‘MATH+ECON+CODE’ MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

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Thursday: “Multinomial choice”
Block 10. Nonparametric multinomial choice

- Emax operator and generalized entropy of choice
- The Daly-Zachary-Williams theorem
- The GEV class

- [OTME], App. E
- McFadden (1981). "Econometric Models of Probabilistic Choice," in C.F. Manski and D. McFadden (eds.), *Structural analysis of discrete data with econometric applications*, MIT Press.
- McFadden (1989). "A Method of Simulated Moments for Estimation of Discrete Response Models WithoutNumerical Integration". *Econometrica*.
- Berry, Levinsohn, and Pakes (1995). "Automobile Prices in Market Equilibrium," *Econometrica*.
- Train. (2009). *Discrete Choice Methods with Simulation*. 2nd Edition. Cambridge University Press.
- G and Salanié (2017). "Cupid's invisible hands". Preprint.
- Chiong, G and Shum, "Duality in Discrete Choice Models". *Quantitative Economics*, 2016.
- Greene and Hensher (1997) Multinomial logit and discrete choice models.

Section 1

EMAX OPERATORS AND DEMAND MAPS

- Assume a consumer is facing a number of options $y \in \mathcal{Y}_0 = \mathcal{Y} \cup \{0\}$, where $y = 0$ is a default option. The consumer is drawing a utility shock which is a vector $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_{|\mathcal{Y}|}) \sim \mathbf{P}$ such that the utility of option y is $U_y + \varepsilon_y$, while the outside option yields utility ε_0.
- U is called vector of *systematic utilities*; ε is called vector of *utility shocks*.
- We assume thoughout that $\mathbf{P}$ has a density with respect to the Lebesgue measure, and has full support.
- The preferred option is the one which attains the maximum in

$$\max_{y \in \mathcal{Y}} \left\{ U_y + \varepsilon_y, \varepsilon_0 \right\}.$$

- Let $s_y = \sigma_y(U)$ be the probability of choosing option y, where σ is given by

$$\sigma_y(U) = \Pr(U_y + \varepsilon_y \geq U_z + \varepsilon_z \text{ for all } z \in \mathcal{Y}_0).$$

  The map σ is called *demand map*, and the vector s is called vector of market shares, or vector of choice probabilities.
- Note that if $s = \sigma(U)$, then $s_y > 0$ for all $y \in \mathcal{Y}_0$ and $\sum_{y \in \mathcal{Y}_0} s_y = 1$.
- Note that because the distribution $\mathbf{P}$ of ε is continuous, the probability of being indifferent between two options is zero, and hence we could have indifferently replaced weak preference $\geq$ by strict preference $>$. Without this, choice probabilities may not have been well defined.

- $\sigma_y(U)$ is increasing in U_y.
- $\sigma_y(U)$ is weakly decreasing in $U_{y'}$ for $y' \neq y$.
- If one replaces (U_y) by $(U_y + c)$, for a constant c, one has $\sigma(U + c) = \sigma(U)$.

- Because of the last property, we can normalize the utility of one of the alternatives. We will normalize the utility of the utility associated to $y = 0$, and hence take
$$U_0 = 0.$$
- Thus in the sequel, σ will be seen as a mapping from $\mathbb{R}^{\mathcal{Y}}$ to the set of $\left(s_y\right)_{y\in\mathcal{Y}}$ such that $s_y > 0$ and $\sum_{y\in\mathcal{Y}} s_y < 1$, and the choice probability of alternative $y = 0$ is recovered by
$$s_0 = 1 - \sum_{y\in\mathcal{Y}} s_y.$$

- Define the expected indirect utility of consumers by

$$G(U) = \mathbb{E}\left[\max_{y \in \mathcal{Y}} (U_y + \varepsilon_y, \varepsilon_0)\right]$$

This is called *Emax operator*, a.k.a. *McFadden's surplus function*.

