Notes taken during class on day 2 (raw and unpolished)

1 The Becker-Shapley-Shubik model / discrete optimal transport

If employee x matches with college y, then

job amenity α_{xy} – GIVEN

production γ_{xy} – GIVEN

Assume that there are n_x employees of type x, and m_y colleges of type y.

Total output generated by an xy pair

$$\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$$

Introduce wages w_{xy} paid by y to x – ENDOGENOUS; DETERMINED AT EQUILIBRIUM.

After transfer,

x gets indidrect utility

$$u_x = \max_y \{\alpha_{xy} + w_{xy}, 0\}$$

and y gets indirect utility v_y

$$v_y = \max_x \{\gamma_{xy} - w_{xy}, 0\}$$

If unmatched, then x and y get 0.

We are looking for an equilibrium / stable matching μ_{xy}=number of xy pairs.

Constraints are given by the populations:

$\sum_y \mu_{xy} \leq n_x$

$\sum_x \mu_{xy} \leq m_y$

Conditions for μ_{xy} to be an equilibrium matching???

$$\mu_{xy} > 0 \implies \left\{ \begin{array}{l} y \in \arg\max_y \{\alpha_{xy} + w_{xy}\} \\ x \in \arg\max_x \{\gamma_{xy} - w_{xy}\} \end{array} \right.$$

$$\begin{aligned} \mu_{xy} > 0 \quad &\implies \quad \left\{ \begin{array}{l} u_x = \alpha_{xy} + w_{xy} \\ v_y = \gamma_{xy} - w_{xy} \end{array} \right. \\ &\implies \quad u_x + v_y = \alpha_{xy} + \gamma_{xy} = \Phi_{xy} \end{aligned}$$

For any x and any y we have $u_x + v_y \geq \Phi_{xy}$.

$u_x \geq 0$

$v_y \geq 0$

To recap, (μ, u, v) is an equilibrium matching iff
(1) μ satisfies the populations constraints
$\sum_y \mu_{xy} \leq n_x$
$\sum_x \mu_{xy} \leq m_y$
(2) pairwise stability:
For any x and any y we have $u_x + v_y \geq \Phi_{xy}$.
$u_x \geq 0$
$v_y \geq 0$
(3) $\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$
$\sum_y \mu_{xy} < n_x \implies u_x = 0$
$\sum_x \mu_{xy} < m_y \implies v_y = 0$

These are optimality conditions associated with the linear programming problem whose primal formulation is

$$\begin{array}{cl} \max\limits_{\mu \geq 0} & \sum\limits_{xy} \mu_{xy} \Phi_{xy} \\ & \sum\limits_{y} \mu_{xy} \leq n_x \ \ [u_x \geq 0] \\ & \sum\limits_{x} \mu_{xy} \leq m_y \ \ [v_y \geq 0] \end{array}$$

and dual formulation

$$\begin{array}{cl} \min\limits_{u \geq 0, v \geq 0} & \sum n_x u_x + \sum m_y v_y \\ s.t. & u_x + v_y \geq \Phi_{xy} \ \ \left[\mu_{xy} \geq 0\right] \end{array}$$

$\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$
$\sum_y \mu_{xy} < n_x \implies u_x = 0 \ \ [u_x > 0 \implies \sum_y \mu_{xy} = n_x]$
$\sum_x \mu_{xy} < m_y \implies v_y = 0$

1.1 Recovering the wages

For any x and y,

$$\begin{array}{rcl} u_x & \geq & \alpha_{xy} + w_{xy} \\ v_y & \geq & \gamma_{xy} - w_{xy} \end{array}$$

this implies

$$u_x - \alpha_{xy} \geq w_{xy} \geq \gamma_{xy} - v_y$$

2 The Choo-Siow model / regularized optimal transport

Choo Siow JPE (2006)
Galichon Salanie (2020)

2.1 Smooth-max

Replace the max

$\max_y \{a_y\}$

By the smooth-max

$T > 0$

$T \log \sum_y \exp(a_y/T) = \max_y \{a_y\} + T \log \sum_y \exp\left(\frac{a_y - \max_y\{a_y\}}{T}\right)$

$0 \leq T \log \sum_{y \in Y} \exp\left(\frac{a_y - \max_y\{a_y\}}{T}\right) \leq T \log |Y|$

2.2 Smooth version of the above model

$u_x = T \log\left(1 + \sum_y \exp\left(\frac{\alpha_{xy} + w_{xy}}{T}\right)\right) = E\left[\max_y \{\alpha_{xy} + w_{xy} + T\varepsilon_y, T\varepsilon_0\}\right]$

