Special, Continuous Parametric Distributions

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see, BH, Chapter 5.

Other useful, parametric distributions:

1. Normal Distribution
2. Chi-squared (χ^2) distribution (with **k** degrees of freedom)
3. Student-t distribution (with **k** degrees of freedom)
4. F-distribution (with $\mathbf{k_1}$ and $\mathbf{k_2}$ degrees of freedom)

Normal Distribution

For a normally distributed variable Y with mean μ and σ^2, the density function is given by

$$\phi(y) = f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right), \quad -\infty < y < \infty$$

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Standard Normal: $\mu = 0, \sigma = 1$

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Standard Normal: $\mu = 0, \sigma = 1$

$$\phi(y) = f(y) = \frac{1}{\sqrt{2\pi \cdot 1^2}} \exp\left(-\frac{(y-0)^2}{2 \cdot 1^2}\right), \quad -\infty < y < \infty$$

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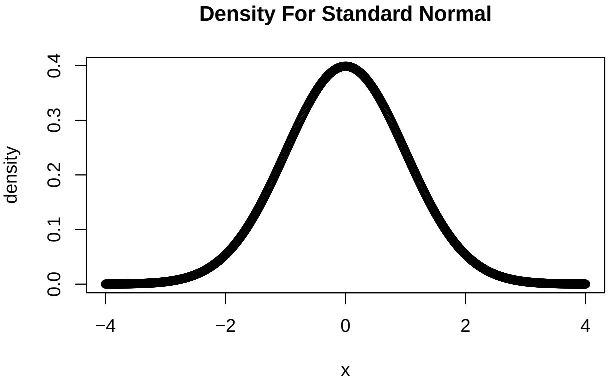
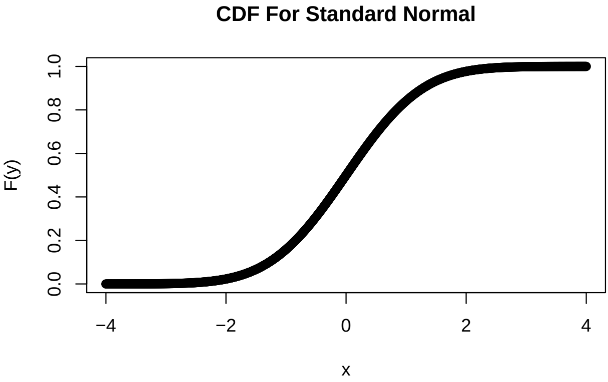
Standard Normal: $\mu = 0, \sigma = 1$

$$\phi(y) = f(y) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{1}{2}y^2}$$

CDF for the standard normal is given by (Note the Greek letters specificially for standard normals)

$$\Phi(y) = \int_{-\infty}^{y} \varphi(t)dt = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp^{-t^2/2} dt$$


Density For Standard Normal
density
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
x


CDF For Standard Normal
F(y)
0.0
0.2
0.4
0.6
0.8
1.0
−4
−2
0
2
4
x

Homework: **Properties of Standard Normal**

1. **Symmetry of PDF (around zero)**

$$\phi(z) = \phi(-z)$$

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2. **Symmetry of tail areas**

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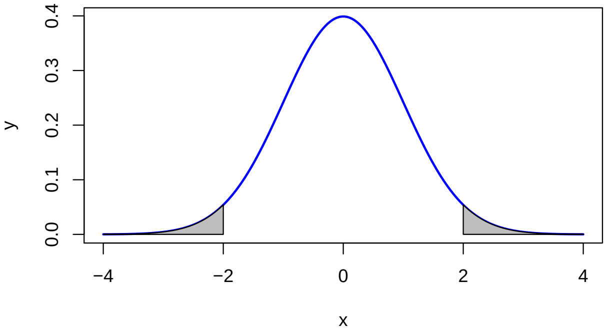
Example:

$$\Pr[Z \leq -2] = \Phi(-2) = 1 - \Phi(2) = \Pr[Z \geq 2]$$



y
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
x

Useful Facts:

1. Skewness $= 0$ [symmetric]
2. Kurtosis $= 3$ [normal tail]
3. A very useful fact regarding the standard normal distribution is that the probability that an observation x is within 2 standard deviations from the mean is 0.95. In other words, a value greater 2 or less than -2 is very unlikely!

