Probabilities

August 2018

Intro: Why should we care about probability?

- One of the most important concepts in statistics and game theory.
- Can be found in any dimension of social sciences.
- Understanding probabilities will help you 1) in your methods classes and 2) in understanding soc. scientific literature.
- We will look at 1) the nature of probability; 2) how to compute probabilities.

Major presuppositions in quantitative social scientific research:

- We cannot determine behavior exactly
- We assume that there is a degree of uncertainty in how any *IV* affects any *DV*, i.e. there is an *error*
- We include a *level of confidence* in any interaction between an *IV* and a *DV*

So, probability is central to understanding the concepts of significance and uncertainty in empirical research

- In general, social-scientific theories (and studies) are NOT deterministic but *stochastic* (probabilistic or *random*)

Classical probability

Core concepts:

- **Outcome** – something that can occur
- **Simple event** – composed of only one outcome
- **Compound event** – the result of more than one outcome
- **Sample space** – the set of all possible outcomes that could occur

Probability deals with issues in which the outcome is not known for certain

$Pr(e) =$ N of ways to get e / N of outcomes in the sample space

More concepts

Empirical probability – observed probability based on an empirical ratio

Bayesian statistics – based on classical probability and subjective probability

- Subjective probability – one's prior perception of probability

Classical probability: The basics

$Pr(A)$ – probability of an outcome occurring (from 0 to 1)

- $Pr(A) \epsilon [0, 1]$
- $Pr(S) = 1$; S – sample space (to put it simply, something must happen)

Different types of events:

Events can be:

- **Independent** – each event is not affected by any other events
- **Dependent** – also called "Conditional", where an event is affected by other events
- **Mutually exclusive** – events can't happen at the same time

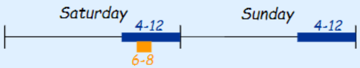
1) Independent events

If two events A and B are **independent**, their **joint** probability equals the product of their probabilities:

$$Pr(A \cap B) = P(A)P(B)$$

Example: your boss (to be fair) randomly assigns everyone an extra 2 hours work on weekend evenings between 4 and midnight.

What are the chances you get Saturday between 6 and 8?



Day: there are two days on the weekend, so **P(Saturday) = 0.5**

Time: you want the **2 hours** of 6-to-8, out of the **8 hours** of 4-to-midnight):

$$\mathbf{P(Your\ Time) = 2/8 = 0.25}$$

And:

$$\begin{aligned}\mathbf{P(Saturday\ and\ Your\ Time)} &= \text{P(Saturday)} \times \text{P(Your Time)} \\ &= 0.5 \times 0.25 \\ &= \mathbf{0.125}\end{aligned}$$

Or a 12.5% Chance

(Note: we could ALSO have worked out that you wanted 2 hours out of a total possible 16 hours, which is 2/16 = 0.125. Both methods work here.)

Figure 1: (*Source*: mathisfun.com)

Notation:

Joint probability – two or more things happen together in some fashion

- $Pr(A \cap B)$ – the probability in which **both** events occur together
- $Pr(A \cup B)$ – the probability in which **one or the other** event occurs

2) Dependent events: Conditional probability

- $Pr(A|B)$ – the probability of a conditional on B –> The probability that A occurs given that B has already occurred
- $P(B|A)$ means "Event B given Event A"
- Example: Winning the lottery with/without buying a ticket
- Central to Bayesian statistics

The formula is:

$$Pr(A \cap B) = Pr(B|A)Pr(A) = Pr(A|B)Pr(B)$$

- 1. The probability that B happens conditional on A's happening, times the chance that A happens.
- 2. The probability that A happens conditional on B's happening, times the chance that B happens

Example: Drawing 2 Kings from a Deck

Event A is drawing a King first, and **Event B** is drawing a King second.

