Mathematical Expectation (I): Basic Definition and Properties

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Alternative Approaches to Characterize a Distribution

> *The probability density function is a natural and familiar way to formulate the distribution of a random variable. But, there are many other functions that are used to identify or characterize a random variable, depending on the setting. (Appendix B. in Greene)*

1. Moment Approach
2. Moment Generating Functions
3. Characteristic Function
4. Entropy Function

Many of such functions involve **Expectation** of a random variable, which we now define.

Mathematical Expectation

Road Map

1. Definitions (Discrete vs. Continuous)
2. Further Discussions on Mathematical Expectation
3. The **distribution** and **expectation** of a function of a random variable (Continuous vs. Discrete)

Expectation (or Mean)

Mathematical Expectation, $\mathbb{E}[X]$ is defined as follows

1. **Discrete Variable** $\mathbb{E}[X] = \sum_{i=1}^{r} x_i p_i$, where $p_i = \Pr[X = x_i]$ if the series is convergent when $r = \infty$
2. **Continuous Variable** $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$

Part : Discrete Case

Example 1 (discrete case)

x	1	2	3
P(X=x)	1/3	1/3	1/3

$\mathbb{E}[X] = \sum_{i=1}^{r} x_i p_i = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$

Example 2 (discrete case) $\mathbb{P}[X = k] = \frac{e^{-1}}{k!}$ for non-negative integer k

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\infty} k \frac{e^{-1}}{k!} \\
&= 0 + \sum_{k=1}^{\infty} k \frac{e^{-1}}{k!} \\
&= \sum_{k=1}^{\infty} \frac{e^{-1}}{(k-1)!} \\
&= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} \\
&= 1
\end{aligned}$$

Important Things about Expectation (or Mean)

1. Expectation is a number representing the distribution of a random variable. Thus, it is a property of the distribution.
2. While the notation used to denote expectations is $\mathbb{E}[X]$, it is **wrong** to think of it as a function of the random variable X. It is rather a **function (property) of the distribution** of X. It is a fixed feature, **Non-random**!!

For example, the **entire PMF** enters how you calculate the expected mean.

$$x_1 \cdot \mathbf{p_1} + x_2\mathbf{p_2} + \cdots + x_r\mathbf{p_r}$$

3. Expectation is just **one of the characteristics** of the distribution of X and, thus, does not provide a complete information about the distribution. It is possible to have many different distributions with the same expectation.

A **complete description** for a discrete random variable is given by the PMF

See next slide for the shortcoming of this concept.

Expectation (or Mean)

Example: Discrete Case

x	1	2	3
P(X=x)	1/4	1/4	1/2

$\mathbb{E}[X] = \sum_{i=1}^{r} x_i p_i = 1 \times 1/4 + 2 \times 1/4 + 3 \times 1/2 = 9/4$, which does not even exist and will **never** in the data! But it tells something about the distribution (with some significant weights on the values greater than 9/4).

But not complete, and indeed needs infinitely many moments to represent and characterize the distribution.

4. While expectation is just ONE of the properties of a distribution, it nevertheless plays a fundamental role in statistics, econometrics and economics. One of the reasons for that is the so called Law of Large Numbers (LLN).

Finitness of Expectation (or Mean): Does Not Have to Exist!

When the support of a random variable is finite, the expectation is always a number (finite).

However, when the support is infinite, the expectation may be infinite since $\sum_{i=1}^{r} x_i p_i$ is a sum of infinitely many terms.

Moreover, when the support of a random variable includes both $-\infty$ and ∞, we can even have an undefined expectation as the formula for expectation can produce $\infty - \infty$. When expectation is infinite ($+\infty$ or $-\infty$) or undefined ($\infty - \infty$), we say that expectation does not exist.

Example $\mathbb{E}[X] = \infty$: **St. Petersburg Paradox**

Part : Continuous Case

Mathematical Expectation, $\mathbb{E}[X]$ is defined as follows

1. **Discrete Variable** $\mathbb{E}[X] = \sum_{i=1}^{r} x_i p_i$, where $p_i = \Pr[X = x_i]$.
2. **Continuous Variable** $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$

Note that in the case of continuous variables

1. PMF is replaced by PDF
2. Summation is replaced by Integration.

Expectation (or Mean)

Example Let $X \sim U(a, b), a < b$.

$$f_X(x) = \frac{1}{b-a} \times \mathbb{I}[a \leq x \leq b]$$

$$\mathbb{E}[X] = \frac{b+a}{2}$$

Expectation (or Mean)

Proof

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x \frac{1}{b-a} \times \mathbb{I}[a \leq x \leq b] dx \\
&= \int_{a}^{b} x \cdot \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \int_{a}^{b} x \cdot dx \\
&= \frac{1}{b-a} \frac{x^2}{2} |_a^b \\
&= \frac{1}{b-a} \frac{b^2 - a^2}{2} \\
&= \frac{b+a}{2}
\end{aligned}$$

Expectation (or Mean)

Note: The previous case is only specific to uniform distribution. It illustrates how we can derive the expecation for *one* type of distribution by applying the definition of mathematical expectation.

Derivations are not necessarily as straightforward for other distributions.

Linear Properties of Expectation

Note that both summation $\sum$ and the integral $\int$ are linear operators. As a result, we can easily show that the following properties hold

1. $\mathbb{E}[c] = c$ where c is a constant.
2. $\mathbb{E}[a \cdot X + b] = a \cdot \mathbb{E}[X] + b$
3. $\mathbb{E}[c_1 \cdot X_1 + c_2 \cdot X_2 + \cdots + c_k \cdot X_k] = c_1\mathbb{E}[X_1] + c_2\mathbb{E}[X_2] + \cdots + c_k \cdot \mathbb{E}[X_k]$

The first two will be left as a homework assignment. The last one is more subtle and will be discussed later due to the nature of multiple variables

More on Expectation Operator

Many previously defined concepts can be expressed as some form of expectation

$$F_X(x) = \Pr[X \leq x] = \mathbb{E}[\mathbb{I}(X \leq x)]$$

Intuition Share of the potential values less than x. This would be left as a homework assignment when we introduce how to analyze **function of a random variable**.