- As the expectation of the maximum of terms which are linear in U, G is convex function in U (strictly convex in fact), and

$$\frac{\partial G}{\partial U_y}(U) = \Pr(U_y + \varepsilon_y \geq U_z + \varepsilon_z \text{ for all } z \in \mathcal{Y}_0).$$

But the right-hand side is simply the probability s_y of chosing option y; therefore, we get:

Theorem (Daly-Zachary-Williams). *The demand map σ is the gradient of the Emax operator G, that is*

$$\sigma(U) = \nabla G(U). \tag{1}$$

Section 2

EXAMPLES

- Assume that $\mathbf{P}$ is the distribution of i.i.d. *centered type I extreme value* a.k.a. *centered Gumbel* terms, which has c.d.f.

$$F(z) = \exp(-\exp(-x+\gamma))$$

  where $\gamma = 0.5772...$ (Euler's constant). The mean of this distribution is zero.
- Basic fact from extreme value theory: if ε_1,...,ε_n are i.i.d. Gumbel distributions, then $\max\{U_y + \varepsilon_y\}$ has the same distribution as $\log\left(\sum_{y=1}^n \exp U_y\right) + \epsilon$, where ϵ is also a Gumbel. (Proof of this fact later).
- Notes:
  - This distribution is sometimes called the "Gumbel max" distribution, to contrast it with the distribution of its opposite, which is then called "Gumbel min".
  - The literature usually calls "standard Gumbel" the distribution with c.d.f. $\exp(-\exp(-x))$; but that distribution has mean γ, which is why we slightly depart from the convention.

- The Emax operator associated with the logit model can be given in closed form as
$$G(U) = \log\left(1 + \sum_{y\in\mathcal{Y}} \exp(U_y)\right)$$
where $s_0 = 1 - \sum_{y\in\mathcal{Y}} s_y$. This is called a *log-partition function*.
- As a result, the choice probability of alternative y is proportional to the exponential of the systematic utility associated with U, that is
$$\sigma_y(U) = \frac{\exp U_y}{1 + \sum_{y'\in\mathcal{Y}} \exp(U_{y'})}$$
which is called a *Gibbs distribution*.
- Assume that the random utility shock is scaled by a factor T. Then
$$\sigma_y(U) = \frac{\exp(U_y/T)}{1 + \sum_{y'\in\mathcal{Y}} \exp(U_{y'}/T)}$$
which is sometimes called the *soft-max operator*, and converges as $T \to 0$ toward
$$\max_{y\in\mathcal{Y}} \{U_y, 0\}.$$

Let $\mathbf{F}$ be a cumulative distribution such that function g defined by

$$g\left(x_1, ..., x_n\right) = -\log \mathbf{F}\left(-\log x_1, ..., -\log x_n\right) \tag{2}$$

is positive homogeneous of degree 1. (This inverts into $\mathbf{F}\left(u_1, ..., u_n\right) = \exp\left(-g\left(e^{-u_1}, ..., e^{-u_n}\right)\right)$). We have by a theorem of McFadden (1978):

THEOREM

Let $\left(\varepsilon_y\right)_{1 \leq y \leq n}$ be a random vector with c.d.f. $\mathbf{F}$, and define

$$Z = \max_{y=1,\ldots,n} \left\{U_y + \varepsilon_y\right\}.$$

Then Z has the same distribution as $\log g\left(e^{U_1}, ..., e^{U_n}\right) + \gamma + \epsilon$, where ϵ is a standard Gumbel. In particular,

$$\mathbb{E}\left[\max_{y=1,\ldots,n} \left\{U_y + \varepsilon_y\right\}\right] = \log g\left(e^{U_1}, ..., e^{U_n}\right) + \gamma$$

where γ is the Euler constant $\gamma \simeq 0.5772$.

PROOF.