For the first card the chance of drawing a King is 4 out of 52 (there are 4 Kings in a dec of 52 cards):

$$P(A) = 4/52$$

But after removing a King from the deck the probability of the 2nd card drawn is **less** likely to be a King (only 3 of the 51 cards left are Kings):

$$P(B|A) = 3/51$$

And so:

$$\mathbf{P(A\ and\ B) = P(A) \times P(B|A)} = (4/52) \times (3/51) = 12/2652 = \mathbf{1/221}$$

So the chance of getting 2 Kings is 1 in 221, or about 0.5%

Figure 2: (*Source*: mathisfun.com)

3) Mutually exclusive events

When two events are mutually exclusive it is impossible for them to happen together:

- For A and B, $Pr(\text{A AND B}) = 0$

If we want to find out the probability of A **or** B happening:

- A or B is the sum of A and B: $Pr(A \cup B) = Pr(A) + Pr(B)$

Example: King OR Queen

In a Deck of 52 Cards:

- the probability of a King is 1/13, so **P(King)=1/13**
- the probability of a Queen is also 1/13, so **P(Queen)=1/13**

When we combine those two Events:

- The probability of a King **or** a Queen is (1/13) + (1/13) = **2/13**

Which is written like this:

P(King or Queen) = (1/13) + (1/13) = 2/13

Figure 3: (*Source*: mathisfun.com)

Non-mutually exclusive events:

Example: “Kings” and “hearts”

Figure 4: (*Source*: mathisfun.com)

The formula is: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

Let's try to solve a few problems:

1. A card is randomly chosen from a pack of 52 playing cards. What is the probability of a Six or a Seven?
2. There are 30 children in a class and they all have at least one cat or dog. 14 children have a cat, 19 children have a dog. What is the probability that a child chosen at random from the class has both a cat and a dog?
3. In a class of 32 children, 16 have a skateboard, 12 have a bicycle and 17 have a scooter.

- 5 of them have a skateboard and a bicycle.
- 7 of them have a skateboard and a scooter.
- 4 of them have a bicycle and a scooter.
- They all have at least one of the three things.

What is the probability that a child chosen at random from the class has a scooter but not a bicycle?

(*Source*: mathopolis)

Six vs. Seven

Choosing a Six and a Seven are mutually exclusive events.

Therefore, use:

Use $Pr(A \cup B) = Pr(A) + Pr(B)$

There are 52 cards, so:

$\text{Pr}(\text{SIX}) = \frac{1}{13}$ and $\text{Pr}(\text{SEVEN}) = \frac{1}{13}$

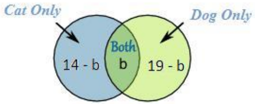
Therefore, $\text{Pr}(\text{SIX or SEVEN}) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$

2) Cats & dogs

Let's say b is how many children have both:

- children having a cat Only must be 14 - b
- children having a dog Only must be 19 - b

And we get:



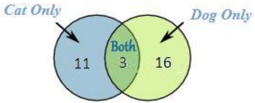
And we know there are 30 children, so:

$\Rightarrow (14 - b) + b + (19 - b) = 30$

$\Rightarrow 33 - b = 30$

$\Rightarrow b = 3$

And we can put in the correct numbers:



So we now know:

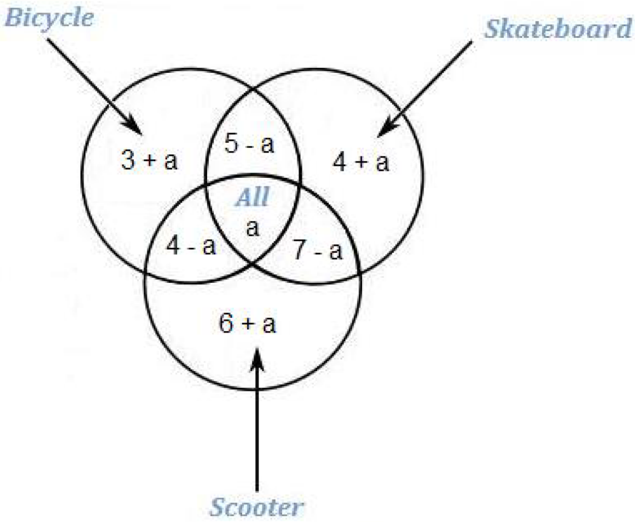
$$\mathrm{P(Both)} = \frac{3}{30} = \frac{1}{10}$$

