Class notes for day 3 (raw and unpolished)

1 A model of matching with progressive taxe

$N(w)$ = net wage if gross wage is w

$N(w) = \min_k \{N_k + (1-\tau_k) w_{xy}\}$

$N_1 = 0, \tau_1 = 1 - 0.1 = .9$

$9.7(1-\tau_1) = N_2 + (1-\tau_2) 9.7$ where $\tau_2 = 1 - 0.12 = .88$

$N_2 = 9.7(\tau_2 - \tau_1) = 9.7 * 2\%$

For each value of w_{xy} (nominal wage), compute

$U_{xy} = \mathcal{U}_{xy}(w_{xy})$=utility of the worker

$V_{xy} = \mathcal{V}_{xy}(w_{xy})$=utility of the firm

In the case of taxes,

$$\begin{aligned} \mathcal{U}_{xy}(w_{xy}) &= \alpha_{xy} + N(w_{xy}) = \alpha_{xy} + \min_k \{N_k + (1-\tau_k) w_{xy}\} \\ \mathcal{V}_{xy}(w_{xy}) &= \gamma_{xy} - w_{xy} \end{aligned}$$

$\mathcal{F}_{xy}$=feasible utility set=$\{u, v : \exists w_{xy}, u \leq \mathcal{U}_{xy}(w_{xy}), v \leq \mathcal{V}_{xy}(w_{xy})\}$.

Transferable utility case: $\mathcal{F}_{xy} = \left\{(u,v) : \exists w_{xy}, u \leq \alpha_{xy} + w_{xy}, v \leq \gamma_{xy} - w_{xy}\right\}$

$[\mathcal{U}_{xy}(w_{xy}) = \alpha_{xy} + w_{xy}, \mathcal{V}_{xy}(w_{xy}) = \gamma_{xy} - w_{xy}]$

Claim: $\mathcal{F}_{xy} = \left\{(u,v) : u + v \leq \alpha_{xy} + \gamma_{xy}\right\}$

$u - \alpha_{xy} \leq \gamma_{xy} - v$ thus one can take $u - \alpha_{xy} \leq w \leq \gamma_{xy} - v$.

Let us define a stable matching.

A stable matching is given by μ_{xy},(u_x, v_y) such that

(1) We have

$\sum_y \mu_{xy} + \mu_{x0} = n_x$

$\sum_x \mu_{xy} + \mu_{0y} = m_y$

(2) $\forall x, y, (u_x, v_y)$ is not in the strict interior of $\mathcal{F}_{xy}$ [Before, $u_x + v_y \geq \alpha_{xy} + \gamma_{xy}$]

If (u_x, v_y) was in the strict interior of $\mathcal{F}_{xy}$ – i.e. if there is $\left(u'_x, v'_y\right) \in \mathcal{F}_{xy}$ such that $u_x \leq u'_x$ and $v_y \leq v'_y$ with at least a strict inequality. Then xy would be a blocking pair.

In other words, there is no point in $\mathcal{F}_{xy}$ that strictly dominates (u_x, v_y)

$u_x \geq 0$

$v_y \geq 0$

(3) $\mu_{xy} > 0 \implies (u_x, v_y) \in \mathcal{F}_{xy}$ [hence it is on the frontier of $\mathcal{F}_{xy}$]

Distance to frontier function. Define the distance-to-frontier (along the diagonal) as

$$D_{xy}(u, v) = \min\{t \in \mathbb{R} : (u - t, v - t) \in \mathcal{F}_{xy}\}$$

$D_{xy}(u,v) > 0$ means $(u,v) \notin \mathcal{F}_{xy}$
$D_{xy}(u,v) \leq 0$ means $(u,v) \in \mathcal{F}_{xy}$
$D_{xy}(u,v) < 0$ means (u,v) is in the strict interior of $\mathcal{F}_{xy}$
$D_{xy}(u,v) = 0$ means (u,v) is on the frontier of $\mathcal{F}_{xy}$

$D_{xy}(u+a, v+a) = a + D_{xy}(u,v)$. Indeed

$$\begin{aligned} D_{xy}(u+a, v+a) &= \min\{t \in \mathbb{R} : (u+a-t, v+a-t) \in \mathcal{F}_{xy}\} \\ &= \min\{t+a \in \mathbb{R} : (u+a-t-a, v+a-t-a) \in \mathcal{F}_{xy}\} \\ &= a + \min\{t \in \mathbb{R} : (u-t, v-t) \in \mathcal{F}_{xy}\} \\ &= D_{xy}(u,v) \end{aligned}$$