The $68 - 95 - 99.7$ **Rule**: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\begin{aligned} P(|X - \mu| < \sigma) &\approx 0.68 \\ P(|X - \mu| < 2\sigma) &\approx 0.95 \\ P(|X - \mu| < 3\sigma) &\approx 0.997 \end{aligned}$$

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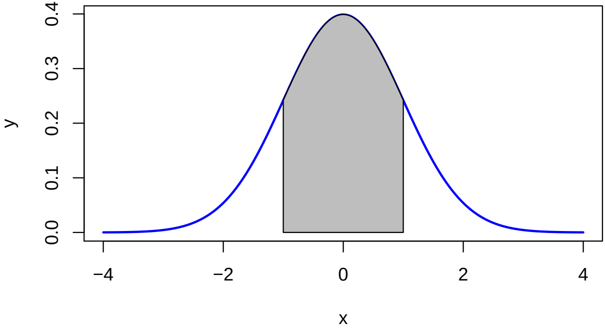
$$P(|Z| < 1) \approx 0.68$$
$$P(|Z| < 2) \approx 0.95$$
$$P(|Z| < 3) \approx 0.997$$

$P(|Z| < 1) \approx 0.68$



$P(|Z| < 2) \approx 0.95$

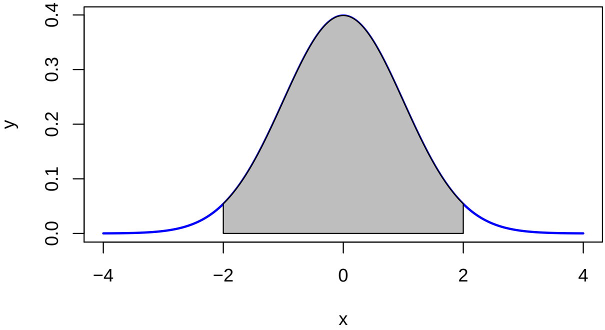


Table 5.1: Normal Probabilities and Quantiles

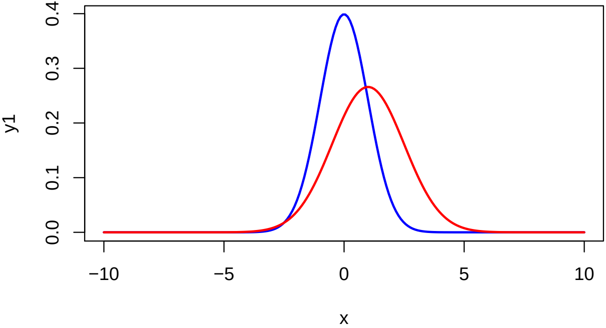
	$\mathbb{P}[Z \leq x]$	$\mathbb{P}[Z > x]$	$\mathbb{P}[\lvert Z \rvert > x]$
$x = 0.00$	0.50	0.50	1.00
$x = 1.00$	0.84	0.16	0.32
$x = 1.65$	0.95	0.005	0.10
$x = 1.96$	0.975	0.025	0.05
$x = 2.00$	0.977	0.023	0.046
$x = 2.33$	0.990	0.010	0.02
$x = 2.58$	0.995	0.005	0.01

1. From Normal to Standard Normal
2. From Standard Normal to Normal

Mulitiplication: You blow up or shrink every value! And even larger values now have positive "probabilities". Cover a larger (or smaller) region.