Let F_Z be the c.d.f. of $Z = \max_{y=1,...,n} \{U_y + \varepsilon_y\}$. One has

$$\begin{aligned} F_Z(z) &= \Pr\left(\max_{y=1,...,n} \{U_y + \varepsilon_y\} \leq z\right) = \Pr(\forall y: \ \varepsilon_y \leq z - U_y) \\ &= \mathbf{F}(z - U_1, ..., z - U_n) = \exp\left(-g\left(e^{U_1 - z}, ..., e^{U_n - z}\right)\right) \\ &= \exp\left(-e^{-z} g\left(e^{U_1}, ..., e^{U_n}\right)\right) = \varphi\left(z - \log g\left(e^{U_1}, ..., e^{U_n}\right) - \gamma\right) \end{aligned}$$

where $\varphi(z) := \exp\left(-e^{-(z-\gamma)}\right)$ is the cdf of the standard Gumbel distribution. Hence Z has the distribution of $\log g\left(e^{U_1}, ..., e^{U_n}\right) + \gamma + \epsilon$, where ϵ is a standard Gumbel. $\square$

- As a result, the choice probability of alternative y is

$$\sigma_y(U) = \frac{\frac{\partial g}{\partial x_y}\left(e^{U_1}, ..., e^{U_n}\right)}{g\left(e^{U_1}, ..., e^{U_n}\right)} e^{U_y}.$$

- The GEV framework has several commonly used examples: logit, nested logit, mixture of logit...
- We just saw the logit model, in which $g(x_1, ..., x_n) = e^{-\gamma} \sum_{y=1}^{n} x_y$. In this case, the distribution of

$$Z = \max_{y=1,...,n} \{U_y + \varepsilon_y\}$$

is $\log \sum_{y=1}^{n} e^{U_y} + \epsilon$, where ϵ is a standard Gumbel.

- The nested logit model is an instance of GEV model where alternatives can be grouped in nests. Eg, people choose their means of transportation (nest), and within this nest, a particular operator.
- Let $\mathcal{X}$ be the set of nests and assume that for each nest x, there is a set $\mathcal{Y}_x$ alternatives. Let U_{xy} be utility from alternative y in nest x, and $\lambda_x \in [0,1]$ and

$$g\left(U_{xy}\right) = e^{-\gamma} \sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}_x} U_{xy}^{1/\lambda_x} \right)^{\lambda_x}.$$

- In this case,

$$G\left(U\right) = \mathbb{E}\left[\max_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}_x} \left\{ U_{xy} + \varepsilon_{xy} \right\} \right] = \log \sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}_x} e^{U_{xy}/\lambda_x} \right)^{\lambda_x}$$

$$\sigma_{xy}\left(U\right) = \frac{\left(\sum_{y \in \mathcal{Y}_x} e^{U_{xy}/\lambda_x} \right)^{\lambda_x}}{\sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}_x} e^{U_{xy}/\lambda_x} \right)^{\lambda_x}} \frac{e^{U_{xy}/\lambda_x}}{\left(\sum_{y \in \mathcal{Y}_x} e^{U_{xy}/\lambda_x} \right)}$$

so the demand map has an interesting interpretation as "choice of nest then choice of alternative".

- Assume that $(\varepsilon_1, \varepsilon_2)$ have a nested logit distribution with two nests, that is, their cdf is given by

$$\mathbf{F}(u_1, u_2) = \exp\left(-e^{-\gamma}\left(e^{-u_1/\lambda} + e^{-u_2/\lambda}\right)^{\lambda}\right).$$

- Particular cases:
    - When $\lambda = 1$, ε_1 and ε_2 are independent and one recovers the logit model.
    - When $\lambda \to 0$, $\mathbf{F}(u_1, u_2) = \exp\left(-e^{-\gamma} e^{\max\{-u_1, -u_2\}}\right) = \min\{\mathbf{F}(u_1), \mathbf{F}(u_2)\}$ and therefore ε_1 and ε_2 are perfectly correlated.
- In general one can show that

$$\lambda = \sqrt{1 - cor(\varepsilon_1, \varepsilon_2)}$$

This formula, due to Tiago de Oliviera, is not straightforward to prove and is the object of an optional problemset.

