Notes taken during class on day 4 (raw and unpolished)

1 Individual stable matching

1.1 Definition of stable matching

Assume that there is one individual per type.

α_{xy}=valuation of an xy match by x
γ_{xy}=valuation of an xy match by y
If x and y remain unmatched, they get utility 0.
Assume strict preferences:
$\alpha_{xy} \neq \alpha_{xy'}$ for $y \neq y'$
$\gamma_{xy} \neq \gamma_{x'y}$ for $x \neq x'$

$\mu_{xy} \in \{0, 1\}$ is a dummy variable for x is matched with y.
μ is a stable matching iff:
1) μ is matching of the populations of x and y
$\sum_y \mu_{xy} \leq n_x = 1$
$\sum_x \mu_{xy} \leq m_y = 1$

2a) There is no blocking pair
for every x and y, we cannot have $\alpha_{xy} > u_x^\mu$ AND $\gamma_{xy} > v_y^\mu$
[xy is a blocking pair if $\alpha_{xy} > u_x^\mu$ AND $\gamma_{xy} > v_y^\mu$]
2b) Individual rationality:
for every x, $u_x^\mu \geq 0$
for every y, $v_y^\mu \geq 0$
Where the utilities of x and y under μ are:
$u_x^\mu = \sum_y \mu_{xy} \alpha_{xy}$
$v_y^\mu = \sum_x \mu_{xy} \gamma_{xy}$

1.2 Existence and computation of a stable matching: the Gale and Shapley algorithm

$\mu_{xy}^A \in \{0, 1\}$=dummy variable=1 if x's offer to y is available (i.e. has not been rejected).

1. x's propose to their most favorite avaibable y,i.e.

$$\mu_{xy}^P = 1 \text{ iff } y \in \arg\max_y \left\{\alpha_{xy} : \mu_{xy}^A = 1\right\}$$

2. y's retain their most favorite offer out of those made, i.e.

$$\mu_{xy}^E = 1 \text{ iff } x \in \arg\max_x \left\{\gamma_{xy} : \mu_{xy}^P = 1\right\}$$

3. We keep track of rejected offers, i.e.

$$\mu_{xy}^A = \mu_{xy}^A - \left(\mu_{xy}^P - \mu_{xy}^E\right)$$

2 Adachi's algorithm

Claim: μ is stable matching iff
(1) μ satisfies the population constraints
(2) there is a (u,v) pair with

$$\begin{aligned} u_x &= \max\left\{\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y\right\}, 0\right\} \\ v_y &= \max\left\{\max_x \left\{\gamma_{xy}, 0 : \alpha_{xy} \geq u_x\right\}, 0\right\} \end{aligned}$$

and $\mu_{xy} = 1 \implies u_x = \alpha_{xy}$ and $v_y = \gamma_{xy}$.

Let us take μ a stable matching, show

$$\begin{aligned} u_x^\mu &= \max\left\{\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\}, 0\right\} \\ v_y^\mu &= \max\left\{\max_x \left\{\gamma_{xy}, 0 : \alpha_{xy} \geq u_x^\mu\right\}, 0\right\} \end{aligned}$$

$\max\left\{\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\}, 0\right\} \geq u_x^\mu$
$u_x^\mu = \alpha_{xY(x)} \leq \max\left\{\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\}, 0\right\}$
Assume $\max\left\{\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\}, 0\right\} > u_x^\mu$
then take $y^* \in \arg\max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\}$, we have
$\alpha_{xy^*} = \max_y \left\{\alpha_{xy} : \gamma_{xy} \geq v_y^\mu\right\} > u_x^\mu$
and $\gamma_{xy^*} \geq v_{y^*}^\mu$,
but y^* is not matched with x, thus (no ties) $\gamma_{xy^*} > v_{y^*}^\mu$,
hence $\alpha_{xy^*} > u_x^\mu$ and $\gamma_{xy^*} > v_{y^*}^\mu$, thus xy^* forms a blocking pair.

Conversely, assume there is a (u,v) pair with

$$\begin{aligned} u_x &= \max_{y \in Y_0} \left\{\alpha_{xy} : \gamma_{xy} \geq v_y\right\} \\ v_y &= \max_{x \in X_0} \left\{\gamma_{xy}, 0 : \alpha_{xy} \geq u_x\right\} \end{aligned}$$

Take μ_{xy} such that $\mu_{xy} = 1$ if $u_x = \alpha_{xy}$, 0 otherwise. Show that μ is a stable matching. $\mu_{xy} = 1$ implies $\gamma_{xy} \geq v_y$, but as $\alpha_{xy} \geq u_x$, one has $v_y \geq \gamma_{xy}$, thus $v_y = \gamma_{xy}$. As a result, $u = u^\mu$ and $v = v^\mu$.

Assume xy is a blocking pair, that is $\alpha_{xy} > u_x^\mu$ and $\gamma_{xy}^\mu > v_y$. Then $\alpha_{xy} > u_x^\mu$, which implies $v_y^\mu \geq \gamma_{xy}$, a contradiction. Therefore there is no blocking pair, and μ is stable.