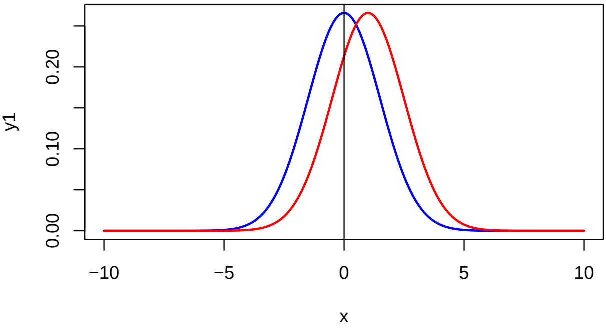
Addition: Your center in this case does not change when you blow it up. It is still zero. How can I make my mean equal to one?

Add one to every value, then
$\frac{(x_1+1)+(x_2+1)+\cdots+(x_N+1)}{N} = \overline{x} + 1 = 0 + 1 = 1$

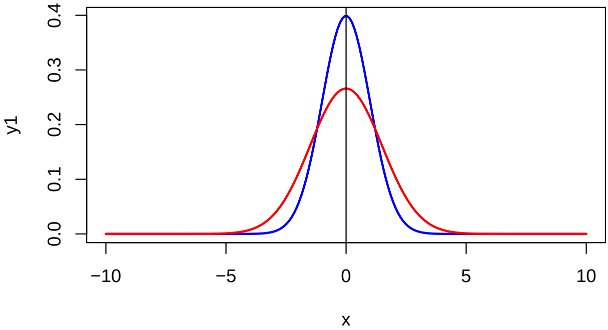


y1
0.0
0.1
0.2
0.3
0.4
−10
−5
0
5
10
x

From Normal to Standard Normal Addition First



From Normal to Standard Normal Mulitiplication Second



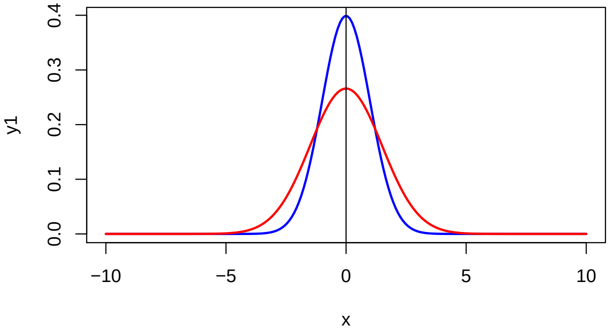
Any normal variable is standardized by

$$Z = \frac{Y - \mu}{\sigma} \sim N(0,1)$$

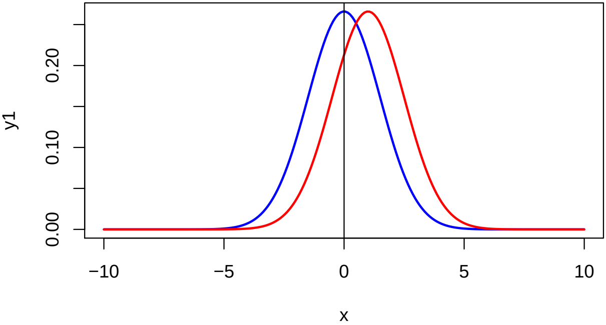
.

This variable has a mean of zero and a standard deviation of one.

From Standard Normal to Any Normal: Multiplication First



From Standard Normal to Any Normal: Addition Second



Also, if $Z \sim N(0, 1)$, we can also generate a random variable distributed from $N(\mu, \sigma^2)$ by doing stretching and shifting the standard normal variable,

$$Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$$

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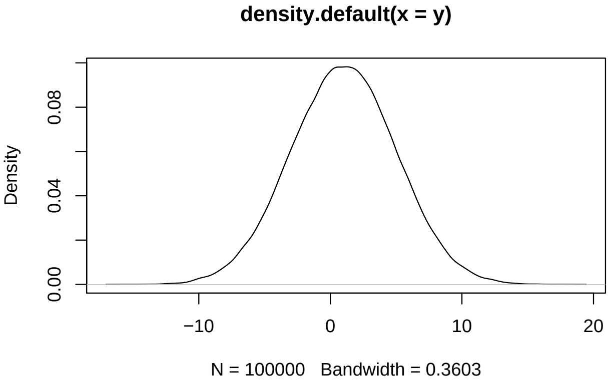
.



