‘math+econ+code’ masterclass on equilibrium transport and matching models in economics

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Day 5: network equilibrium

- Equilibrium flow problem and min-cost flow problem
- Equilibrium transport problem and optimal transport problem
- Perfect matching, min-cut max-flow theorem and Strassen's theorem
- Reduced network and Bellman-Ford algorithm

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Section 1

Equilibrium flows

The reference for the following is Galichon, Samuelson and Vernet (2020).

- Consider a trading network $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is the set of nodes and $\mathcal{A}$ is the set of directed arcs.
- Consider ∇ a $\mathcal{A} \times \mathcal{Z}$ matrix such that $\nabla_{az} = 1$ if z is the endpoint of a, and -1 if z is the starting point of a. For $f \in \mathbb{R}^{\mathcal{Z}}$ and $xy \in \mathcal{A}$ one has $(\nabla f)_{xy} = f_y - f_x$.
- Let $p \in \mathbb{R}^{\mathcal{Z}}$ be the price vector of a commodity, such that p_z is the price at node z.
- Let $R : \mathbb{R}^{\mathcal{Z}} \mapsto \mathbb{R}^{\mathcal{A}}$ be a function $R_{xy}(p)$ be the rent of the strategy that consists in buying at x at price p_x and selling at y at price p_y. $R_{xy}(p)$ is decreasing in p_x, increasing in p_y, and does not depend on the other entries of p. Examples:
  - Additive case: $R_{xy}(p) = p_y - p_x - c_{xy}$ (no tax). Note that in that case, $R(p) = \nabla p - c$.
  - Linear case: $R_{xy}(p) = p_y - (1 + \tau) p_x - c_{xy}$ (import tax)
  - More generally, $R_{xy}(p) = p_y - C_{xy}(p_x)$.

- Pairwise stability: Because there is free entry, the prices are such that there is no positive rent on any arc, that is:

$$R_{xy}(p) \leq 0 \ \forall xy \in \mathcal{A}$$

- Note that the set of p such that $R_{xy}(p) \leq 0$ for all $xy \in \mathcal{A}$ is a sublattice of $\mathbb{R}^{\mathcal{Z}}$.
- One may want to normalize the prices at some "ground" node. In that case, we will denote the set of nodes by $\mathcal{Z}_0$ instead of $\mathcal{Z}$, where $\mathcal{Z}_0 = \tilde{\mathcal{Z}} \cup \{0\}$ is the full set of nodes, including the ground node which is 0, and $\tilde{\mathcal{Z}}$, the set of non-ground nodes.
- In the additive case, this writes

$$p_y - p_x \leq c_{xy} \ \forall xy \in \mathcal{A}.$$

- Let s_z be the exit flow, i.e. the flow leaving the network at z, and let μ_{xy} be the flow of commodity through arc xy. One has for all nodes $z \in \mathcal{Z}_0$

$$\sum_{x:xz\in\mathcal{A}} \mu_{xz} - \sum_{y:zy\in\mathcal{A}} \mu_{zy} = s_z$$

which can be rewritten

$$\nabla^\intercal \mu = s.$$

- Note that we for a feasible flow to exist, one must have

$$\sum_{z\in\mathcal{Z}_0} s_z = 0$$

for this reason it is enough to specify the exit flow for the non-ground nodes, and deduce $s_0 = -\sum_{z\in\tilde{\mathcal{Z}}} s_z$.

- No trader will operate trades between x and y at a loss. Hence

$$\mu_{xy} > 0 \implies R_{xy}(p) \geq 0$$

which combining with the requirement that p should be stable prices, yields

$$\mu_{xy} > 0 \implies R_{xy}(p) = 0$$

- This is a **complementary slackess** condition, which can be written

$$\sum_{xy \in \mathcal{A}} \mu_{xy} R_{xy}(p) = 0$$

GSV (2020) define

Definition. (μ, p) is a called an equilibrium flow when the following conditions are met:

(i) $\mu \geq 0$ and $\nabla^{\intercal}\mu = s$

(ii) $R(p) \leq 0$

(iii) $\sum_{xy \in \mathcal{A}} \mu_{xy} R_{xy}(p) = 0$

- The problem above is called an equilibrium flow (EQF) problem. As we shall see, μ and p are jointly determined by a pair of coupled problems
  - Given p, μ is the solution to a linear programming problem (max flow problem)
  - Given μ, p is the solution to a dynamic programming problem (generalized shortest path problem)
- However, in the additive case, these two problems become decoupled.

- In the additive case ($R_{xy}(p) = p_y - p_x - c_{xy}$), both μ and p solve linear programming problems that are dual of eachother.
- μ solves the primal problem

$$\min_{\mu \geq 0} \sum_{xy \in \mathcal{A}} \mu_{xy} c_{xy}$$
$$s.t.\ \nabla^{\mathsf{T}} \mu = s$$

  while p solves the dual problem

$$\max_{s} \sum_{z \in \mathcal{Z}} s_z$$
$$s.t.\ s_y - s_x \leq c_{xy}$$

- Cf. m+e+c_optim lectures (http://alfredgalichon.com/mec_optim/).

- Given a map $f\colon \mathbb{R}^d \to \mathbb{R}^d$, a nonlinear complementarity problem consists of finding

$$z \geq 0, f(z) \geq 0 \text{ and } z^\intercal f(z) = 0.$$

- This generalizes the complementarity slackness from linear programming, $\max_{x\geq 0} c^\intercal x : Ax \leq d = \min_{y\geq 0} d^\intercal y : A^\intercal y \geq c$, where

$$A^\intercal y - c \geq 0 \;\; [x \geq 0], d - Ax \geq 0 \;\; [y \geq 0]$$

Therefore, in that case, $z = (x, y)$, and $f(z) = \begin{pmatrix} 0 & A^\intercal \\ -A & 0 \end{pmatrix} + \binom{-c}{d}$.

- In the present case, $z = (\mu, p^+, p^-)$ and

$$f(z) = \left(-R\left(p^+ - p^-\right), s - \nabla^\intercal \mu, \nabla^\intercal \mu - s\right).$$

- Let $f\colon \mathbb{R} \to \mathbb{R}_{+}$ be a continuous function such that $f(-\infty) = 0$ and $f(+\infty) = +\infty$, and let $T > 0$ be a temperature parameter.
- One can look for $\mu_{xy}^{T} = f(R_{xy}(p) / T)$ as an approximation of the solution to the EQF problem. That is, look for p such that

$$\sum_{x:xz\in\mathcal{A}} f(R_{xz}(p) / T) - \sum_{y:zy\in\mathcal{A}} f(R_{zy}(p) / T) = s_s$$

- Therefore the system writes $E^{T}(p) = s$, where

$$E_z^{T}(p) = \sum_{x:xz\in\mathcal{A}} f(R_{xz}(p) / T) - \sum_{y:zy\in\mathcal{A}} f(R_{zy}(p) / T)$$

