Quantile Function and Its Properties: Part (I)

Le Wang

Alternative Approaches to Characterize a Distribution

We are now looking for the **inverse function** of the CDF. And obviously, that is an equivalent approach to characterize the distribution as well.

Alternative Approaches to Characterize a Distribution

Quantile (rank variable)

From your undergrad econometrics, τ^{th} quantile is often loosely defined as the value that is greater than τ percent of the population values.

For example, median (50^{th} percentile).

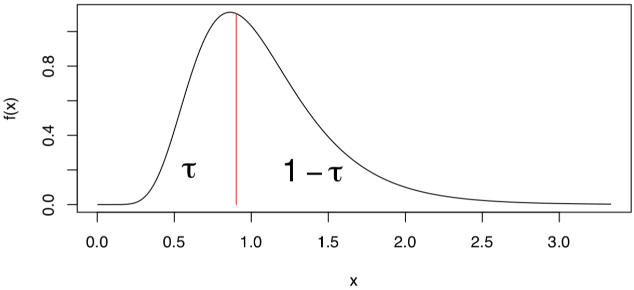
Quantile and CDF

Quantile is closely related to **CDF**

$$F[Q_X(.50)] = \Pr[X \leq Q_X(.5)] = .5$$

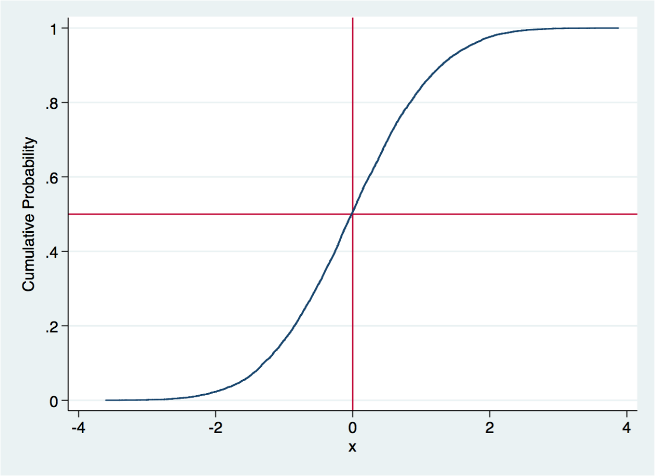
Quantile and PDF

From the perspective of PDF, the τ^{th} quantile splits the area under the density into two parts: one with area τ below the τ^{th} quantile and the other with area $1 - \tau$ above it.



$$\Pr[X \leq q_\tau] = \int_{-\infty}^{q_\tau} f(x)dx = \tau$$

Quantile (Graphial Example)



Quantile and Mean

Homework Question

Note that the following result holds if X is strictly continuously distributed

$$\mathbb{E}[X] = \int_0^1 Q_X(\tau) d\tau$$

I put it here, but the property (Skorohod representation) needed for proving this result will be shown later.

Quantile and CDF

Case 1: $F(\cdot)$ is continuous and strictly monotonic (X is continuously distributed).

We again know that there exists an inverse function, $Q_X(\tau) \equiv F^{-1}(\tau)$,such that

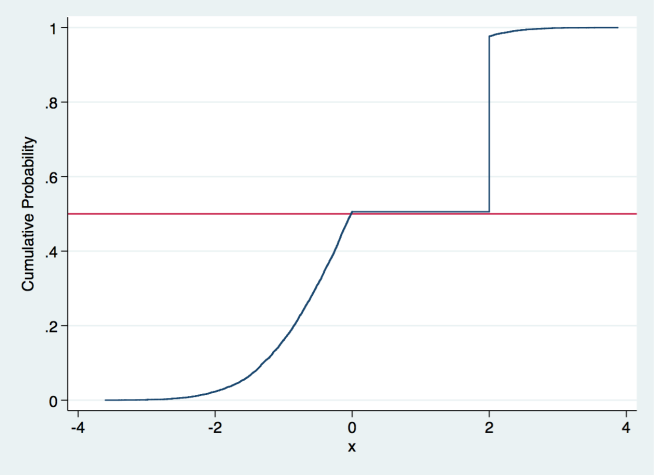
$$F^{-1}(\tau) = F^{-1}(F(x)) = x$$

Quantile

Case 2: $F(\cdot)$ is neither continuous nor strictly monotonic (i.e., X is NOT continuously distributed)

Quantile (Graphial Example)

Any values between 0 and 2 can be 50^{th} percentile



Quantile Function

Since not all distribution functions are strictly increasing, not all distribution functions have proper inverses. The following **generalized inverse function** allows for retaining many of the proerties that are useful for various probabilistic arguments

