

Day 1: Basics

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Why Do I Need to Study Mathematics?

- Statistical analysis/Formal modeling
 - ▶ Standard tools for political science research
 - ▶ Applied field of mathematics
- We study mathematics to...
 - ▶ read textbooks
 - ▶ read articles using statistics/formal modeling

Real Stats

Specifically, the expression for the sum of squared residuals for any given estimates of $\hat{\beta}_0$ and $\hat{\beta}_1$ is

$$\sum_{i=1}^N \hat{\epsilon}_i^2 = \sum_{i=1}^N (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2$$

Game Theory for Applied Economists

where $y_i(e_i) = e_i + \varepsilon_i$. The first-order condition for (2.2.4) is

$$(w_H - w_L) \frac{\partial \text{Prob}\{y_i(e_i) > y_j(e_j^*)\}}{\partial e_i} = g'(e_i). \quad (2.2.5)$$

That is, worker i chooses e_i such that the marginal disutility of extra effort, $g'(e_i)$, equals the marginal gain from extra effort, which is the product of the wage gain from winning the tournament, $w_H - w_L$, and the marginal increase in the probability of winning.

By Bayes' rule,¹²

$$\begin{aligned} \text{Prob}\{y_i(e_i) > y_j(e_j^*)\} &= \text{Prob}\{\varepsilon_i > e_j^* + \varepsilon_j - e_i\} \\ &= \int_{\varepsilon_j} \text{Prob}\{\varepsilon_i > e_j^* + \varepsilon_j - e_i \mid \varepsilon_j\} f(\varepsilon_j) d\varepsilon_j \\ &= \int_{\varepsilon_j} [1 - F(e_j^* - e_i + \varepsilon_j)] f(\varepsilon_j) d\varepsilon_j, \end{aligned}$$

Elements of Statistical Learning

form. The least squares estimate of $a^T\beta$ is

$$\hat{\theta} = a^T \hat{\beta} = a^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}. \quad (3.17)$$

Considering \mathbf{X} to be fixed, this is a linear function $\mathbf{c}_0^T \mathbf{y}$ of the response vector \mathbf{y} . If we assume that the linear model is correct, $a^T \hat{\beta}$ is unbiased since

$$\begin{aligned} \mathbb{E}(a^T \hat{\beta}) &= \mathbb{E}(a^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}) \\ &= a^T (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \beta \\ &= a^T \beta. \end{aligned} \quad (3.18)$$

No Worries!

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= Today

No Worries!

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No Worries!

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Note

- The mathematics you need depends on what kinds of research you want to do.
- This lecture will cover basic mathematics that you're likely to encounter

Do I Need to Memorize Formulas?

- NO!
- Rather focus on understanding the meanings of the formulas and when/why/how you should use them.

Today

- Basics
 - ▶ Algebra review
 - ▶ Set
 - ▶ Variable
 - ▶ Functions
 - ★ Graphing equations/inequalities
 - ★ Exponential and log functions
 - ▶ Summation & product operators
- (Today's lecture is a bit all over the place...)

Algebra Review: Order of Calculation

- Perform the following calculations.

1. $3 + 5 \times 2 - 17 =$

2. $5 \times (7 - 4) + (4 - 2) =$

3. $(8 - 3) - (9 - 13) \times 4 =$

Algebra Review: Fraction

- Perform the following calculations.

1. $\frac{2}{3} - \frac{1}{5} =$

2. $\frac{4}{5} \times \frac{2}{3} =$

3. $\frac{2}{5} \div \frac{3}{8} =$

4. $\frac{\frac{2}{7}}{\frac{3}{4}} =$

Algebra Review: Power, Root, Factorial, & Absolute value

- Perform the following calculations.