- Note that, $E^{T}(p)$ satisfies the weak gross substitutes property as $E_z^{T}(p)$ is weakly decreasing with respect to p_x for $x \neq z$.
- In particular in the differentiable case,

$$\frac{\partial E_z^T(p)}{\partial p_x} = \frac{f'\left(R_{xz}(p)/T\right)}{T}\frac{\partial R_{xz}}{\partial p_x} - \frac{f'\left(R_{zz}(p)/T\right)}{T}\frac{\partial R_{zx}}{\partial p_x} \leq 0.$$

- Consider $\left(\mu^T, p^T\right)$ where $\mu^T = f\left(R_{xy}\left(p^T\right)/T\right)$ and p^T a solution of

$$E^T\left(p^T\right) = 0$$

  and assume $\mu^T \to \mu^*$ and $p^T \to p^*$ as $T \to 0$.
- Therefore, μ remains bounded, and we have $f\left(R_{xy}\left(p^T\right)/T\right) \leq K$, thus $R_{xy}\left(p^T\right) \leq Tf^{-1}\left(K\right)$, and as a result $R_{xy}\left(p^*\right) \leq 0$.
- Further, $\mu_{xy} > 0$ implies $\lim R_{xy}\left(p^T\right) = 0$, thus $R_{xy}\left(p^*\right) = 0$.

- In the additive case, $R_{xy}(p) = p_y - p_x - c_{xy}$, and

$$E_z^T(p) = \sum_{x:xz\in\mathcal{A}} f\left(\frac{p_y - p_x - c_{xy}}{T}\right) - \sum_{y:zy\in\mathcal{A}} f\left(\frac{p_y - p_x - c_{xy}}{T}\right)$$

- Let $F(z) = \int^z f(t)\,dt$, which is a convex function. We have $E_z(p) = \partial W(p) / \partial p_z$, where

$$W(p) = \sum_{xy\in\mathcal{A}} TF\left(\frac{p_y - p_x - c_{xy}}{T}\right)$$

- In particular, when $f(z) = \exp(z)$, $F(z) = \exp(z)$, and

$$W(p) = \sum_{xy \in \mathcal{A}} T \exp\left(\frac{p_y - p_x - c_{xy}}{T}\right).$$

- Similarly, when $f(z) = z^{+}$, we get $F(z) = z^2 1\{z \geq 0\}/2$, and

$$W(p) = \sum_{xy \in \mathcal{A}} T \left(\left(\frac{p_y - p_x - c_{xy}}{T}\right)^{+}\right)^2 /2$$

Section 2

From dual to primal and conversely

- Let Γ be a subset of $\mathcal{A}$. A flow $\mu \geq 0$ is a perfect matching along Γ whenever (i) it is a feasible flow, i.e.

$$\nabla^\intercal \mu = s,$$

  and (ii) there is now flow outisde of Γ, i.e. $\mu_a > 0 \implies a \in \Gamma$.
- Clearly, the problem of recovering the primal solution (i.e. the flow μ) based on the dual solution (i.e. the prices p) is a perfect matching – simply define

$$\Gamma = \left\{ a \in \mathcal{A} : R_a\left(p\right) = 0 \right\}.$$

- The perfect matching problem is a linear programming problem: indeed, it can be solved using

$$\min_{\mu \geq 0} \sum_a \mu_a 1\left\{a \notin \Gamma\right\}$$
$$s.t. \; \nabla^\intercal \mu = s$$

- Assume that we know $\mu_{xy} > 0$ and we would like to recover the equilibrium prices $p \in \mathbb{R}^{\mathcal{Z}_0}$ such that $p_0 = 0$, $R_{xy}(p) \leq 0$ for all xy, and $\mu_{xy} > 0$ implies $R_{xy}(p) \leq 0$.
- From the lattice representation theorem, we know that this set is a sublattice of $\mathbb{R}^{\mathcal{Z}_0}$. We would like to get the largest element of this set.
- As we shall see, this is a *dynamic programming problem*.

- Extend the set of arcs by adding the reverse of the arcs where there is a positive amount of flow, i.e. $\mathcal{A}^r = \mathcal{A} \cup \{yx : xy \in \mathcal{A},\ \mu_{xy} > 0\}$. For such reverse arcs yx, define $R_{yx}(p) = -R_{xy}(p)$. Such a network is called *reduced network*.
- See textbook treatments in Ahuja, Magnuti and Orlin (1993) and Bertsekas (1998).

- We shall restrict ourselves to the case $R_{xy}(p) = p_y - C_{xy}(p_x)$. In that case, for reverse arcs yx, we define $C_{yx}(p) = C_{xy}^{-1}(p)$.

Lemma. The set of equilibirum prices are the fixed points of an isotone map

$$T(p)_y = \min\left\{p_y, \min_{xy \in \mathcal{A}^r} C_{xy}(p_x)\right\}.$$

Proof. $T(p) = p$ if and only if $p_y \leq C_{xy}(p_x)$ for all x such that $xy \in \mathcal{A}^r$, that is

$$\begin{aligned} p_y &\leq C_{xy}(p_x),\ \forall xy \in \mathcal{A} \\ p_y &\leq C_{yx}^{-1}(p_x),\ \forall yx \in \mathcal{A} : \mu_{xy} > 0 \end{aligned}$$

that is

$$\begin{aligned} p_y &\leq C_{xy}(p_x),\ \forall xy \in \mathcal{A} \\ p_y &\geq C_{xy}(p_x),\ \forall xy \in \mathcal{A} : \mu_{xy} > 0. \end{aligned}$$

QED.

This suggests to iterate map T in order to converge to the lattice upper bound of the set of fixed points. This method is known as the Bellman-Ford algorithm, and it is an early instance of dynamic programming.

Algorithm (Bellman-Ford).
At period 1, set $p_0^1 = 0$ and $p_z^1 = +\infty$.
At period $t \geq 2$, set $p_y^t = \min \left\{ p_y^{t-1}, \min_{xy \in \mathcal{A}^r} C_{xy} \left(p_x^{t-1} \right) \right\}$
Repeat until convergence.

- In the additive case, recall that $C_{xy}\left(p_x\right) = c_{xy} + p_x$. In this case, following the approach above, we construct the reduced network by adding the reverse arcs yx to $\mathcal{A}$ whenever $\mu_{xy} > 0$. One associates these with cost $c_{yx} = -c_{xy}$.
- One seeks the largest element of the set

$$\left\{p : p_y - p_x \leq c_{xy} \; \forall xy \in \mathcal{A}^r, p_0 = 0\right\}$$

which formulates as a linear programming problem

$$\begin{aligned} &\max p_y - p_0 \\ &s.t.\ p_y - p_x \leq c_{xy} \end{aligned}$$

- The Bellman-Ford algorithm consists of deducing the optimal solution in t steps from an optimal solution in $t-1$ steps using Bellman's equation $p_y^t = \min\left\{p_y^{t-1}, \min_{xy \in \mathcal{A}^r}\left\{c_{xy} + p_x^{t-1}\right\}\right\}$.

