x)$ and set it to 0,

$$\begin{aligned} f'(x) = 3x^2 - 2x = x(3x-2) &= 0 \\ &\Rightarrow x = 0, \frac{3}{2} \end{aligned}$$

Then compute $f''(x)$ and evaluate these points.

$$\begin{aligned} & f''(x) = 6x - 2 \\ \Rightarrow\ & f''(0) < 0, f''(\frac{3}{2}) > 0 \end{aligned}$$

Finally evalute $f(x)$ at these points and boundary points, revealing that $(-1, -2)$ is the global minimum, $(\frac{3}{2}, -\frac{4}{27})$ is the local minimum, $(0, 0)$ and $(1, 0)$ are the global maxima.

Optimization: Exercises

- Find all the extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain.
  1. $f(x) = x^3 - x + 1 \ (x \in [0, 1])$
  2. $f(x) = x^3 - 3x \ (x \in [-2, 2])$

Taylor Series Expansion/Approximation

- Derivatives describe how the function is shaped.
  - $f'(x)$: increasing/decreasing
  - $f''(x)$: convex/concave
  - ...

- Can we approximate $f(x)$ with its derivatives? $\rightarrow$ Taylor series expansion

- By applying the Taylor series expansion, we can approximate weird(!?) functions (e.g., $\exp(x)$) using polynomials.

Taylor Series Expansion/Approximation (cont.)

- **Sequence** is an ordered list of numbers.
- **Series** is a sum of numbers is a sequence.
- Example: Fibonacci sequence is an ordered list of numbers satisfying the relationship $x_i = x_{i-1} + x_{i-2}$ $(i > 1)$, which looks like
$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots,$$

Taylor Series Expansion/Approximation (cont.)

- **Taylor series**: If $f(x)$ is infinitely differentiable at the neighborhood of a, we can approximate $f(x)$ as

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= f(a) + \sum_{i=1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i \end{aligned}$$

- When we set $a = 0$, the series is especially called the **Maclaurin series**:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Taylor Series Expansion/Approximation: Example

- Question: Find the Maclaurin series for $f(x) = e^x$.
  - Answer:

$$
\begin{aligned}
f(x) &= \exp(0) + \frac{\exp(0)}{1!}x + \frac{\exp(0)}{2!}x^2 + \frac{\exp(0)}{3!}x^3 + \cdots \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= \sum_{i=0}^{\infty} \frac{1}{i!}x^i
\end{aligned}
$$

Taylor Series Expansion/Approximation: Exercise

- Question: Find the Maclaurin series for $f(x) = \log(1+x)$.

Tomorrow

- Problem set 2 $\rightarrow$ review in the morning
- Tomorrow
  - Partial derivative
  - Integral
  - Moore & Siegel, Chapters 7 & 15.2.1