Functions

August 2018

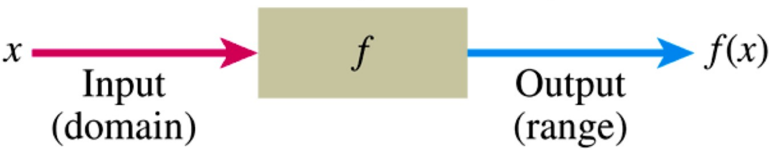
Functions: Intro

Definition:

- A function describes the relationship between two or more concepts or variables (a mapping of one set to another).

$f(x)$ – function *of* x.

Commonly functions are visualized as producing an output (usually outputs are represented with y, $f(x)$, $g(x)$, etc.) when given an input (often x is used to represent the inputs).

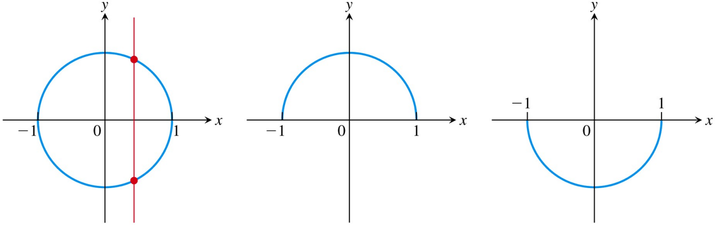


Functions are used in:

- Game theory
    - Preferences and payoffs (utility functions).
- Social sciences in general
    - Hypotheses building and testing.
    - Very common.

Examples

1. $y = 2x$ is a function because there is exactly one output, y, for every input, x.
2.



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Note: The circle (a) is *not* a function as each x-value is assigned to two y values (as indicated by the red vertical line). The semicircles (b) and (c) are both functions.

Functions: Basic rules and concepts

A function assigns only one value of y to any x. Thus, for any value of x, there is one y value that corresponds. However, any given value of y can correspond to multiple values of x. For example:

$y = x^2$

Each value of x produces one value of y. However, each value of y (except $y = 0$) corresponds to two values of x: for instance, when $y = 4$, $x = 2$ or $x = -2$.

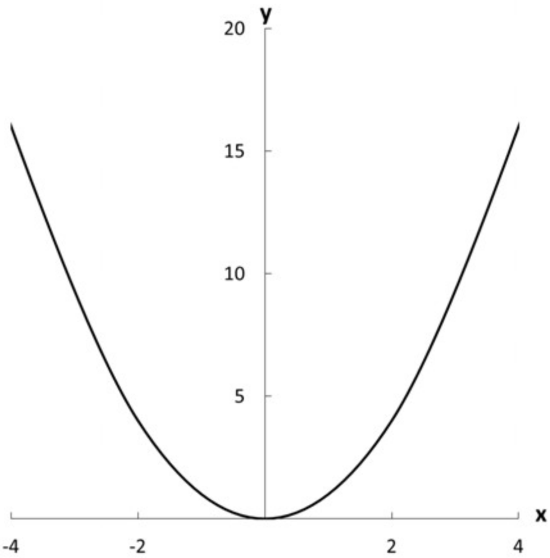


Figure 1: Graph of $y = x^2$ (*Source*: M&S:59)

Functions: Basic rules and concepts

$f(x) : A \rightarrow B$, where:

- A is the **domain** – the set of elements over which the function is defined
- B is the the **codomain** – the set from which values of $f(x)$ may be drawn

$f(x) = x + 1$

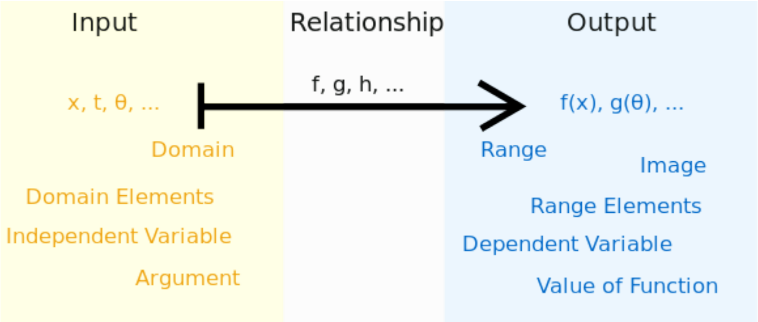
or (equivalently)

$y = x + 1$

If you input 4 as x, your output is 5.