Reformulate the matching problem – matching with imperfectly transferable utility

(1) We have
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
(2) $\forall x, y, D_{xy}(u_x, v_y) \geq 0$
$u_x \geq 0$
$v_y \geq 0$
(3) $\mu_{xy} > 0 \implies D_{xy}(u_x, v_y) = 0$

Examples.
1. TU case $\mathcal{F}_{xy} = \{(u,v) : u + v \leq \Phi_{xy}\}$
$D_{xy}(u_x, v_y) = \frac{u+v-\Phi_{xy}}{2}$
2. Taxation with progressive taxes.
In this case

$$\mathcal{F}_{xy} = \bigcap_k \mathcal{F}_{xy}^k$$

where $\mathcal{F}_{xy}^k = \left\{(u,v) : \lambda_k u + \nu_k v \leq \Phi_{xy}^k\right\}$, with $\lambda_k + \nu_k = 1$ and $\lambda_k, \nu_k > 0$.

Then $D_{xy}^k(u_x, v_y) = \lambda_k u + \nu_k v - \Phi_{xy}^k$.

Claim

$$D_{xy}(u,v) = \max_k D_{xy}^k(u,v)$$

Indeed,

$$D_{xy}(u,v) = t$$

such that $(u-t, v-t) \in \bigcap_k \mathcal{F}_{xy}^k$ and for any $t' < t$, there exists k with $(u-t', v-t') \notin \mathcal{F}_{xy}^k$.

$D_{xy}^k(u-t, v-t) \leq 0$
and there exists k with $D_{xy}^k(u-t, v-t) = 0$
thus $\max_k D_{xy}^k(u-t, v-t) = 0$
thus $\max_k D_{xy}^k(u,v) - t = 0$

hence $t = \max_k D^k_{xy}(u,v)$.

More specifically, in the case of matching with taxes

$$\begin{aligned} u &= \alpha_{xy} + \min_k \{N_k + (1-\tau_k) w_{xy}\} \\ v &= \gamma_{xy} - w_{xy} \end{aligned}$$

this becomes

$$\begin{aligned} u - \alpha_{xy} - \min_k \left\{N_k + (1-\tau_k)\left(\gamma_{xy} - v\right)\right\} &= 0 \\ \max_k \left\{u - \alpha_{xy} - N_k + (1-\tau_k)\left(v - \gamma_{xy}\right)\right\} &= 0 \end{aligned}$$

this is to say

$$\frac{u - \alpha_{xy} - N_k + (1-\tau_k)\left(v - \gamma_{xy}\right)}{2-\tau_k} \leq 0 \text{ with equality for some } k$$

therefore

$$D^k_{xy}(u,v) = \frac{u - \alpha_{xy} - N_k + (1-\tau_k)\left(v - \gamma_{xy}\right)}{2-\tau_k}$$

Parameterize the frontier, that is the set of (u,v) such that $D(u,v) = 0$ by $w = v - u$
Express u and v on the frontier as a function of w.
$D(u,v) = 0$
$w = v - u$

$D(u, u+w) = 0$
$u + D(0,w) = 0$
therefore $u = -D(0,w)$

$D(v-w, v) = 0$
$v + D(-w, 0) = 0$
$v = -D(-w,0)$

$u = -D(0,w)$
$v = -D(-w,0)$

1.1 Matching problem with regularization

$U_{xy} = \alpha_{xy} + N(w_{xy})$
$V_{xy} = \gamma_{xy} - w_{xy}$

We get

$$\begin{aligned}\mu_{xy} &= \mu_{x0}\exp\left(\frac{U_{xy}}{T}\right)\\ \mu_{xy} &= \mu_{0y}\exp\left(\frac{V_{xy}}{T}\right)\end{aligned}$$

