The Derivative

August, 2018

Intro

Definition:

- The derivative helps to understand the slope of the line, the instantaneous rate of change of a function.
- The derivative of a function f at a point x, written $f'(x)$, is given by:

$$f'(x) = \lim_{\triangle x \to \infty} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

if the limit exists.

Sometimes the derivative is written like this:

$$\frac{dy}{dx} = \lim_{\triangle x \to \infty} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

The process of finding a derivative is called **differentiation**.

Why is it useful?

Differentiation, i.e. calculating the rate of change, has applications to nearly all quantitative disciplines, whether it's natural or social science.

- Social scientists use differentiation and rate to determine how people, goods, and processes change due to the change of an independent variable (boundless.com).
- For example, differentiation is useful when we study the changes in 1) the GDP; 2) the flow of traffic; 3) the fertility rate affect any kind of social-economic dimensions.

Financial derivatives are contracts between two parties that specify conditions (especially the dates, resulting values and definitions of the underlying variables, the parties' contractual obligations, and the notional amount) under which payments are to be made between the parties (wiki).

- Some of the more common derivatives include forwards, futures, options, swaps, and variations of these.

What does it all mean in mathematical terms?

To put it simply, to find the derivative of a function $y = f(x)$ we use the slope formula:

Slope = Change in Y / Change in $X = \frac{\triangle y}{\triangle x}$

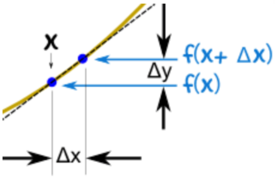


Figure 1: Change in slope (*Source*: www.mathsisfun.com)

Here we see that:

x changes from x to $x + \triangle x$

y changes from $f(x)$ to $f(x + \triangle x)$

Now follow these steps:

- Fill in this slope formula: $\frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$
- Simplify it as best as you can
- Then make $\triangle x$ shrink towards zero.

Let's try an example:

Problem: the function $f(x) = x^2$, let's find the derivative of x^2 (or $\frac{d}{dx}x^2$).

We know $f(x) = x^2$, and can calculate $f(x + \triangle x)$:

Start with: $f(x + \triangle x) = (x + \triangle x)^2$

Expand $(x + \triangle x)^2$: $f(x + \triangle x) = x^2 + 2x\triangle x + (\triangle x)^2$

The slope formula is:

$$\frac{f(x + \triangle x) - f(x)}{\triangle x}$$

1) Put in $f(x + \triangle x)$ and $f(x)$: $\frac{x^2 + 2x\triangle x + (\triangle x)^2 - x^2}{\triangle x}$
2) Simplify (x^2 and $-x^2$ cancel each other out): $\frac{2x\triangle x + (\triangle x)^2}{\triangle x}$
3) Simplify more (divide through by $\triangle x$): $= 2x + (\triangle x)^2$
4) as $\triangle x$ heads towards 0: $= 2x$

Result: the derivative of x^2 is 2

Too put it simply, this means that, for the function x^2, the slope or "rate of change" at any point is $2x$.

For example, when $x = 2$ the slope is $2x = 4$

Let's represent it graphically:

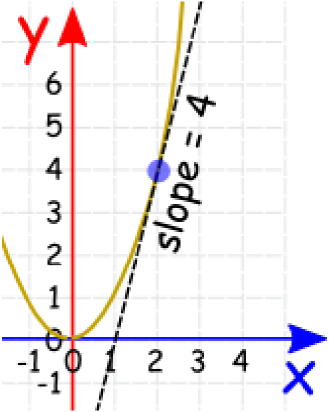


Figure 2: Change in slope (*Source*: www.mathsisfun.com)

Let's use the https://www.mathsisfun.com/calculus/derivative-plotter.html for more visualizations

Let's practice:

Problem: the function $f(x) = x^3$, let's find the derivative of x^3 (or $\frac{d}{dx}x^3$).