How we can mathematically show that $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$?

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$$\begin{aligned} P(Z \leq z) &= P\left(\frac{X-\mu}{\sigma} \leq z\right) \\ &= P(X \leq z\sigma + \mu) \end{aligned}$$

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\begin{aligned}
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&= P(X \leq z\sigma + \mu) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{z\sigma+\mu} e^{-(x-\mu)^2/\left(2\sigma^2\right)} dx
\end{aligned}
$$

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&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{z} e^{-t^2/2} \cdot \sigma dt, \quad \left(\text{substitute } t = \frac{x-\mu}{\sigma}\right)
\end{aligned}
$$

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\end{aligned}
$$

Note that the previous result is built on **Integration by substitution**:

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f(\varphi(t))\varphi'(t)dt$$

In Leibniz notation, the substitution $x = \varphi(t)$ yields

$$\frac{dx}{dt} = \varphi'(t)$$

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$$\frac{dx}{dt} = \varphi'(t)$$

In our case

$$\begin{aligned} x = \varphi(t) &= \sigma \cdot t + \mu \\ \varphi'(t) &= \sigma \\ b &= \varphi^{-1}(t) = \frac{x-\mu}{\sigma} = \frac{z\sigma + \mu - \mu}{\sigma} = z \\ a &= -\infty \end{aligned}$$

Homework:

Using the strategy above, show that $X = Z \cdot \sigma + \mu \sim N(\mu, \sigma^2)$, where Z is standard normal.

Additional Results We can show that the standard normal density integrates to 1 over the whole real line.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$$

The proof is based on the result

$$\int_{0}^{\infty} e^{-z^2/2} dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$$

See Casella and Berger, p.103 for a complete proof.

An important result: **Central Limit Theorem** Casella and Berger Theorem 5.5.14, page 236:

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{N}} \sim \mathcal{N}(0,1)$$

Other useful, parametric distributions:

1. Log Normal Distribution
2. Chi-squared (χ^2) distribution (with **k** degrees of freedom)
3. Student-t distribution (with **k** degrees of freedom)
4. F-distribution (with $\mathbf{k_1}$ and $\mathbf{k_2}$ degrees of freedom)

All these distributions can be constructed from a Normal Distribution!

Summary of Relationships among Four Distributions

1. $Z = N(0,1) \xrightarrow{\sum_{j=1}^{k} Z_j} \chi_k^2$
2. $t_k \xrightarrow{(t_k)^2} F_{1,k}$
3. $t_k \xrightarrow{k \to \infty} Z = N(0,1)$
4. $F_{k_1,k_2} \xrightarrow{k_2 \to \infty} k_1 \cdot \chi_{k_1}^2$

Things to pay attention when discussing a parametric distribution:

1. What is the probability mass/density function? The relationship between a potential value and the (relative) probability
2. How does this density function look like?
3. What are the features of this distribution? Moments (mean, variance, skewness and kurtosis)
4. What kind of things can be characterized by this distribution?

Note: In what follows, we will present the relationships among normal, Chi-square, t, and F distributions.

How we can be defined one distribution as a function of another. NOT the actual density or CDF functions, even though they are parametric.

Normal

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right), \quad -\infty < y < \infty$$

Any normal variable is standardized by

$Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$.

