‘MATH+ECON+CODE’ MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

Alfred Galichon (New York University)

Thursday: “Multinomial choice”
Block 12. Demand models, old and new

- Beyond GEV: the pure characteristics models, the random coefficient logit model, the probit model
- Simulation methods: Accept-Reject and SARS
- Demand inversion (ctd): The inversion theorem

- [OTME], Ch. 9.2
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- Berry, Levinsohn, and Pakes (1995). "Automobile Prices in Market Equilibrium," *Econometrica*.
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- Train (2009). *Discrete Choice Methods with Simulation*. 2nd Edition. Cambridge University Press.
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Section 1

Choice models beyond GEV

- The GEV models are convenient analytically, but not very flexible.
  - The logit model imposes zero correlation across alternatives
  - The nested logit allows for nonzero correlation, but in a very rigid way (needs to define nests).
- A good example is the probit model, where ε is a Gaussian vector. For this model, there is no close-form solution neither for G nor for G^*.
- More recently, a number of modern models don't have closed-form either. These models require simulation methods in order to approximate them by discrete models.

- The pure characteristics model (Berry and Pakes, 2007) can be motivated as follows. Assume y stands for the number of bedrooms. The logit model would assume that the random utility associated with a 2-BR is uncorrelated with a 3-BR, which is not realistic.
- Let ξ_y is the typical size of a bedroom of size y, one may introduce ϵ as the valuation of size; in which case the utility shock associated with y should be $\varepsilon_y = \epsilon \xi_y$. More generally, the characteristics ξ_y is a d-dimensional (deterministic) vector, and $\epsilon \sim \mathbf{P}_\epsilon$ is a (random) vector of the same size standing for the valuations of the respective dimensions, so that
$$\varepsilon_y = \epsilon^\intercal \xi_y.$$
- For example, if each alternative y stands for a model of car, the first component of ξ_y may be the price of car y; the other components may be other characteristics such as number of seats, fuel efficiency, size, etc. In that case, for a given dimension $y \in \mathcal{Y}_0$, ϵ_y is the (random) valuation of this dimension by the consumer with taste vector ϵ.

- Assume without loss of generality that $\varepsilon_y = 0$, that is $\xi_0 = 0$ as we can always reduce the setting to this case by replacing ξ_y by $\xi_y - \xi_0$.
- Letting Z be the $|\mathcal{Y}| \times d$ matrix of (y, k)-term ξ_y^k, this rewrites as

$$\varepsilon = Z\epsilon.$$

- Hence, we have

$$G(U) = \mathbb{E}\left[\max\left\{U + Z\epsilon, 0\right\}\right].$$

and

$$\sigma_y(U) = \Pr\left(U_y - U_z \geq (Z\epsilon)_y - (Z\epsilon)_z \;\; \forall z \in \mathcal{Y}_0 \backslash \{y\}\right).$$

- When $d = 1$ (scalar characteristics), one has $\sigma_y(U) = \Pr\left(U_y - U_z \geq (\xi_y - \xi_z)\epsilon \; \forall z \in \mathcal{Y}_0 \backslash \{y\}\right)$, and thus

$$\sigma_y(U) = \Pr\left(\max_{z:\xi_y > \xi_z}\left\{\frac{U_y - U_z}{\xi_y - \xi_z}\right\} \leq \epsilon \leq \min_{z:\xi_y < \xi_z}\left\{\frac{U_y - U_z}{\xi_y - \xi_z}\right\}\right)$$

  with the understanding that $\max_{z\in\emptyset} f_z = -\infty$ and $\min_{z\in\emptyset} f_z = +\infty$.
- Therefore, letting $\mathbf{F}_\epsilon$ be the c.d.f. associated with the distribution of ϵ, one has a closed-form expression for σ_y:

$$\sigma_y(U) = \mathbf{F}_\epsilon\left(\left[\max_{z:\xi_y > \xi_z}\left\{\frac{U_y - U_z}{\xi_y - \xi_z}\right\}, \min_{z:\xi_y < \xi_z}\left\{\frac{U_y - U_z}{\xi_y - \xi_z}\right\}\right]\right)$$