OTHER POPULAR EXAMPLES

- Probit model (later)
- Berry-Pakes' pure characteristics model (later)
- Berry-Levinsohn-Pakes' mixed logit coefficient model (later)

Section 3

Demand inversion

- In many settings, the econometrician observes the market shares s_y and wants to deduce the corresponding vector of systematic utilities. That is, we would like to solve:
  Problem. *Given a vector s with positive entries satisfying $\sum_{y \in \mathcal{Y}} s_y < 1$, characterize and compute the set*

$$\sigma^{-1}(s) = \left\{ U \in \mathbb{R}^{\mathcal{Y}} : \sigma(U) = s \right\}.$$

- This problem is called "demand inversion," or "conditional choice probability inversion," or "identification problem." It is a central issue in econometrics/industrial organization and will be a key building block for matching models.

- We saw in Lecture 3 how to invert gradient of convex functions: if G is strictly convex and C^1, then

$$\sigma^{-1}(s) = \nabla G^{-1}(s) = \nabla G^*(s).$$

- G^* is the Legendre-Fenchel transform of G; we call it the *entropy of choice*, defined by

$$G^*(s) = \max_U \left\{ \sum_{y \in \mathcal{Y}} s_y U_y - G(U) \right\}. \tag{3}$$

- Hence, $\sigma^{-1}(s)$ is the vector U such that

$$U \in \arg\max_U \left\{ \sum_{y \in \mathcal{Y}} s_y U_y - G(U) \right\}.$$

- Convex duality implies that if s and U are related by $s \in \partial G(U)$, then

$$G(U) = \sum_{y \in \mathcal{Y}} s_y U_y - G_x^*(s). \tag{4}$$

- But letting $Y = \arg\max_y \{U_y + \varepsilon_y\}$, $G(U) = \mathbb{E}[U_Y + \varepsilon_Y]$ implies

$$G(U) = \sum_{y \in \mathcal{Y}} s_y U_y + \mathbb{E}[\varepsilon_Y],$$

thus one has

$$G^*(s) = -\mathbb{E}[\varepsilon_Y]. \tag{5}$$

Hence, the entropy of choice $G^*(s)$ is interpreted as minus the expected amount of heterogeneity needed to rationalize the choice probabilities s.

- Then
$$G^*(s) = s_0 \log(s_0) + \sum_{y \in \mathcal{Y}} s_y \log s_y$$
where $s_0 = 1 - \sum_{y \in \mathcal{Y}} s_y$. Hence, G^* is a bona fide entropy function when $\mathbf{P}$ is Gumbel–hence the name of *entropy of choice* in general.
- As a result,
$$\sigma_y^{-1}(s) = \log \frac{s_y}{s_0}$$
which is the celebrated *log-odds ratio formula*: the log of the odds of alternatives y and 0 identify the difference between the systematic utilities of these alternatives.

- The entropy of choice G^* in the nested logit model is given by

$$G^*(s) = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}_x} \lambda_x s_{xy} \ln s_{xy} + \sum_{x\in\mathcal{X}} (1-\lambda_x) \left(\sum_{z\in\mathcal{Y}_x} s_{xz}\right) \ln \left(\sum_{z\in\mathcal{Y}_x} s_{xz}\right) \tag{6}$$

  if $s_{xy} \geq 0$ and $\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}_x} s_{xy} = 1$, $G^*(s) = +\infty$ otherwise.
- Identification in the nested logit model: with normalization $\sum_{x\in\mathcal{X}} \left(\sum_{y\in\mathcal{Y}_x} e^{U_{xy}/\lambda_x}\right)^{\lambda_x} = 1$, one has $s_{xy} = \left(\sum_{y\in\mathcal{Y}_x} e^{U_{xy}/\lambda_x}\right)^{\lambda_x - 1} e^{U_{xy}/\lambda_x}$, thus $\sum_{y\in\mathcal{Y}_x} e^{U_{xy}/\lambda_x} = \left(\sum_{y\in\mathcal{Y}_x} s_{xy}\right)^{1/\lambda_x}$, therefore

$$U_{xy} = \lambda_x \log s_{xy} - (\lambda_x - 1) \log \sum_{y\in\mathcal{Y}_x} s_{xy}.$$

- **And now, let's code!**