Matching with NTU's

Octavia Ghelfi

NYU

Introduction

- "Large Matching Markets as Two-Sided Demand Systems"
  Konrad Menzel
- Differences with Choo-Siow
  - NTU instead of TU
  - Individual specific chemistry-shocks rather than type-specific chemistry shock

Results

1. Highly tractable asymptotic approximation to the distribution of matched characteristics
2. Type-specific matching frequencies depend on agents' preferences only through a joint surplus measure and inclusive values
3. Without further assumptions it is not possible to identify preferences on the male and female side of the market separately
4. Extension: individuals are aware of only a random subset of potential matching partners

Results

1. Many common RU models exhibit IIA as a limiting property when the set of choice alternatives is large:
   1.1 CCP converge to their Logit analog for a broad class of RU models
   1.2 $\implies$ matching probabilities can be summarized by the Inclusive Value
2. The main object of interest is the distribution of matched characterstics $\mu^\star$

Model

- Types
  - woman i has type $x_i \in \mathcal{X}$
  - men j has type $y_j \in \mathcal{Y}$
  - $\mathcal{X}$ and $\mathcal{Y}$ are bounded subsets of $\mathbb{R}^d$
- Distribution of types (pdf's)
  - $w(x)$
  - $m(y)$

Market Size

- Consider a sequence of markets of increasing size indexed by n
- Size of market n
  - $n_w = [n \exp\{\gamma_w\}]$ women
  - $n_m = [n \exp\{\gamma_m\}]$ men

Model

- Utility of a Match

  Systematic utility + individual-specific chemistry shock

$$U_{ij} = U(x_i, y_j) + \sigma\varepsilon_{ij}$$

$$V_{ji} = V(y_j, x_i) + \sigma\eta_{ji}$$

- Utility of a Single

$$U_{i0} = 0 + \sigma \max_{k=1,\ldots,[\sqrt{n}]} \{\varepsilon_{i0,k}\}$$

$$V_{j0} = 0 + \sigma \max_{k=1,\ldots,[\sqrt{n}]} \{\eta_{j0,k}\}$$

- ε and η are Gumbel (26)

Technical Difficulties

1. Ensure share of unmatched agents stable along the sequence
    1.1 Intuition: make the outside option more attractive at the right rate as the market grows large
    1.2 Economical justification: As the market grows agent can choose from an increasing number of potential spouses
    1.3 Reason: Shocks ε and η unbounded support implies any alternative with a fixed utility level will eventually be dominated by one of the largest draws for the increasing set of potential matching partners.
2. In the limit the distribution of matched characteristics of men and women does not degenerate to a matching rule that is deterministic or independent of observed characteristics
    2.1 Solution: technical assumption on σ (27)

Stable Matching

Definition

A matching $\mu^\star$ is stable given preferences U_{ij} and V_{ji} if every individual prefers his/her spouse under $\mu^\star$ to any other achievable partner. Formally $\mu^\star$ should satisfy
(i) if $U_{ij} > U_{i\mu^\star(i)}$ then $V_{j\mu^\star(j)} \geq V_{ji}$
(ii) if $V_{ji} > V_{j\mu^\star(j)}$ then $U_{i\mu^\star(i)} \geq U_{ij}$

Remark

The number of distinct stable matchings for a realization of preferences increases exponentially in the number of individuals on each side of the market

Stable Matching

- If preferences are strict, a stable matching always exists
- The set of stable matchings is a lattice
  - Maximal element μ^W
  - Minimal element μ^M

  wrt preferences on the female side

Stable Matching

- The $W-$preferred stable matching satisfies

$$U_{i\mu^W(i)} \geq U_{i\mu^\star(i)}$$

$$V_{j\mu^M(j)} \leq V_{j\mu^\star(j)}$$

For any other stable matching $\mu^\star$

- The number of distinct stable matchings increases exponentially in the number of individuals on each side of the market