Section 3

Bipartite case: the equilibrium transport problem

- Consider the case where $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$, and $\mathcal{A} = \mathcal{X} \times \mathcal{Y}$. $\mathcal{X}$ are the source nodes, $\mathcal{Y}$ are the destination ones, and each source is connected to a destination.
- $n_x \geq 0$ is the mass at source $x \in \mathcal{X}$ and $m_y \geq 0$ is the mass at destination $y \in \mathcal{Y}$. Assume that the total source mass and total destination mass are the same: $\sum_x n_x = \sum_y m_y$. Set $s_z = -n_z 1\left\{z \in \mathcal{X}\right\} + m_y 1\left\{z \in \mathcal{Y}\right\}$.
- Then (μ, p) is a solution to the equilibrium transport (ET) problem if:

$$\begin{cases} \sum_{y \in \mathcal{Y}} \mu_{xy} = n_x, \sum_{x \in \mathcal{X}} \mu_{xy} = m_y \\ R_{xy}\left(p\right) \leq 0 \\ \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \mu_{xy} R_{xy}\left(p\right) = 0 \end{cases}$$

- In the bipartite case, it will often make sense to set $u_x = p_x$ and $v_y = -p_y$, and $\Psi_{xy}(u_x, v_y) = -R_{xy}(u_x, -v_y)$, so that $\Psi_{xy}(u, v)$ is increasing in u and v, and the problem becomes

$$\begin{cases} \sum_{y\in\mathcal{Y}} \mu_{xy} = n_x, \sum_{x\in\mathcal{X}} \mu_{xy} = m_y \\ \Psi_{xy}(u_x, v_y) \geq 0 \\ \sum_{\substack{x\in\mathcal{X} \\ y\in\mathcal{Y}}} \mu_{xy}\Psi_{xy}(u_x, v_y) = 0 \end{cases}$$

- Interpretation: if x and y match, they can bargain over the feasible sets of utilities (u_x, v_y) such that $\Psi_{xy}(u_x, v_y) \leq 0$.

- Note that if $R_{xy}(p) = p_y - C_{xy}(p_x)$, then $\Psi_{xy}(u_x, v_y) = C_{xy}(u_x) + v_y = v_y - \mathbb{V}_{xy}(u_x)$ where $\mathbb{V}_{xy}(u_x) = -C_{xy}(u_x)$ is continuous and decreasing.
- If (μ, u, v) is a solution to the ET problem in the previous formulation, then the following conjugation relation holds

$$\begin{cases} v_y = \max_{x \in \mathcal{X}} \mathbb{V}_{xy}(u_x) \\ u_x = \max_{y \in \mathcal{Y}} \mathbb{U}_{xy}(v_y) \end{cases}$$

- This relation is called a Galois connection, see Noeldeke and Samuelson (2017). In particular, if $\mathbb{V}_{xy}(u_x) = \Phi_{xy} - u_x$, then v is the Φ-convex conjugate of u, as studied in Villani (2008), and if $\mathcal{X} = \mathcal{Y} = \mathbb{R}^d$ and $\Phi_{xy} = x^\intercal y$, then v is the Legendre-Fenchel transform of u_x.

- Assuming everything is smooth, and letting f_P and f_Q be the densities of P and Q we have under some conditions that the equilibrium transportation plan is given by $y = T(x)$, where mass balance yields

$$\left|\det DT(x)\right| = \frac{f_P(x)}{f_Q(T(x))}$$

and optimality in $\max_{x \in \mathcal{X}} \mathbb{V}_{xy}(u(x))$ yieds

$$\partial_x \mathbb{V}_{xT(x)}(u(x)) + \partial_u \mathbb{V}_{xT(x)}(u(x)) \nabla u(x) = 0$$

which thus inverts into

$$T(x) = e(x, u(x), \nabla u(x)).$$

- In the case when $\mathbb{V}_{xy}(u(x)) = x^{\intercal} y - u(x)$, we get $e(x, u(x), \nabla u(x))$; in the case when $\mathbb{V}_{xy}(u(x)) = \Phi(x, y) - u(x)$, we get $e(x, u(x), \nabla u(x)) = \nabla_x \Phi(x, .)^{-1}(\nabla u(x))$.
- Trudinger (2014) studies Monge-Ampere equations in u of the form

$$\left|\det De(., u, \nabla u)\right| = \frac{f_P}{f_Q(e(., u, \nabla u))}.$$

- When $\Psi_{xy}(u_x, v_y) = u_x + v_y - \Phi_{xy}$, the problem writes

$$\begin{cases} \sum_{y\in\mathcal{Y}} \mu_{xy} = n_x, \sum_{x\in\mathcal{X}} \mu_{xy} = m_y \\ u_x + v_y \geq \Phi_{xy} \\ \mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy} \end{cases}$$

- This are the complementary slackness conditions associated with the optimal transport problem, namely

$$\max_{\mu\geq 0} \sum \mu_{xy}\Phi_{xy}$$
$$s.t. \sum_{y\in\mathcal{Y}} \mu_{xy} = n_x, \sum_{x\in\mathcal{X}} \mu_{xy} = m_y$$

which has dual

$$\min_{u,v} \sum_{x\in\mathcal{X}} n_x u_x + \sum_{y\in\mathcal{Y}} m_y v_y$$
$$s.t.\ u_x + v_y \geq \Phi_{xy}$$

- Many result extend beyond $\mathcal{X}$ and $\mathcal{Y}$ discrete; the theory is called optimal transport theory.

- Consider now the case when $\sum_x n_x \neq \sum_y m_y$. Then define $\tilde{\mathcal{Z}} = \mathcal{X} \cup \mathcal{Y}$, and add a ground node 0. Let $\mathcal{Z}_0 = \mathcal{X} \cup \mathcal{Y} \cup \{0\}$, and let

$$s_z = -n_z 1\{z \in \mathcal{X}\} + m_y 1\{z \in \mathcal{Y}\} + \left(\sum_{y\in\mathcal{Y}} m_y - \sum_{x\in\mathcal{X}} n_x\right) 1\{z = 0\}.$$

- The set of arcs is now $\mathcal{A} = \mathcal{X} \times \mathcal{Y} \cup \mathcal{X} \times \{0\} \cup \{0\} \times \mathcal{Y}$. We set $p_0 = 0$, so that and the problem becomes

$$\begin{cases} \sum_{y\in\mathcal{Y}} \mu_{xy} + \mu_{x0} = n_x, \sum_{x\in\mathcal{X}} \mu_{xy} + \mu_{0y} = m_y \\ C_{xy}(p) \leq 0, C_{x0}(p_x, 0) \leq 0, C_{0y}(0, p_y) \leq 0 \\ \sum_{\substack{x\in\mathcal{X} \\ y\in\mathcal{Y}}} \mu_{xy} C_{xy}(p) = 0 \end{cases}$$

