Intro + Basic Algebra

August 2018

Workshop objectives

- To go over major mathematical concepts used in graduate-level statistics courses.
- To see how different mathematical concepts are used in social scientific research.

This will allow you:

- To be better prepared for taking stats courses.
- To understand the math behind different types of statistical analysis.
- To better understand academic literature.
- To integrate and use statistics in your own research projects.

Overview

The workshop will cover:

- Basic rules of algebra (first day).
- Functions.
- Limits.
- Derivatives.
- Integrals.
- Probabilities.
- Linear algebra.

Since it is tailored to your needs, the schedule is somewhat flexible.

Structure

One hr. morning session followed by three 1.5 hrs. sessions:

- Brief introduction (40% of time).
- Solving problems (60% of time).

We will have three breaks (see the schedule).

Why do we need to know basic algebra?

- To communicate ideas clearly.
- To study general concepts.
- To understand and use statistics.
- To be able to grasp more advanced math.
- Finally, to use it in our research projects.

Sets

Notation	Meaning
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real (rational and irrational) numbers
$\mathbb{C}$	Complex numbers
Subscript: $\mathbb{N}_+$	Positive (negative) values of the set
Superscript: $\mathbb{N}^d$	Dimensionality (number of dimensions)

Preliminaries

A set is a collection of elements. Finite or infinite, countable or uncountable, bounded or unbounded.

The number of elements in a **finite set** is finite; there is no limit to the number of elements in an **infinite set** - N.

A countable set is one whose elements can be counted. **An uncountable set** does not have this property. N vs. $x \in [0, 1]$, $x \in R$.

A bounded set has finite size (but may have infinite elements), while an **unbounded set** does not.

$A \subset B$ - A is a proper subset of B.

$A \subseteq B$ - A is a subset of B.

The cardinalility of a set is the number of elements in that set. $|A|$

A singleton is a set with only one element (i.e. cardinality of one).

An **empty set** (or **null set**) is a set with nothing in it and is written $\emptyset$.

An **universal set** is the set that contains all elements.

Set can be ordered or unordered. {a, b, c} vs. {c, a, b}.

Operators

The **difference** between two sets - $A \setminus B$ (A difference B).
$x \in A \setminus B$ if $x \in A$ and $x \notin B$.

The **the complement of a set** - A' or A^c - contains the elements that are not contained in A.

The intersection of two sets, $A \cap B$ (A intersection B), is the set of elements common to both series. **The union of two sets** is written $A \cup B$ and it the set of all the elements of contained in either set.

Mutually exclusive sets are sets with an intersection equal to the empty set, i.e., sets with no elements in their intersection.

A partition is is the collection of subsets whose union forms the set.

A Cartesian product: two sets A and B, with $x \in A$ and $y \in B$; $A \times B$ is the set consisting of all possible ordered pairs (x, y).

Exercise

Let A = $\{1, 5, 10\}$ and B = $\{1, 2, \ldots, 10\}$.

1. Is $A \subset B$, $B \subset A$, both or neither?
2. What is $A \cup B$?
3. What is $A \cap B$?
4. Partition B into two sets, A and everything else. Call everything else C. What is C?
5. What is $A \cup C$?
6. What is $A \cap C$?

Solution

1. $A \subset B$ because all the elements of A are contained in B, but the reverse is not true.
2. $A \cup B = \{1, 1, 2, 3, 4, 5, 5, 6, 7, 8, 9, 10, 10\}$
3. $A \cap B = \{1, 5, 10\} = \text{A}$
4. $\text{C} = \{2, 3, 4, 6, 7, 8, 9\}$
5. B because B was partitioned into the sets A and C, implying that the elements of A and C together make up the entirety of B.
6. $\emptyset$.

Exercise

Write an element of the Cartesian product $[0;1] \times (1;2)$.

Solution

Anything with two components, e.g., (0:5; 1:5). The Cartesian product creates a larger two-dimensional space from the two one-dimensional spaces, and all components of the space (i.e., the elements of the ordered pair) must be in the constituent sets. Thus the first element must be in [0; 1] and the second in (1; 2).