Opportunity Sets

Definition

Given a matching μ let the *Opportunity Set* $M_i = M_i[\mu] \subset \{1, ..., n_M\}$ be the set of men j preferring woman i to their current match $\mu(j)$, that is

$$j \in M_i[\mu] \Longleftrightarrow V_{ji} \geq V_{j\mu(j)}$$

Similarly

$$i \in W_j[\mu] \Longleftrightarrow U_{ij} \geq U_{i\mu(i)}$$

Note: the size of a typical participant opportunity set grows at a rate proportional to $\sqrt{n}$

Stability

Define the indirect random utility

$$U_i^\star(M) := \max_{j \in M \cup \{0\}} U_{ij} \quad \textit{and} \quad V_j^\star(W) := \max_{i \in W \cup \{0\}} V_{ji}$$

Lemma

A matching μ is pairwise stable if and only if

$$U_{i\mu(i)} \geq U_i^\star\left(M_i\left[\mu\right]\right) \ \textit{and} \ V_{j\mu(j)} \geq V_j^\star(W_j\left[\mu\right])$$

Inclusive Value

The inclusive values for opportunity sets M_i and W_j are

$$I_w\left[M_i\right] := \frac{1}{\sqrt{n}} \sum_{j \in M_i} \exp\left\{U(x_i, y_j)\right\}$$

$$I_m\left[W_j\right] := \frac{1}{\sqrt{n}} \sum_{i \in W_j} \exp\left\{V(y_j, x_i)\right\}$$

Intuition: sample average over potential spouses
Two women of the same type will face very similar matching opportunities as the market grows large.

Inclusive Value: Relationship with Indirect Utility

$$\mathbb{E}\left[\max_{j\in M\cup\{0\}} U_{ij}\,|x_i\right] = \log\left(1 + I_w\left[M_i\right]\right) + \frac{1}{2}\log n + \kappa$$

κ is Euler's constant
Note: This relationship makes the inclusive value a good measure of welfare
Preview of results:

1. The inclusive value is a sufficient statistic with respect to the Conditional Choice Probabilities
2. Stability equivalent to a fixed point condition on the inclusive values

Feasibility Condition

$$\int_{\mathcal{Y}} f(x,y)dy + f(x,*) = w(x)\exp\{\gamma_w\}$$

$$\int_{\mathcal{X}} f(x,y)dx + f(*,y) = m(y)\exp\{\gamma_m\}$$

Law of Large Numbers for Inclusive Values

- Inclusive values associated with extremal matching

$$I_w[M_i] := \frac{1}{\sqrt{n}} \sum_{j \in M_i^M} \exp\{U(x_i, y_j)\}$$

$$I_m[W_j] := \frac{1}{\sqrt{n}} \sum_{i \in W_j^M} \exp\{V(y_j, x_i)\}$$

- The opportunity sets M_j^M and W_j^M are random and endogenous in the finite agent market

Law of Large Numbers for Inclusive Values

- Approximate the probability that a given man j is available to woman i under the $M-$optimal stable matching with

$$\sqrt{n} Pr\left\{j \in M_i^M\right\} = \frac{\exp\left\{V(y,x)\right\}}{1 + I_{mj}^M} + o_p(1)$$

- Define

$$\Gamma_w^m = \frac{1}{n}\sum_{j=1}^{n_m} \frac{\exp\left\{U(x,y) + V(y,x)\right\}}{1 + I_{mj}^M}$$

Law of Large Numbers for Inclusive Values

$$I_{w_i}^M - \Gamma_w^M(x_i) = \frac{1}{\sqrt{n}} \sum_{j \in M_i^M} \exp\{U(x,y)\} - \frac{1}{n} \sum_{j=1}^{n_m} \frac{\exp\{U(x,y) + V(y,x)\}}{1 + I_{mj}^M}$$

$$= \frac{1}{n} \sum_{j=1}^{n_m} \exp\{U(x,y)\} \left[n^{\frac{1}{2}} \mathbb{I} \left\{ V_{ji} \geq V_j^\star(W_j^M) - \frac{\exp\{V(y,x)\}}{1 + I_{m_j}^M} \right\} \right]$$

Lemma

$$I_{wi}^M \geq \Gamma_w^M(x_i) + o_p(1) \quad \textit{and} \quad I_{mj}^M \leq \Gamma_m^M(z_j) + o_p(1)$$

$$I_{wi}^W \leq \Gamma_w^W(x_i) + o_p(1) \quad \textit{and} \quad I_{mj}^W \geq \Gamma_m^W(z_j) + o_p(1)$$

This result allows us to approximate inclusive values as a function of observable characteristics alone.