$v_y = T \log\left(1 + \sum_x \exp\left(\frac{\gamma_{xy} - w_{xy}}{T}\right)\right)$

$$
\begin{aligned}
\Pr\left(y \in \arg\max_y \{\alpha_{xy} + w_{xy} + T\varepsilon_y, T\varepsilon_0\}\right) &= \frac{\exp\left(\frac{\alpha_{xy} + w_{xy}}{T}\right)}{1 + \sum_y \exp\left(\frac{\alpha_{xy} + w_{xy}}{T}\right)} \\
&= \exp\left(\frac{\alpha_{xy} + w_{xy} - u_x}{T}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Pr\left(0 \in \arg\max_y \{\alpha_{xy} + w_{xy} + T\varepsilon_y, T\varepsilon_0\}\right) &= \frac{1}{1 + \sum_y \exp\left(\frac{\alpha_{xy} + w_{xy}}{T}\right)} \\
&= \exp(-u_x/T)
\end{aligned}
$$

But $\Pr(y \in \arg\max_y \{\alpha_{xy} + w_{xy} + T\varepsilon_y, T\varepsilon_0\}) = \mu_{xy}/n_x$, thus

$$
\begin{aligned}
\mu_{xy} &= \exp\left(\frac{\alpha_{xy} + w_{xy} - u_x + T \ln n_x}{T}\right) \\
\mu_{x0} &= \exp(-u_x + T \ln n_x)
\end{aligned}
$$

similarly,

$$
\begin{aligned}
\mu_{xy} &= \exp\left(\frac{\gamma_{xy} - w_{xy} - v_y + T \ln m_y}{T}\right) \\
\mu_{0y} &= \exp(-v_y + T \ln m_y)
\end{aligned}
$$

Introduce

$$
\begin{aligned}
a_x &= u_x - T \ln n_x \\
b_y &= v_y - T \ln m_y
\end{aligned}
$$

so that we have

$$\mu_{xy} = \exp\left(\frac{\alpha_{xy} + w_{xy} - a_x}{T}\right)$$

similarly,

$$\mu_{xy} = \exp\left(\frac{\gamma_{xy} - w_{xy} - b_y}{T}\right)$$

Multiplying the previous two equations pairwise, we get

$$\mu_{xy}^2 = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{T}\right)$$

where it is recalled that $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$ is the total output. Similarly,

$$\mu_{x0} = \exp\left(-\frac{a_x}{T}\right) \text{ and } \mu_{0y} = \exp\left(-\frac{b_y}{T}\right)$$

To summerize, we arrive at Choo and Siow's formula

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2T}\right).$$

Recall the population equations:
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$

Plug in Choo-Siow's formula into the population equations

$$\left\{ \begin{array}{l} \sum_y \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2T}\right) + \mu_{x0} = n_x \\ \sum_x \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2T}\right) + \mu_{0y} = m_y \end{array} \right.$$

3 Computation issues

Given α_{xy} and γ_{xy}, compute w_{xy} and μ_{xy}.
We will see 2 directions to solve this system

- the "optimization way" – see this system as foc of a convex optimization problem

  $$\min_{a,b} F(a, b)$$

  where F is convex

- the "gross substitute way" – see this system as an equilibrium problem with gross substitutes

  $$e(p) = q$$

  where e has the GS property – and hence Coordinate Update algorithms (such as Gauss-Seidel or Jacobi) can be used.

3.1 Convex optimization formulation

Recall the system to be solved

$$\begin{aligned}
\mu_{xy} &= \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \\
\mu_{x0} &= \exp\left(-\frac{a_x}{T}\right) \text{ and } \mu_{0y} = \exp\left(-\frac{b_y}{T}\right)
\end{aligned}$$

we have

$$\begin{cases}
-\sum_y \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right) + n_x = 0 \\
-\sum_x \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{b_y}{T}\right) + m_y = 0
\end{cases}$$

I want to interpret as

$$\begin{aligned}
\frac{\partial F(a,b)}{\partial a_x} &= 0 \\
\frac{\partial F(a,b)}{\partial b_y} &= 0
\end{aligned}$$