1. $2^3 \times 4^2 =$

2. $(2^3)^2 =$

3. $3^{-1} =$

4. $\sqrt[3]{8} =$

5. $5! =$

6. $|-3| =$

Set

- **Set:** a collection of distinct objects
- **Element:** an object consisting a set

Set: Notation

- We write elements of a set within $\{\}$.
 - ▶ $A = \{1, 2, 4, 6, 7\}$
- When we can write elements of a set using a general form, we use the notation $\{\text{general form}|\text{definition}\}$
 - ▶ e.g., $B = \{x^2|x \text{ is an integer}, 1 \leq x \leq 4\}$
- If x is an element of the set A , we write $x \in A$.
- If x is not an element of the set A , we write $x \notin A$.

Set: Subset

- If every element of A is also in B , A is called the **subset** of B , and denoted as $A \subseteq B$.
- Example: Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{1, 2, 7\}$. Then, $B \subseteq A$.

Set Operation

- **Intersection:** the intersection of A and B , denoted as $A \cap B$, is the set of common elements to both sets.
- **Union:** the union of A and B , denoted as $A \cup B$, is the set of all elements contained in either set.
- **Complement:** the complement of A , denoted as A^c , is the set that contains elements that are not contained in A .
- **Difference:** the difference of sets A and B , denoted as A/B , is the set of elements in A but not in B .
 - ▶ A/B is the same as $A \cap B^c$.
 - ▶ A/B is generally different from B/A .

Set: Number System

- **Natural numbers** (\mathbb{N}): set of 0 and positive integers ($\{0, 1, 2, \dots\}$)
- **Integers** (\mathbb{Z}): set of all integers ($\{\dots, -2, -1, 0, 1, 2, \dots\}$)
- **Rational numbers** (\mathbb{Q}): set of all numbers which can be represented as the fraction of integers
- **Real numbers** (\mathbb{R}): set of all numbers whose squares are larger than or equal to 0.
 - ▶ Real numbers which cannot be represented as the fraction of integers (e.g., $\sqrt{2}$) are called the **irrational numbers**
- Using the set notation, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Set: Additional Notations/Terms

- **Empty set:** a set with no elements, denoted as \emptyset
- If sets A and B have no elements in common (i.e., $A \cap B = \emptyset$), we say A and B are **disjoint**
- We denote the union and the intersection of a sequence of sets (A_1, A_2, \dots, A_n) as follows.

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

Set: Open/Closed Intervals

- **Interval:** a set of real numbers that the any numbers between the two points in the set are also included in the set.
- **Open interval** is an interval that does not include the end points as elements. e.g., $A = \{a < i < b | i \in \mathbb{R}\}$
 - ▶ We denote open intervals using parentheses $()$.
 - ▶ e.g., $A = (a, b)$
- **Closed interval** is an interval that include the end points as elements. e.g., $A = \{a \leq i \leq b | i \in \mathbb{R}\}$
 - ▶ We denote closed intervals using square brackets $[]$.
 - ▶ e.g., $A = [a, b]$

Set: Exercises

1. List all the elements of the set $A = \{x^3 | x \in \mathbb{Z}, -2 \leq x \leq 2\}$
2. Let $A = \{-3, 1, 4, 6, 8, 13\}$ and $B = \{-5, -3, 1, 8, 11, 13\}$.
Then what is
 - 2.1 $A \cap B$
 - 2.2 $A \cup B$
 - 2.3 A/B
 - 2.4 B/A

Variable/Constant

- **Variable:** a symbol which represent an arbitrary number
 - ▶ value of a variable is not fully specified, so it can change/vary
- **Constant:** a quantity whose value do not change

Basic Number Properties

- **Commutative Property:** the order of addition/multiplication does not affect the outcome

$$a + b = b + a$$

$$a \times b = b \times a$$

- **Associative Property:** the order of addition/multiplication does not matter as long as the sequence of operation is not changed

$$(a + b) + c = a + (b + c)$$

$$(a \times b) \times c = a \times (b \times c)$$

- **Distributive Property:** distribution of multiplication over addition/subtraction

$$a(b + c) = ab + ac$$

Expansion

- Remove parentheses $()$ from the product of polynomials.
- Monomial/Polynomial
 - ▶ **Monomial**: product of a constant and a variable raised to some value, which looks like ax^k
 - ★ The constant (a) is called the **coefficient**, and the number of exponent (k) is called the **degree/order** of the monomial
 - ▶ **Polynomial**: sum of monomials
 - ★ e.g., $2x + 3$, $x^3 - 4x + 5$
 - ★ The degree of a polynomial is the highest degree of its monomials
- How to perform expansion?
 - ▶ Repeatedly apply the distribution rule!