Symbols and notation

Symbol	Meaning
$+$	Addition
$-$	Subtraction
$*$ or $\times$ or $\cdot$	Multiplication
$/$ or $\div$	Division
$\pm$	Plus or minus
x^n	Exponentiation ("to the nth power")
$\sqrt[n]{x}$	Radical or nth root
!	Factorial
∞	Infinity
$\sum_{i=k}^{l} x_i$	Sum of x_i from index $i = k$ to $i = l$
$\prod_{i=k}^{l} x_i$	Product of x_i from index $i = k$ to $i = l$
$\ldots$	Continued progression
$\frac{d}{dx}$	Total derivative with respect to x
$\frac{\partial}{\partial x}$	Partial derivative with respect to x
$\int dx$	Integral over x
$\cup$	Set union
$\cap$	Set intersection
$\times$	Cartesian product of sets
$\setminus$	Set difference
A^c	Complement of set A
$\emptyset$	Empty (or null) set
$\in$	Set membership
$\notin$	Not member of set
$\mid$ or : or $\ni$	Such that
$\subset$	Proper subset
$\subseteq$	Subset
$<$	Less than
$\leq$	Less than or equal to
$=$	Equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\neq$	Not equal to
$\equiv$	Equivalent to or Defined as
$f()$ or $f(\cdot)$	Function
$\{\ \}$	Delimiter for discrete set
$(\ )$	Delimiter for open set
$[\ ]$	Delimiter for closed set
$\forall$	For all (or for every or for each)
$\exists$	There exists
$\Rightarrow$	Implies
$\Leftrightarrow$	If and only if
$\neg C$ or $\sim C$	Negation (not C)

Greek letters

Upper-case	Lower-case	English	Upper-case	Lower-case	English
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

Let's go over some basic algebraic rules:

- Basic formulas:

$$(a+b)=(b+a).$$

$$ab=ba \text{ (commutative property)}.$$

$$1a=a.$$

$$a+0=a \text{ (identity property)}.$$

$$a+(-a)=0 \text{ (inverse property)}.$$

$$x^{-1}\times x=1.$$

$$(ab)c=a(bc).$$

$$(a+b)+c=a+(b+c) \text{ (associative property)}.$$

$$(-a)b = a(-b).$$

$$(-a)(-b) = ab.$$

$$-(a + b) = -a - b.$$

$$a(b + c) = ab + ac \text{ (distributive property).}$$

$$(a + b)(c + d) = ac + ad + bc + bd.$$

$$a \div 0 = \infty.$$

$$a^0 = 1.$$

$$a = 1a = a^1 = 1a^1.$$

$$-1 \times a = -a.$$

Special binomial products:

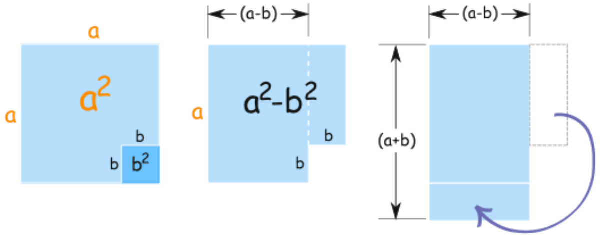
$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a-b)(a+b) = a^2 - b^2$ – the "difference of squares".

FOIL vs. factor.

We can illustrate that:



Source: mathisfun.com

Order of operations (reminder)

In arithmetic and algebra the order of operations is parentheses, exponents, multiplication, division, addition, subtraction - PEMDAS.

Exercises

1. $x^1 =$.
2. $-a \times (-b^2) =$.
3. $\sum_{i=1}^{4} x_i =$.
4. $\prod_{m=6}^{9} x_m =$.
5. $4! =$.
6. $\sqrt[2]{9} =$.
7. $\sqrt[3]{27} =$.
8. $(\frac{3(2-4)}{2+3})^3 =$.

Ratios, proportions, percentages

The ratio of two quantities is one divided by the other - $\frac{x}{y}$ - is the ratio of x to y. Ratios are also written as $x : y$.

The proportion of two variables is the amount one variable represents of the sum of itself and a second variable: $\frac{x}{x+y}$.

The percentage one variable represents of a total is the proportion represented over the range from 0 to 100. In other words, the percentage is a linear transformation of the proportion $|\frac{x}{x+y}| \times 100$.

The percentage change in a variable: $\frac{x_{t+1} - x_t}{x_t}$.

Fractions

$\frac{Numerator}{Denominator}$.

$2\frac{3}{4} = \frac{11}{4}$.

$\frac{10x}{2} = 5$.

Addition - same denominator $\frac{\beta}{4} + \frac{\alpha}{3} = \frac{3\beta+4\alpha}{12}$.

Factoring - $\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1, x \neq 1$.