- Hint: keep in mind the slope formula: $\frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$

Solution:

We know $f(x) = x^3$, and can calculate $f(x + \triangle x)$:

Start with: $f(x + \triangle x) = (x + \triangle x)^3$

Expand $(x + \triangle x)^3$: $f(x + \triangle x) = x^3 + 3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3$

Again, we use the slope formula:

$$\frac{f(x + \triangle x) - f(x)}{\triangle x}$$

1) Put in $f(x + \triangle x)$ and $f(x)$: $\frac{x^3+3x^2\triangle x+3x(\triangle x)^2+(\triangle x)^3-x^3}{\triangle x}$
2) Simplify (x^3 and $-x^3$ are cancelled): $\frac{3x^2+3x(\triangle x)^2+(\triangle x)^3}{\triangle x}$
3) Simplify more (divide through by $\triangle x$): $= 3x^2 + 3x\triangle x + (\triangle x)^2$
4) as $\triangle x$ heads towards 0: $= 3x^2$

Result: The derivative of x^3 equals $= 3x^2$

Problems:

1) Find $\frac{d}{dx}(x^2 + 5x)$
2) Find $\frac{d}{dx}(2x^3)$
3) Find $\frac{d}{dx}(x^3 - 3x)$

1) Find $\frac{d}{dx}(x^2+5x)$

So
$f(x+\triangle x) = (x+\triangle x)^2+5(x+\triangle x) = x^2+2x\triangle x+(\triangle x)^2+5x+5\triangle x$

The slope formula: $\frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$

$= \frac{x^2+2x\triangle x+(\triangle x)^2+5x+5\triangle x-(x^2+5x)}{\triangle x}$

$= \frac{x^2+2x\triangle x+(\triangle x)^2+5x+5\triangle x-x^2+5x}{\triangle x}$

$= \frac{2x\triangle x+(\triangle x)^2+5\triangle x}{\triangle x}$

$= 2x + \triangle x + 5$

As $\triangle x$ heads towards 0, the result is: $\frac{d}{dx}(x^2+5x) = 2x+5$

2) Find $\frac{d}{dx}(2x^3)$

$f(x) = (2x^3)$

So $f(x + \triangle x) = 2(x + \triangle x)^3 = 2(x^3 + 3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3)$

$= 2x^3 + 6x^2\triangle x + 6x(\triangle x)^2 + 2(\triangle x)^3$

Let's bring in the slope formula: $\frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$

$= \frac{2x^3+6x^2\triangle x+6x(\triangle x)^2+2(\triangle x)^3-2x^3}{\triangle x}$

Now, let's simply this: $= 6x^2 + 6x\triangle x + 2(\triangle x)^2$

As $\triangle x$ heads towards 0, the result is: $\frac{d}{dx}(2x^3) = 6x^2$

3) Find $\frac{d}{dx}(x^3 - 3x)$

$f(x) = x^3 - 3x$

So $f(x + \triangle x) = (x + \triangle x)^3 - 3(x + \triangle x)$

$= x^3 + 3x^2\triangle x + 3x(\triangle x)^2 + (\triangle x)^3 - 3x - 3\triangle x =$

The slope formula: $\frac{\triangle y}{\triangle x} = \frac{f(x+\triangle x)-f(x)}{\triangle x}$

$= \frac{x^3+3x^2\triangle x+3x(\triangle x)^2+(\triangle x)^3-3x-3\triangle x-(x^3-3x)}{\triangle x}$

$= \frac{x^3+3x^2\triangle x+3x(\triangle x)^2+(\triangle x)^3-3x-3\triangle x-x^3+3x}{\triangle x}$

$= \frac{3x^2\triangle x+3x(\triangle x)^2+(\triangle x)^3-3\triangle x}{\triangle x}$

$3x^2 + 3x\triangle x + (\triangle x)^2 - 3$

As $\triangle x$ heads towards 0, the result is: $\frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$

Multivariate functions and the partial derivative

Derivation applies to equations with more than a variable. Consider the equation $f(x,z) = 3z^3 - 3z^2 + \sqrt{z} + x$. Say we want to know how y changes with x, holding z constant. This is knows as taking the **the partial derivative** - $\frac{\partial}{\partial_x} f(x,y)$ or simply ∂_x. It means the following: treat every variable other than x as a constant, and just take the derivative with respect to x. $\frac{\partial}{\partial_x} f(x,z) = 1$.

Marginal effects. $y = \beta_0 + \beta_1 x + \beta_2 z + \beta_3 xz$. Partial derivative with respect to x: $\frac{\partial}{\partial_x} f(x,z) = \beta_1 + \beta_3 z$.