Expansion (cont.)

- Question: Multiply out $(2x + 3)(x^2 - 8x + 5)$

► Answer:

$$\begin{aligned} & (2x + 3)(x^2 - 8x + 5) \\ = & 2x(x^2 - 8x + 5) + 3(x^2 - 8x + 5) \\ = & 2x^3 - 16x^2 + 10x + 3x^2 - 24x + 15 \\ = & 2x^3 - 13x^2 - 14x + 15 \end{aligned}$$

Expansion (cont.)

- Question: Multiply out $(x + 2)^3$

▶ Answer:

$$\begin{aligned}(x + 2)^3 &= (x + 2)(x + 2)^2 \\&= (x + 2) \{x(x + 2) + 2(x + 2)\} \\&= (x + 2)(x^2 + 4x + 4) \\&= x(x^2 + 4x + 4) + 2(x^2 + 4x + 4) \\&= x^3 + 6x^2 + 12x + 8\end{aligned}$$

Factoring

- **Factoring:** writing a polynomial as a product of polynomials of lower degrees.
- Factoring is the reverse of expanding parentheses.
- (Factoring is a bit harder than expansion, and you need to get used to it...)
- Tips
 - ▶ If you find common factors, try grouping the terms containing them
 - ★ e.g., $x^3 - 5x^2 = x^2 \times x + x^2 \times (-5) = x^2(x - 5)$
 - ▶ Try applying the rules on the next page

Factoring/Expansion Rules

- Some of the rules you often encounter:

1. $(x + a)(x + b) = x^2 + (a + b)x + ab$

2. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

3. $(x + a)^2 = x^2 + 2ax + a^2$

4. $(x - a)^2 = x^2 - 2ax + a^2$

5. $(x + a)(x^2 - ax + a^2) = x^3 + a^3$

6. $(x - a)(x^2 + ax + a^2) = x^3 - a^3$

7. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

- (Again, you don't need to memorize them...)

Factoring/Expansion: Exercises

1. Multiply out:

1.1 $(x + 6)(x - 7) =$

1.2 $(x^2 + 3)(x^2 + x + 7) =$

2. Factorize:

2.1 $x^2 - 2x - 3 =$

2.2 $6x^2 - x - 1 =$

2.3 $x^3 - 8 =$

2.4 $x^3 - 3x^2 - 10x =$

Solving Equations/Inequalities

- This is when we want find values of a variable satisfying an equation/range of values of a variable satisfying an inequality
- Tips
 - ▶ Rearrange terms so that the variable of interest is isolated
 - ▶ Make sure to perform the same operations on both sides of equality/inequality
 - ★ Be careful when multiplying by negative numbers in solving inequalities!
 - ▶ Check answer

Solving Equations/Inequalities (cont.)

- Question: Temperatures in Celsius (say x) is converted to those in Fahrenheit (y) using the following equation.

$$y = \frac{9}{5}x + 32$$

Then,

1. Find the temperature when both are equal.
2. When do temperatures measured in Fahrenheit are higher than those in Celsius?