Let's practice:

1. Simplify $(z \times x) \times y \times \frac{1}{z} =$.
2. Expand and simplify $a^2(5b - a) - a(3b^2 - a^2) =$.
3. Simplify $(3x + 2y)^2 - (3x - 2y)^2 =$.
4. Simplify $(a + b)^2 + (a - b)^2 + 2(a + b)(a - b) =$.
5. Simplify $\frac{2x^2+20x+50}{2x^2-50} =$.
6. Add these fractions $\frac{2g+13}{3g} + \frac{4g-5}{4g} =$.
7. FOIL: $(2x - 3)(5x + 7) =$.
8. Factor: $q^2 - 10q + 9 =$.
9. Factor and reduce: $\frac{\beta-\alpha}{\beta^2-\alpha^2} =$.

1) Expand and simplify $a^2(5b - a) - a(3b^2 - a^2)$

First, we need to solve:

$a^2(5b - a) = 5a^2b - a^3$

and

$a(5b^2 - a^2) = 3ab^2 - a^3$

So: $a^2(5b - a) - a(3b^2 - a^2)$

$= 5a^2b - a^3 - (3ab^2 - a^3)$

$= 5a^2b - a^3 - 3ab^2 + a^3$ – two minus signs make a plus, so $-(-a^3)$ becomes $+a^3$

$= 5a^2b - 3ab^2$

2) Simplify $(3x + 2y)^2 - (3x - 2y)^2$

Let's expand both parts first:

For $(3x + 2y)^2$:

$$= (3x)^2 + 2 \cdot 3x \cdot 2y + (2y)^2$$

$$= 9x^2 + 12xy + 4y^2$$

For $(3x - 2y)^2$:

$$= (3x)^2 - 2 \cdot 3x \cdot 2y + (2y)^2$$

$$= 9x^2 - 12xy + 4y^2$$

Finally:

$$(3x + 2y)^2 - (3x - 2y)^2$$

$$= 9x^2 + 12xy + 4y^2 - (9x^2 - 12xy + 4y^2)$$

$$= 9x^2 + 12xy + 4y^2 - 9x^2 + 12xy - 4y^2$$

$$= 24xy$$

3) Simplify $(a+b)^2 + (a-b)^2 + 2(a+b)(a-b)$

Strategy 1:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$2(a+b)(a-b) = 2(a^2 - b^2) = 2a^2 - 2b^2$

So:

$(a+b)^2 + (a-b)^2 + 2(a+b)(a-b)$

$= a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + 2a^2 - 2b^2$

$= 4a^2$

An alternative strategy for $(a+b)^2 + (a-b)^2 + 2(a+b)(a-b)$:

$= (a+b)^2 + 2(a+b)(a-b) + (a-b)^2$

$= [(a+b) + (a-b)]^2$

$= [2a]^2$

$= 4a^2$

Equations with a single unknown:

The easiest way to solve an equation with a single unknown is to:

1. Isolate all of the terms containing the unknown on one side of the equation.
2. Isolate all of the terms that do not contain the unknown on the other side.
3. Combine all of the terms containing the unknown, and all those not containing the unknown.

Example:

Solve $3x - 6 = 12$

Start with: $3x - 6 = 12$

Add 6 to both sides: $3x = 12 + 6$

Divide by 3: $x = \frac{12+6}{3} = 6$

The result: $x = 6$

Problems:

1. Solve $5x + 2 = -13$.
2. Solve $\frac{1}{4}x - 3 = 4$.
3. Solve $\frac{3x}{x+1} + 6 = \frac{-3}{x+1}$.
4. Solve $.30\Omega + .05 = .25$.
5. Solve $15\delta + 45 - 6\delta = 36$.

1) Solve $5x + 2 = -13$

Start with $5x + 2 = -13$

Subtract 2 from both sides: $5x = -13 - 2 = -15$

Divide by 5: $x = \frac{-15}{5}$

The result: $x = -3$

2) Solve $\frac{1}{4}x - 3 = 4$

First, we add 3 to both sides:

$\frac{1}{4}x - 3 + 3 = 4 + 3$

$= \frac{1}{4}x = 7$

Next, we multiply both sides by 4:

$= \frac{1}{4}x \cdot 4 = 7 \cdot 4$

Hence, the result is $x = 28$

3) Solve $\frac{3x}{x+1} + 6 = \frac{-3}{x+1}$

Multiply all terms by $\frac{x+1}{1}$

$$\frac{3x}{x+1} \times \frac{x+1}{1} + 6 \times \frac{x+1}{1} = \frac{-3}{x+1} \times \frac{x+1}{1}$$

$\Rightarrow$ 3x + 6(x + 1) = -3

Expand the bracket

3x + 6x + 6 = -3

$\Rightarrow$ 9x + 6 = -3

Subtract 6 from both sides:

9x + 6 - 6 = -3 - 6

$\Rightarrow$ 9x = -9

Divide both sides by 9

Therefore x = -1

However, replacing x by -1 in the original equation gives

$$\frac{3x}{0} + 6 = \frac{-3}{0}$$

But division by zero is undefined

Therefore x = -1 does not actually work, and so:

There is no solution!