Exercises

1) Find the partial derivative with respect to x:
$f(x, z) = 3z^2 - z + 1.$

2) Find the partial derivative with respect to x:
$f(x, z) = 3zx + 2z + 2z^2x^3.$

3) Find the partial derivative with respect to z:
$f(x, y, z) = 11z + 3x^2y + 5x^2z + 7z^2y.$

4) Find the partial derivative with respect to z:
$f(x, y, z) = 4x^2y^2z^2 + 8xyz + 12xy + 14x.$

5) Find the partial derivative with respect to z:
$f(x, y, z) = 8xyz^2 + 10x^2y^2 + 12x^2y + 14x^2z^2.$

Derivative Rules

Luckily, we don't have to use the slope formula all the time!

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	2x
Square Root	√x	$(\frac{1}{2})x^{-\frac{1}{2}}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	ln(x)	1/x
	$\log_a(x)$	1 / (x ln(a))

Figure 3: Common functions and their derivatives (*Source*: www.mathsisfun.com)

Derivative Rules

Rules	Function	Derivative
Multiplication by constant	cf	cf′
Power Rule	x^n	nx^{n-1}
Sum Rule	f + g	f′ + g′
Difference Rule	f - g	f′ − g′
Product Rule	fg	f g′ + f′ g
Quotient Rule	f/g	$(f' g - g' f)/g^2$
Reciprocal Rule	1/f	$-f'/f^2$

Figure 4: Key derivative rules (*Source*: www.mathsisfun.com)

How can we use the derivative rules?

Example: What is the derivative ($\frac{d}{dx}$) of $x^2 + x^3$?

The sum rule says:

The derivative of $f + g = f' + g'$

So we can take care of each derivative separately and then add them.

We use the power rule:

$\frac{d}{dx}x^2 = 2x$

$\frac{d}{dx}x^3 = 3x^2$

The result:

The derivative of $x^2 + x^3 = 2x + 3x^2$

Let's practice:

1) What is $\frac{d}{dx}(4x^2)$?
2) What is $\frac{d}{dx}(x^3 - 3x^2)$?
3) What is $\frac{d}{dx}(6x^4 - 3x^5 + 5x^6)$?
4) Find the derivative of y with respect to x:
   $y = 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1.$
5) Find the derivative of y with respect to x:
   $y = 7x^4 - 9x^3 + 5x + 117.$
6) Find the derivative of y with respect to x:
   $y = 27x^3 + 5x^2 - x + 13.$

1) What is $\frac{d}{dx}(4x^2)$?

The multiplication by constant rule: $cf = cf'$

This means that the derivative of $4f$ is $4f'$

Next, we know from the power rule:

$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x$

The results is:

$\frac{d}{dx}(4x^2) = 4X2x = 8x$

2) What is $\frac{d}{dx}(x^3 - 3x^2)$?

The difference rule:

The derivative of $f - g = f' - g'$

Now let's use the power rule:

1) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$
2) $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$ meaning that $\frac{d}{dx}(3x^2) = 3X2x = 6x$

Now we can solve the problem:

$\frac{d}{dx}(x^3 - 3x^2) = 3x^2 - 6x$

3) What is $\frac{d}{dx}(6x^4 - 3x^5 + 5x^6)$?

Use the power rule:

$\frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3$, therefore $\frac{d}{dx}(6x^4) = 6X4x^3 = 24x^3$

$\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$, therefore $\frac{d}{dx}(3x^5) = 3X5x^4 = 15x^4$

$\frac{d}{dx}(x^6) = 6x^{6-1} = 6x^5$, therefore $\frac{d}{dx}(5x^6) = 5X6x^5 = 30x^5$

The final answer: $\frac{d}{dx}(6x^4 - 3x^5 + 5x^6) = 24x^3 - 15x^4 + 30x^5$

More on the rules for differentiation

1) The derivative is a linear operator: $\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$. $\frac{d(cf)}{dx} = c\frac{df}{dx}$.
2) Chain rule: $(g(f(x)))\prime = g\prime(f(x))f\prime(x)$.