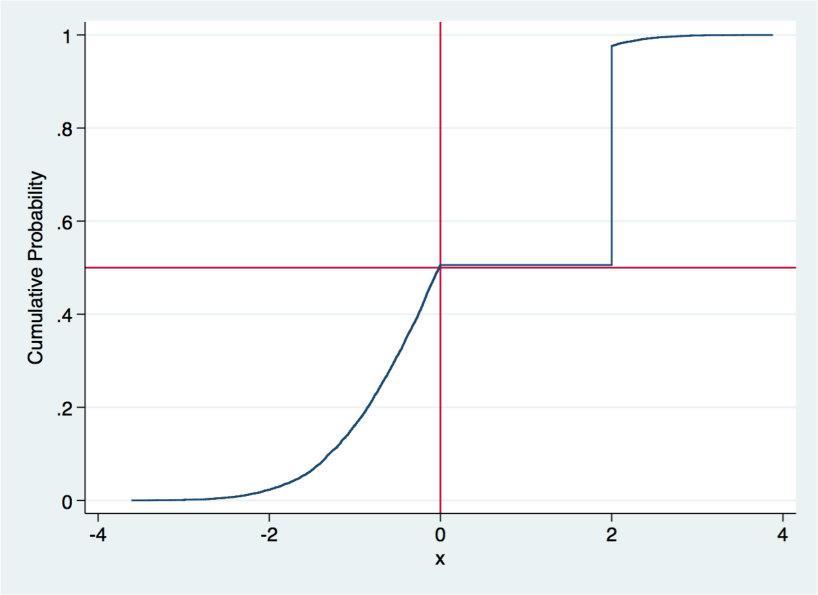
Definition: Let $F(x)$ be a distribution function (a nondecreasing and right continuous function). The quantile function or the generalized inverse function of F is defined as

$$F^{-1}(\tau) \equiv \inf\{x : F(x) \geq \tau, 0 < \tau < 1\}$$

Quantile Function

This definition also allows us to study the properties of quantile function, and modern estimation of quantile function that can be readily extended to quantile regression.

Quantile and CDF



Numerical Example of Quantile Function

Example:

$\{0, 1, 2\}$

$$\Pr[X = 0] = \frac{1}{3}, \Pr[X = 1] = \frac{1}{2}, \Pr[X = 2] = \frac{1}{6}$$

Numerical Example of Quantile Function

Consider an example of discrete random variable with

$$\Pr[X = 0] = \frac{1}{3}, \Pr[X = 1] = \frac{1}{2}, \Pr[X = 2] = \frac{1}{6}$$

median, i.e., $\tau = \frac{1}{2}$, then

$$F^{-1}(.5) = \inf\{x : F(x) \geq .5\}$$

Numerical Example of Quantile Function

$$F^{-1}(.5) = \inf\{x : F(x) \geq .5\}$$

For all the x,

$$F[0] = \Pr[X \leq 0] = \frac{1}{3}$$
$$F[1] = \Pr[X \leq 1] = \Pr[X = 0] + \Pr[X = 1] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$
$$F[2] = \Pr[X \leq 2] = 1$$

Numerical Example of Quantile Function

$$F^{-1}(.5) = \inf\{x : F(x) \geq .5\}$$

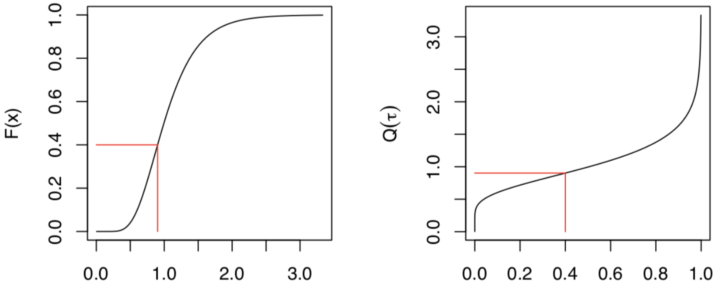
For all the x,

$$F[0] = \Pr[X \leq 0] = \frac{1}{3}$$
$$F[1] = \Pr[X \leq 1] = \Pr[X = 0] + \Pr[X = 1] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$
$$F[2] = \Pr[X \leq 2] = 1$$

$$\begin{aligned} F^{-1}(.5) &= \inf\{x : F(x) \geq .5\} \\ &= \inf\{1, 2\} \\ &= 1 \end{aligned}$$

Properties of Quantile Function

Further Properties of Quantile Functions regarding Continuity



Properties of Quantile Function

Continuity at a point and Properties of Quantile Function

1. If F is continuous at $x = F^{-1}(\tau)$, then $F(F^{-1}(\tau)) = \tau$.
2. If $F^{-1}(\tau)$ is continuous at $\tau = F(x)$, then $F^{-1}(F(x)) = x$.