- We can always redefine the problem by setting $u_x = -R_{x0}(p_x, 0)$ and $v_y = -R_{0y}(0, p_y)$, and $\Psi_{xy}(u_x, v_y) = -R_{xy}\left(R_{x0}(., 0)^{-1}(-u_x), R_{0y}(0, .)^{-1}(-v_y)\right)$, which becomes

$$\begin{cases} \sum_{y\in\mathcal{Y}} \mu_{xy} + \mu_{x0} = n_x, \sum_{x\in\mathcal{X}} \mu_{xy} + \mu_{0y} = m_y \\ \Psi_{xy}(u_x, v_y) \geq 0, u_x \geq 0, v_y \geq 0 \\ \sum_{xy} \mu_{xy} \Psi_{xy}(u_x, v_y) + \sum_x \mu_{x0} u_x + \sum_y \mu_{0y} v_y = 0 \end{cases}$$

Section 4

Strassen's theorem

- Consider $\mathcal{X}$ and $\mathcal{Y}$ two open subsets of respectively $\mathbb{R}^d$ and $\mathbb{R}^{d'}$. Let Γ be a closed subset of $\mathcal{X} \times \mathcal{Y}$, which stand for the set of pairs (x, y) that are compatible.
- For $x \in \mathcal{X}$, denote $\Gamma(x) == \{y \in \mathcal{Y} : (x, y) \in \Gamma\}$ the subset of receivers $y \in \mathcal{Y}$ that are compatible with donor x. Γ is a *set-valued function*, or *correspondence*. For $B \subseteq \mathcal{X}$, denote

$$\Gamma(B) = \{y \in \mathcal{Y} : \exists x \in B, (x, y) \in \Gamma\}.$$

- The problem of maximizing the number of compatible pairs is given by

$$\max_{\pi \in \mathcal{M}(P,Q)} \Pr_\pi((X, Y) \in \Gamma)$$

or equivalently

$$\min_{\pi \in \mathcal{M}(P,Q)} \Pr_\pi((X, Y) \notin \Gamma).$$

This is an optimal transport problem with with $0-1$ cost (or $0-1$ surplus).

- By the Monge-Kantorovich theorem, the previous problem coincides with

$$= \sup \int a(x)\, dP(x) - \int p(y)\, dQ(y)$$
$$s.t.\ a(x) - b(y) \leq 1\{(x,y) \notin \Gamma\}$$

- We will see that we can take a and b valued in $\{0,1\}$. Then $a(x) = 1\{x \in A\}$ and $b(y) = 1\{y \in B\}$, so that the constraint rewrites

$$1\{y \notin B\} \leq 1\{(x,y) \notin \Gamma\} + 1\{x \notin A\}$$

which means that if $y \in \Gamma(x)$ and $x \in A$ implies $y \in B$, that is $\Gamma(A) \subseteq B$. Therefore,

$$= \sup_{A,B} \{P(A) - Q(B) : \Gamma(A) \subseteq B\},$$

hence:

- **Theorem** (Strassen). *Let P and Q be two probability measures on $\mathcal{X}$ and $\mathcal{Y}$, and let $\Gamma : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a closed correspondence. Then*

$$\min_{\pi \in \mathcal{M}(P,Q)} \Pr_\pi((X,Y) \notin \Gamma) = \sup_{A \subseteq \mathcal{X}} \{P(A) - Q(\Gamma(A))\}. \qquad (1)$$

where the sup on the right hand-side extends to all closed subsets of $\mathcal{X}$

- Let a and b a pair of solutions to the dual problem. Then

$$a(x) = \min_{y \in \mathcal{Y}} \{ 1\{(x,y) \notin \Gamma\} + b(y) \}$$
$$b(y) = \max_{x \in \mathcal{X}} \{ a(x) - 1\{(x,y) \notin \Gamma\} \}$$

- Step 1: a and b valued in $[0,1]$. One can take wlog $\min_y b(y) = 0$. It follows from $0 \leq 1\{(x,y) \notin \Gamma\} \leq 1$ and the first equality that

$$0 \leq \min_y \{ 1\{(x,y) \notin \Gamma\} \} \leq a(x) \leq 1 + \min_y b(y) = 1$$

Simlarily, it follows from $a(x) \leq 1$ and the second inequality that

$$b(y) \leq 1.$$

- Step 2: a and b can be taken valued in $\{0,1\}$. Indeed, $a(x) = \int_0^1 1\{t \leq a(x)\}\, dt$ and $b(y) = \int_0^1 1\{t \leq b(y)\}\, dt$. Let us show that $1\{t \leq a(x)\} - 1\{t \leq b(y)\} \leq 1\{(x,y) \notin \Gamma\}$. By contradiction, if not, then $1\{(x,y) \notin \Gamma\} = 0$, $b(y) > t$ and $t \leq a(x)$. But this implies $a(x) - b(y) > 0$, yet $a(x) - b(y) \leq 1\{(x,y) \notin \Gamma\} = 0$, a contradiction.
- Next, each of $a_t(x) = 1\{t \leq a(x)\}$ and $b_t(y) = 1\{t \leq b(y)\}$ are feasible, and their convex combination is optimal for the dual; thus each of them is optimal. QED.

- Hall's marriage lemma: assume there are n donors $i \in \{1, ..., n\}$ and receivers $j \in \{1, ..., n\}$. Let $\Gamma(i) \subseteq \{1, ..., n\}$ be the set of receivers which are compatible with donors i, and for $A \subseteq \{1, ..., n\}$, define $\Gamma(A) = \cup_{i \in A} \Gamma(i)$. A (pure) matching is a permutation σ such that $j = \sigma(i)$ means that i donates to j. A matching is perfect if $\sigma(i) \in \Gamma(ij)$ for all $i \in \{1, ..., n\}$. Hall's theorem says that there is a perfect matching if and only if

$$\forall A \subseteq \{1, ..., n\}, \ \ |A| \leq |\Gamma(A)| .$$

- Follows from the previous result by taking $\mathcal{X} = \mathcal{Y} = \{1, ..., n\}$ and P and Q the uniform distributions on these sets. To do this, note that the value of the dual is zero if and only if $P(A) \leq Q(\Gamma(A))$ for all $A \subseteq \mathcal{X}$.
- As for the primal, we'll need to show it has a Monge solution.

- There is a perfect matching iff the value of the (primal) problem is zero:
  - $\Longrightarrow$ is obvious.
  - For $\Leftarrow$, if the value of the problem is zero, there exists $\pi \in \mathcal{M}(P, Q)$ such that $\sum \pi_{ij} 1\{i \notin \Gamma(j)\} = 0$. One can show that w.l.o.g. π can be taken such that $\pi_{ij} = 1\{i = \sigma(j)\} / n$.
- To show the latter, consider among the matrices $\pi \in \mathcal{M}(P, Q)$ with $\sum \pi_{ij} 1\{i \notin \Gamma(j)\} = 0$ the one such that $n\pi$ has the smallest number of noninteger cells.
  - Assume that this number is > 0. Then start with one noninteger cell. There is another noninteger cell on the same line; on the same column of that cell, there is another one; on the line of the latter, another one; etc. At some point, we'll get a cycle. It's possible to strictly decrease the number of noninteger entries of $n\pi$ by removing enough mass on that cycle.
- The previous argument is (in disguise) the Birkhoff-von Neumann theorem: any coupling between the uniform distribution over $\{1, ..., n\}$ and itself can be written as a convex combination of Monge couplings between these distributions.