Examples: $f(x) = 2x^2 \quad g(x) = e^x, \quad h(x) = g(f(x)). \quad h\prime(x) = g\prime(u)f\prime(x) = e^u(4x) = 4xe^{2x^2}$.
$g(x) = x^2 \quad f(x) = 2x - a, \quad h(x) = g(f(x)) = (2x-a)^2. \quad h\prime(x) = g(f(x))\prime = g\prime(f(x))f\prime(x) = 2(2x-a)(2) = 8x - 4a$.

3) Products and quotients: $(fg)\prime = f\prime g + fg\prime$. $(\frac{f}{g})\prime = \frac{f\prime g - fg\prime}{g^2}$.

Example: $f(x) = 3x - 7, \quad g(x) = x^3 + 6$. $(\frac{f}{g})\prime =$

$$\frac{\frac{d(3x-7)}{dx}(x^3+6)-(3x-7)(\frac{d(x^3+6)}{dx})}{(x^3+6)^2} = \frac{(x^3+6)(3)-(3x-7)(3x^2)}{36+12x^3+x^6} = \frac{-6x^3+21x^2+18}{36+12x^3+x^6}.$$

Sum rule	$(f(x)+g(x))' = f'(x)+g'(x)$
Difference rule	$(f(x)-g(x))' = f'(x)-g'(x)$
Multiply by constant rule	$f'(ax) = af'(x)$
Product rule	$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$
Quotient rule	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain rule	$(g(f(x)))' = g'(f(x))f'(x)$
Inverse function rule	$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$
Constant rule	$(a)' = 0$
Power rule	$(x^n)' = nx^{n-1}$
Exponential rule 1	$(e^x)' = e^x$
Exponential rule 2	$(a^x)' = a^x(\ln(a))$
Logarithm rule 1	$(\ln(x))' = \frac{1}{x}$
Logarithm rule 2	$(\log_a(x))' = \frac{1}{x(\ln(a))}$
Trigonometric rules	$(\sin(x))' = \cos(x)$
	$(\cos(x))' = -\sin(x)$
	$(\tan(x))' = 1+\tan^2(x)$
Piecewise rules	Treat each piece separately

Exercises

1) Find the derivative of y with respect to x: $y = (\frac{x^2+1}{x+1})^2$.
2) Find the derivative of y with respect to x: $y = e^{x - ln(x) + 5}$
3) $f(x) = x^2 g(x) + 6x^2, \quad where \quad g(x) = \log_a(x) + x^7$.
4) $y = ax^{n-1}$.
5) $f(x) = \frac{x^2-4}{x^5-x^3+x}$.
6) $f(x) = a_n x^n + a_{n-1} x^{n-1} ... + a_0$. Try also expressing the derivative as a series.

Application of derivatives - optimization theory

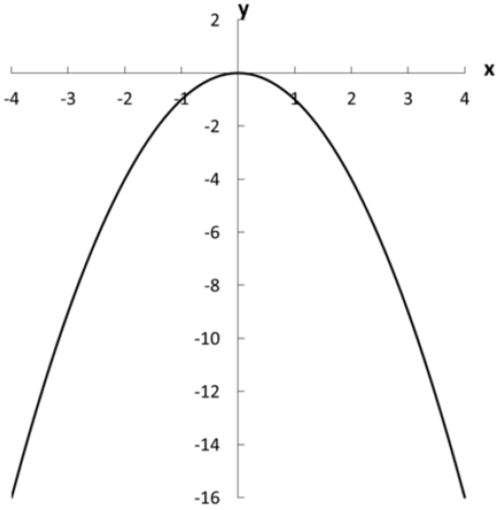
Find the maximum and the minimum of a function - maxima and minima as types of *extrema*.

Maximum and minimum are concepts just as straightforward as you might expect. They are the high and low points of a function, or the "peaks" and "valleys" in the graph of a function. **A high point** is called a maximum and a low point is called a minimum. Together these two points are referred to as **the extrema of a function**.

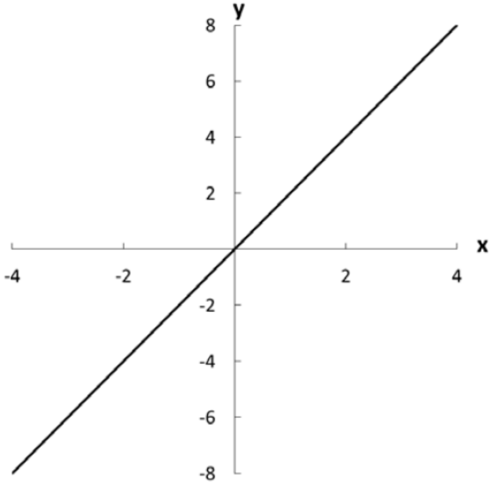
$f(x) = x^2$ - no maximum; $g(x) = -x^2$ - no minimum.