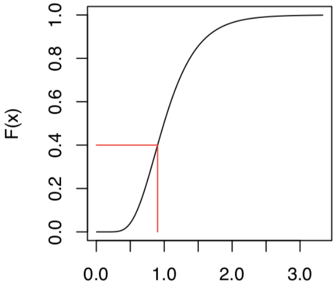
Continuity and Properties of Quantile Function

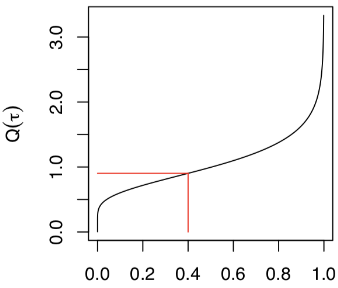
3. F is continuous and strictly increasing $\iff$ F^{-1} is continuous and strictly increasing.

Properties of Quantile Function

Continuity everywhere and Properties of Quantile Function

4. The following statements are equivalent:
   a. $F(F^{-1}(\tau)) = \tau, \forall \tau \in (0,1)$
   b. F is continuous
   c. $F^{-1}(\tau)$ is strictly increasing.





Quantile Functions and Uniform Distribution

Quantile Functions

Quantile Function and Uniform Distribution

1. **Inverse Transformation Method** (for Simulation or Reproduction Property)
2. **Probability Integral Transform:** $Y = F(X)$
3. **Skorohod Representation**

Quantile and Uniform Distribution

There is an interesting connection between quantiles and the uniform $(0, 1)$ distribution, $\mathcal{U}(0, 1)$.

Properties of Quantile Function

Reproduction Property Let X be a real-valued random variable with distribution function F, and let U be uniformly distributed between zero and one. Then,

$$F^{-1}(U) \sim F_X$$

Properties of Quantile Function

$$F^{-1}(U) \sim F_X$$

Proof We can show that

$$\begin{aligned}
\Pr[F^{-1}(U) \leq x] &= \Pr[U \leq F(x)] \\
&= F_U(F(x)) \\
&= F(x)
\end{aligned}$$

Recall that Property 4.

$$F^{-1}(\tau) \leq x \iff \tau \leq F(x)$$

Quantiles and Uniform Distribution

$$F^{-1}(U) \sim F_X$$

The reproduction property enables the **inversion method** in Monte-Carlo simulation to generate random numbers of an arbitrary distribution from uniformly distributed random numbers. The method will work whenever the generalized inverse can be computed explicitly.

Numerical Example: Exponential Variable, $X \sim Expo(\frac{1}{2})$ for $X > 0$

$$\text{PDF:} \quad f(x) = \frac{1}{2}\exp\{-\frac{1}{2}x\}$$

$$\text{CDF:} \quad F(x) = 1 - \exp\{-\frac{1}{2}x\} = u$$

Numerical Example: Exponential Variable, $X \sim Expo(\frac{1}{2})$ for $X > 0$

$$\text{PDF:} \quad f(x) = \frac{1}{2}\exp\{-\frac{1}{2}x\}$$
$$\text{CDF:} \quad F(x) = 1 - \exp\{-\frac{1}{2}x\} = u$$

This implies that

$$F^{-1}(u) = -2 \cdot \log(1 - u)$$

Example: $u = 0.843$, then $Expo(2) = -2 \cdot \log(1 - .843) = 3.703$

Quantiles and Uniform Distribution

The reverse is true as well.

Theorem Let X be a continuously distributed with a **strictly increasing** CDF, $F_X(x)$. Suppose that $Y = F_X(X)$. Then,

$$Y \sim \mathcal{U}(0,1)$$

Quantiles and Uniform Distribution

The reverse is true as well.

Theorem Let X be a continuously distributed with a **strictly increasing** CDF, $F_X(x)$. Suppose that $Y = F_X(X)$. Then,

$$Y \sim \mathcal{U}(0, 1)$$

Note: This is not necessarily true if X has mass point (no jump in the CDF). For example, X is a Bernoulli distribution with p, then

$$\begin{aligned} F(0) &= 1 - p \quad \text{with} \quad p^* = 1 - p \\ F(1) &= 1 \qquad\quad\;\; \text{with} \quad p^* = p \end{aligned}$$

Note that $F(\cdot)$ is known as the *probability integral transform* (PIT). The result presented here is itself a direct consequence of the transformation theorem that we have shown earlier. Remember that $F(\cdot)$ in this case is a monotonically increasing function, and we obtain its density by using the following formula

$$f_Y(y) = f_X(x)\frac{1}{\mid dF_X(x)/dx \mid} = 1$$

Quantiles and Uniform Distribution

$$Y = F_X(x) \sim \mathcal{U}(0, 1)$$

Sketch Proof

$\Pr[Y \leq y] = \Pr[F_X(X) \leq y]$ (by definition)