Exercise. What is F such that

$$\begin{aligned}
\frac{\partial F(a,b)}{\partial a_x} &= n_x - \sum_y \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right) \\
\frac{\partial F(a,b)}{\partial b_y} &= m_y - \sum_x \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) - \exp\left(-\frac{b_y}{T}\right)
\end{aligned}$$

it is

$$\begin{aligned}
F(a,b) &= \sum_x n_x a_x + \sum_y m_y b_y \\
&\quad +2T \sum_{x,y} \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) \\
&\quad +T \sum_x \exp\left(-\frac{a_x}{T}\right) + T \sum_y \exp\left(-\frac{b_y}{T}\right)
\end{aligned}$$

when $T \to 0$?
the smooth penalization $2T \sum_{x,y} \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right)$
becomes a hard penalization that is equal to zero if $\Phi_{xy} - a_x - b_y \leq 0$
and $+\infty$ else
therefore enforcing the constraint $a_x + b_y \geq \Phi_{xy}$.
Similarly, the smooth penalization $T \sum_x \exp\left(-\frac{a_x}{T}\right)$
becomes $a_x \geq 0$
Therefore the problem becomes
$\min \sum_x n_x a_x + \sum_y m_y b_y$
s.t. $a_x + b_y \geq \Phi_{xy}$, $a_x \geq 0$ $b_y \geq 0$

3.2 Equilibrium with gross substitutes formulation

How can we reformulate the system of equations in (a, b)

$$\begin{cases} -\sum_y \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right) + n_x = 0 \\ -\sum_x \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{b_y}{T}\right) + m_y = 0 \end{cases}$$

$$\begin{cases} -\sum_y \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right) = -n_x \\ \sum_x \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) + \exp\left(-\frac{b_y}{T}\right) = m_y \end{cases}$$

as

$$e_z\left(p\right) = q_z$$

where $e_z\left(p\right)$ is increasing in p_z and weakly decreasing in $p_{z'}$, for $z' \neq z$?

$Z = X \cup Y$

$p_x = a_x$ for $x \in X$

$e_x\left(p\right) = -\sum_y \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right)$

$$\begin{aligned} q_x &= -n_x \\ q_y &= m_y \end{aligned}$$

3.3 Initialization

Let's initialize $p_x^0 = -T \ln n_x$ and $p_y^0 = -\infty$.

Then update p_y^1 so that $e_y\left(\left(p_x^0\right)_x, p_y^1\right) = q_y (= m_y)$.

we solve $\sum \exp\left(\Phi_{xy} - p_x^0 + p_y^1\right) + \exp\left(p_y^1\right) = m_y$

Because the value of p_y^1 is finite,

we have

$$p_y^0 = -\infty \leq p_y^1$$

Then update p_x^1 so that $e_x\left(p_x^1, \left(p_y^1\right)_y\right) = q_x (= -n_x)$. We have

$\sum_y \exp\left(\Phi_{xy} - p_x^1 + p_y^1\right) + \exp\left(-\frac{p_x^1}{T}\right) = n_x \geq \exp\left(-\frac{p_x^1}{T}\right)$

thus $\exp\left(-\frac{p_x^0}{T}\right) = n_x \geq \exp\left(-\frac{p_x^1}{T}\right)$

thus

$$p_x^0 \leq p_x^1$$

We have $m_y \geq \exp\left(p_y/T\right)$, thus $p_y \leq T \ln m_y$.

$p_y = -b_y$ for $y \in Y$

$$
\begin{aligned}
e_x\left(p\right) &= -\sum_y \exp\left(\frac{\Phi_{xy} - p_x + p_y}{2T}\right) - \exp\left(-\frac{p_x}{T}\right) \\
e_y\left(p\right) &= \sum_x \exp\left(\frac{\Phi_{xy} - p_x + p_y}{2T}\right) + \exp\left(\frac{p_y}{T}\right)
\end{aligned}
$$

We have

$$
De = \begin{pmatrix} \frac{\partial e_x}{\partial p_x} & \frac{\partial e_x}{\partial p_y} \\ \left(\frac{\partial e_x}{\partial p_y}\right)^\top & \frac{\partial e_y}{\partial p_y} \end{pmatrix} = \begin{pmatrix} diag\left(\frac{1}{2T}\exp\left(\frac{\Phi_{xy}-p_x+p_y}{2T}\right) + \frac{1}{T}\exp\left(-\frac{p_x}{T}\right)\right) & \\ -\frac{1}{2T}\exp\left(\frac{\Phi_{xy}-p_x+p_y}{2T}\right) & diag\left(\frac{1}{2T}\exp\left(\frac{\Phi_{xy}-p_x+p_y}{2T}\right) + \frac{1}{T}\exp\right.
\end{pmatrix}
$$

Let's run the Gauss-Seidel algorithm

- Solve for a_x in the first set of equations

$$
-\sum_y \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) - \exp\left(-\frac{a_x}{T}\right) + n_x = 0
$$

- Solve for b_y in the second set of equations

$$
\sum_x \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) + \exp\left(-\frac{b_y}{T}\right) = m_y
$$