Section 5

Congestion and capacity constraints

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Section 6

Congestion externalities

- In the min-cost flow problem problem, we were minimizing a linear transportation cost $\mathcal{W}(\pi)$ under feasibility constraints, i.e.

$$\min \mathcal{W}(\pi)$$
$$s.t.\ \pi_{ij} \geq 0$$
$$\mathcal{N}\pi = b$$

- We now would like to relax the assumption that our total total cost function $\mathcal{W}$ should be linear with respect to π. We shall take $\mathcal{W}$ as a separable function

$$\mathcal{W}(\pi) = \sum_{(i,j)\in A} K_{ij}(\pi_{ij})$$

where $K_{ij}(.)$ are real valued functions, one for each arc.

- This allows us to model *positive network spillovers*, which is the case where there are positive externalities, captured by the choice of $K_{ij}(x)$ as concave function, which means that path from i to j becomes less and less costly the more people go through it.
- Negative externalities, or *congestion effect*, are captured by a choice of convex function for $K_{ij}(x)$. Throughout the sequel, we shall assume that this is the case.

- Assume that $\mathcal{W}$ is a convex function. Then the primal value of the optimal transportation problem on the network

$$\begin{aligned} &\min \mathcal{W}(\pi) \\ &s.t.\ \pi \geq 0 \\ &\mathcal{N}\pi = b \end{aligned} \tag{2}$$

coincides with its dual value, which is

$$\max_{w} \sum_{i} w_i b_i - \mathcal{W}^* \left(w' \mathcal{N} \right) \tag{3}$$

where

$$\left(w' \mathcal{N} \right)_{ij} = w_j - w_i$$

and $\mathcal{W}^*$ is the convex conjugate function to $\mathcal{W}$, i.e.

$$\mathcal{W}^*(\kappa) = \sup_{\pi_{ij} \geq 0} \left(\sum_{(i,j) \in A} \pi_{ij} \kappa_{ij} - \mathcal{W}(\pi) \right). \tag{4}$$

- This follows from a min-max argument, as one has

$$\begin{aligned}
&\min_{\pi\geq 0}\max_{w}\mathcal{W}\left(\pi\right)+w'\left(b-\mathcal{N}\pi\right)\\
&=\max_{w}w'b+\min_{\pi\geq 0}\mathcal{W}\left(\pi\right)-w'\mathcal{N}\pi\\
&=\max_{w}w'b-\max_{\pi\geq 0}w'\mathcal{N}\pi-\mathcal{W}\left(\pi\right)\\
&=\max_{w}w'b-\mathcal{W}^{\ast}\left(w'\mathcal{N}\right).
\end{aligned}$$

- First, this problem is a generalization of the min-cost flow problem. Take
$$\mathcal{W}\left(\pi\right) = \sum_{(i,j)\in A} \pi_{ij} k_{ij}.$$
- Then, one has
$$\begin{aligned}\mathcal{W}^{*}\left(\kappa\right) &= 0 \text{ if } \kappa_{ij} \leq k_{ij} \text{ for all } \left(i,j\right)\in A \\ &= +\infty \text{ otherwise.}\end{aligned}$$

  Hence, Equation (3) becomes
$$\begin{aligned}&\max_{w} w^{\prime} b \\ &s.t.\ w^{\prime}\mathcal{N} \leq k\end{aligned}$$
  recovering the min cost flow problem.

- We now give a more interesting important example. Consider the case where

$$\mathcal{W}(\pi) = \sum_{(i,j)\in A} \pi_{ij} k_{ij} + \sigma \sum_{(i,j)\in A} \pi_{ij} \ln \pi_{ij}.$$

- In that case, there is a vector $(w_i)_{i\in V}$ such that for each $(i,j) \in A$, the optimal flow π_{ij} satisfies the Schrödinger equation

$$\pi_{ij} = \exp\left(\frac{-k_{ij} + w_j - w_i - 1}{\sigma}\right), \tag{5}$$

where the w's exist, are unique up to an additive constant, and are a solution of

$$\max_w \sum_i w_i b_i - \sum_{(i,j)\in A} \sigma \exp\left(\frac{k_{ij} - w_j + w_i - \sigma}{\sigma}\right)$$

and the flow defined by Equation 5 is automatically feasible.

- The interpretation of this theorem is very interesting. The log-likelihood of a transition from i to j is proportional to minus the direct transportation cost $-k_{ij}$. Hence, all other things equal, all transitions are possible, but less costly transitions will be more likely than others. The potential w_i, on the other hand, adjusts π_{ij} so that it satisfies the feasibility constraint. Hence a terminal node with a high outgoing flow should "pump in" more mass, and therefore transitions to this node should receive higher probability.

- Proof: equation (4) becomes

$$\mathcal{W}^*(\kappa) = \sup_{\pi_{ij} \geq 0} \left(\sum_{(i,j) \in A} \pi_{ij} \left(\kappa_{ij} - k_{ij} - \sigma \ln \pi_{ij} \right) \right),$$

hence by first order conditions,

$$\kappa_{ij} - k_{ij} - \sigma \ln \pi_{ij} - \sigma = 0,$$

hence

$$\pi_{ij} = \exp\left(\frac{\kappa_{ij} - k_{ij} - \sigma}{\sigma} \right).$$

- Therefore

$$\mathcal{W}^*(\kappa) = \sum_{(i,j) \in A} \sigma \exp\left(\frac{\kappa_{ij} - k_{ij} - \sigma}{\sigma} \right)$$

and when $\kappa = w \nabla \mathcal{N}$, one has $\kappa_{ij} = w_j - w_i$, thus

$$\pi_{ij} = \exp\left(\frac{w_j - w_i - k_{ij} - \sigma}{\sigma} \right),$$

The first order conditions associated to Equation (3), one gets

$$b_k = \frac{\partial \mathcal{W}^* \left(w/\mathcal{N} \right)}{\partial w_k}$$

thus

$$b_k = \sum_{a \in A} \frac{\partial \mathcal{W}^*}{\partial \kappa_a} \left(w/\mathcal{N} \right) \mathcal{N}_{ka},$$

hence

$$\begin{aligned} b_k = & \sum_{a \text{ arrives at } k} \exp\left(\frac{\kappa_a - k_a - \sigma}{\sigma} \right) \\ & - \sum_{a \text{ leaves from } k} \exp\left(\frac{\kappa_a - k_a - \sigma}{\sigma} \right) \end{aligned}$$

which is exactly the feasibility equation.