Quantiles and Uniform Distribution

$$Y = F_X(x) \sim \mathcal{U}(0, 1)$$

Sketch Proof

$$\begin{aligned} \Pr[Y \leq y] &= \Pr[F_X(X) \leq y] \text{ (by definition)} \\ &= \Pr[F^{-1}(F(X)) \leq F^{-1}(y)] \text{ (non-decreasing)} \end{aligned}$$

Quantiles and Uniform Distribution

$$Y = F_X(x) \sim \mathcal{U}(0,1)$$

Sketch Proof

$$\begin{aligned}
\Pr[Y \leq y] &= \Pr[F_X(X) \leq y] \text{ (by definition)} \\
&= \Pr[F^{-1}(F(X)) \leq F^{-1}(y)] \text{ (non-decreasing)} \\
&= \Pr[X \leq F^{-1}(y)] \text{ (property of the inverse function)}
\end{aligned}$$

Quantiles and Uniform Distribution

$$Y = F_X(x) \sim \mathcal{U}(0,1)$$

Sketch Proof

$$\begin{aligned}
\Pr[Y \leq y] &= \Pr[F_X(X) \leq y] \text{ (by definition)} \\
&= \Pr[F^{-1}(F(X)) \leq F^{-1}(y)] \text{ (non-decreasing)} \\
&= \Pr[X \leq F^{-1}(y)] \text{ (property of the inverse function)} \\
&= F(F^{-1}(y)) \text{ (definition)}
\end{aligned}$$

Quantiles and Uniform Distribution

$$Y = F_X(x) \sim \mathcal{U}(0,1)$$

Sketch Proof

$$\begin{aligned}
\Pr[Y \leq y] &= \Pr[F_X(X) \leq y] \text{ (by definition)} \\
&= \Pr[F^{-1}(F(X)) \leq F^{-1}(y)] \text{ (non-decreasing)} \\
&= \Pr[X \leq F^{-1}(y)] \text{ (property of the inverse function)} \\
&= F(F^{-1}(y)) \text{ (definition)} \\
&= y \text{ (property of the inverse function)}
\end{aligned}$$

Note: Many of the proofs in this course are NOT complete because the assumptions are not necessarily stated and the logic clearly explained. You need to complete these proofs on your own by filling in these details.

Quantile and Uniform Distribution

Skorokhod Representation

If X is distributed with CDF $F_X(x)$, then there is $U \sim \mathcal{U}(0,1)$ such that $X = F_X^{-1}(U) = m(U)$ holds almost surely.

Quantile and Uniform Distribution

Skorokhod Representation

If X is distributed with CDF $F_X(x)$, then there is $U \sim \mathcal{U}(0,1)$ such that $X = F_X^{-1}(U) = m(U)$ holds almost surely.

Note: The proof is rather trivial for the continuous variable where you can just define $U = F(X)$, which exists.

Note: You will use this in your homework.

Differentiation of Quantile Function

Theorem Let $F(x)$ be a distribution function. If F has a positive continuous $f(x)$ density f in a neighborhood of $F^{-1}(\tau_0)$ where $0 < \tau_0 < 1$, then the derivative $\frac{d}{d\tau}F^{-1}(\tau)$ exists at $\tau = \tau_0$ and

$$\frac{d}{d\tau}F^{-1}(\tau)|_{\tau_0} = \frac{1}{f(F^{-1}(\tau_0))}$$

Quantiles of A Function of Random Variable

Theorem Let X be a real-valued random variable with distribution $F_X(x) = \Pr[X \leq x]$. If $g(X)$ is a **nondecreasing, left continuous** function, then

$$F^{-1}_{g(X)}(\tau) = g(F_X^{-1}(\tau))$$

Proof: See, e.g., Reza Hosseini **Quantiles Equivariance** for an example of the proof and a counterexample that the quantile function is equivariant under increasing transformations (claimed in Koenker's classic book on Quantile Regression).

This is particularly useful and carries to the quantile regression framework. Suppose that $Y = \ln W$ (log wages) and $y = F_Y^{-1}(\tau)$, then

$$F_W^{-1}(\tau) = \exp(y)$$

Note: You cannot use this for expectation.

Future Applications

A variant of the result can be generalized and applied to many situations (e.g., censored quantile regression, binary quantile regression, maximum score model).