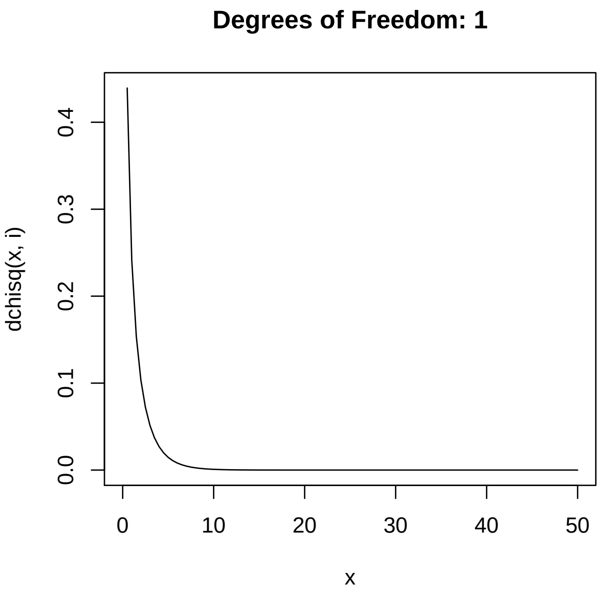
Chi-squared χ^2 with k degrees of freedom

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_k^2$$

$Z_j \sim N(0,1), j = 1, \ldots, k$ is a standard normal variable.

Examples:

1. $Y = Z_1^2$ is distributed as χ_1^2
2. $Y = Z_1^2 + Z_2^2$ is distributed as χ_2^2
3. $\ldots$


Degrees of Freedom: 1
dchisq(x, i)
0.0
0.1
0.2
0.3
0.4
0
10
20
30
40
50
x

1. It is non-negative [**Question:** Why?]. So, your forecast for an outcome following a χ^2 distribution should never be negative.
2. Asymmetrical and skewed to the right.
3. The Larger the degrees of freedom, the less skewed the density becomes. In particular, skewness $= \sqrt{\frac{8}{k}}$

Let's look at Stata example: `Stata_example_special_dist03.do`

The χ^2 distributions arises from the need in estimtaion of variance. It is associated with many test statistics, as most of them are about the variances under alternative specifications.

It also leads to some other distributions, e.g., those involving both mean and variance!

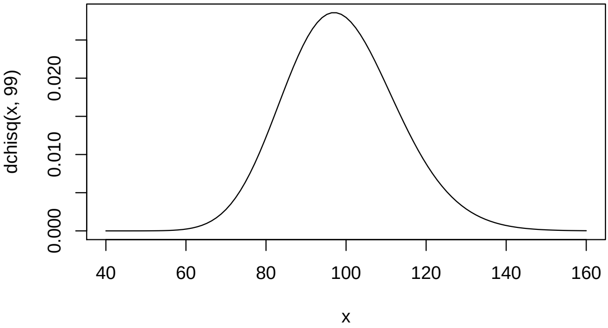
Example: (Under Certain Assumptions (e.g., X is normally distributed))

$$\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} = \frac{(N-1)\frac{\sum(X_i - \overline{X})^2}{N-1}}{\sigma^2} \sim \chi^2_{N-1}$$

See Casella and Berger, Theorem 5.3.1 (p. 218)

Again, how to visualize this?

Let's program it. Suppose $N = 100$. First, let's look at what $\chi^2_{100-1=99}$ will look like.



$$\frac{(N-1)\widehat{\sigma}^2}{\sigma^2}$$

Let's look at Stata example: `Stata_example_special_dist02.do`

Let's accumulate what we know!