Fixed Point Representation

- Convergence of I_{mj}^M to Γ_m^M and Continuous Mapping Theorem imply that we can write

$$\Gamma_w^M(x_i) \geq \frac{1}{n}\sum_{j=1}^{n_m} \frac{\exp\{U(x,y)+V(y,x)\}}{1+\Gamma_m^M(z_j)}$$

- Define the mapping

$$\Psi_w[\Gamma_m](x) = \frac{1}{n}\sum_{j=1}^{n_m} \frac{\exp\{U(x,y)+V(y,x)\}}{1+\Gamma_m^M(z_j)}$$

$$\Psi_m[\Gamma_w](z) = \frac{1}{n}\sum_{j=1}^{n_w} \frac{\exp\{U(x,y)+V(y,x)\}}{1+\Gamma_w^M(x_j)}$$

- Stability conditions imply

$$\Gamma_w^M \geq \Psi_w[\Gamma_m] + o_p(1) \quad \textit{and} \quad \Gamma_m^M \leq \Psi_m[\Gamma_w] + o_p(1)$$

Fixed Point Representation

Also

$$\Gamma_w^M \leq \Gamma_w^\star \leq \Gamma_w^W \quad \textit{and} \quad \Gamma_m^W \leq \Gamma_m^\star \leq \Gamma_m^M$$

where $\Gamma^\star$ is the inclusive value funtion associated with a random stable matching $\mu^\star$.

Fixed Point Representation

Theorem

The mapping $\log \Gamma \longmapsto \log \Psi[\Gamma]$ *is a contraction mapping. Specifically a solution to the problem*

$$\Gamma_w^\star = \Psi_w[\Gamma_m^\star] \quad \textit{and} \quad \Gamma_m^\star = \Psi_m[\Gamma_w^\star]$$

exists and it is unique.

Identification

The expected matching frequencies are

$$f(x,z) = \frac{\exp\left\{U(x,y)+V(y,x)\right\} w(x)m(y)}{\left(1+\Gamma_w^\star(x)\right)\left(1+\Gamma_m^\star(y)\right)}$$

Taking logs and rearranging

$$\log f(x,z) - [\log w(x) + \log m(y)] + [\log\left(1+\Gamma_w^\star(x)\right) + \log\left(1+\Gamma_m^\star(y)\right)] = W(x,z$$

Also since we normalize $U(x,0)=0$ and $V(y,0)=0$

$$f(x,*) = \frac{w(x)}{\left(1+\Gamma_w^\star(x)\right)} \quad \mathit{and} \quad f(*,y) = \frac{m(y)}{\left(1+\Gamma_m^\star(y)\right)}$$

Using a differencing argument to eliminate $\Gamma^\star$ we obtain

$$\log \frac{f(x,z)}{f(x,*)f(*,z)} = W(x,z)$$

Identification

We can identify the expected inclusive value from the conditional probabilties of remaining single given different values of x

$$\Gamma_w(x) = \frac{w(x) \exp\{\gamma_w\}}{f(x, *)} - 1$$

Results

1. Inclusive Value sufficient statistics of agent's choice set
2. The distribution of matches converges
3. Identification and Estimation of payoff parameters

Gumbel

The upper tail of the distribution $G(\varepsilon)$ is of type I if there exists an auxiliary function $a(s) \geq 0$ such that the cdf satisfies

$$\lim_{s \to \infty} \frac{1 - G(s + a(s)v)}{1 - G(s)} = e^{-v}$$

$\forall v \in \mathbb{R}$
(7)

Assumption on σ_n

$\sigma_n = \frac{1}{a(b_n)}$ where $b_n = G^{-1}(1 - \frac{1}{\sqrt{n}})$ and $a(s)$ is the auxiliary function specified in the Gumbel distribution assumption.