Now instead of having $U_{xy}+V_{xy}=\Phi_{xy}=\alpha_{xy}+\gamma_{xy}$, as in the TU model, we replace by

$$D_{xy}\left(U_{xy},V_{xy}\right)=0$$

We have

$$\begin{aligned}U_{xy} &= T\ln\frac{\mu_{xy}}{\mu_{x0}}\\ V_{xy} &= T\ln\frac{\mu_{xy}}{\mu_{0y}}\end{aligned}$$

thus

$$D_{xy}\left(T\ln\mu_{xy}-T\ln\mu_{x0},T\ln\mu_{xy}-T\ln\mu_{0y}\right)=0$$

that is

$$T\ln\mu_{xy}+D_{xy}\left(-T\ln\mu_{x0},-T\ln\mu_{0y}\right)=0$$

thus

$$\begin{aligned}\mu_{xy} &= \exp\left(-\frac{1}{T}D_{xy}\left(-T\ln\mu_{x0},-T\ln\mu_{0y}\right)\right)\\ &= M_{xy}\left(\mu_{x0},\mu_{0y}\right)\end{aligned}$$

Recall the population equations:
$\sum_y \mu_{xy}+\mu_{x0}=n_x$
$\sum_x \mu_{xy}+\mu_{0y}=m_y$

Plug in Choo-Siow's formula into the population equations

$$\begin{cases}\sum_y M_{xy}\left(\mu_{x0},\mu_{0y}\right)+\mu_{x0}=n_x\\ \sum_x M_{xy}\left(\mu_{x0},\mu_{0y}\right)+\mu_{0y}=m_y\end{cases}$$

1.2 The problem is not an optimization problem

Question: Can we see this set of equations as $\min_{\mu_{x0},\mu_{0y}} F\left(\left(\mu_{x0}\right),\left(\mu_{0y}\right)\right)$?
Set $T=1$ for convenience

$$\begin{aligned}\mu_{xy} &= \exp\left(-D_{xy}\left(a_x,b_y\right)\right)\\ \mu_{x0} &= \exp\left(-a_x\right)\\ \mu_{0y} &= \exp\left(-b_y\right)\end{aligned}$$

$$\left\{\begin{array}{l}\sum_y \exp\left(-D_{xy}\left(a_x,b_y\right)\right)+\exp\left(-a_x\right)=n_x\\ \sum_x \exp\left(-D_{xy}\left(a_x,b_y\right)\right)+\exp\left(-b_y\right)=m_y\end{array}\right.$$

Yesterday (TU case)

$$\left\{\begin{array}{l}\frac{\partial F}{\partial a_x}:=\sum_y \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2}\right)+\exp\left(-a_x\right)=n_x\\ \frac{\partial F}{\partial b_y}:=\sum_x \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2}\right)+\exp\left(-b_y\right)=m_y\end{array}\right.$$

$$\begin{aligned}\frac{\partial^2 F}{\partial a_x \partial b_y} &= \frac{\partial^2 F}{\partial b_y \partial a_x}?\\ -\frac{1}{2}\exp\left(\frac{\Phi_{xy}-a_x-b_y}{2}\right) &= -\frac{1}{2}\exp\left(\frac{\Phi_{xy}-a_x-b_y}{2}\right)\end{aligned}$$

Do we have

$$\partial_{b_y} D_{xy}\left(a_x,b_y\right)\exp\left(-D_{xy}\left(a_x,b_y\right)\right)=\partial_{a_x} D_{xy}\left(a_x,b_y\right)\exp\left(-D_{xy}\left(a_x,b_y\right)\right)$$

yes iff $\partial_{b_y} D_{xy}\left(a_x,b_y\right)=\partial_{a_x} D_{xy}\left(a_x,b_y\right)$

$D\left(c+a,c+b\right)=c+D\left(a,b\right)$

$\partial_a D\left(a,b\right)+\partial_b D\left(a,b\right)=1$

That implies $\partial_{b_y} D_{xy}\left(a_x,b_y\right)=\partial_{a_x} D_{xy}\left(a_x,b_y\right)=1/2$

1.3 The problem is an equilibrium problem with GS

Take $p_x=a_x$ and $p_y=-b_y$ and reformulate.

$$\left\{\begin{array}{l}-\sum_y \exp\left(-D_{xy}\left(p_x,-p_y\right)\right)-\exp\left(-p_x\right)=-n_x\\ \sum_x \exp\left(-D_{xy}\left(p_x,-p_y\right)\right)+\exp\left(p_y\right)=m_y\end{array}\right.$$

$$\left\{\begin{array}{l}\sum_y M_{xy}\left(\mu_{x0},\mu_{0y}\right)+\mu_{x0}=n_x\\ \sum_x M_{xy}\left(\mu_{x0},\mu_{0y}\right)+\mu_{0y}=m_y\end{array}\right.$$