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$
2. $\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1} \rightarrow \frac{\widehat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{N-1}}{(N-1)}$

Student t distribution with k degrees of freedom

$$t_k = \frac{Z}{\sqrt{\chi_k^2/k}}$$

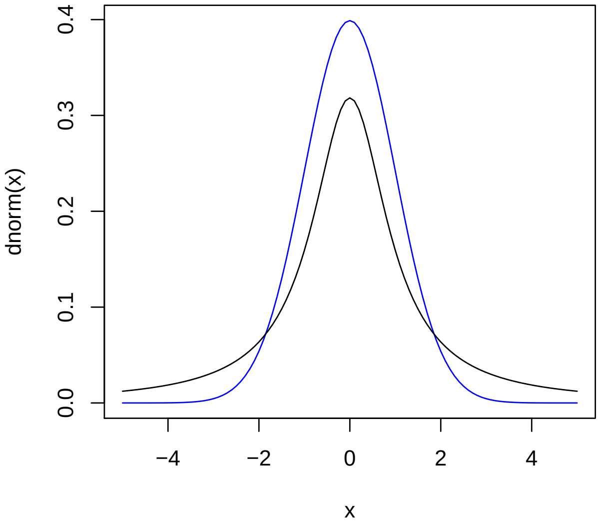
where Z is a standard normal, and χ_k^2 is a chi-squared variable with k degrees of freedom. It will be similarly defined below.

F distribution with k_1, k_2 degrees of freedom

$$F_{k_1,k_2} = \frac{\frac{\chi_{k_1}^2}{k_1}}{\frac{\chi_{k_2}^2}{k_2}}$$

$$(t_k)^2 = \left(\frac{Z}{\sqrt{\chi_k^2/k}}\right)^2 = \frac{Z^2}{\chi_k^2/k} = \frac{Z^2/1}{\chi_k^2/k} = \frac{\chi_1^2/1}{\chi_k^2/k} = F_{1,k}$$

t Distribution with Degrees of Freedom: 1

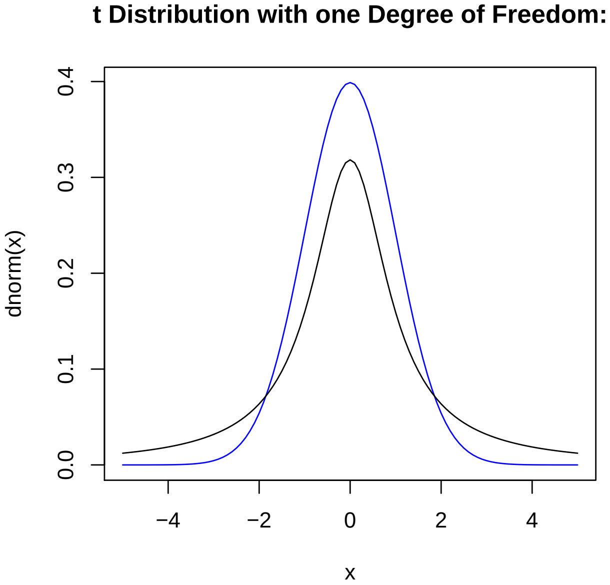


Let's look at Stata example: `Stata_example_special_dist03.do`

Some useful facts of t- distribution

1. Symmetric [skewness $= 0$]
2. Fat tail [kurtosis > 3] (very useful for modelling financial variables)
3. As $k \to \infty$, $t_k \to N(0, 1)$

Fat tail further examined



Fat tail further examined: Let's examine the actual "probability" of extreme values (those at both ends)

Let's look at Stata example: `Stata_example_special_dist03.do`

$$t_k = \frac{Z}{\sqrt{\frac{\chi_k^2}{k}}}$$

As k increases, $t_k \to Z$. Intuitively, you can think that the numeratore is multiplied by k. The standard normal variable prevails as k increases.

Let's look at Stata example: `Stata_example_special_dist03.do`

Example: What is this?

$$\frac{\overline{X} - \mu}{\widehat{\sigma}/\sqrt{N}}$$

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Example: What is this?

$$\frac{\overline{X} - \mu}{\widehat{\sigma}/\sqrt{N}} \quad = \quad \frac{(\overline{X} - \mu)}{(\sigma/\sqrt{N})} \bigg/ \frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})}$$

Example: What is this?

$$\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} = \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})}$$
$$= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{\widehat{\sigma}}{\sigma}$$

Let's accumulate what we know!