2.1 Reformulate as an equilibrium problem with gross substitutes

$Y_0 = Y \cup \{0\}$ and $X_0 = X \cup \{0\}$

$$
\begin{aligned}
u_x &= \max_{y\in Y_0}\left\{\alpha_{xy} : \gamma_{xy} \geq v_y\right\} \\
v_y &= \max_{x\in X_0}\left\{\gamma_{xy} : \alpha_{xy} \geq u_x\right\}
\end{aligned}
$$

Reformulate as

$$
\begin{aligned}
\max_{y\in Y_0}\left\{\alpha_{xy} : \gamma_{xy} \geq v_y\right\} - u_x &= 0 \\
\max_{x\in X_0}\left\{\gamma_{xy} : \alpha_{xy} \geq u_x\right\} - v_y &= 0
\end{aligned}
$$

Take $p_x = u_x$ and $p_y = -v_y$ and reformulate as

$$
\begin{aligned}
p_x - \max_{y\in Y_0}\left\{\alpha_{xy} : p_y \geq -\gamma_{xy}\right\} &= 0 \\
\max_{x\in X_0}\left\{\gamma_{xy} : \alpha_{xy} \geq p_x\right\} + p_y &= 0
\end{aligned}
$$

Introduce $c_{xy} = -\gamma_{xy}$ and we have

$$
\begin{aligned}
p_x - \max_{y\in Y_0}\{\alpha_{xy} : p_y \geq c_{xy}\} &= 0 \\
p_y - \min_{x\in X_0}\{c_{xy} : \alpha_{xy} \geq p_x\} &= 0
\end{aligned}
$$

Order-preserving mapping:

$$
\left((p_x)_x, (p_y)_y\right) = T\left((p_x)_x, (p_y)_y\right)
$$

where

$$
\begin{aligned}
T_x(p) &= \max_{y\in Y_0}\{\alpha_{xy} : p_y \geq c_{xy}\} \\
T_x(p) &= \min_{x\in X_0}\{c_{xy} : \alpha_{xy} \geq p_x\}
\end{aligned}
$$

and T preserves ordering. Hence by Tarski's fixed point theorem, the set of fixed points of T is a lattice.

Claim. Start from the lower bound. We have $p^0 \leq T\left(p^0\right)$. Then $p^1 \leq T^2\left(p^0\right) = p^2$ etc.

$p^0 \leq p^1 \leq p^2$.... and p will converge to p^* such that $T(p^*) = p^*$

Claim: p^* is the smallest fixed point of T: for any fixed point $\tilde{p}$, one has $p^* \leq \tilde{p}$. Indeed, $p^0 \leq \tilde{p}$, thus by applying T^k, one has $p^k \leq \tilde{p}$ and therefore $p^* \leq \tilde{p}$.

Similarly, if one starts from p^0=the upper bound, $p^0 \geq T\left(p^0\right)$, then p^t converges to the highest fixed point of T.

Adachi's algorithm.
Iterate

$$\begin{aligned} u_x &= \max_{y \in Y_0} \left\{ \alpha_{xy} : \gamma_{xy} \geq v_y \right\} \\ v_y &= \max_{x \in X_0} \left\{ \gamma_{xy} : \alpha_{xy} \geq u_x \right\} \end{aligned}$$

Let's start with very large p, i.e. large u_x and small v_y
$u_x = \max_{y \in Y_0} \{\alpha_{xy}\}$
and
$v_y = \min_{x \in X_0} \left\{\gamma_{xy}\right\}$

2.2 A reinterpretation of Gale and Shapley as coordinate update

Now assume that there is a discrete set of prices.
Start with p^0 with $e\left(p^0\right) \leq 0$
For each z, if $e_z\left(p^t\right) < 0$, move p_z up by 1 unit.
if $e_z\left(p^t\right) = 0$, do not move p_z.

3 Decentralized stable matching

Assume n_x of passengers of type x
and m_y cars of type y.
Assume that if x and y match then:

- x gets at most α_{xy}
- y gets at most γ_{xy}

That is u_x and v_y are such that, $(u_x, v_y) \in \mathcal{F}_{xy}$ if

$$u_x - \alpha_{xy} \leq 0 \text{ and } v_y - \gamma_{xy} \leq 0$$

that is $\max\left\{u_x - \alpha_{xy}, v_y - \gamma_{xy}\right\} \leq 0$.
The distance function $D_{xy}(u, v)$ is the real number t such that

$$\max\left\{u_x - t - \alpha_{xy}, v_y - t - \gamma_{xy}\right\} = 0.$$

that is

$$D_{xy}(u, v) = \max\left\{u_x - \alpha_{xy}, v_y - \gamma_{xy}\right\}.$$

1. Define "aggregate stable matching" under NTU
Look for $\left(\mu_{xy}, u_x, v_y\right)$ such that
(1) population count
$\sum_y \mu_{xy} + \mu_{x0} = n_x$

$\sum_x \mu_{xy} + \mu_{0y} = m_y$

(2) absence of a blocking pair
$D_{xy}(u, v) \geq 0$
$u_x \geq 0$
$v_y \geq 0$

3) $\mu_{xy} > 0 \implies D_{xy}(u, v) \geq 0$
$\mu_{x0} > 0 \implies u_x = 0$
$\mu_{0y} > 0 \implies v_y = 0$

2. Define aggregate matching with NTU with entropic regularization
$M_{xy}(\mu_{x0}, \mu_{0y}) = \exp\left(-\frac{1}{T} D_{xy}\left(-T \ln \mu_{x0}, -T \ln \mu_{0y}\right)\right)$, hence
$M_{xy}(\mu_{x0}, \mu_{0y}) = \min\left\{\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right\}$. – NTU analog of Choo Siow's formula.