In this function, x is called the **input**, **independent** or **exogenous** variable. y or $f(x)$ is the **output**, **dependent** or **endogenous** variable.

Notation matters:



Source: mathisfun.com

Identity and inverse function terms

Term	Meaning
Identity function	Elements in domain are mapped to identical elements in codomain
Inverse function	Function that when composed with original function returns identity function
Surjective (onto)	Every value in codomain produced by value in domain
Injective (one-to-one)	Each value in range comes from only one value in domain
Bijective (invertible)	Both surjective and injective; function has an inverse

A function is **surjective or onto** if every value in the codomain is produced by some value in the domain. A is surjective if $\forall b \in B, \exists a \in A \ni f(a) = b$ (for all b in B there exists an a in A such that the function of a is b). $f : R \rightarrow R, f(x) = x$ vs. $f : (0,1) \rightarrow R, f(x) = x$. First function - surjective, because every point in R was reached by some point in the domain (the same point). The second was not surjective, as nothing outside (0, 1) in the codomain was reached.

A function is **injective or one-to-one** if each value in the range comes from only one value in the domain. $\forall a, c \in A, \quad \forall b \in B, \quad if \quad f(a) = b \quad and \quad f(c) = b, \quad then \quad a = c$. $f : R \rightarrow R, f(x) = x^2$ - not injective. For example, $y = 4$ is the result of plugging both $x = 2$ and $x = -2$ into the function (it would be injective if we confined ourselves to real numbers no less than zero, though).

If a function is both injective and surjective (one-to-one and onto), then it is **bijective**. A bijective function is **invertible**, and so has an inverse.

Monotonic function

- Definition: a function of one variable, defined on a subset of the real numbers, whose increment $\triangle f(x) = f(x\prime) - f(x\prime)$, for $\triangle x = x\prime - x > 0$, does not change sign, that is, is either always negative or always positive.
- Strictly/weakly increasing or decreasing functions.
- Examples of monotonic and non-monotonic functions.

Monotonic Function Terms

Term	Meaning
Increasing	Function increases on subset of domain
Decreasing	Function decreases on subset of domain
Strictly increasing	Function always increases on subset of domain
Strictly decreasing	Function always decreases on subset of domain
Weakly increasing	Function does not decrease on subset of domain
Weakly decreasing	Function does not increase on subset of domain
(Strict) monotonicity	Order preservation; function (strictly) increasing over domain

Example

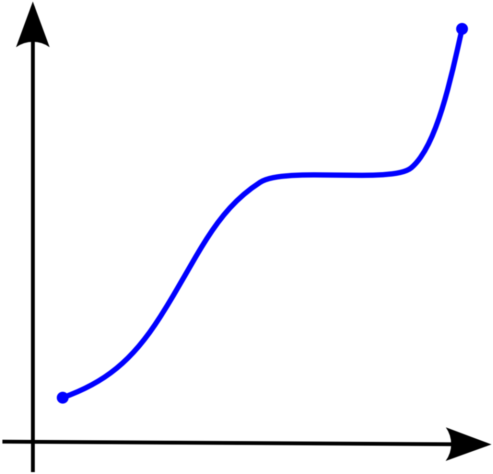


Figure 2: Graph of a monotonically weakly increasing function (*Source*: Wikipedia)

Linear function

- Definition: The linear equation (affine equation) states that the size of the impact of x on y is constant across all values of x.
- Linear functions are those whose graph is a straight line.