We now consider the individual decision problem, sometimes called "selfish routing problem". Consider the cost of adding transporting one incremental amount of mass δb in the network from source nodes S to terminal ones T. Let $\delta\pi$ the incremental flow generated.
Assume that the transportation cost of shipping $\delta\pi_{ij}$ through arc (i,j) is a function of the degree of saturation of the network: $k_{ij}(\pi_{ij})\,\delta\pi_{ij}$, where $k_{ij}(.)$ are functions defined over each arcs and assumed to be increasing (in order to model congestion). Clearly, any incremental shipper will face a linear optimization cost with cost $k_{ij} = K'_{ij}(\pi_{ij})$. This rules out cycles, and suboptimal paths in the network flow decomposition and this motivates the notion of a Wardrop equilibrium.

Definition. π is a Wardrop equilibrium if given any flow decomposition of π

$$\pi = \sum_{\rho \in \mathcal{P}} h_\rho 1 \{a \in \rho\} + \sum_{\mu \in \mathcal{C}} g_\mu 1 \{a \in \mu\},$$

then:
(i) $g_\mu = 0$ for all cycles μ, and
(ii) any path ρ with $h_\rho > 0$ from a source to a terminal node is optimal with respect to cost $k_{ij}(\pi_{ij})$.

π is a Wardrop equilibrium if and only if it solves problem (2)

$$\min_{\pi \geq 0} \sum_{ij} K_{ij}\left(\pi_{ij}\right) \tag{6}$$
$$s.t.\ \mathcal{N}\pi = b$$

where K_{ij} is a primitive of k_{ij}, i.e. $K'_{ij}\left(x\right) = k_{ij}\left(x\right)$.
The first oder conditions of problem (6), coincide with those of

$$\min_{\hat{\pi} \geq 0} \sum_{ij} k_{ij}\hat{\pi}_{ij}$$
$$s.t.\ \mathcal{N}\hat{\pi} = b$$

where $k_{ij} = K'_{ij}\left(\pi_{ij}\right)$. Thus Wardrop equilibria and optimizers of problem (2) coincide.

Note that π is not optimal. Indeed, the optimal π minimizes instead

$$\min_{\hat{\pi}\geq 0}\sum_{ij} k_{ij}\left(\hat{\pi}_{ij}\right)$$
$$s.t.\ \mathcal{N}\hat{\pi}=b$$

which is a different problem, unless the cost functions k_{ij} are linear.
The function

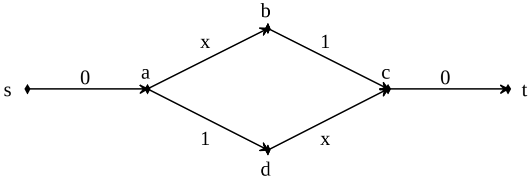
$$l_{ij}\left(x\right)=\frac{k_{ij}\left(x\right)}{x}=\frac{K_{ij}'\left(x\right)}{x}$$

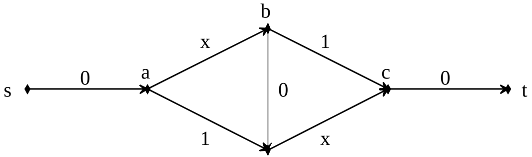
which captures the cost per unit of traffic is called the *latency function*.
With this definition, the optimal π minimizes

$$\min_{\hat{\pi}\geq 0}\sum_{ij}\hat{\pi}_{ij}l_{ij}\left(\hat{\pi}_{ij}\right) \tag{7}$$
$$s.t.\ \mathcal{N}\hat{\pi}=b.$$

which is clearly analoguous to (6), but l_{ij} is in general different from K_{ij}. The loss of social welfare due to the difference between the optimal π and the equilibrium π is called in the literature the *price of anarchy (*Koutsoupias and Papadimitriou, 1999). It can be theoretically bounded.

Consider Figure 54, where the functions $k_{ij}(x)$ are indicated along the arcs. Thus there is no congestion effect in arcs (a, d) which costs one whatever the traffic is; and there is congestion effect in arcs (a, b) which cost π_{ab} when π_{ab} is the flow through that arc.





One would like to move one unit from node s to node t. In the first picture, the unique Wardrop equilibrium consists in splitting the flow into two halves, one on the path (s, a, b, c, t). Total cost per infinitesimal unit of mass is $3/2$ either way, hence total cost is $3/2$ and coincides with the optimum.
Let us now consider the second picture, where one has simply added a free arc to the network from b to d. This obviously does not change the optimal flow, and one would anticipate that expanding possibilities has no reverse effect. It turns out that it actually *worsens* the situation. Indeed, irrespective of $x < 1$, the path (s, a, b, d, c, t) is now a shortest path, thus the only Wardrop equilibrium has now all traffic through that path – with a cost of 2.

Section 7

Capacity constraints

- One has

$$\min_{\mu \geq 0} \sum_{z \in \mathcal{Z}} \left(s_z - \nabla^\top \mu \right)^+ = \max_{0 \leq \varphi \leq 1} \sum_z s_z \varphi_z : \nabla \varphi \leq 0$$

Indeed, $\min_{\mu \geq 0} \sum_{z \in \mathcal{Z}} \left(s_z - \nabla^\top \mu \right)^+$

$$\begin{aligned}
&= \min_{\mu_a \geq 0, P_z \geq 0} \left\{ \sum_z P_z : P_z \geq s_z - \nabla^\top \mu \right\} \\
&= \min_{\mu_a \geq 0, P_z \geq 0} \max_{\varphi_z \geq 0} \sum_z P_z \left(1 - \varphi_z \right) + \sum_z \varphi_z s_z - \sum \varphi_z \left(\nabla^\top \mu \right)_z \\
&= \max_{\varphi_z \geq 0} \sum_z \varphi_z s_z : \varphi \leq 1, \nabla \varphi \leq 0.
\end{aligned}$$

- More generally,

$$\min_{\underline{\mu} \leq \mu \leq \overline{\mu}} \sum_{z \in \mathcal{Z}} \left(s_z - \nabla^\top \mu \right)^+ = \max_{\substack{U_a, \tau_a \geq 0, \\ 1 \geq \varphi_z \geq 0}} \left\{ \sum_z \varphi_z s_z + \sum_a U_a \underline{\mu}_a - \sum_a \tau_a \overline{\mu}_a \right\}$$

$$s.t. \ \nabla \varphi \leq \tau_a - U_a$$

- Indeed,

$$= \min_{\underline{\mu} \leq \mu \leq \overline{\mu}, P_z \geq 0} \left\{ \sum_z P_z : P_z \geq s_z - \nabla^\top \mu \right\}$$

$$= \min_{\mu_a, P_z \geq 0} \max_{U_a, \tau_a \geq 0, \varphi_z \geq 0} \left\{ \begin{array}{c} \sum_z P_z \left(1 - \varphi_z\right) + \sum_z \varphi_z s_z - \sum \varphi_z \left(\nabla^\top \mu\right)_z \\ + \sum_a \tau_a \left(\mu_a - \overline{\mu}_a\right) + \sum U_a \left(\underline{\mu}_a - \mu_a\right) \end{array} \right\}$$

$$= \max_{U_a, \tau_a \geq 0, \varphi_z \geq 0} \sum_z \varphi_z s_z + \sum_a U_a \underline{\mu}_a - \sum_a \tau_a \overline{\mu}_a : \varphi \leq 1, \nabla \varphi \leq \tau_a - U_a.$$

(1) Hoffman's theorem

- Hoffman's theorem provides a set of necessary and sufficient conditions so that there exists a feasible flow whithin some bounds. Let $\mathcal{O}(B) = \{xy \in \mathcal{A} : x \in B, y \notin B\}$, and $\mathcal{I}(B) = \{xy \in \mathcal{A} : x \notin B, y \in B\}$ be respectively the set of outward and inward arcs of B.