Solving Quadratics

- After you rearrange the terms in the form
(i) $ax^2 + bx + c = 0$ / (ii) $ax^2 + bx + c \leq 0$ / (iii) $ax^2 + bx + c \geq 0$...
- If you can easily factorize as $a(x + \alpha)(x + \beta)$ ($\alpha > \beta, a > 0$), then
 - ▶ (i): $(x + \alpha) = 0$ and/or $(x + \beta) = 0 \Rightarrow x = -\alpha, -\beta$
 - ▶ (ii): $(x + \alpha) \geq 0$ and $(x + \beta) \leq 0 \Rightarrow -\alpha \leq x \leq -\beta$
 - ▶ (iii): $(x + \alpha) \leq 0$ or $(x + \beta) \geq 0 \Rightarrow x \leq -\alpha, x \geq -\beta$
 - ▶ How should we do when $a < 0$?
- When we cannot easily factorize... → **Complete the square**
 - ▶ Transform the quadratic into the form $a(x \pm \alpha) = \beta$ and solve for x by taking the square root of those terms.

Solving Quadratics: Complete the Square

1. Move the constant to the RHS, and divide the equation by the coefficient on the squared term

▶ $x^2 + \frac{b}{a}x = -\frac{c}{a}$

2. Divide the coefficient on x by 2, square the value, and add to both sides

▶ $x^2 + 2 \cdot \frac{b}{2a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$

3. Factor the LHS into the form $(x \pm \alpha)^2$ and simplify the RHS

▶ $(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$

4. Take the square root of both sides

▶ $x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

5. Solve for x

▶ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Solving Quadratics: Exercises

- Solve the following equations/inequalities for x
 1. $2x^2 - 4x - 3 = 0$
 2. $x^2 + x - 6 \leq 0$
 3. $x^3 = x^2 + 12x$

Function

- **Function** is a relation between sets which associate every element of the first set to exactly one element of the second set.
 - ▶ describes how the values of the LHS variable (inputs) changes with values of RHS variables (outputs)
 - ▶ input variables are often called the *independent* variables
 - ▶ output variable is often called the *dependent* variable
- Domain, image, range
 - ▶ **Domain:** set over which a function is defined
 - ▶ **Image:** output value of the function
 - ▶ **Range:** set of the images of all elements of the domain

Function: Notation

- Let f be a function which relates every element $x \in X$ to exactly one element $y \in Y$. Then we denote the relationship as

$$y = f(x)$$

or

$$f : X \rightarrow Y$$

- ▶ Other letters often used to denote functions: $g, h, u, v \dots$

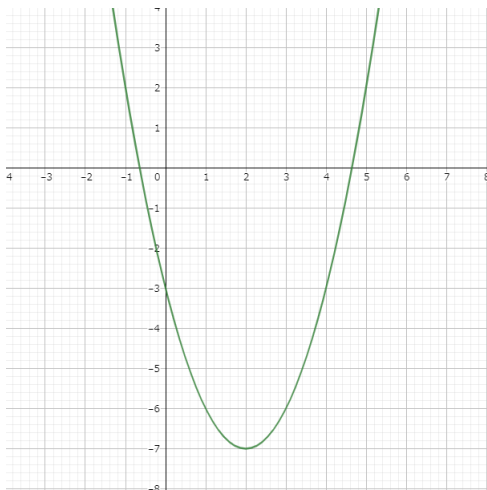
Function: Examples

- $f(x) = x + 4$ ($x \in \mathbb{R}$)
- $f(x) = x^2 + 2$ ($x \in \mathbb{R}$)
 - ▶ What is the range of this function?
- $f(x_1, x_2) = 2x_1 - x_2 + 7$ ($x_1, x_2 \in \mathbb{R}$)
 - ▶ f maps/associates every element $(x_1, x_2) \in X$ to exactly one element of $y \in Y$.
- $x^2 + y^2 = 1$ is not a function!
 - ▶ e.g., when $x = 1$, $y = \pm 1$

Graphing Functions

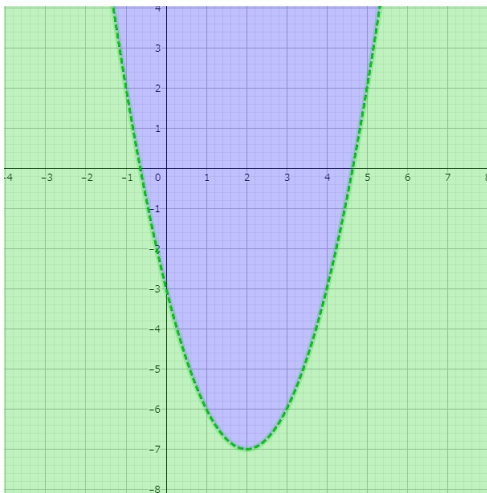
- Why do you want to graph a function?
 - ▶ Is it increasing/decreasing?
 - ▶ How fast is it increasing/decreasing?
 - ▶ ...
- Intersection of algebra and geometry

Graphing Functions (cont.)