For example, censored outcomes such as consumption

$$y = \max\{x'\beta + \epsilon, 0\}$$

$$\begin{aligned} & median(\epsilon \mid x) = 0 \\ \Longrightarrow\ & median(y \mid x) = \max\{x'\beta, 0\} \end{aligned}$$

Quantile and Symmetric Distribution

Theorem Let X be distributed with a PDF f_X, which is symmetric around μ:

$$f_X(\mu + u) = f_X(\mu - u)$$

for any $u \in \mathbb{R}$. Then the following properties hold

1. $\mathbb{E}[X] = \mu$
2. $Q_X(0.5) = \mu$
3. $Q_X(1 - \tau) - \mu = \mu - Q_X(\tau)$

Optional Reading: Properties for General Quantile Function

Math Preliminaries

1. Infimum and its properties regarding subsets
2. Limit point and infimum
3. Order Properties of a function
4. Right continuity of a function

mathematical preliminaries:

1. **infimum (**inf**):** Greatest lower bound

Example

1. $\inf\{1, 2, 3, \dots\} = 1$ (in the set)
2. $\inf\{x \in \mathbb{R} \mid 0 < x < 1\} = 0$ (not in the set)

Proposition: Suppose that A, B are subsets of $\mathbb{R}$ such that $A \subset B$. If $\inf A, \inf B$ both exist, then

$$\inf A \geq \inf B$$

mathematical preliminaries:

2. A point a is a *limit* (or accumulation or cluster) point of a set A if every $\epsilon-$ neighborhood $V_\epsilon(a)$ of a intersects the set A in some point other than a.

mathematical preliminaries:

2. A point a is a *limit* (or accumulation or cluster) point of a set A if every $\epsilon-$ neighborhood $V_\epsilon(a)$ of a intersects the set A in some point other than a.

3. If $A \subset \mathbb{R}$, $a = \inf\{A\}, a \notin A$, then the infimum a is a limit point of A.

mathematical preliminaries:

2. A point a is a *limit* (or accumulation or cluster) point of a set A if every $\epsilon-$ neighborhood $V_\epsilon(a)$ of a intersects the set A in some point other than a.

3. If $A \subset \mathbb{R}$, $a = \inf\{A\}, a \notin A$, then the infimum a is a limit point of A.

4. **Theorem** A point a is a limit point of a set A if and only if $a = \lim x_i$ for some sequence x_i contained in A such that $x_i \neq a$ for all $i \in \mathbb{N}$.

mathematical preliminaries:

2. A point a is a *limit* (or accumulation or cluster) point of a set A if every $\epsilon-$ neighborhood $V_\epsilon(a)$ of a intersects the set A in some point other than a.

3. If $A \subset \mathbb{R}$, $a = \inf\{A\}, a \notin A$, then the infimum a is a limit point of A.

4. **Theorem** A point a is a limit point of a set A if and only if $a = \lim x_i$ for some sequence x_i contained in A such that $x_i \neq a$ for all $i \in \mathbb{N}$.

mathematical preliminaries: The above theorems and definitions together imply that for an **infimum**

There exists a sequence $\{x_i\}_{i=1}^{\infty} \in A$ with $\lim_{i \to \infty} x_i = x^0$ and $x_i > x_0$ for all i.

mathematical preliminaries:

1. **Order Properties of Functions:** Suppose that $f, g : A \to \mathbb{R}$ and x^0 is a limit point of A. If

$$f(x) \geq g(x) \quad \forall x \in A$$

and $\lim_{x \to x^0} f(x)$ and $\lim_{x \to x^0} g(x)$ exist, then

$$\lim_{x \to x^0} f(x) \geq \lim_{x \to x^0} g(x)$$

2. **Right Continuity** If $f : A \to \mathbb{R}$ is right continuous at a point x^0 if

$$\lim_{x \to (x^0)^+} f(x) = f(x^0)$$

mathematical preliminaries:

Order properties and **Right continuity** imply that if $f : A \to \mathbb{R}$ is right continuous at a point x^0 that is an limit point of A, and $f(x) \geq \tau = g(x) \quad \forall x \in A$, then

$$f(x^0) = \lim_{x \to x^0} f(x) \geq \tau \quad (= \lim_{x \to x^0} \tau)$$
$$f(x^0) \geq \tau$$

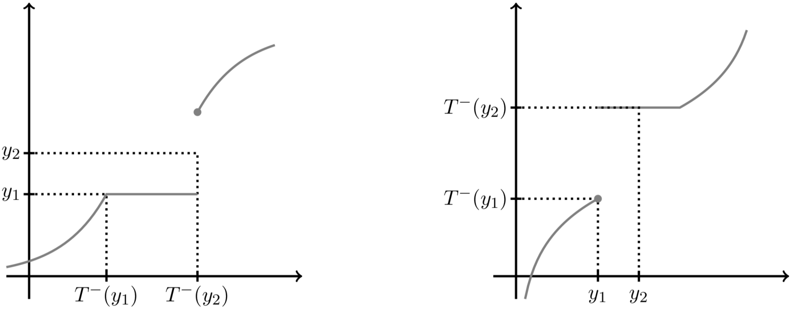


Figure 1 An increasing function (left) and its corresponding generalized inverse (right).