Theorem (Hoffman). Given $s \in \mathbb{R}^{\mathcal{Z}}$, and $\underline{\mu}$ and $\overline{\mu}$ in $\mathbb{R}_{+}^{\mathcal{A}}$ there exists $\mu \in \mathbb{R}_{+}^{\mathcal{A}}$ such that

$$\begin{aligned} \nabla^{\mathsf{T}} \mu &= s \\ \underline{\mu} \leq \mu &\leq \overline{\mu} \end{aligned}$$

if and only if for all subsets B of $\mathcal{Z}$, one has

$$\sum_{z \in B} s_z \leq \sum_{a \in \mathcal{I}(B)} \overline{\mu}_a - \sum_{a \in \mathcal{O}(B)} \underline{\mu}_a.$$

Proof. Such a μ exists if the value of the linear programming problem above is zero, thus, by duality, if and only if

$$0 = \max_{\substack{U_a, \tau_a \geq 0, \\ 1 \geq \varphi_z \geq 0}} \left\{ \sum_z \varphi_z s_z + \sum_a U_a \underline{\mu}_a - \sum_a \tau_a \overline{\mu}_a \right\}$$

- A capacitated network consists of a set of nodes $\mathcal{Z}$, a set of arcs $\mathcal{A}$, and a capacity vector $\bar{\mu} \in \mathbb{R}^{\mathcal{A}}$ such that a capacity $\bar{\mu}_a$ is associated with each arc a in the network. Given the vector of outgoing flow s, a feasible flow is a vector $\mu \geq 0$ that should not only satisfy the mass balance equation $\nabla^\intercal \mu = s$, but also the capacity constraint $\mu_a \leq \bar{\mu}_a$ for all $a \in \mathcal{A}$.
- Consider the problem

$$\min_{\mu \geq 0} \sum_{a \in \mathcal{A}} \mu_a c_a$$
$$s.t.\ \nabla^\intercal \mu = s$$
$$\mu \leq \bar{\mu}$$

- One sees that the dual to this problem is

$$\max_{\substack{z_a^+, z_a^- \geq 0 \\ p}} \sum_{z \in \mathcal{Z}} s_z p_z + \sum z_a^+ \overline{\mu}_a$$
$$s.t.\ z^+ - z^- = \nabla p - c$$

- By complementary slackness, one has:

$$\begin{cases} \mu_{xy} = 0 \implies p_y - p_x \leq c_{xy} \\ \mu_{xy} \in (0, \bar{\mu}_{xy}) \implies p_y - p_x = c_{xy} \\ \mu_{xy} = \bar{\mu}_{xy} \implies p_y - p_x \geq c_{xy} \end{cases}$$

- The dual rewrites

$$\max_p \sum_{z \in \mathcal{Z}} s_z p_z + \sum \overline{\mu}_a \left(\nabla p - c \right)_a^+$$

- Assume w.l.o.g. that the total mass of source nodes (and hence the total mass of target nodes) is one, that is $\sum_{z:s_z>0} s_z = 1$. The max-flow problem is the problem of determining the highest $t \in \mathbb{R}$ such that there exists a feasible flow associated with ts. That is

$$\max_{t,\mu \geq 0} \{t\}$$
$$s.t.\ \nabla^{\intercal} \mu = ts$$
$$\mu \leq \bar{\mu}$$

- Consider $\Gamma \subseteq \mathcal{X} \times \mathcal{Y}$, and look for a perfect matching μ_{xy} along Γ between marginal distributions (n_x) and (m_y) such that $\sum_x n_x = \sum_y m_y$, i.e. such that

$$\sum_{y \in \mathcal{Y}} \mu_{xy} = n_x, \ \sum_{x \in \mathcal{X}} \mu_{xy} = m_y, \ \mu_{xy} > 0 \implies xy \in \Gamma.$$

- Create one origin node o and one destination node d, and set arcs ox such that $\mu_{ox} = n_x$, $\mu_{yd} = m_y$, and $s_z = 1\{z = d\} - 1\{z = 0\}$. Then it is easy to see that there is a perfect matching if and only if the maximum flow from o to d is one.

Theorem 1. The max-flow problem has dual expression

$$\begin{aligned} \min_{p,\tau\geq 0} & \bar{\mu}^{\intercal}\tau \\ s.t.\ & p^{\intercal}s = 1 \\ & \tau \geq \nabla p \end{aligned}$$

that is $\min_p \bar{\mu}^{\intercal} \left(\max\left\{\nabla p, 0\right\} : p^{\intercal}s = 1\right)$.
Proof. Rewrite the max-flow problem as

$$\begin{aligned} & \max_{t,\mu\geq 0} \min_{p,\tau\geq 0} t + p^{\intercal}\nabla^{\intercal}\mu - tp^{\intercal}s + \bar{\mu}^{\intercal}\tau - \mu^{\intercal}\tau \\ & = \min_{p,\tau\geq 0} \bar{\mu}^{\intercal}\tau + \max_{t,\mu\geq 0} t\left(1 - p^{\intercal}s\right) + \mu^{\intercal}\left(\nabla p - \tau\right), QED. \end{aligned}$$

- **Complementary slackness**. Recall that $\tau_{xy} = \max\{p_y - p_x, 0\}$. By complementary slackness, one has $\tau_{xy} > 0 \implies \mu_{xy} = \bar{\mu}_{xy}$. As $\tau = \max\{\nabla p, 0\}$, this implies that $p_y > p_x \implies \mu_{xy} = \bar{\mu}_{xy}$. Still by complementary slackness, one has $\mu_{xy} > 0 \implies \tau_{xy} = p_y - p_x$, that is $p_y \geq 0$. To summarize

$$\begin{cases} p_y > p_x \implies \mu_{xy} = \bar{\mu}_{xy} \\ p_y < p_x \implies \mu_{xy} = 0 \end{cases}.$$

- As a result, on any path from a source point to a destination point where mass actually flow, the price weakly increases. But the price does not strictly increase across an arc unless the arc capacity is saturated.