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$
2. $\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1} \rightarrow \frac{\widehat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{N-1}}{(N-1)}$

$$\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} = \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})}$$

$$
\begin{aligned}
\frac{\overline{X} - \mu}{\widehat{\sigma}/\sqrt{N}} &= \frac{(\overline{X} - \mu)}{(\sigma/\sqrt{N})} \Bigg/ \frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})} \\
&= \frac{(\overline{X} - \mu)}{(\sigma/\sqrt{N})} \Bigg/ \frac{\widehat{\sigma}}{\sigma}
\end{aligned}
$$

$$\begin{aligned}\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} &= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})} \\ &= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{\widehat{\sigma}}{\sigma}\end{aligned}$$

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$

$$
\begin{aligned}
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&= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Bigg/\frac{\widehat{\sigma}}{\sigma}
\end{aligned}
$$

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$$
\begin{aligned}
\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} &= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})} \bigg/ \frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})} \\
&= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})} \bigg/ \frac{\widehat{\sigma}}{\sigma} \\
&= \frac{N(0,1)}{\sqrt{\frac{\chi^2_{N-1}}{(N-1)}}}
\end{aligned}
$$

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$
2. $\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1} \to \frac{\widehat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{N-1}}{(N-1)}$

$$\begin{aligned}
\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} &= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Big/\frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})} \\
&= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\Big/\frac{\widehat{\sigma}}{\sigma} \\
&= \frac{N(0,1)}{\sqrt{\frac{\chi^2_{N-1}}{(N-1)}}}
\end{aligned}$$

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3. $t_k = \frac{Z}{\sqrt{\frac{\chi^2_k}{k}}}$

$$
\begin{aligned}
\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} &= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\bigg/\frac{(\widehat{\sigma}/\sqrt{N})}{(\sigma/\sqrt{N})} \\
&= \frac{(\overline{X}-\mu)}{(\sigma/\sqrt{N})}\bigg/\frac{\widehat{\sigma}}{\sigma} \\
&= \frac{N(0,1)}{\sqrt{\frac{\chi^2_{N-1}}{(N-1)}}} \\
&\sim t_{N-1}
\end{aligned}
$$

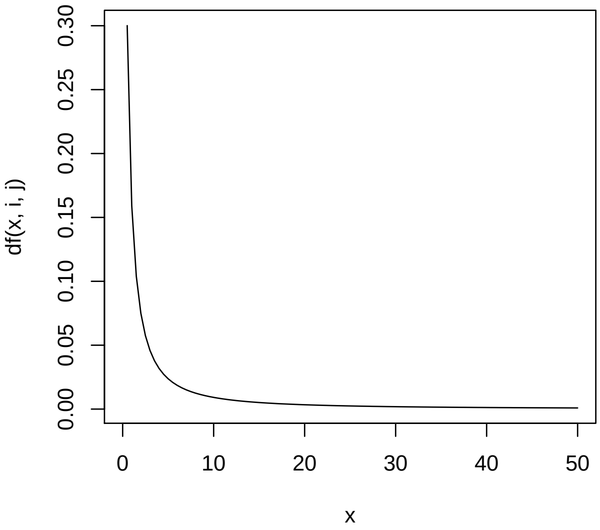
1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$
2. $\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1} \rightarrow \frac{\widehat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{N-1}}{(N-1)}$
3. $t_k = \frac{Z}{\sqrt{\frac{\chi^2_k}{k}}}$

What else do we know? $N \to \infty$, the degrees of freedom for the t_{N-1} distribution increase... it becomes the standard normal distribution!

Let's accumulate what we know!