Figure 5: (*Source*: mathopolis.com)

3) Skaterboards and bycicles

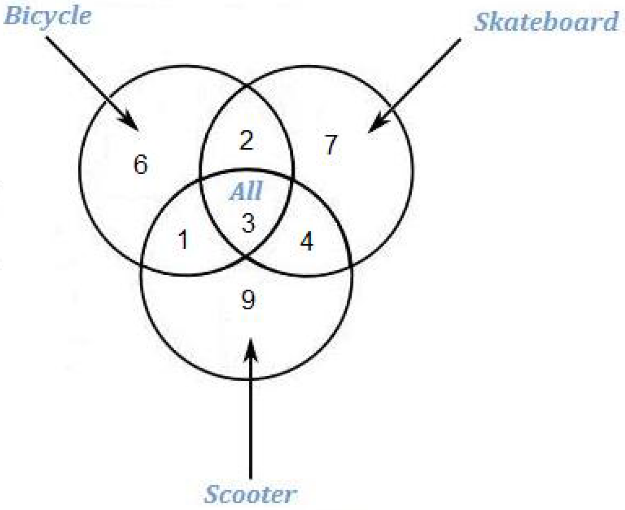
Let's say a is the number of children who have all three things: - children who have a skateboard and a bicycle but not a scooter must be 5 - a. - children who have a skateboard and a scooter but not a bicycle must be 7 - a. - children who have a bicycle and a scooter but not a skateboard must be 4 - a.

- children who have a skateboard, but not a bicycle or a scooter, must be 16 - $[(5 - a) + a + (7 - a)] = 4 + a$.
- children who have a bicycle, but not a skateboard or a scooter, must be 12 - $[(5 - a) + a + (4 - a)] = 3 + a$.
- children who have a scooter, but not a bicycle or a skateboard must be 17 - $[(7-a) + a + (4-a)] = 6 + a$



Source: www.mathopolis.com

And we know there are 32 children altogether, so: $(4 + a) + (5 - a) + a + (7 - a) + (3 + a) + (4 - a) + (6 + a) = 32$ $29 + a = 32$ $a = 3$



Source: www.mathopolis.com

Result: Pr(A child has a scooter but no bicycle) $= \frac{13}{32}$.

Combinations and permutations

A combination is a way of choosing k objects from n objects when one does not care about the order in which one chooses the objects.

$C_n^k = \frac{n!}{k!(n-k)!}$.

A permutation is a way of choosing k objects from n objects when one does care about the order in which one chooses the objects.

$P_n^k = \frac{n!}{(n-k)!}$.

Bayes' Rule

The bedrock of multiple types of quantitative analysis

Takes a prior belief about some event's occurrence and transforms into a posterior belief about that event

- Prior belief –> New data –> posterior belief
- $P(A)$ –> $P(A/B)$

Used in statistics and game theory:

- In statistics, it forms the basis of what's known as Bayesian statistics
- Prior distribution –> new data –> posterior distribution
- In Signaling theory: How the receiver goes from prior to posterior belief based on the signaler's actions/ new information

The Bayes' rule can be very helpful in medicine (diagnostics) – how to properly update prior beliefs based on new information

How does it work?

Recall that:

$$Pr(A \cap B) = Pr(B|A)Pr(A) = Pr(A|B)Pr(B)$$

So, the Bayes' rule (or theorem) states:

$$Pr(A|B) = Pr\frac{Pr(B|A)Pr(A)}{Pr(B)}$$

The probability of some event A conditional on some event B is equal to the probability of B given A devided by the probability of B,

- P(A|B) is "Probability of A given B", the probability of A given that B happens.
- P(A) is Probability of A.
- P(B|A) is "Probability of B given A", the probability of B given that A happens.
- P(B) is Probability of B.