Algorithm. Start with $\mu_{x0}^0=n_x$ and $\mu_{0y}^0=0$ [i.e. $p_y^0=-\infty$ and $p_x^0=-\ln n_x$]

Update μ_{0y}: set μ_{0y}^1 such that

$\sum_x M_{xy}\left(\mu_{x0}^0,\mu_{0y}^1\right)+\mu_{0y}^1=m_y$

Update μ_{x0}: set μ_{x0}^1 such that

$\sum_y M_{xy}\left(\mu_{x0}^1,\mu_{0y}^1\right)+\mu_{x0}^1=n_x$

It is easy to see that $\mu_{x0}^0>\mu_{x0}^1$ and $\mu_{0y}^0<\mu_{0y}^1$. Let's see that these monotonicities carry on forever.

Claim: μ_{0y}^{t+1} is an decreasing function of μ_{x0}^t.

Indeed, assume $\mu_{x0}^t\leq\tilde{\mu}_{x0}^t$ for all x, we want to show that $\mu_{0y}^{t+1}\geq\tilde{\mu}_{0y}^{t+1}$.

$\sum_x M_{xy}\left(\mu_{x0}^t,\mu_{0y}^{t+1}\right)+\mu_{0y}^{t+1}=m_y$

$\sum_x M_{xy}\left(\tilde{\mu}_{x0}^t, \tilde{\mu}_{0y}^{t+1}\right) + \tilde{\mu}_{0y}^{t+1} = m_y$

By contradiction, assume $\mu_{0y}^{t+1} < \tilde{\mu}_{0y}^{t+1}$.

$n_x = \sum_x M_{xy}\left(\mu_{x0}^t, \mu_{0y}^{t+1}\right) + \mu_{0y}^{t+1} < \sum_x M_{xy}\left(\tilde{\mu}_{x0}^t, \mu_{0y}^{t+1}\right) + \tilde{\mu}_{0y}^{t+1} \leq \sum_x M_{xy}\left(\tilde{\mu}_{x0}^t, \tilde{\mu}_{0y}^{t+1}\right) + \tilde{\mu}_{0y}^{t+1} = n_x$

contradiction. Hence $\mu_{0y}^{t+1} \geq \tilde{\mu}_{0y}^{t+1}$.

$\mu_{x0}^0 > \mu_{x0}^1$ implies by the order-preserving property that $\mu_{x0}^1 \geq \mu_{x0}^2$ and hence

μ_{x0}^t is decreasing.

Similarly, μ_{0y}^t is increasing.

But these quantities are bounded by 0 and m_y respectively, thus they converge.

2 Uniqueness of equilibrium

$$\begin{cases} e_x(p) = -\sum_y \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) - \exp\left(-p_x\right) \\ e_y(p) = \sum_x \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) + \exp\left(p_y\right) \end{cases}$$

Berry, Gandhi, Haile

Assume $e : R^Z \to R^Z$

such that

(1) the domain of e is a cartesian product

(2) $e_z(p)$ is weakly increasing in p_z and weakly decreasing in $p_{z'}$ for $z' \neq z$.

and

$e_0(p) = -\sum_{z\in Z} e_z(p)$ is weakly decreasing in each of the p_z.

in other words $\sum_{z\in Z} e_z(p)$ is weakly increasing in each of the p_z

$e_z(p) + \sum_{z'\neq z} e_{z'}(p)$ is weakly increasing in p_z

$Z_0 = Z \cup \{0\}$

(3) For $z \in Z$, there is a sequence z_k, $k = 0, ..., K$ with $z_0 = z$,...,$z_K = 0$ such that $e_{z^{k+1}}$ is strictly decreasing in p_{z^k}.

Then $e(p)$ is inverse isotone, that is $e_z(p) \leq e_z(p')$ for all $z \in Z$.

In particular, $e(p) = e(p') \implies p = p'$.

$$\begin{cases} e_x(p) = -\sum_y \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) - \exp\left(-p_x\right) \\ e_y(p) = \sum_x \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) + \exp\left(p_y\right) \end{cases}$$

$$\begin{aligned} e_0(p) &= -\sum_y \sum_x \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) - \sum_y \exp\left(p_y\right) \\ &\quad + \sum_x \sum_y \exp\left(-D_{xy}\left(p_x, -p_y\right)\right) + \sum_x \exp\left(-p_x\right) \\ &= \sum_x \exp\left(-p_x\right) - \sum_y \exp\left(p_y\right) \end{aligned}$$