Solve μ_{x0} and μ_{0y} using
$\mu_{x0} + \sum_y \min\left\{\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right\} = n_x$
$\mu_{0y} + \sum_x \min\left\{\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right\} = m_y$

3. Relate with Gale-Shapley
Theorem. If $n_x = 1$ and $m_y = 1$, then
consider μ_{xy} a stable matching in the Gale-Shapley sense. Then (μ, u^μ, v^μ) is an aggregate stable matching.

Let us show that $D_{xy}(u^\mu, v^\mu) \geq 0$. Assume by contrast that $D_{xy}(u^\mu, v^\mu) < 0$. Then $\alpha_{xy} > u_x^\mu$ and $\gamma_{xy} > v_y^\mu$ thus xy would be a blocking pair.

$\mu_{xy} > 0$ then x and y are matched, thus $\alpha_{xy} = u_x^\mu$ and $\gamma_{xy} = v_y^\mu$ and $D_{xy}(u^\mu, v^\mu) = 0$.

Conversely, lets assume that (μ, u, v) is an aggregate stable matching. We will show that μ is stable in the Gale-Shapley sense. Assume it were not. Then there would exist a blocking pair xy with $\alpha_{xy} > u_x^\mu = \alpha_{xY^\mu(x)}$ and $\gamma_{xy} > v_y^\mu = \gamma_{X^\mu(y)y}$

We have
$\max\left\{u_x - \alpha_{xY^\mu(x)}, v_{Y^\mu(x)} - \gamma_{xY^\mu(x)}\right\} = 0$, thus $u_x \leq \alpha_{xY^\mu(x)} < \alpha_{xy}$
Similarly,
$\max\left\{u_{X^\mu} - \alpha_{xY^\mu(x)}, v_y - \gamma_{X^\mu(y)y}\right\} = 0$, thus $v_y \leq \gamma_{X^\mu(y)y} < \gamma_{xy}$
hence
$\max\left\{u_x - \alpha_{xy}, v_y - \gamma_{xy}\right\} < 0$
which is a contradiction.

4. Obtain as an outcome of a stationary equilibrium with waiting lines.
Consider a stationary model where
Candidates for emigration from country x arrive at rate n_x per period
Visas for country y are delivered at rate m_y per period

Let us call τ_{xy}^{α} the waiting time by citizen of country x waiting for a visa for country y

Let us call τ_{xy}^{γ} the waiting time by country y for citizen of type x

$$\begin{aligned} L_x(t+1) &= L_x(t) + n_x - \sum_{y \in Y_0} \mu_{xy} \\ L_x(t+1) &= L_x(t) + m_y - \sum_{x \in X_0} \mu_{xy} \end{aligned}$$

where

$$\mu_{xy} = n_x \frac{\exp\left(\alpha_{xy} - \tau_{xy}^{\alpha}\right)}{1 + \sum_{y'} \exp\left(\alpha_{xy'} - \tau_{xy'}^{\alpha}\right)} = m_y \frac{\exp\left(\gamma_{xy} - \tau_{xy}^{\gamma}\right)}{1 + \sum_{x'} \exp\left(\gamma_{x'y} - \tau_{x'y}^{\gamma}\right)}$$

that is

$$\mu_{xy} = \mu_{x0} \exp\left(\alpha_{xy} - \tau_{xy}^{\alpha}\right) = \mu_{0y} \exp\left(\gamma_{xy} - \tau_{xy}^{\gamma}\right)$$

Stationarity: $L_y(t+1) = L_y(t)$, that is

$$\begin{aligned} \sum_{x \in X_0} \mu_{xy} &= m_y \\ \sum_{y \in Y_0} \mu_{xy} &= n_x \end{aligned}$$

and we have

$$\min\left\{\tau_{xy}^{\alpha}, \tau_{xy}^{\gamma}\right\} = 0$$

Introduce

$$\begin{aligned} U_{xy} &= \alpha_{xy} - \tau_{xy}^{\alpha} \\ V_{xy} &= \gamma_{xy} - \tau_{xy}^{\gamma} \end{aligned}$$

this reformulates

$$\max\left\{\ln \frac{\mu_{xy}}{\mu_{x0}} - \alpha_{xy}, \ln \frac{\mu_{xy}}{\mu_{0y}} - \gamma_{xy}\right\} = 0$$

thus

$$\mu_{xy} = \min\left\{\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}\right\}.$$