Example:

If some infectious disease is rare (1%) but high fever is fairly common (10%), and the 90% of cases involving this disease exhibit high fever, then:

$$Pr(\text{Disease}|\text{High fever}) = \frac{Pr(\text{Disease})Pr((\text{High fever}|\text{Disease})}{Pr(\text{High Fever})} = \frac{1X90}{10} = 9$$

Result: In this case, 9% of the time expect high fever to mean an infectious disease.

False positives and false negatives:

One of the most common situations when we can use the Bayes' rule

- However, the formula needs to be updated

Example (*Source*: mathopolis.com):

At the Fairtown High Court, people on trial are judged as follows:

- For people who really are guilty, the judgement says "Yes" 97% of the time
- For people who are in fact innocent, the judgement says "Yes" 4% of the time ("false positive")

If 72.5% of people on trial really are guilty, and the judgement for a randomly selected person says "Guilty", what are the chances that the person really is guilty?

How to solve it:

This puzzle can be represented as a table:

	Judgement says "Yes"	Judgement says "No"
Is guilty	**97%**	**3% "False Negative"**
Is innocent	**4% "False Positive"**	**96%**

And as a tree diagram:

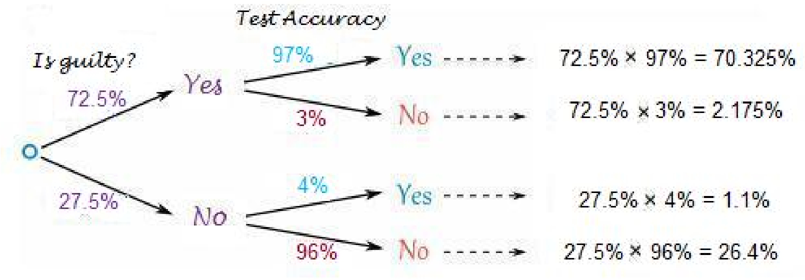


Figure 6: (*Source*: mathopolis.com)

The formula

For false positives/negatives, we use an updated formula:

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{P(A|B)Pr(B) + Pr(A|\text{NOT B})Pr(\text{NOT B})}$$

It reads as follows: "the posterior probability of B given A is the product of the prior probability of B and the probability of A given B divided by the product of the prior probability of B and the probability of A given B plus the product of the prior probability of not B and the probability of A given not B."

Our example:

- Pr(B|A) = x – Chance of being “really guilty”
- Pr(A|B) = 97% – Chance of being guilty when the judgement says “Yes” (True positive)
- Pr(B) = 72.5% – Chance of being guilty
- Pr(A NOT B) = 4% – Chance of recieving a “Yes” judgment while being innocent (False positive)
- Pr (Not B) = 27.5% – Chance of recieving a “No” judgement

$$Pr(B|A) = \frac{Pr(A|B)Pr(B)}{P(A|B)Pr(B) + Pr(A|\text{NOT B})Pr(\text{NOT B})}$$

So:

$$x = \frac{0.97 * 0.725}{0.97 * 72.5 + 0.04 * 0.275} = \frac{0.70325}{0.70325 + 0.011} = \frac{0.70325}{0.71425} = 0.9845$$

Result: In 98.46% of cases, those who receive the “Yes” judgement are really guilty.

Let's practice:

- 1% of women have breast cancer.
- 80% of mammograms detect breast cancer when it is there.
- 9.6% of mammograms detect breast cancer when it's **not** there.

How likely is it to have cancer with a positive result?

Solution:

	Cancer (1%)	No Cancer (99%)
Test Pos	80%	9.6%
Test Neg	20%	90.4%

Figure 7: (*Source*: betterexplained.com)

The formula is:

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{P(B|A)Pr(A) + Pr(B|\text{NOT A})Pr(\text{NOT A})}$$

- Pr(A|B) = Chance of having cancer (A) given a positive test (B). This is what we want to know: How likely is it to have cancer with a positive result? This is our x.
- Pr(B|A) = Chance of a positive test (B) given that one had cancer (A). This is the chance of a true positive, 80% in our case.
- Pr(A) = Chance of having cancer (1%).
- Pr(not A) = Chance of not having cancer (99%).
- Pr(B|not A) = Chance of a positive test (B) given that one didn't have cancer (~A). This is a false positive, 9.6% in our case.