Now assume that there is only one source node z^o and one destination node z^d. Then $s = 1\{z = z^o\} - 1\{z = z^d\}$, so that the problem reformulates as

$$\begin{aligned} &\min_p \sum_{xy \in \mathcal{A}} \bar{\mu}_{xy} \max\{p_y - p_x, 0\} \\ &s.t.\ p_{z^o} = 0, p_{z^d} = 1. \end{aligned} \tag{8}$$

The max-flow min-cut theorem expresses that one can take $p \in \{0, 1\}$, and so the problem becomes a min-cut problem.

Theorem 2. Consider problem (8).
(i) The set of solutions is a nonempty convex lattice.
(ii) The subset made of the solutions valued in $\{0,1\}$ is a nonempty lattice.

Proof. (i) The problem reexpresses as the minimization of $\sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} \max\{p_y, p_x\} - \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} p_x$ subject to the constraints $p_{z^o} = 0$ and $p_{z^d} = 1$. The constraints form a lattice, and the function fo be minimized is submodular, so the set of solutions is a lattice.

Proof (ctd). (ii) First, let us show that it is enough to consider solutions in $[0,1]$. Consider $\mathcal{Z}_0 = \arg\min_{z\in\mathcal{Z}}\{p_z\}$ and $\mathcal{Z}_1 = \arg\max_{z\in\mathcal{Z}}\{p_z\}$. There can only be outward flow leaving $\mathcal{Z}_0$, and inward flow entering $\mathcal{Z}_1$, and thus $z^o \in \mathcal{Z}_0$ and $z^d \in \mathcal{Z}_1$. Consider p a solution in $[0,1]$. For $t \in (0,1)$, define $p_z^t = 1\{t \leq p_z\}$. One has $\max\left\{p_x^t, p_y^t\right\} = 1\{t \leq \max\{p_x, p_y\}\}$, thus, letting V be the value of (8), one gets

$$\begin{aligned} V &\leq \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} \max\left\{p_y^t, p_x^t\right\} - \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} p_x^t \\ &= \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} 1\{t \leq \max\{p_x, p_y\}\} - \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} 1\{t \leq p_x\} \end{aligned}$$

but integration over t, we get $V \leq \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} \max\{p_y, p_x\} - \sum_{xy\in\mathcal{A}} \bar{\mu}_{xy} p_x = V$, thus the inequalities above hold as equalities, and therefore the problem has integer solutions.

A cut is a partition of $\mathcal{Z}$ into $\mathcal{Z}_0$ and $\mathcal{Z}_1$ such that $z^o \in \mathcal{Z}_0$ and $z^d \in \mathcal{Z}_1$. The capacity of the cut is given by

$$\sum_{\substack{x \in \mathcal{Z}_0 \\ y \in \mathcal{Z}_1 \\ xy \in \mathcal{A}}} \bar{\mu}_{xy}$$

A cut can be identified to a vector p such that $p_z \in \{0,1\}$ with $p_{z^o} = 0$ and $p_{z^d} = 1$.
The set of cuts is a lattice under the operations
$(\mathcal{Z}_0, \mathcal{Z}_1) \vee (\mathcal{Z}_0', \mathcal{Z}_1') := (\mathcal{Z}_0 \cup \mathcal{Z}_0', \mathcal{Z}_1 \cap \mathcal{Z}_1')$ and
$(\mathcal{Z}_0, \mathcal{Z}_1) \wedge (\mathcal{Z}_0', \mathcal{Z}_1') := (\mathcal{Z}_0 \cap \mathcal{Z}_0', \mathcal{Z}_1 \cup \mathcal{Z}_1')$.
By theorem 1, the value of the minimum cut is equal to the value of the maximum flow. By theorem 2, the set of minimum cuts is a lattice.

Given a flow μ, consider the reduced capacitated network, which is the network $(\mathcal{Z}, \mathcal{A}^r, \bar{\mu}^r)$ where $\mathcal{A}^r = \mathcal{A} \cup \{yx : xy \in \mathcal{A} \text{ and } \mu_{xy} > 0\}$, and $\bar{\mu}^r = \bar{\mu} - \mu$.
The Ford-Fulkerson algorithm consists of the following:
If there is a path z_k from $z_0 = z^o$ to $z_K = z^d$ with $c = \min_{k=0\ldots K-1} \bar{\mu}^r_{x_k x_{k+1}} > 0$, then increase μ by c along this path; update $\bar{\mu}^r$ and continue;
else, stop.
Theorem. If the capacities are integer-valued, then the Ford-Fulkerson algorithm converges towards a maximum flow.

Corollary. Consider a directed graph $(\mathcal{Z}, \mathcal{A})$. The following statements are equivalent:
(i) there is a directed path from x to y.
(ii) for any partition $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ such that $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there must be at least an arc from a node in $\mathcal{X}$ to a node $\mathcal{Y}$.

Download the NYC subway data from the Gihtub repository (subrepository `mec_equil\data\NYC_subway`). The '`arcs`' file lists for each arc represented by a line, the origin node (column 1), the destination node (column 2) and the length of the arc (column 3). You can ignore the other colums. The '`nodes`' file lists for each node represented by line, the name of the node (column 1). You can ignore the other columns.
Assume that the capacity of arc a is given by

$$\mu_a = 10^5 / \left(d_a + 10^3 \right)$$

where d_a is the distance of the arc indicated in column 3 of the '`arcs`' file. You should verify that this number should be a number between 3.84 and 100.
Your origin point z^0 will be the "14 St - Union Sq" station in Manhattan (node #452), and your destination point z^d will be the "59 St (R/N)" station in Brooklyn (node #471).

The max flow problem is given by

$$\max_{t,\mu\geq 0} t$$
$$s.t.\ \nabla^{\intercal}\mu = ts$$
$$\mu \leq \bar{\mu}$$

where $s_z = 1\left\{z = z^d\right\} - 1\left\{z = z^o\right\}$, and d_a is the lenght of arc a.

Q1. Compute the max flow using Gubori.
Q2. Compute the max flow using Ford-Fulkerson.

Assume that the cost per unit though arc a is equal to $c\left(\mu_a\right) = 1 + \mu_a^{2/3}$. As before, your origin point z^0 will be the "14 St - Union Sq" station in Manhattan (node #452), and your destination point z^d will be the "59 St (R/N)" station in Brooklyn (node #471), and $s_z = 1\left\{z = z^d\right\} - 1\left\{z = z^o\right\}$.
Q3. Compute the social welfare

$$\max_{\mu \geq 0} \sum_a \mu_a c\left(\mu_a\right)$$
$$s.t.\ \nabla^{\mathsf{T}} \mu = s.$$

Q4. Compute the Wardrop equilibrium $\left(\mu_a^{eq}\right)$, a solution of

$$\max_{\mu \geq 0} \sum_a k\left(\mu_a\right)$$
$$s.t.\ \nabla^{\mathsf{T}} \mu = s.$$

where $k\left(\mu\right) = \int_0^{\mu} c\left(t\right) dt = \mu_a + \frac{3}{5}\mu_a^{5/3}$. Compute the social welfare associated with $\left(\mu_a^{eq}\right)$.