A linear function has the following form:

$$f(x) = a + bx$$

A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y.

a is the constant term or the y intercept. It is the value of the dependent variable when $x = 0$.

b is the coefficient of the independent variable. It is also known as the slope and gives the rate of change of the dependent variable.

Example:

According to Shaffer (1981), the probability that voters participate in a US presidential election is a linear function of education:

Probability of voting $= a + b(edu.)$

Probability of voting $= 1.215 + 0.134 * edu$

Shaffer uses four levels of education: 0-8 years of education, 9-11 years, 12 years, and more than 12 years

Conclusion: with each level of education, the probability of voting increases by $\sim$.13.

Graphing a linear function

To graph a linear function:

1. Find 2 points which satisfy the equation.
2. Plot them.
3. Connect the points with a straight line.

Example:

$y = 25 + 5x$

let $x = 1$

then

$y = 25 + 5(1) = 30$

let $x = 3$

then

$y = 25 + 5(3) = 40$

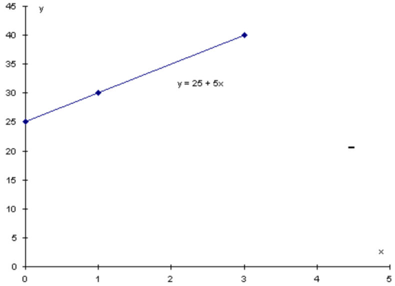


Figure 3: Graph of $y = 25 + 5x$ (*Source*: www.columbia.edu/itc/sipa/math/linear.html)

Another example:

A company receives \$45 for each unit of output sold. It has a variable cost of \$25 per item and a fixed cost of \$1600.

What is its profit if it sells (a) 75 items, (b)150 items, and (c) 200 items?

$R(x) = 45x$
$C(x) = 1600 + 25x$

$P(x) = 45x - (1600 + 25x) = 20x - 1600$

- let $x = 75$ $P(75) = 20(75) - 1600 = -100$ a loss
- let $x = 150$ $P(150) = 20(150) - 1600 = 1400$
- let $x = 200$ $P(200) = 20(200) - 1600 = 2400$

Source: www.columbia.edu

Linear functions: features

- Defined by additivity and scaling

Additivity (superposition): $f(x_1 + x_2) = f(x_1) + f(x_2)$

- The impact of a sum of variables is equivalent to the sum of the impacts of those variables

Scaling (homogeneity): $f(cx) = cf(x)$ (c is constant)

- The size of the input is proportional to the size of the output

$f(x) = ax$ (where a is the slope of the function)

Linear equation vs. linear function

A linear equation is an equation that contains only terms of order x^1 and $x^0 = 1$.

A linear function is characterised by additivity and scaling.

Linear function: $y = f(x) = \beta x$.

$$f(x_1 + x_2) = \beta(x_1 + x_2) = \beta x_1 + \beta x_2, \quad \beta x_1 + \beta x_2 = f(x_1) + f(x_2).$$

Linear equation: $y = f(x) = \alpha + \beta x$.

$$f(x_1 + x_2) = \alpha + \beta(x_1 + x_2) = \alpha + \beta x_1 + \beta x_2,$$
$$f(x_1) + f(x_2) = (\beta x_1 + \alpha) + (\beta x_2 + \alpha),$$
$$\alpha + \beta x_1 + \beta x_2 \neq 2\alpha + \beta x_1 + \beta x_2$$

Nonlinear functions: exponents

Exponents (aka power functions) are a shorthand for expressing the multiplication of a number by itself:
$x^n = x \times x \times x ... \times x \quad (n \quad times)$.

$x^{-n} = \frac{1}{x^n}$.

$x^{\frac{2}{3}} = \sqrt[3]{x^2}$. Consider $f(x) = x^2$. Non-linear relation: the slope of the curve for $y = x^2$ is not constant: the impact of x on y changes as we move along the x-axis (i.e., consider different values of x).

Multiplication: $x^m \times x^n = x^{m+n}$.

