'MATH+ECON+CODE' MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

Alfred Galichon (NYU)

Monday: Linear programming
Block 1. Linear programming duality

Preliminary stuff

- Schedule: Mon 6/17 – Fri 6/21, 2019, 8am-12pm noon and 2pm-4pm.
- Location: NYU Paris, 59 bd St-Germain.
- Office hours: by appointment.
- Course webpage: http://alfredgalichon.com/mec_optim
- Text (optional): Galichon (2016). *Optimal Transport Methods in Economics*, Princeton.

Introducing ourselves

- Introducing me: A. Galichon, professor of economics and of mathematics at NYU (ag133@nyu.edu)
- Introducing veteran students:
    - Pauline Corblet, Sciences Po graduate student (pauline.corblet@sciencespo.fr)
    - Octavia Ghelfi, NYU graduate student (ofg208@nyu.edu)
    - James Nesbit, NYU graduate student (jmn425@nyu.edu)
- Introducing you

Remarks on this course

- Targeting Math and Econ students. Self-contained for both audiences
- Teaching format: 15 "blocks"; each block = 50 minutes of theory + 1 hour of coding
  - coding most often based on an empirical application related to the theory just seen
  - students are expected to write their own code; we'll ensure that it is operational at the end of each block
  - Pauline, Octavia and James will lead course-related discussions after 4pm.
- Programming: our demos will be done in R and the support will be in R only, but you are welcome to use the language of your choice e.g. Matlab, C++, Python, Julia... Solvers used will be Gurobi (for LP) and NLOPT (for nonlinear optimization), so make sure your language of choice has a convenient interface to these.
- Questions?

This course

- This course is focused on models of demand, matching models, and optimal transport methods, with various applications pertaining to labor markets, economics of marriage, industrial organization, matching platforms, networks, and international trade, from the crossed perspectives of theory, empirics and computation.
- It will introduce tools from economic theory, mathematics, econometrics and computing, on a needs basis, without any particular prerequisite other than the equivalent of a first year graduate sequence in econ or in applied math.

Outline

- Part I: Tools
  - Monday 6/17: linear programming
  - Tuesday 6/18: optimal transport toolbox 1 - duality
  - Wednesday 6/19: optimal transport toolbox 2 - convex analysis
- Part II: Models
  - Thursday 6/20: static and dynamic multinomial choice
  - Friday 6/21: statistical estimation of models of matching with transfers

Software installation

- See the setup.md document in the github repository

Knowing you

Please fill out:
https://goo.gl/forms/5WLfkpc3GNCecd6A3

Assessment

- Students needing a grade have a choice between:
  - Either a take-home exam (24 hours) available e.g. from Saturday Jan 26 noon, until Sunday Jan 27 noon (**to be confirmed with the class**),
  - Or a short paper (12 pages or more), to be discussed with the instructor. The paper will bear some connections, in a broad sense, with the topics of the course. Many papers are considered acceptable: original research paper, survey paper, report on numerical experiments, replication of existing empirical results… are all acceptable. The requirement is to be innovative on a theoretical, empirical, or computational level. This work should be submitted before June 30, 2019.
- Email me to indicate which of these you are opting for.

Block 1: Linear programming

Learning objectives: block 1

- Linear programming duality
- Economic interpretation of the dual
- Numerical computation

References for block 1

- [OTME], App. B
- Stigler (1945), The cost of subsistence. *Journal of Farm Economics*.
- Dantzig (1990), The diet problem. *Interface*.
- Complements:
  - Gale (1960), *The theory of linear economic models*.
  - Vohra (2011), *Mechanism Design: A Linear Programming Approach*.
- www.gurobi.com
- www.gnu.org/software/glpk/