Plug in the numbers:

$$x = \frac{0.8 * 0.01}{0.8 * 0.01 + 0.096 * 0.99} = \frac{0.008}{0.008 + 0.09504} = 0.0776$$

Result: The chance of cancer with a positive result is 7.76%

Odds and odds ratios

The odds of an event is defined as the ratio of the probability of the event's occurring and the probability that it does not occur: $\frac{Pr(y)}{Pr(\neg y)}$.

The odds ratio of two events, x_1 and x_2, then, is the ratio of the individual odds: $\frac{(\frac{Pr(x_1)}{Pr(\neg x_1)})}{(\frac{Pr(x_2)}{Pr(\neg x_2)})}$.

More exercises

1. Characterize the following as independent, mutually exclusive, and/or collectively exhaustive:

1. 33 year-old, middle income, Asian American, male.
2. Vote share, size of the economy, education level.
3. Less, same, more.

1. 33 year-old, middle income, Asian American, male.

Independent (well, largely: gender, race, and age are associated with income level, but the other three are independent).

2. Vote share, size of the economy, education level.

Independent (again, largely: an incumbent political party's vote share is associated with macroeconomic activity).

3. Less, same, more.

Mutually exclusive and collectively exhaustive values of a variable.

Let $P(A) = 0.3$ and $P(A \cup B) = 0.5$. Find P(B), assuming both events are independent.

In general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $P(A \cap B) = P(A|B)P(B)$, which, when combined, yield: $P(A \cup B) = P(A) + P(B) - P(A|B)P(B)$. If the two events are independent, then P(A|B) = P(A), giving $P(A \cup B) = P(A) + P(B) - P(A)P(B) = P(B)(1 - P(A)) + P(A)$.

We solve for P(B) to get $P(B) = \frac{P(A \cup B) - P(A)}{1 - P(A)} = \frac{0.5 - 0.3}{1 - 0.3} = \frac{2}{7}$.

Let P(A) $= 0.4$ and $P(A \cup B) = 0.6$. Find P(B), assuming both events are independent.

$$P(B) = \frac{0.6-0.4}{1-0.4} = \frac{1}{3}.$$

Compute each of the following:

1. $\frac{5!}{6!}$.
2. C_{12}^{5}.
3. P_{7}^{2}.

A committee contains fifteen legislators with ten men and five women. Find the number of ways that a delegation of six: a) Can be chosen. b) With an equal number of men and women can be chosen. c) With a proportional number of men and women can be chosen.

1. This is the number of ways 6 elements can be chosen from 15, or C_{15}^{6}.
2. Now we have the joint probability of two independent events: choosing 3 women from 5 and 3 men from 10. This is: $C_{5}^{3} \times C_{10}^{3}$.
3. Finally, we have the joint probability of two independent events: choosing 2 women from 5 and 4 men from 10, since there are twice as many men as women in the full group. This is: $C_{10}^{4} \times C_{5}^{2}$.

In a certain city, 30% of the citizens are conservatives, 30% are liberals, and 40% are independents. In a recent election, 50% of conservatives voted, 40% of liberals voted, and 30% of independents voted.

1. What is the probability that a person voted?
2. If the person voted, what is the probability that the voter is conservative?

1. $\Pr(V) = \Pr(V|C)\Pr(C) + \Pr(V|L)\Pr(L) + \Pr(V|I)\Pr(I) = 0.39.$

2. $\Pr(C|V) = (\Pr(V|C)\Pr(C))/\Pr(V) = \frac{0.5 \cdot 0.3}{.39} = 0.38.$

If the odds of x_1 are 3:1 and the odds of x_2 are 1:2, what is the odds ratio of $x_1 : x_2$?

$$\frac{3/1}{1/2} = \frac{3}{0.5} = 6.$$