‘MATH+ECON+CODE’ MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

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Tuesday: “Optimal transport I”
Block 6. Semidiscrete matching

- Hotelling's characteristics model
- Voronoi tessations, power diagrams
- Semi-discrete transport, Aurenhammer's method

References for block 6

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- Anderson, de Palma, and Thisse (1992). *Discrete Choice Theory of Product Differentiation*. MIT.
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- Feenstra, Levinsohn (1995). *Estimating Markups and Market Conduct with Multidimensional Product Attributes*. Restud.
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Section 1

MOTIVATION AND SETTING

- Today we'll consider a version of the transportation problem where we seek to match a continuous distribution on $\mathbb{R}^d$ with a discrete distribution. This problem is called a *semi-discrete transportation problem*.
- Actually, we will introduce this problem not as a matching problem, but as a demand problem. We'll model the demand for facilities (such as schools, stores) in the physical space. The same approach applies to the demand for products (e.g. cars) in the characteristics space, see e.g. Lancaster (1966), Feenstra and Levinsohn (1995), and Berry and Pakes (2007).
- We'll simulate fountain locations on a city represented by the two dimensional square.

- Consider inhabitants of a city whose geographic coordinates are $x \in \mathcal{X} = [0,1]^2$. More generally, $\mathcal{X}$ will be a convex subset of $\mathbb{R}^d$ ($d = 2$ is only to fix ideas). The location of inhabitants is distributed according to a absolutely continuous distribution P whose support is included on $\mathcal{X}$, normalized to 1, so P is a probability distribution.
- There are M fountains, located at points $y_j \in \mathbb{R}^d$, $1 \leq j \leq M$. Fountain j is assumed to have capacity q_j, and it is assumed that $\sum_j q_j = 1$, which means that total supply equals the total demand.
- An inhabitant at x has a transportation cost associated with using fountain j which is proportional to the square distance to the fountain

$$\tilde{\Phi}(x, y) := -|x - y|^2 / 2. \tag{1}$$

- Let $\tilde{v}_j$ be the price charged by fountain j. The utility of the consumer at location x is therefore $\tilde{\Phi}(x, y_j) - \tilde{v}_j$, and the indirect surplus of the consumer at x is given by

$$\tilde{u}(x) = \max_{j \in \{1,\ldots,M\}} \left\{ \tilde{\Phi}(x, y_j) - \tilde{v}_j \right\}. \tag{2}$$

- Without loss of generality, one can replace the quadratic surplus $\tilde{\Phi}(x, y) = -|x - y|^2 / 2$ by the scalar product surplus

$$\Phi(x, y) := x^\intercal y. \tag{3}$$

Indeed, note that $\tilde{\Phi}(x, y) = \Phi(x, y) + |x|^2 / 2 + |y|^2 / 2$, and introduce the *reduced indirect surplus* $u(x)$ and the v_j's the *reduced prices* as

$$u(x) = \tilde{u}(x) + |x|^2 / 2, \text{ and } v_j = \tilde{v}_j + |y_j|^2 / 2, \tag{4}$$

one immediately sees that $\tilde{u}(x) + \tilde{v}_j \geq \tilde{\Phi}(x, y_j)$ if and only if $u(x) + v_j \geq \Phi(x, y_j)$. It follows that the consumer at location x chooses fountain j that maximizes

$$u(x) = \max_{j \in \{1,\dots,M\}} \left\{ \Phi(x, y_j) - v_j \right\}. \tag{5}$$

- Hence the problem can be reexpressed so that the surplus of consumer x at fountain j is simply $x^\intercal y_j - v_j$. It is clear from (5) that (unlike $\tilde{u}$), the reduced surplus u is a piecewise affine and convex function from $\mathbb{R}^d$ to $\mathbb{R}$. The connection with convex and piecewise affine functions is the reason for reformulating the problem as we did.

- The demand set of fountain j is

$$\mathcal{X}_j^v := \left\{ x \in \mathcal{X} : \tilde{\Phi}\left(x, y_j\right) - \tilde{v}_j \geq \tilde{\Phi}\left(x, y_k\right) - \tilde{v}_k,\ \forall k \right\}$$

which is equivalent to

$$\mathcal{X}_j^v = \left\{ x \in \mathcal{X} : x^\intercal \left(y_j - y_k\right) \geq v_j - v_k,\ \forall k \right\}. \tag{6}$$

- Basic properties:
  - $\mathcal{X}_j$ is a convex polyhedron;
  - the intersection of $\mathcal{X}_j$ and $\mathcal{X}_k$'s lies in the hyperplane of equation $\left\{ x : x^\intercal \left(y_j - y_k\right) + v_k - v_j = 0 \right\}$;
  - the set $\mathcal{X}_j$ weakly increases when v_k $(k \neq j)$ increases, and strictly decreases when v_j decreases.
- The system of sets $\left(\mathcal{X}_j^v\right)_j$ is called *power diagram* associated to the price system v.

- If fountains do not charge any fee, that is, if $\tilde{v}_j = 0$, or equivalently if $v_j = |y_j|^2/2$, then $\mathcal{X}_j^0$ is the set of consumers who are closer to fountain j than to any other fountain. The cells $\mathcal{X}_j^0$ form a partition of $\mathcal{X}$ called *Voronoi tesselation*, which is a very particular case of a power diagram. See Figure 1. The Voronoi diagrams have the property that fountain j belongs to cell $\mathcal{X}_j^0$; when $\tilde{v} \neq 0$, this property may no longer hold for more general power diagrams.