Section 1

Motivation: the diet problem

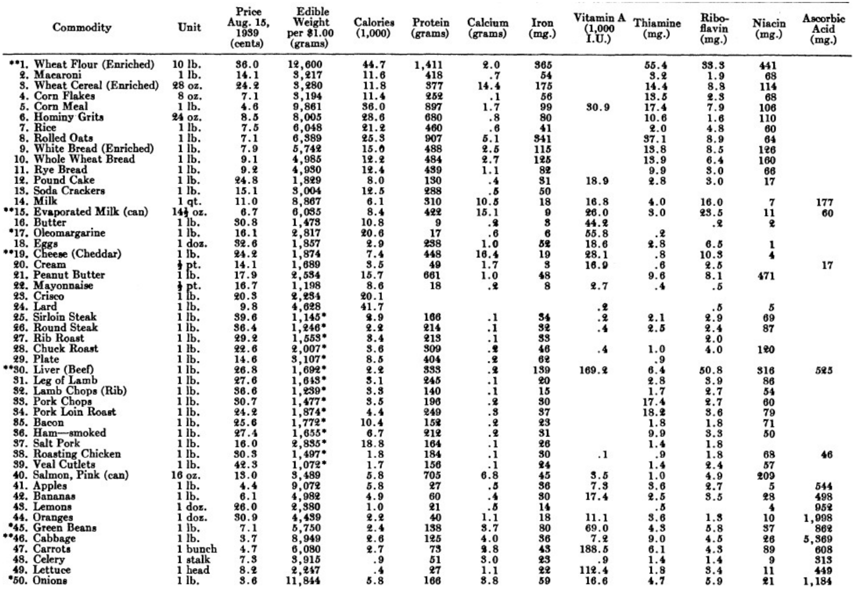
Stigler's diet problem

- During World War II, engineers in US Army were wondering how to feed their personnel at minimal cost, leading to what is now called the "optimal diet problem".
    - Nutritionists have identified a number of vital nutrients (calories, protein, calcium, iron, etc.) that matter for a person's health, and have determined the minimum daily intake of each nutrient
    - For each basic food (pasta, butter, bread, etc), nutritionists have characterized the intake in each of the various nutrients
    - Each food has a unit cost, and the problem is to find the optimal diet = combination of foods that meet the minimal intake in each of the nutrients and achieves minimal cost
- The problem was taken on by G. Stigler, who published a paper about it in 1945, giving a first heuristic solution, exhibiting a diet that costs \$39.93 per year in 1939 dollars. Later (in 1947) it was one of the first application of G.B. Dantzig's method (the simplex algorithm), which provided the exact solution (\$39.67). It then took 120 man-day to perform this operation. At the end of this block, the computer will perform it for us in a fraction of second.
- However, don't try this diet at home! Dantzig did so and almost died from it...

A look at our data

- Our dataset (directory ‘01-appli-diet’) was directly taken from Stigler’s article. It is a csv file called ‘StiglerData1939.txt’:

A look at our data



Commodity	Unit	Price Aug. 15, 1939 (cents)	Edible Weight per $1.00 (grams)	Calories (1,000)	Protein (grams)	Calcium (grams)	Iron (mg.)	Vitamin A (1,000 I.U.)	Thiamine (mg.)	Ribo-flavin (mg.)	Niacin (mg.)	Ascorbic Acid (mg.)
**1. Wheat Flour (Enriched)	10 lb.	36.0	12,600	44.7	1,411	2.0	365		55.4	33.3	441	
2. Macaroni	1 lb.	14.1	3,217	11.6	418	.7	54		3.2	1.9	68	
3. Wheat Cereal (Enriched)	28 oz.	24.2	3,280	11.8	377	14.4	175		14.4	8.8	114	
4. Corn Flakes	8 oz.	7.1	3,194	11.4	252	.1	56		13.5	2.3	68	
5. Corn Meal	1 lb.	4.6	9,861	36.0	897	1.7	99	30.9	17.4	7.9	106	
6. Hominy Grits	24 oz.	8.5	8,005	28.6	680	.8	80		10.6	1.6	110	
7. Rice	1 lb.	7.5	6,048	21.2	460	.6	41		2.0	4.8	60	
8. Rolled Oats	1 lb.	7.1	6,389	25.3	907	5.1	341		37.1	8.9	64	
9. White Bread (Enriched)	1 lb.	7.9	5,742	15.0	488	2.5	115		13.8	8.5	126	
10. Whole Wheat Bread	1 lb.	9.1	4,985	12.2	484	2.7	125		13.9	6.4	160	
11. Rye Bread	1 lb.	9.2	4,930	12.4	439	1.1	82		9.9	3.0	66	
12. Pound Cake	1 lb.	24.8	1,829	8.0	130	.4	31	18.9	2.8	3.0	17	
13. Soda Crackers	1 lb.	15.1	3,004	12.5	288	.5	50					
14. Milk	1 qt.	11.0	8,867	6.1	310	10.5	18	16.8	4.0	16.0	7	177
**15. Evaporated Milk (can)	14½ oz.	6.7	6,035	8.4	422	15.1	9	26.0	3.0	23.5	11	60
16. Butter	1 lb.	30.8	1,473	10.8	9	.2	3	44.2		.2	2	
*17. Oleomargarine	1 lb.	16.1	2,817	20.6	17	.6	6	55.8	.2			
18. Eggs	1 doz.	32.6	1,857	2.9	238	1.0	52	18.6	2.8	6.5	1	
**19. Cheese (Cheddar)	1 lb.	24.2	1,874	7.4	448	16.4	19	28.1	.8	10.3	4	
20. Cream	½ pt.	14.1	1,689	3.5	49	1.7	3	16.9	.6	2.5		17
21. Peanut Butter	1 lb.	17.9	2,534	15.7	661	1.0	48		9.6	8.1	471	
22. Mayonnaise	½ pt.	16.7	1,198	8.6	18	.2	8	2.7	.4	.5		
23. Crisco	1 lb.	20.3	2,234	20.1								
24. Lard	1 lb.	9.8	4,628	41.7				.2		.5	5	
25. Sirloin Steak	1 lb.	39.6	1,145*	2.9	166	.1	34	.2	2.1	2.9	69	
26. Round Steak	1 lb.	36.4	1,246*	2.2	214	.1	32	.4	2.5	2.4	87	
27. Rib Roast	1 lb.	29.2	1,553*	3.4	213	.1	33			2.0		
28. Chuck Roast	1 lb.	22.6	2,007*	3.6	309	.2	46	.4	1.0	4.0	120	
29. Plate	1 lb.	14.6	3,107*	8.5	404	.2	62		.9			
**30. Liver (Beef)	1 lb.	26.8	1,692*	2.2	333	.2	139	169.2	6.4	50.8	316	525
31. Leg of Lamb	1 lb.	27.6	1,643*	3.1	245	.1	20		2.8	3.9	86	
32. Lamb Chops (Rib)	1 lb.	36.6	1,239*	3.3	140	.1	15		1.7	2.7	54	
33. Pork Chops	1 lb.	30.7	1,477*	3.5	196	.2	30		17.4	2.7	60	
34. Pork Loin Roast	1 lb.	24.2	1,874*	4.4	249	.3	37		18.2	3.6	79	
35. Bacon	1 lb.	25.6	1,772*	10.4	152	.2	23		1.8	1.8	71	
36. Ham—smoked	1 lb.	27.4	1,655*	6.7	212	.2	31		9.9	3.3	50	
37. Salt Pork	1 lb.	16.0	2,835*	18.8	164	.1	26		1.4	1.8		
38. Roasting Chicken	1 lb.	30.3	1,497*	1.8	184	.1	30	.1	.9	1.8	68	46
39. Veal Cutlets	1 lb.	42.3	1,072*	1.7	156	.1	24		1.4	2.4	57	
40. Salmon, Pink (can)	16 oz.	13.0	3,489	5.8	705	6.8	45	3.5	1.0	4.9	209	
41. Apples	1 lb.	4.4	9,072	5.8	27	.5	36	7.3	3.6	2.7	5	544
42. Bananas	1 lb.	6.1	4,982	4.9	60	.4	30	17.4	2.5	3.5	28	498
43. Lemons	1 doz.	26.0	2,380	1.0	21	.5	14		.5		4	952
44. Oranges	1 doz.	30.9	4,439	2.2	40	1.1	18	11.1	3.6	1.3	10	1,998
*45. Green Beans	1 lb.	7.1	5,750	2.4	138	3.7	80	69.0	4.3	5.8	37	862
**46. Cabbage	1 lb.	3.7	8,949	2.6	125	4.0	36	7.2	9.0	4.5	26	5,369
47. Carrots	1 bunch	4.7	6,080	2.7	73	2.8	43	188.5	6.1	4.3	89	608
48. Celery	1 stalk	7.3	3,915	.9	51	3.0	23	.9	1.4	1.4	9	313
49. Lettuce	1 head	8.2	2,247	.4	27	1.1	22	112.4	1.8	3.4	11	449
*50. Onions	1 lb.	3.6	11,844	5.8	166	3.8	59	16.6	4.7	5.9	21	1,184