Definitions: - First, the slope of the line tangent to an extremum itself will always be zero, and hence the first derivative of the function at a point that is an extremum will always equal zero as well. - Second, if we were to graph the first derivative, an extremum will always be a point on that graph that crosses the x-axis, i.e., a point where the slope of the tangents to points in the original function changes signs.

Graph of $f(x) = -x^2$



Graph of first derivative of $f(x) = -x^2$



An extremum is local whenever it is the largest (or smallest) value of the function over some interval of values the domain of the function.

A function with a rate of increase that slows as the value of the function gets bigger is an example of **a concave function**.

f is **concave** if for any points x_1, x_2 in its domain and any weight $\lambda \in [0, 1]$ $f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2)$.

A function with a rate of increase that speeds up as the value of the function gets bigger is an example of **a convex function**.

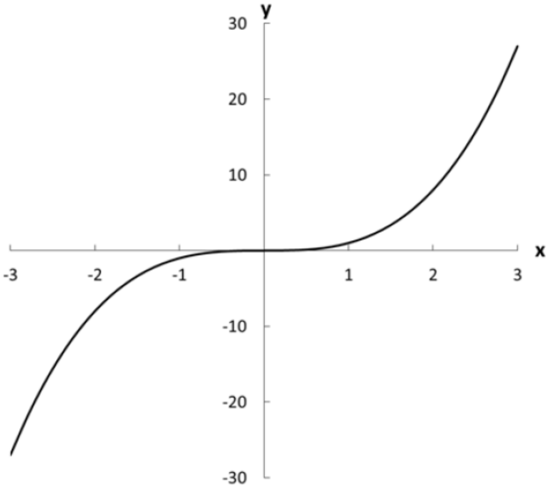
f is **convex** if for any points x_1, x_2 in its domain and any weight $\lambda \in [0, 1]$ $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.

Points at which the first and second derivatives change from positive to negative or vice versa, implying a stopover at zero, have substantive meaning.

A critical point is any point x^* such that either $f\prime(x^*) = 0$ or $f\prime(x^*)$ doesn't exist. Loosely, critical points are points in the function's domain at which things happen. Either the function blows up, or it jumps, or it is stationary. Fermat: local extrema occur at critical points, and we are interested in finding extrema.

Not all critical points are extrema. Some are instead **inflection points**, which are points at which the graph of the function changes from concave to convex or vice versa.

Graph of first derivative of $f(x) = x^3$



Procedure for finding the global minimum and maximum of a function (M&S).

1. Find $f'(x)$.
2. Set $f'(x^*) = 0$ and solve for all x^*. These are stationary points of the function.
3. Find $f''(x)$.
4. For each stationary point x^*, substitute x^* into $f''(x)$.
   - If $f''(x^*) < 0$, $f(x)$ has a local maximum at x^*.
   - If $f''(x^*) > 0$, $f(x)$ has a local minimum at x^*.
   - If $f''(x^*) = 0$, x^* may be an inflection point. To check this:
     a) Calculate higher-order derivatives ($f'''(x), f^{(4)}(x)$, etc.) until you find the first one that is non-zero at x^*. Call the order of this derivative n.
     b) If n is odd, then this x^* is an inflection point and not an extremum. Do not include it in further steps.
     c) If n is even and $f^{(n)}(x^*) < 0$, $f(x)$ has a local maximum at x^*.
     d) If n is even and $f^{(n)}(x^*) > 0$, $f(x)$ has a local minimum at x^*.
5. Substitute each local extremum into $f(x)$ to find the function's value at that point.
6. Substitute the lower and upper bounds of the domain over which you are attempting to find the extrema into $f(x)$ to find the function's values at those points.
7. Find the smallest value of the function from those computed in the previous two steps. This is the global minimum, and the function attains this at the corresponding x^* or boundary point. Find the largest value of the function from those computed in the previous two steps. This is the global maximum, and the function attains this at the corresponding x^* or boundary point.