1. $\frac{\overline{X}-\mu}{\sigma/\sqrt{N}} \sim N(0,1)$
2. $\frac{(N-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2_{N-1} \rightarrow \frac{\widehat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{N-1}}{(N-1)}$
3. $\frac{\overline{X}-\mu}{\widehat{\sigma}/\sqrt{N}} \sim t_{N-1}$

F Distribution with Degrees of Freedom: 1 and 1



1. It is non-negative [**Question:** Why?]. So, your forecast for an outcome following a F distribution should never be negative.
2. Asymmetrical and skewed to the right.

Sounds familiar? χ^2!!

F and χ^2 statistics are really the same thing in that, after a normalization, chi-squared is the limiting distribution of the F as the denominator degrees (k_2) of freedom goes to infinity. The normalization is

$$\chi^2_{k_1} = k_1 * F_{k_1,k_2}, \quad k_2 \to \infty$$

Let's look at Stata example: `Stata_example_special_dist03.do`

Summary of Relationships among Four Distributions

1. $Z = N(0,1) \xrightarrow{\sum_{j=1}^{k} Z_j} \chi_k^2$
2. $t_k \xrightarrow{(t_k)^2} F_{1,k}$
3. $t_k \xrightarrow{k \to \infty} Z = N(0,1)$
4. $F_{k_1,k_2} \xrightarrow{k_2 \to \infty} k_1 \cdot \chi_{k_1}^2$

Log Normal Distribution (Optional)

The distributions of many economic and financial variables are lognormal instead of normal in their original forms.

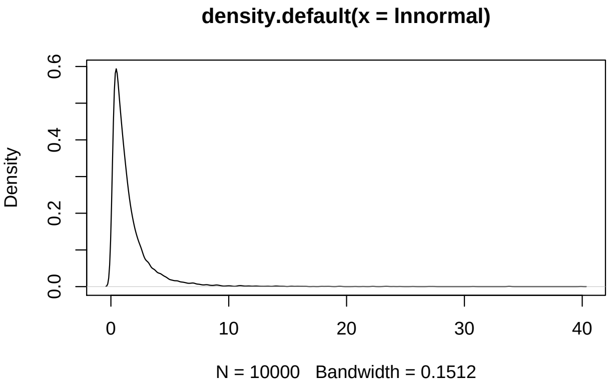
Normal

$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right), -\infty < y < \infty$

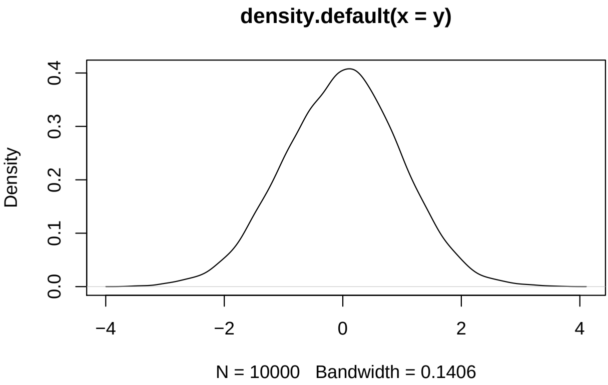
Log Normal Distribution

$f(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), 0 < y < \infty$

```
lnnormal<-rlnorm(10000)
plot(density(lnnormal))
```



```r
y<-log(lnnormal)
plot(density(y))
```



We often transform economic and financial variables by taking the log of the original variable of interest.

1. Many economic and financial variables grow exponentially, so their path is non-linear. Transforming these variables with a logarithm operation achieves linearity.
2. The transformation through logarithm operations changes the variables in concern from absolute terms to relative terms, so comparison can be made cross-sections and over time.
3. The logarithm transformation may help achieve stationarity in time series data, though this statement may be controversial.

Probably, one of the most convenient reasons for many economic and financial variables to follow lognormal distributions is the **non-negative constraint**.

That is, these variables can only take values that are greater than or equal to zero. Due partly to this, values closer to zero are compressed and those far away from zero are stretched out. Lognormal distributions possess these features.