This graph highlights two difficulties when thinking about the generalized inverse function in general

1. When there is a **flat** region in the CDF

2. When there is a **jump** (flat region in the quantile function)

Properties of Quantile Function

1. $F^{-1}(\tau)$ is nondecreasing and left continuous. (the **opposite** of the CDF)
2. $F^{-1}(F(x)) \leq x$
3. $F(F^{-1}(\tau)) \geq \tau$ (for $F^{-1}(\tau) < \infty$ implicitly assumed below)
4. $F^{-1}(\tau) \leq x \iff \tau \leq F(x)$ (a result of 2. and 3.)

Properties of Quantile Function

1. $F^{-1}(\tau)$ is nondecreasing and left continuous. (the **opposite** of the CDF)

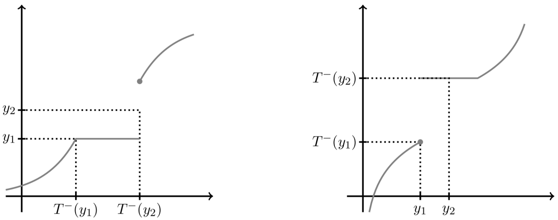


Figure 1 An increasing function (left) and its corresponding generalized inverse (right).

Properties of Quantile Function

Nondecreasing means that

$$\tau < F(x) \text{ only implies } F^{-1}(\tau) \leq x$$

Properties of Quantile Function

Property: $F^{-1}(\tau)$ is nondecreasing

Proof: Suppose that $\tau_1 < \tau_2$. Then we would like to show that $x_1 = F^{-1}(\tau_1) \leq x_2 = F^{-1}(\tau_2)$.

Properties of Quantile Function

Property: $F^{-1}(\tau)$ is nondecreasing

Proof: Suppose that $\tau_1 < \tau_2$. Then we would like to show that $x_1 = F^{-1}(\tau_1) \leq x_2 = F^{-1}(\tau_2)$.

Properties of Quantile Function

Property: $F^{-1}(\tau)$ is nondecreasing

Proof: Suppose that $\tau_1 < \tau_2$. Then we would like to show that $x_1 = F^{-1}(\tau_1) \leq x_2 = F^{-1}(\tau_2)$.

By definition, $x_1 = \inf\{x : F(x) \geq \tau_1\}$ and $x_2 = \inf\{x : F(x) \geq \tau_2\}$.

Properties of Quantile Function

Property: $F^{-1}(\tau)$ is nondecreasing

Proof: Suppose that $\tau_1 < \tau_2$. Then we would like to show that $x_1 = F^{-1}(\tau_1) \leq x_2 = F^{-1}(\tau_2)$.

By definition, $x_1 = \inf\{x : F(x) \geq \tau_1\}$ and $x_2 = \inf\{x : F(x) \geq \tau_2\}$.

We know that $F(x) \geq \tau_2 \implies F(x) \geq \tau_1$. In other words, $\{x : F(x) \geq \tau_2\} \subset \{x : F(x) \geq \tau_1\}$.

Properties of Quantile Function

Property: $F^{-1}(\tau)$ is nondecreasing

Proof: Suppose that $\tau_1 < \tau_2$. Then we would like to show that $x_1 = F^{-1}(\tau_1) \leq x_2 = F^{-1}(\tau_2)$.

By definition, $x_1 = \inf\{x : F(x) \geq \tau_1\}$ and $x_2 = \inf\{x : F(x) \geq \tau_2\}$.

We know that $F(x) \geq \tau_2 \implies F(x) \geq \tau_1$. In other words, $\{x : F(x) \geq \tau_2\} \subset \{x : F(x) \geq \tau_1\}$.

Thus, it follows that
$x_1 = \inf\{x : F(x) \geq \tau_1\} \leq x_2 = \inf\{x : F(x) \geq \tau_2\}$.

Numerical Example of Quantile Function

Consider our previous example,

$$F[0] = \frac{1}{3}$$
$$F[1] = \frac{5}{6}$$
$$F[2] = 1$$

$\tau = \frac{1}{2} < F(1) = \frac{5}{6}$, but $F^{-1}(.5) = F^{-1}(F(x)) = 1$.

Properties of Quantile Function

More Difficult Properties of Quantile Functions

These properties are about the **inverse** of the **inverse**. Intuition does not work quite straightforward for the **discrete** or **mixed** distribution for which you have mass points.

1. $F^{-1}(F(x)) >=< x$?
2. $F(F^{-1}(\tau)) >=< \tau$?

Properties of Quantile Function

Let's use a graph to illustrae our intuition and guide our proof.