$x^m \times z^n = (xz)^m$.

$x^m + z^n \neq (xz)^{m+n}$.

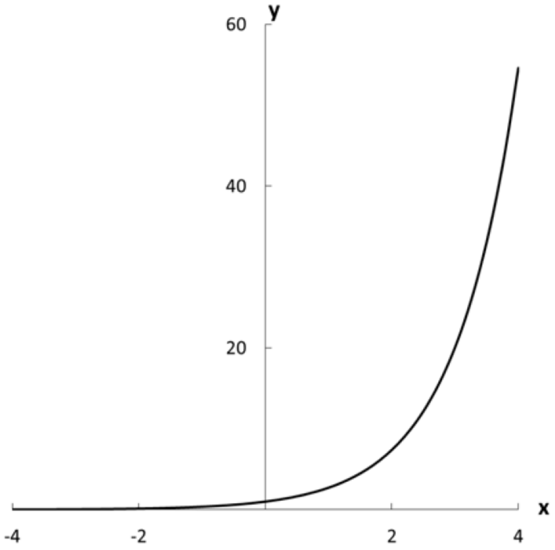
$(x^m)^n = x^{mn}$.

$\frac{x^m}{x^n} = x^{m-n} = x^m \times x^{-n}$

$\frac{x^m}{z^m} = (\frac{x}{z})^m$.

The exponential function: not x^a, but a^x. The most commonly used exponential function: $a = e$, where e is the base of the natural logarithm, $e \approx 2.7183$. $y = exp(x) = e^x$.

Graph of the exponential function



Exercises

1. Simplify $x^{-2} \times x^3$.
2. Simplify $(b \cdot b \cdot b) \times c^{-3}$.
3. Simplify $((qr)^\gamma)^\delta$.
4. Solve $x - xe^{5x+2} = 0$.
5. Solve $5(x^2 - 4) = (x^2 - 4)e^{7-x}$.

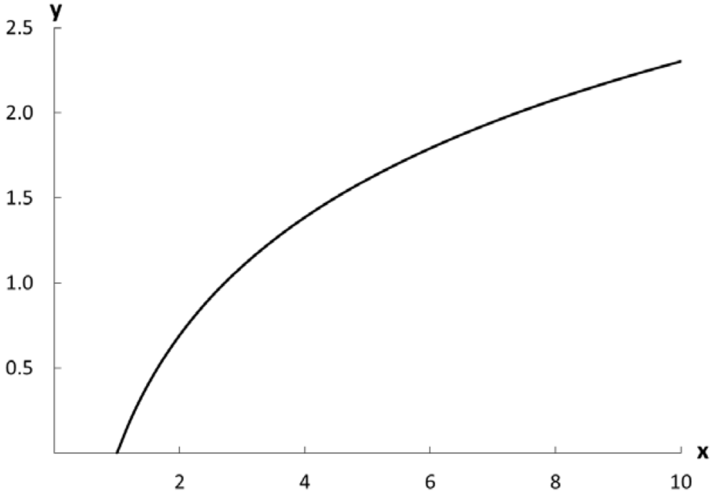
Logarithms

Logarithms can be understood as the inverses of exponents (and vice versa). They can be used to transform an exponential function to a linear one, or a linear function to a nonlinear one in which the impact of one variable on another declines as the first variable rises in value. $\log_a x, \quad a > 0, \quad a \neq 1.$ $a^{\log_a x} = x \quad \textit{and} \quad \log_a a^x = x.$

Logs can be written in any base, though the most common are base 10 and the natural log ($ln(x)$).