Motivation: the diet problem

- Problem setup:
  - Assume there are nutrients $i \in \{1, ..., m\}$ (calories, protein, calcium, iron, etc.) that matter for a person's health, in such way that the minimum daily intake of nutrient i should be d_i.
  - Nutrients do not come as standalone elements, but are combined into various foods. Each unit of food $j \in \{1, ..., n\}$ yields a quantity N_{ij} of nutrient $i \in \{1, ..., m\}$. The dollar cost of food j is c_j.
- The problem is to find the diet that achieves the minimal intake of each nutrient at a cheapest price. If $q \in \mathbb{R}^n$ is a vector such that $q_j \geq 0$ is the quantity of food i purchased, the quantity of nutrient i ingested is $\sum_{j=1}^n N_{ij} q_j$, and the cost of the diet is $\sum_{j=1}^n q_j c_j$. The optimal diet is therefore given by

$$\min_{q \geq 0} c^\top q \\ s.t.\ Nq \geq d. \tag{1}$$

Section 2

A crash course on linear programming

Linear programming in standard form

- Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, A be a $m \times n$ matrix, and consider the following problem

$$V_P = \max_{x \in \mathbb{R}^n_+} c^\top x \qquad (2)$$
$$s.t.\ Ax = d$$

  This problem is a *linear programming problem*, as the objective function, namely $x \to c^\top x$ is linear, and as the constraint, namely $x \in \mathbb{R}^n_+$ and $Ax = d$ are also linear (or more accurately, affine). Problem (2) is called *primal program*, for reasons to be explained soon. The set of x's that satisfy the constraint are called *feasible solutions*; the set of solutions of problem (2) are called *optimal solutions*.

- Remarks:
  - The previous diet problem can be reformulate into this problem – why?
  - A problem does not necessarily have a feasible solution (e.g. if $A = 0$ and $d \neq 0$), in which case (by convention) $V_P = -\infty$.
  - The whole space may be solution (e.g. if $A = 0$ and $d = 0$), in which case $V_P = +\infty$.