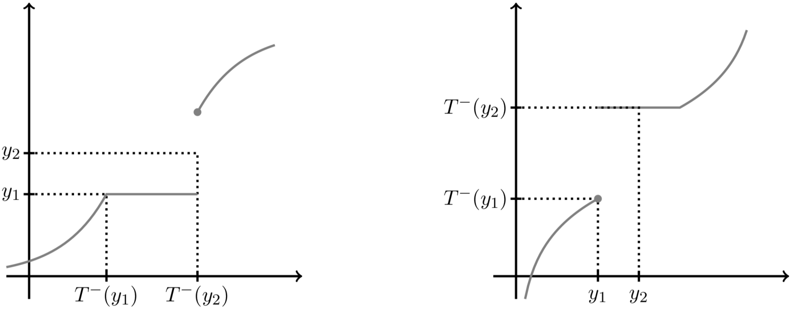


Figure 1 An increasing function (left) and its corresponding generalized inverse (right).

Properties of Quantile Function (More Difficult)

1. $F^{-1}(\tau)$ is nondecreasing and left continuous. (the **opposite** of the CDF)
2. $F^{-1}(F(x)) \leq x$
3. $F(F^{-1}(\tau)) \geq \tau$
4. $F^{-1}(\tau) \leq x \iff \tau \leq F(x)$ (a result of 2. and 3.)

Properties of Quantile Function

2. $F^{-1}(F(x)) \leq x$
3. $F(F^{-1}(\tau)) \geq \tau$
4. $F^{-1}(\tau) \leq x \iff \tau \leq F(x)$ (a result of 2. and 3.)

Properties of Quantile Function

Property:

$$F^{-1}(F(x^0)) \leq x^0$$

Proof $F^{-1}(F(x^0)) = \inf\{x : F(x) \geq F(x^0)\}$. And because $x^0 \in \{x : F(x) \geq F(x^0)\}$, $F^{-1}(F(x)) \leq x^0$.

Properties of Quantile Function

Property:

$$F(F^{-1}(\tau)) \geq \tau$$

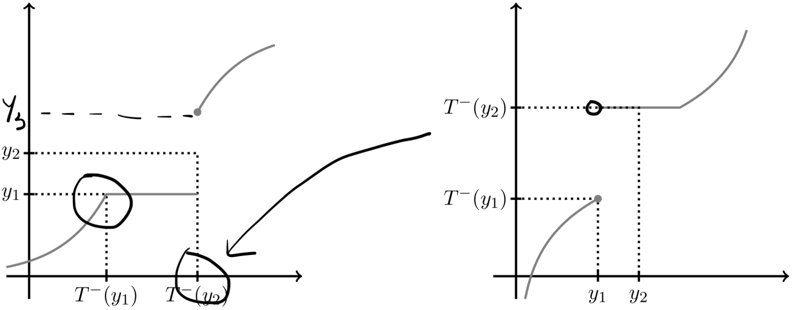


Figure 1 An increasing function (left) and its corresponding generalized inverse (right).

Properties of Quantile Function

Property:

$$F(F^{-1}(\tau)) \geq \tau$$

Properties of Quantile Function

Property:

$$F(F^{-1}(\tau)) \geq \tau$$

Proof Suppose that $x^0 = F^{-1}(\tau) = \inf\{x : F(x) \geq \tau\}$. Define Set $A = \{x : F(x) \geq \tau\}$

Case 1 $x^0 \in \{x : F(x) \geq \tau\}$. Then $F(F^{-1}(\tau)) = F(x_0) \geq \tau$.

Properties of Quantile Function

Property:

$$F(F^{-1}(\tau)) \geq \tau$$

Proof Suppose that $x^0 = F^{-1}(\tau) = \inf\{x : F(x) \geq \tau\}$. Define Set $A = \{x : F(x) \geq \tau\}$

Case 1 $x^0 \in \{x : F(x) \geq \tau\}$. Then $F(F^{-1}(\tau)) = F(x_0) \geq \tau$.

Case 2 $x^0 \notin \{x : F(x) \geq \tau\}$. There exists a sequence $\{x_i\}_{i=1}^{\infty} \in A$ with $\lim_{i \to \infty} x_i = x^0$ and $x_i > x_0$ for all i.

Right continuity of F implies $F(x^0) = \lim_{i \to \infty} F(x_i) \geq \tau$.

Properties of Quantile Function

2. $F^{-1}(F(x)) \leq x$
3. $F(F^{-1}(\tau)) \geq \tau$
4. $F^{-1}(\tau) \leq x \iff \tau \leq F(x)$ (a result of Properties 2. and 3.)

Proof

$$F^{-1}(\tau) \leq x \implies \tau \leq F(x)$$

Since F is non-decreasing, we know that $F(F^{-1}(\tau)) \leq F(x)$. Due to Property 3. above, $\tau \leq F(F^{-1}(\tau)) \leq F(x)$.

The second part is simiarily shown using Property 2.