$ln(1) = 0, \quad ln(x) = 0 \quad \textit{when} \quad x = 1, \quad ln(x) < 0 \quad \textit{when} \quad 0 < x < 1.$ $ln(x_1 \times x_2) = ln(x_1) + ln(x_2), \quad x_1, x_2 > 0.$
$ln(\frac{x_1}{x_2}) = ln(x_1) - ln(x_2), \quad x_1, x_2 > 0.$
$ln(x_1 + x_2) \neq ln(x_1) + ln(x_2), \quad \textit{for} \quad x_1, x_2 > 0.$
$ln(x_1 - x_2) \neq ln(x_1) - ln(x_2), \quad \textit{for} \quad x_1, x_2 > 0.$
$ln(x^b) = b ln(x), \quad \textit{for} \quad x > 0.$
$ln(1 + x) \approx x, \quad x > 0 \quad \textit{and} \quad x \approx 0.$

Graph of the natural logarithm function



Exercises

1. Simplify into one term $ln(3x) - 2ln(x+2)$.
2. Rewrite the following by taking the log of both sides. Is the result a linear (affine) function? $y = \alpha \times {x_1}^{\beta 1} \times {x_2}^{\beta 2} \times {x_3}^{\beta 3}$.
3. Solve $2ln(\sqrt{x}) - ln(1-x) = 2$.
4. Solve $\log_{10}(x) + \log_{10}(x-3) = 1$.
5. Solve $3 + 2ln(\frac{x}{7} + 3) = (-4)$.

Solution to the second exercise

$ln(y) = ln(\alpha) + \beta 1 ln(x_1) + \beta 2 ln(x_2) + \beta 3 ln(x_3)$. *Yes*. Two rules were used: $ln(xy) = ln(x + ln(y)$ $\quad ln(x^a) = aln(x)$.

Radicals (or Roots)

Roots (sometimes called radicals) are those numbers represented by the radical symbol: $\sqrt[n]{x}$. They are (almost) the inverse functions of x raised to the power n:
$\sqrt[n]{x^n} = x, \quad as \quad long \quad as \quad n = 2k \quad or \quad x \geq 0$. Functions with radicals are nonlinear: $y = \sqrt[n]{x}$.

$\sqrt[n]{x^p} = x^{\frac{p}{n}}$. $\sqrt[n]{x} + \sqrt[n]{x} \neq \sqrt[n]{x + x}, \quad n > 1$.
$\sqrt[a]{x} + \sqrt[b]{y} \neq \sqrt[a+b]{x + y}, \quad a, b > 1$. $\sqrt[n]{x} + \sqrt[n]{z} = \sqrt[n]{xz}, \quad n > 1$.
$\sqrt[a]{x} + \sqrt[b]{z} \neq \sqrt[ab]{xz}, \quad a, b > 1$. $\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}}, \quad for \quad n > 1$.
$\frac{\sqrt[a]{x}}{\sqrt[b]{z}} \neq \sqrt[a+b]{\frac{x}{z}}, \quad for \quad a \neq b \quad and \quad a, b > 1$.

Exercises

1. Simplify $\sqrt{x} \quad \times \quad \sqrt[5]{x}$.
2. Simplify $\sqrt[2]{294x^4}$.
3. Simplify $\sqrt[5]{-486x^2y^6}$.
4. Simplify $\sqrt[3]{a^4} - \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^{-1}}} + \sqrt[3]{\frac{27}{a^{-1}}}$.
5. Write as a simple radical and simplify: $\frac{\sqrt[2]{(2x-3)^3} \cdot \sqrt[4]{2x-3}}{\sqrt[3]{(2x-3)^4}}$.

Nonlinear functions: quadratic

$$f(x) = ax^2 + bx + c$$

- Quadratic functions are generally used when you believe that the rate of increase in some variable is itself increasing