- Graph of $y = f(x)$ is the collection of points satisfying the relationship.
- Compute the points $(x_0, f(x_0))$ for multiple x_0 and connect them!

Graphing Functions (cont.)



- Areas above the curve (areas in blue): set of points satisfying the relationships $y > f(x)$
- Areas below the curve (areas in green): set of points satisfying the relationships $y < f(x)$

Graphing Functions (cont.)

- Linear function (polynomial of degree 1): $y = a + bx$
 - ▶ Must pass the point $(0, a)$.
- Quadratic function
 - ▶ If we can factorize to $a(x + b)(x + c) \rightarrow$ must pass the points $(-b, 0)$ and $(-c, 0)$
 - ▶ Complete the square and get $a(x \pm b)^2 + c \rightarrow$ must pass the point $(-b, c)$
 - ★ If $a > 0$: $(-b, c)$ is the global minimum
 - ★ If $a < 0$: $(-b, c)$ is the global maximum

Graphing Functions: Exercises

- Graph the following relationships.

1. $y = x^2 - 2x + 5$

2. $y = 2x^2 + 2x - 12$

3. $y \geq \frac{3}{2}x - 4$

Functions Often Used in Social Science

- Exponential function
- Logarithmic function
- (Trigonometric function)

Exponential Function

- An exponential function of x is some constant (say a) raised to x : $f(x) = a^x$
 - ▶ $f(x) = 4^x$
 - ▶ $f(x) = (-2)^x$
 - ▶ What is the range of an exponential function?
- Euler's constant e
 - ▶ $e = \lim_{x \rightarrow \infty} (1 + \frac{1}{h})^h = 2.71828\dots$
 - ▶ $e^x = \exp(x)$

Exponential Function: Basic Rules

1. $a^{x_1} \times a^{x_2} = a^{x_1+x_2} \quad (x_1, x_2 \in \mathbb{R})$

2. $(a^{x_1})^{x_2} = a^{x_1 x_2}$

3. $a^0 = 1$

4. $a^{-x} = \frac{1}{a^x}$

5. $a^{\frac{1}{x}} = \sqrt[x]{a}$

Logarithmic Function

- Logarithm is the inverse of an exponential function. Logarithm of x to base a is the number such that a power of it equals to x . Formally,

$$y = \log_a x \Leftrightarrow x = a^y$$

- Log of x to base e is called the **natural log**, and often denoted as

$$\log_e x = \log x = \ln(x)$$

- Question: what is the domain of a log function?

Logarithmic Funtion: Basic Rules

1. $\log_a x^b = b \log_a x$
2. $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$
3. $\log_a xy = \log_a x + \log_a y$
4. $\log_a \frac{x}{y} = \log_a (x \times y^{-1}) = \log_a x - \log_a y$
5. $\log_x x = 1$
6. $\log_a 1 = \log_a a^0 = 0$
7. (Change of base) $\log_b x = \frac{\log_a x}{\log_a b}$