- When $\mathbf{P}_\epsilon$ is the $\mathcal{N}(0, S)$ distribution, then the pure characteristics model is called a Probit model; in this case,

$$\varepsilon \sim \mathcal{N}(0, \Sigma) \text{ where } \Sigma = ZSZ^\intercal.$$

- Note the distribution ε will not have full support unless $d \geq |\mathcal{Y}|$ and Z is of full rank.
- Computing σ in the Probit model thus implies computing the mass assigned by the Gaussian distribution to rectangles of the type

$$[l_y, u_y].$$

When Σ is diagonal (random utility terms are i.i.d. across alternatives), this is numerically easy. However, this is computationally difficult in general (more on this later).

- The random coefficient logit model (Berry, Levinsohn and Pakes, 1995) may be viewed as an interpolant between the random characteristics model and the logit model. In this case,

$$\varepsilon = (1 - \lambda) Z\epsilon + \lambda\eta$$

where $\epsilon \sim \mathbf{P}_\epsilon$, η is an EV1 distribution independent from the previous term, and λ is a interpolation parameter ($\lambda = 1$ is the logit model, and $\lambda = 0$ is the pure characteristics model).

- In this case, one may compute the Emax operator as

$$\begin{aligned}
G(U) &= \mathbb{E}\left[\max_{y\in\mathcal{Y}_0}\left\{U_y + (1-\lambda)(Z\epsilon)_y + \lambda\eta_y\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\max_{y\in\mathcal{Y}_0}\left\{U_y + (1-\lambda)(Z\epsilon)_y + \lambda\eta_y\right\}|\epsilon\right]\right] \\
&= \mathbb{E}\left[\lambda \log \sum_{y\in\mathcal{Y}_0} \exp\left(\frac{U_y + (1-\lambda)(Z\epsilon)_y}{\lambda}\right)\right]
\end{aligned}$$

- Recall

$$G(U) = \mathbb{E}\left[\lambda \log \sum_{y \in \mathcal{Y}_0} \exp\left(\frac{U_y + (1-\lambda)(Z\epsilon)_y}{\lambda}\right)\right].$$

- The demand map in the random coefficients logit model is obtained by derivation of the expression of the Emax, i.e.

$$\sigma_y(U) = \mathbb{E}\left[\frac{\exp\left(\frac{U_y + (1-\lambda)(Z\epsilon)_y}{\lambda}\right)}{\sum_{y' \in \mathcal{Y}_0} \exp\left(\frac{U_{y'} + (1-\lambda)(Z\epsilon)_{y'}}{\lambda}\right)}\right].$$

Section 2

Simulation methods

- In a number of cases, one cannot compute the choice probabilities $\sigma(U)$ using a closed-form expression. In this case, we need to resort to simulation to compute G, G^*, σ and σ^{-1}.
- The idea is that:
  - one is able to compute G and G^* for discrete distributions (more on this later)
  - the sampled versions of G, G^*, σ and σ^{-1} converge to the populations objects when the sample size is large.

- One simulates N points $\varepsilon^i \sim P$. The Emax operator associated with the empirical sample distribution P_N is

$$G_N = N^{-1} \sum_{i=1}^{N} \max_{y \in \mathcal{Y}} \left\{ U_y + \varepsilon_y^i \right\}$$

and the demand map is given by

$$\sigma_{N,y}\left(U\right) = N^{-1} \sum_{i=1}^{N} 1 \left\{ U_y + \varepsilon_y^i \geq U_z + \varepsilon_z^i \ \forall z \in \mathcal{Y}_0 \right\}$$

- In the literature, σ_N is called the *accept-reject simulator.*

- McFadden's smoothed accept-reject simulator (SARS) consists in sampling $\varepsilon \sim P$: $\varepsilon^1, ..., \varepsilon^N$, and replacing the max by the smooth-max

$$\sigma_{N,T,y}(U) = \sum_{i=1}^{N} \frac{1}{N} \frac{\exp\left(\left(U_y + \varepsilon_y^i\right)/T\right)}{\sum_z \exp\left(\left(U_z + \varepsilon_z^i\right)/T\right)}$$

- One seeks U so that the induced choice probabilities are s, that is

$$s_y = \sum_{i=1}^{N} \frac{1}{N} \frac{\exp\left(\left(U_y + \varepsilon_y^i\right)/T\right)}{\sum_z \exp\left(\left(U_z + \varepsilon_z^i\right)/T\right)}.$$

- The associated Emax operator is

$$G_{N,T}(U) = \mathbb{E}_{\mathbf{P}_N}\left[G_{\text{logit}}\left(U + \varepsilon^i\right)\right]$$

so the underlying random utility structure is a random coefficient logit.