Example: Muller and Seligson (1987) on rebellion and government control

- Rebellion will be low in countries that exert very little or very high government coercion

$$r = \alpha + \beta_1 c - \beta_2 c^2$$

where, r is rebellion and c is coercion

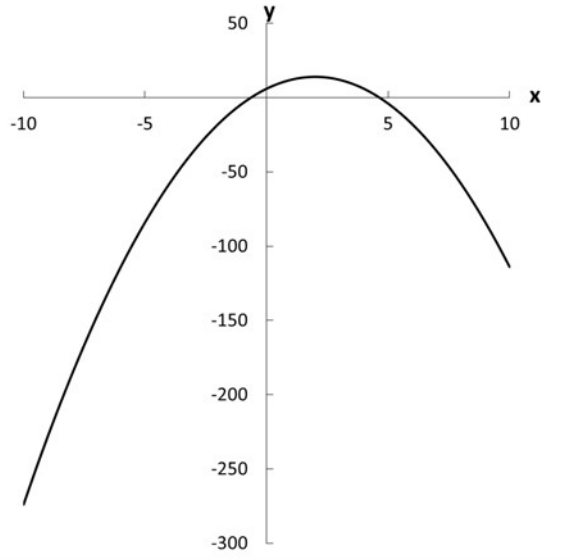
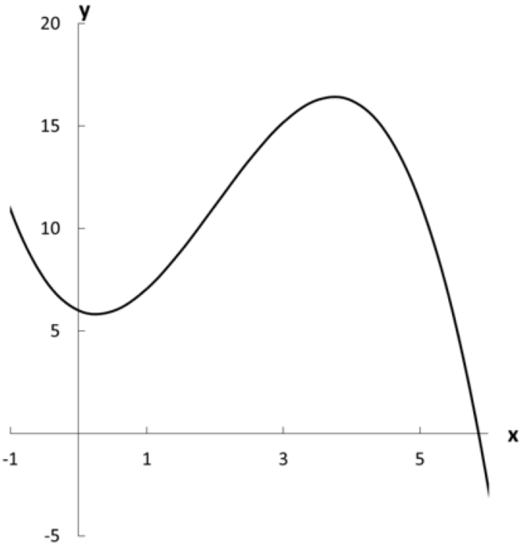


Figure 4: Graph of $y = 6 + 5x - 2x^2$ (*Source*: M&S:63)

Graph of a cubic polinomial



Function transformation:

Addition:

$$(f+g)(x) = f(x) + g(x)$$

Example: $f(x) = 2x + 4$ and $g(x) = x^2$

$(f+g)(x) = (2x+4) + (x^2) = x^2 + 2x + 3$

Subtraction:

$$(f-g)(x) = f(x) - g(x)$$

Multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Example: $f(x) = 2x + 3$ and $g(x) = x^2$

$(f \cdot g)(x) = (2x+3) \cdot (x^2) = 2x^3 + 3x^2$

Inverse function

- Definition: the inverse of a function has all the same points as the original function, except that the x's and y's have been reversed

$f^{-1}(f(x)) = x$

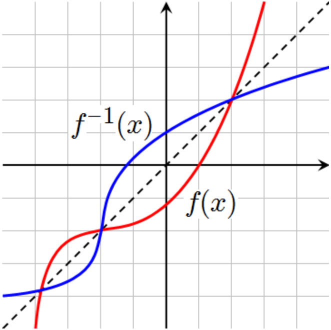


Figure 5: Graph of an inverse function (*Source*: Wikipedia)

Let's practice

Find the inverse of $y = 3x - 2$.

The inverse of $y = 3x - 2$:

$$y = 3x - 2$$

$$y + 2 = 3x$$

$$\frac{y+2}{3} = x$$

$$x = \frac{y+2}{3}$$

$$f^{-1}(y) = \frac{y+2}{3}$$

Problems:

1. What is the inverse of the function $f(x) = 5x - 4$?
2. What is the inverse of the function $f(x) = \frac{1}{2}(3x + 4)$?
3. Find the inverse of $f(x) = 3x + 4$.

1) What is the inverse of the function $f(x) = 5x - 4$

Change "y" for "f(x)": $y = 5x - 4$

Add 4 to both sides: $y + 4 = 5x$

Divide both sides by 5: $\frac{y+4}{5} = x$

Swap sides: $x = \frac{y+4}{5}$

The result: $f^{-1}(y) = \frac{y+4}{5}$

2) What is the inverse of the function $f(x) = \frac{1}{2}(3x + 4)$?