Exponential and Logarithmic Functions: Exercises

1. Simplify the following expressions.

1.1 $2^5 \times 4^{\frac{3}{2}} =$

1.2 $2\log x - \log(3x) =$

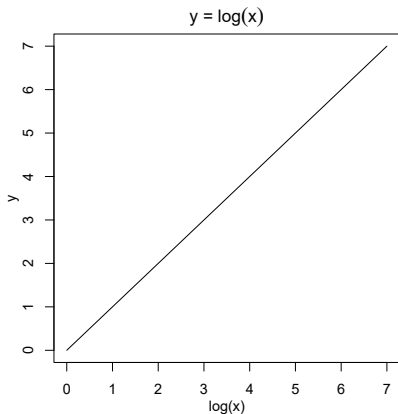
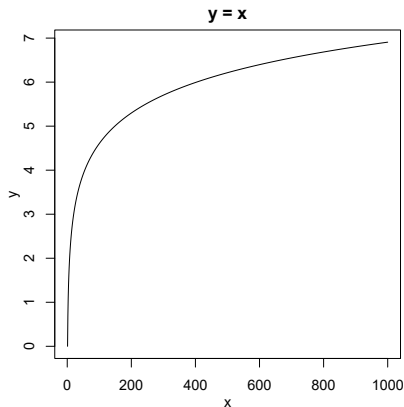
1.3 $a^{\log_a x} =$

2. Derive the change of base formula. (Hint: by definition, x is represented as...)

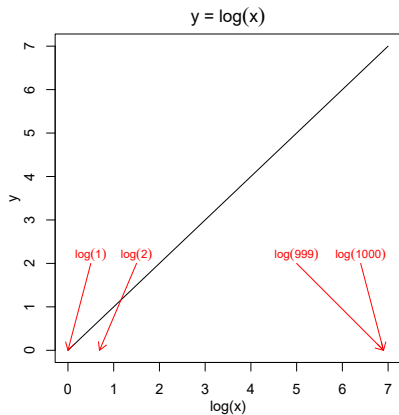
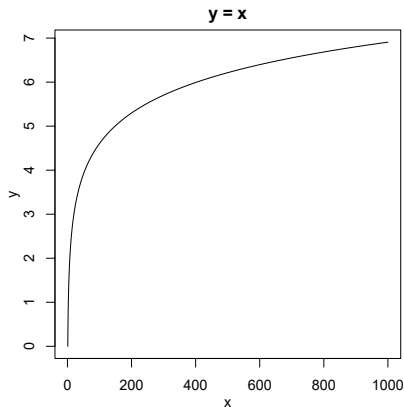
Why Do People Use Logs?

- Log function is (one of) the most frequently used function(s) in political science.
- But why?
- Log function stretches out the scale when the value of the input is small while compresses the scale when it's large.
 - ▶ Dealing with skewness of the data
 - ▶ Express nonlinear relationship
 - ▶ ...

Why Do People Use Logs? (cont.)



Why Do People Use Logs? (cont.)



Summation and Product Operators

- Assume a sequence of variables/values

$$x_m, x_{m+1}, \dots, x_n \quad (m < n)$$

- We denote their sum as follows.

$$\sum_{i=m}^n x_i = x_m + x_{m+1} + \dots + x_n$$

- We denote their product as follows.

$$\prod_{i=m}^n x_i = x_m \times x_{m+1} \times \dots \times x_n$$

Summation and Product Operators: Examples

- $\sum_{i=1}^{10} i = 1 + 2 + \cdots + 10 = 55.$
- $\prod_{i=1}^7 i = 7! = 840.$
- $\sum_{i=1}^6 x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6.$
- $\prod_{j=2}^5 x_j = x_2 \times x_3 \times x_4 \times x_5.$

Summation and Product Operators: Exercises

1. Perform the following calculation.

1.1 $\sum_{i=3}^7 i^2$

1.2 $\sum_{i=1}^n a$

2. Show that

▶ $\sum_{i=1}^n (x_i + y_i) = \sum_i^n x_i + \sum_i^n y_i$

▶ $\sum_{i=1}^n (ax_i + b) = a \sum_{i=1}^n x_i + nb$

Tomorrow

- Problem set 1 \rightarrow review in the morning
- Tomorrow
 - ▶ Limits
 - ▶ Derivative
 - ▶ Unconstrained optimization
 - ▶ Moore & Siegel, Chapters 4-6 & 8