Source: mathopolis.com

Inequalities

Addition: If $x > y$, then $x + a > y + a$.

Multiplication: If $a > 0$, then $x > y | \times a$ is $ax > ay$. If $a < 0$, then $x > y | \times a$ is $ax < ay$ (the sign flips).

Example:

Solve for y: $-4y > 2x + 12$.

$-4y > 2x + 12 | \div (-4)$. $\Rightarrow y < (-\frac{x}{2}) - 3$.

Exercises

1. $-\Gamma > \frac{\Gamma+4}{7}$.
2. $5(x+2) - 3x \leq 4 + 2x + 3(x-1)$.
3. $z + z[3 - 2(1+z)] < 5[1 + 2(z-3)] - 2z^2$.
4. $\frac{3x+1}{x+4} \geq 1$.
5. $\frac{x-8}{x} \leq 3 - x$.

Quadratic equations

$ax^2 + bx + c = 0$

where:

- a, b and c are known values. a can't be 0.
- "x" is the variable or *unknown* (we don't know it yet).

In $2x^2 + 5x + 3 = 0$ a=2, b=5, and c=3

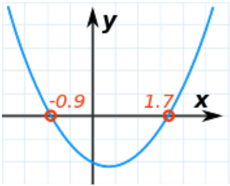
Let's use http://www.mathopenref.com/quadraticexplorer.html for some visualizations

How to solve quadratic equations?

We must use a special formula:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

- This means that quadratic equations have two answers:



Let's practice: Solve $5x^2 + 6x + 1 = 0$

Coefficients: a = 5, b = 6, c = 1

Quadratic formula:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Plug in a, b and c:

$$x = \frac{-6 \pm \sqrt{(6^2 - 4 \cdot 5 \cdot 1)}}{2 \cdot 5}$$

Now we can solve it:

$$x = \frac{-6 \pm \sqrt{(16)}}{10}$$

Which gives us:

$$x = \frac{-6 \pm 4}{10}$$

So: x = -0.2 **or** -1

Let's use http://www.mathopenref.com/quadraticexplorer.html to check it out

Problems:

1. Solve the quadratic equation $6x^2 + 7x - 3 = 0$.
2. Solve the equation $x(10x - 1) = 2$.
3. Solve the equation $x^2 + 14x - 14 = 0$.
4. Complete the square and solve for y: $\frac{1}{3}y^2 + \frac{2}{3}y - 16 = 0$.
5. Solve using the quadratic formula: $2x^2 + 5x - 7$.
6. Solve the equation $x^2 + x - 4 = 0$.
7. $6x^2 + 11x - 35 = 0$.
8. $x^2 - 48 = 0$.
9. $x^2 - 7x = 0$.

1) Solve the quadratic equation $6x^2 + 7x - 3 = 0$

Coefficients are: a = 6, b = 7, c = -3

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a,b,c: $x = \dfrac{-7 \pm \sqrt{7^2 - 4 \times 6 \times (-3)}}{2 \times 6}$

Solve: $x = \dfrac{-7 \pm \sqrt{49 + 72}}{12}$

$$= \frac{-7 \pm \sqrt{121}}{12}$$

$$= \frac{-7 \pm 11}{12}$$

$$= \frac{1}{3} \text{ or } -1.5$$

Source: mathopolis.com

2) Solve the equation $x(10x - 1) = 2$

First, we need to transform this into a quadratic equation:

What is $x(10x - 1) = 2$?

It is: $10x^2 - x - 2 = 0$

Now we can solve it as a quadratic equation:

Coefficients: a=10, b= -1, c = -2

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Plug in a, b, c:

$$x = \frac{-(-1) \pm \sqrt{((-1)^2 - 4 \cdot 10 \cdot (-2))}}{2 \cdot 10}$$

Solve it:

$$x = \frac{1 \pm \sqrt{1 + 80)}}{20}$$