Voronoi tesselations: Illustration

Figure: A Voronoi diagram, with ten cells.

Section 2

Decentralized solution: dual problem

- The demand for fountain j is given by $P\left(\mathcal{X}_j\right) = \Pr\left(X \in \mathcal{X}_j\right)$ where $X \sim P$, which is the mass of consumers who prefer fountain j over the others.
- Note that in general $x^\intercal y_j - u\left(x\right) \leq v_j$; yet if consumer x chooses fountain j, then this inequality holds as an equality. Hence, the set of consumer who prefer fountain j is given by

$$\mathcal{X}_j = \arg\max_{x \in \mathcal{X}} \left\{x^\intercal y_j - u\left(x\right)\right\} \tag{7}$$

- By first order conditions $x \in \mathcal{X}_j$ if and only if $\nabla u\left(x\right) = y_j$ (assuming u is differentiable at x). Therefore

$$\mathcal{X}_j := \nabla u^{-1}\left(\left\{y_j\right\}\right). \tag{8}$$

- Introduce the social welfare of producers and consumers as

$$S(v) := \sum_j q_j v_j + \mathbb{E}_P \left[\max_{j \in \{1,\dots,M\}} \left\{ X^\intercal y_j - v_j \right\} \right]. \tag{9}$$

- We have

$$\frac{\partial S(v)}{\partial v_k} = q_k - \mathbb{E}_P \left[1 \left\{ \nabla u(X) = y_k \right\} \right] = q_k - P\left(\mathcal{X}_k^v \right).$$

  Thus, the excess supply for fountain j is given by

$$q_k - P\left(\mathcal{X}_k^v \right) = \frac{\partial S(v)}{\partial v_k} \tag{10}$$

  where S is defined by (9) above.

- Hence, market clearing prices, or equilibrium prices are prices v such that demand and supply clear, that is, such that $q_k = P\left(\mathcal{X}_k^v \right)$ for each k; in other words

$$\frac{\partial S(v)}{\partial v_k} = 0.$$

Section 3

Central planner's solution: primal problem

- The central planner may decide arbitrarily on assigning to each inhabitant x a fountain $T(x) \in \{y_1, ..., y_M\}$, in a such way that each fountain j is used to its full capacity, that is

$$P\left(T\left(X\right) = y_j\right) = q_j, \ \forall j \in \{1, ..., M\}. \tag{11}$$

- The planner seeks to maximize the total surplus subject to capacity constraints; hence

$$\begin{aligned} &\max \mathbb{E}_P\left[X^\intercal T\left(X\right)\right] \\ &s.t.\ (11) \end{aligned} \tag{12}$$

- This is a Monge problem, whose Kantorovich relaxation is

$$\max_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi\left[X^\intercal Y\right]. \tag{13}$$

- By the Monge-Kantorovich theorem, the dual problem is

$$\begin{aligned} &\min_{u,v} \mathbb{E}_P \left[u\left(X\right) \right] + \mathbb{E}_Q \left[v\left(Y\right) \right] \\ &s.t.\ u\left(x\right) + v\left(y\right) \geq x^{\intercal} y, \end{aligned} \tag{14}$$

  where the constraint should hold almost surely with respect to P and Q.

- The constraint should be verified for $y \in \left\{ y_1, ..., y_M \right\}$, and the constraint+optimality implies $u\left(x\right) = \max_{j \in \{1,...,M\}} \left\{ \Phi\left(x, y_j\right) - v_j \right\}$. Thus, the dual problem (14) rewrites as

$$\min_{v \in \mathbb{R}^M} \mathbb{E}_P \left[\max_{j \in \{1,...,M\}} \left\{ X^{\intercal} y_j - v_j \right\} \right] + \sum_{j=1}^{M} q_j v_j \tag{15}$$

  which is the minimum of S over $v \in \mathbb{R}^M$.

- As a result:
  1. There exist equilibrium prices, which are the minimizers of S.
  2. The total welfare at equilibrium coincides with the optimal welfare.

- Note that

$$\arg\max_{j\in\{1,\ldots,M\}}\left\{\Phi\left(x,y_j\right)-v_j\right\}$$

is a singleton for almost every x (it is not a singleton when x is at the boundary between two cells). The assumption that P is absolutely continuous is crucial here.

- Hence the map

$$T\left(x\right)=\nabla u\left(x\right)$$

is defined almost everywhere and coincides with arg max whenever it is defined. Thus the Monge and the Kantorovich problems have the same solutions.

- We turn to a discussion on the numerical determination of the prices (we discuss the determination of the v's, as the expression for the w's immediately follows). The function F to minimize being convex, we can use a standard gradient descent algorithm in which the increase in prices is given by

$$v_j^{t+1} - v_j^t = \varepsilon \left(P \left(\nabla u \left(X \right) = y_k \right) - q_k \right), \tag{16}$$

which has immediately an economic interpretation: the fountains that are over-demanded *raise* their prices, while the fountains that are under-demanded *lower* their prices. This a *tâtonnement process*.

ALGORITHM

Take an initial guess of v^0. At step t, define v^{t+1} by

$$v_j^{t+1} = v_j^t - \varepsilon_t \partial S / \partial v_j \left(v^t \right),$$

for ε_t small enough. Stop when $\partial S / \partial v_j \left(v^{t+1} \right)$ is sufficiently close to zero.

- **And now, let's code!**