$y = \frac{1}{2}(3x + 4)$

Multiply by 2: $2y = 3x + 4$

We get to: $3x = 2y - 4$

Divide by 3: $x = \frac{1}{3}(2y - 4)$

Optional: extract 2: $x = \frac{2}{3}(y - 2)$

So: $f^{-1}(y) = \frac{2}{3}(y - 2)$

Function composition

Definition: function composition is the pointwise application of one function to the result of another to produce a third function.

- To put it simply: **function composition** is applying one function to the results of another

$(gof)(x)$ or $g(f(x))$ – g composed with f or more commonly g of f of x.

How do you find the composition of two functions?

Here are the steps we can use to find the composition of two functions:

Step 1: Rewrite the composition in a different form. For example, the composition $(fog)(x)$ needs to rewritten as $f(g(x))$.

Step 2: Replace each occurrence of x found in the outside function with the inside function. For example, in the composition of $(fog)(x) = f(g(x))$, we need to replace each x found in $f(x)$, the outside function, with $g(x)$, the inside function.

Step 3: Simplify the answer.

- How does it work?

Lets say, $f(x) = 5x + 3$ and $g(x) = x^2$, then

$g(f(x)) = (5x + 3)^2$

Practice

1. Given $f(x) = 2x + 3$ and $g(x) = -x^2 + 6$, find (g o f)(x).
2. Given $f(x) = \frac{x}{x+1}$ and $g(x) = 9x - 3$, find (g o f)(x).
3. Given $f(x) = 3x^2 + 2x - 5$ and $g(x) = 2x - 3$, find (f o g) (x) and (g o f) (x).
4. Simplify $h(x) = g(f(x)), \quad where \quad f(x) = x^2 + 2 \quad and \quad g(x) = \sqrt[2]{x - 4}$.

1) Given $f(x) = 2x + 3$ and $g(x) = -x^2 + 6$, find (g o f)(x):

$$(gof)(x) = g(f(x))$$
$$= g(2x + 3)$$
$$= -(...)2 + 6$$
$$= -(2x + 3)^2 + 6$$
$$= -(4x^2 + 12x + 9) + 5$$
$$= -4x^2 - 12x - 9 + 6$$
$$= -4x^2 - 12x - 3$$

2) Given $f(x) = \frac{x}{x+1}$ and $g(x) = 9x - 3$, find $(g \circ f)(x)$:

To find $(g \circ f)(x)$ we will substitute $f(x) = \frac{x}{x+1}$ into every variable that occurs in g.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g\left(\frac{x}{x+1}\right) \\
&= 9\left(\frac{x}{x+1}\right) - 3 \\
&= \frac{9x}{x+1} - 3 \\
&= \frac{9x}{x+1} - \frac{3(x+1)}{x+1} \\
&= \frac{9x - 3(x+1)}{x+1} \\
&= \frac{9x - 3x - 3}{x+1} \\
&= \frac{6x - 3}{x+1}
\end{aligned}$$

3) Given $f(x) = 3x^2 + 2x - 5$ and $g(x) = 2x - 3$, find (f o g) (x) and (g o f) (x)

To find $(f \circ g)(x)$ we will substitute g in for every variable that occurs in f.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x-3) \\ &= 3(2x-3)^2 + 2(2x-3) - 5 \\ &= 3(4x^2 - 12x + 9) + 4x - 6 - 5 \\ &= 12x^2 - 36x + 27 + 4x - 6 - 5 \\ &= 12x^2 - 32x + 16\end{aligned}$$

To find $(g \circ f)(x)$ we will substitute f into every variable that occurs in g.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 + 2x - 5) \\ &= 2(3x^2 + 2x - 5) - 3 \\ &= 6x^2 + 4x - 10 - 3 \\ &= 6x^2 + 4x - 13\end{aligned}$$