Section 3

The inversion theorem

THEOREM (G AND SALANIÉ)

Consider a solution $(u(\varepsilon), v_y)$ *to the dual Monge-Kantorovich problem with cost* $\Phi(\varepsilon, y) = \varepsilon_y$, *that is:*

$$\min_{u,v} \int u(\varepsilon)\, d\mathbf{P}(\varepsilon) + \sum_{y \in \mathcal{Y}_0} v_y s_y \tag{1}$$

$$s.t.\ u(\varepsilon) + v_y \geq \Phi(\varepsilon, y)$$

Then:
(i) $U = \sigma^{-1}(s)$ *is given by* $U_y = v_0 - v_y$.
(ii) The value of Problem (1) is $-G^*(s)$.

PROOF.

$\sigma^{-1}(s) = \arg\max_{U:U_0=0} \left\{ \sum_{y\in\mathcal{Y}} s_y U_y - G(U) \right\}$, thus, letting $v = -U$, v is the solution ot

$$\min_{v:v_0=0} \left\{ \sum_{y\in\mathcal{Y}_0} s_y v_y + G(-v) \right\}$$

which is exactly problem (1). □

- It follows from the inversion theorem that the problem of demand inversion in the pure characteristics model is a semi-discrete transport problem, a point made in Bonnet, G and Shum (2017).
- Indeed, the correspondence is:
    - an alternative y is a fountain
    - the characteristics of an alternative is a fountain location
    - the systematic utility associated with alternative y is minus the price of fountain y
    - the market share of altenative y coindides with the capacity of fountain y
    - the random vector ϵ is the location of an inhabitant

- Cf. Bonnet, G. and Shum (2017). Let $u_i = T \log \sum_z \exp\left((U_z + \varepsilon_z^i) / T \right)$. One has

$$\left\{ \begin{array}{l} s_y = \sum_{i=1}^N \frac{1}{N} \exp\left((U_y - u_i + \varepsilon_y^i) / T \right) \\ \frac{1}{N} = \sum_y \frac{1}{N} \exp\left((U_y - u_i + \varepsilon_y^i) / T \right) \end{array} \right. .$$

- As a result, (u_i, U_y) are the solution of the regularized OT problem

$$\min_{u,U} \sum_{i=1}^N \frac{1}{N} u_i - \sum s_y U_y + \sum_{i,y} \frac{1}{N} \exp\left((U_y - u_i + \varepsilon_y^i) / T \right).$$

- Consider the IPFP algorithm for solving the latter problem:

$$\begin{cases} \exp\left(u_i^{k+1}/T\right) = \sum_z \exp\left(\left(U_z^k + \varepsilon_z^i\right)/T\right) \\ \exp U_y^{k+1}/T = \frac{N s_y}{\sum_{i=1}^N \exp\left(\left(-u_i^{k+1} + \varepsilon_y^i\right)/T\right)} \end{cases}$$

- This rewrites as

$$\exp U_y^{k+1}/T = \frac{N s_y}{\sum_{i=1}^N \frac{\exp\left(\varepsilon_y^i/T\right)}{\sum_z \exp\left(\left(U_z^k + \varepsilon_z^i\right)/T\right)}}, \text{ i.e.}$$

$$U_y^{k+1} = T \log s_y - T \log \sum_{i=1}^N \frac{1}{N} \frac{\exp\left(\varepsilon_y^i/T\right)}{\sum_z \exp\left(\left(U_z^k + \varepsilon_z^i\right)/T\right)}$$

which is exactly the contraction mapping algorithm of Berry, Levinsohn and Pakes (1995, appendix 1).

- **And now, let's code!**