Introduce new unknowns $A_x = \exp\left(-\frac{a_x}{2T}\right)$ and $B_y = \exp\left(-\frac{b_y}{2T}\right)$ and $K_{xy} = \exp\left(\frac{\Phi_{xy}}{2T}\right)$, we have

$$
\begin{aligned}
\sum_y K_{xy} A_x B_y + A_x^2 &= n_x \\
\sum_x K_{xy} A_x B_y + B_y^2 &= m_y
\end{aligned}
$$

thus

$$
A_x^2 + 2A_x\left(\frac{1}{2}\sum_y K_{xy} B_y\right) + \left(\frac{1}{2}\sum_y K_{xy} B_y\right)^2 = n_x + \left(\frac{1}{2}\sum_y K_{xy} B_y\right)^2
$$

thus

$$
\begin{aligned}
A_x &= \sqrt{n_x + \left(\frac{1}{2}\sum_y K_{xy} B_y\right)^2} - \frac{1}{2}\sum_y K_{xy} B_y \\
B_y &= \sqrt{m_y + \left(\frac{1}{2}\sum_x K_{xy} A_y\right)^2} - \frac{1}{2}\sum_x K_{xy} A_y
\end{aligned}
$$

4 Identification issues

Given μ_{xy} and possibly w_{xy}, can we compute α_{xy} and γ_{xy}.

Recall Choo and Siow's formula

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2T}\right).$$

Scale $T = 1$, and we get

$$\alpha_{xy} + \gamma_{xy} = \Phi_{xy} = \ln \frac{\mu_{xy}^2}{\mu_{x0}\mu_{0y}}.$$

Assume μ_{xy} is observed, then one can identify $\alpha + \gamma$ only.

Now assume μ_{xy} and w_{xy} are observed, we have

$$\begin{aligned} \alpha_{xy} &= \ln \frac{\mu_{xy}}{\mu_{x0}} - w_{xy} \\ \gamma_{xy} &= w_{xy} + \ln \frac{\mu_{xy}}{\mu_{0y}} \end{aligned}$$

5 Taxes

Assume "flat tax" ie

x gets indidrect utility

$$u_x = \max_y \{\alpha_{xy} + (1-\tau) w_{xy}, 0\}$$

and y gets indirect utility v_y

$$v_y = \max_x \left\{\gamma_{xy} - w_{xy}, 0\right\}$$

Equilibrium:
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
$u_x \geq 0, v_y \geq 0$
$u_x \geq \alpha_{xy} + (1-\tau) w_{xy}$ with equality if $\mu_{xy} > 0$
$v_y \geq \gamma_{xy} - w_{xy}$ with equality if $\mu_{xy} > 0$.

Rewrite as:
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
$u_x \geq 0, v_y \geq 0$
$u_x \geq \alpha_{xy} + (1-\tau) w_{xy}$ with equality if $\mu_{xy} > 0$
$(1-\tau) v_y \geq (1-\tau) \gamma_{xy} - (1-\tau) w_{xy}$ with equality if $\mu_{xy} > 0$.

Denote $\tilde{v}_y = (1-\tau) v_y$ and $\tilde{\gamma}_{xy} = (1-\tau)\gamma_{xy}$ and $\tilde{w}_{xy} = (1-\tau) w_{xy}$ the indirect utility of the firm and the output measured in post-tax dollars, then we have

$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
$u_x \geq 0, v_y \geq 0$
$u_x \geq \alpha_{xy} + \tilde{w}_{xy}$ with equality if $\mu_{xy} > 0$
$\tilde{v}_y \geq \tilde{\gamma}_{xy} - \tilde{w}_{xy}$ with equality if $\mu_{xy} > 0$.
THEREFORE
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
$u_x \geq 0, v_y \geq 0$
$u_x + \tilde{v}_y \geq \alpha_{xy} + \tilde{\gamma}_{xy}$ with equality if $\mu_{xy} > 0$

Thus μ_{xy} and $(u_x, \tilde{v}_y)$ are solution to

$$
\begin{aligned}
& \max \sum_{xy} \mu_{xy} \left(\alpha_{xy} + (1-\tau)\gamma_{xy}\right) \\
s.t. \quad & \sum_y \mu_{xy} \leq n_x \\
& \sum_x \mu_{xy} \leq m_y
\end{aligned}
$$

and

$$
\begin{aligned}
& \min \sum_x n_x u_x + \sum_y m_y \tilde{v}_y \\
s.t. \quad & u_x + \tilde{v}_y \geq \alpha_{xy} + \tilde{\gamma}_{xy} \\
& u_x \geq 0, \tilde{v}_y \geq 0
\end{aligned}
$$