Duality

There is a powerful tool called duality which provides much insight into the analysis of problem (2). The idea is to rewrite the problem as

$$V_P = \max_{x \in \mathbb{R}^n_+} \left\{ c^\top x + L_P\left(d - Ax\right) \right\}$$

where $L_P\left(z\right)$ is a penalty function whose value is zero if the constraint is met, that is if $z = 0$, and $-\infty$ if it is not, namely if $z \neq 0$. The simplest choice of such penalty function is given by $L_P\left(z\right) = \min_{y \in \mathbb{R}^m} \left\{ z^\top y \right\}$. One has

$$V_P = \max_{x \in \mathbb{R}^n_+} \min_{y \in \mathbb{R}^m} \left\{ c^\top x + \left(d - Ax\right)^\top y \right\}.$$

Duality (ctd)

However, the minimax inequality $\max_x \min_y \leq \min_y \max_x$ always holds, thus

$$V_P \leq \min_{y\in\mathbb{R}^m} \max_{x\in\mathbb{R}^n_+} \left\{ c^\top x + (d - Ax)^\top y \right\} = \min_{y\in\mathbb{R}^m} \max_{x\in\mathbb{R}^n_+} \left\{ x^\top \left(c - A^\top y \right) + d^\top y \right\}$$

$$\leq \min_{y\in\mathbb{R}^m} \left\{ d^\top y + L_D \left(c - A^\top y \right) \right\} =: V_D$$

where $L_D(z) = \max_{x\in\mathbb{R}^n_+} \left\{ x^\top z \right\}$ is equal to 0 if $z \in \mathbb{R}^n_-$, and to $+\infty$ if not. Therefore, the value V_D is expressed by the *dual program*

$$V_D = \min_{y\in\mathbb{R}^m} d^\top y, \qquad (3)$$
$$s.t.\ A^\top y \geq c$$

and the weak duality inequality $V_P \leq V_D$ holds. It turns out that as soon as either the primal or dual program has an optimal solution, then both programs have an optimal solution and the values of the two programs coincide, so the weak duality becomes an equality $V_P = V_D$ called strong duality. Further, if $x^* \in \mathbb{R}^n_+$ is an optimal primal solution, and $y^* \in \mathbb{R}^m$ is an optimal dual solution, then complementary slackness holds, that is $x^*_i > 0$ implies $\left(A^\top y^* \right)_i = c_i$.

Duality theorem

We summarize these results into the following statement.
Theorem. In the setting described above:
(i) The weak duality inequality holds:

$$V_P \leq V_D.$$

(ii) As soon as the primal or the dual program have an optimal solution, then both programs have an optimal solution, and strong duality holds:

$$V_P = V_D.$$

(iii) If $x^* \in \mathbb{R}^n_+$ is an optimal primal solution, and $y^* \in \mathbb{R}^m$ is an optimal dual solution, then complementary slackness holds:

$$x_i^* > 0 \text{ implies } \left(A^\top y^*\right)_i = c_i.$$

Back to Stigler's problem

- Recall the optimal diet problem

$$\min_{q\geq 0} c^\top q$$
$$s.t.\ Nq \geq d.$$

which has minimax formulation $\min_{q\geq 0}\max_{\pi\geq 0} c^\top q + d^\top \pi - q^\top N^\top \pi$, so the dual is

$$\max_{\pi\geq 0} d^\top \pi$$
$$s.t.\ N^\top \pi \leq c$$

- Interpretation: imagine that there is a new firm called Nutrient Shoppe, who sells raw nutrients. Let π_i be the price of nutrient i. The cost of the diet is $d^\top \pi$. Consumer purchase raw nutrients and can generate "synthetic" foods. The cost of the synthetic version of food j is $\sum_{i=1}^m N_{ij}\pi_i = \left(N^\intercal \pi\right)_j$. The constraint thus means that each "synthetic" food is more affordable than its natural counterpart.

Back to Stigler's problem (ctd)

- The duality means that it is possible to price the nutrients so that the synthetic foods are cheaper than the natural ones, in such a way that the price of the synthetic diet equals the price of the natural diet.
- Complementary slackness yields:
  - $q_j > 0$ implies $(N^\intercal \pi)_j = c_j$; that is, if natural food j is actually purchased, then the prices of its synthetic and natural versions coincide
  - $\pi_i > 0$ implies $(Nq)_i = d_i$; that is, if nutrient i has a positive price, then the natural diet has the "just right" amount.
- **And now, let's code!**