Properties of Quantile Function

Proposition Let X be a real-valued random variable with distribution function $F(x) = \Pr[X \leq x]$. Then for any $\tau \in (0,1)$,

1. $\Pr[X \leq F^{-1}(\tau)] \geq \tau$ and $\Pr[X < F^{-1}(\tau)] \leq \tau$
2. $\Pr[X > F^{-1}(\tau)] \leq 1 - \tau$ and $\Pr[X \geq F^{-1}(\tau)] \geq 1 - \tau$.

More General Case (Skip)

We actually can allow the CDF of X to be **continuous** but non-decreasing. In other words, we will see a flat region in the CDF, but **no jumps**. The proof in Casella's book is a bit difficult to follow. So, I present a slightly different version here.

Proof. The key is again to show that for the general case, the following continues to hold

$$\Pr[F_X(X) \leq y] = \Pr[X \leq F^{-1}(y)]$$

Proposition (general case):

$$Y = F_X(x) \sim \mathcal{U}(0, 1)$$

More General Case (Skip)

Remember that

$$\Pr[F_X(X) \leq y] = \Pr[X \in \{x : F_X(x) \leq y\}]$$

Lets define $Q_X(y) = F^{-1}(y)$ the smallest value satisfying $F(x) = y$. Note that since the CDF is continuous and there exists no jump, we will not have $F(Q_X(y)) > y$.

$$\Pr[X \in \{x : F_X(x) \leq y\}] = \Pr[\{x : X \leq Q_X(y)\} \cup \{x : F_X(x) = y\}]$$

More General Case (Skip)

Remember that

$$\Pr[F_X(X) \leq y] = \Pr[X \in \{x : F_X(x) \leq y\}]$$

Lets define $Q_X(y) = F^{-1}(y)$ the smallest value satisfying $F(x) = y$. Note that since the CDF is continuous and there exists no jump, we will not have $F(Q_X(y)) > y$.

$$\begin{aligned}
\Pr[X \in \{x : F_X(x) \leq y\}] &= \Pr[\{x : X \leq Q_X(y)\} \cup \{x : F_X(x) = y\}] \\
&= \Pr[\{X \leq Q_X(y)\}] + \Pr[\{x : F_X(x) = y\}]
\end{aligned}$$

More General Case (Skip)

Remember that

$$\Pr[F_X(X) \leq y] = \Pr[X \in \{x : F_X(x) \leq y\}]$$

Lets define $Q_X(y) = F^{-1}(y)$ the smallest value satisfying $F(x) = y$. Note that since the CDF is continuous and there exists no jump, we will not have $F(Q_X(y)) > y$.

$$\begin{aligned}
\Pr[X \in \{x : F_X(x) \leq y\}] &= \Pr[\{x : X \leq Q_X(y)\} \cup \{x : F_X(x) = y\}] \\
&= \Pr[\{X \leq Q_X(y)\}] + \Pr[\{x : F_X(x) = y\}] \\
&= \Pr[X \leq Q_X(y)]
\end{aligned}$$

More General Case (Skip)

Remember that

$$\Pr[F_X(X) \leq y] = \Pr[X \in \{x : F_X(x) \leq y\}]$$

Lets define $Q_X(y) = F^{-1}(y)$ the smallest value satisfying $F(x) = y$. Note that since the CDF is continuous and there exists no jump, we will not have $F(Q_X(y)) > y$.

$$\begin{aligned}
\Pr[X \in \{x : F_X(x) \leq y\}] &= \Pr[\{x : X \leq Q_X(y)\} \cup \{x : F_X(x) = y\}] \\
&= \Pr[\{X \leq Q_X(y)\}] + \Pr[\{x : F_X(x) = y\}] \\
&= \Pr[X \leq Q_X(y)] \\
&= F(Q_X(y))
\end{aligned}$$

More General Case (Skip)

Remember that

$$\Pr[F_X(X) \leq y] = \Pr[X \in \{x : F_X(x) \leq y\}]$$

Lets define $Q_X(y) = F^{-1}(y)$ the smallest value satisfying $F(x) = y$. Note that since the CDF is continuous and there exists no jump, we will not have $F(Q_X(y)) > y$.

$$\begin{aligned}
\Pr[X \in \{x : F_X(x) \leq y\}] &= \Pr[\{x : X \leq Q_X(y)\} \cup \{x : F_X(x) = y\}] \\
&= \Pr[\{X \leq Q_X(y)\}] + \Pr[\{x : F_X(x) = y\}] \\
&= \Pr[X \leq Q_X(y)] \\
&= F(Q_X(y)) \\
&= y (\text{note the definition of } Q_X(y))
\end{aligned}$$

I recently used this result in one of my proofs to characterize the **rank-rank** model in the mobility literature.

see may paper