

## PS3, Ex. 5: Luxembourg as a rogue state

Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican ( $V$ ) and Denmark ( $D$ ) are preparing an attack on Luxembourg's nuclear research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_V s_D,$$

where  $s_i \in [0, 1]$  is the share of its military capacity that country  $i$  ( $i \in \{V, D\}$ ) uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country  $i$  is

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost.

- (a) Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium (NE) of this game.
- (b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the  $\bar{s}_V = \bar{s}_D = \bar{s}$  that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

## PS3, Ex. 5: Luxembourg as a rogue state

(a) Find the NE in the static game:

Expected payoff for player  $i \neq j$ :

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for  $i$ :

$$\begin{aligned} FOC : \frac{\delta u_i}{\delta s_i} &= 1 + 0 - s_j - 2s_i = 0 \\ s_i &= \frac{1 - s_j}{2} \end{aligned}$$

Take advantage of symmetry:

$$\begin{aligned} s_i &= \frac{1 - s_i}{2} \\ 2s_i &= s_i = 1 \\ s_i^* &= \frac{1}{3} \end{aligned}$$

$$\text{i.e. } NE = \left\{ (s_D^*, s_V^*) = \left( \frac{1}{3}, \frac{1}{3} \right) \right\}$$

(b) Find the SO given shares are equal:

Expected payoff for  $i$ ,  $\bar{s}_D = \bar{s}_V = \bar{s}$ :

$$\begin{aligned} u_i(\bar{s}) &= \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}} \\ &= 2\bar{s} - 2\bar{s}^2 \end{aligned}$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$\begin{aligned} FOC : \frac{\delta \pi^S}{\delta s_i} &= 4 - 8\bar{s} = 0 \\ \bar{s} &= \frac{4}{8} = \frac{1}{2} > \frac{1}{3} \end{aligned}$$

i.e. the SO is higher than the NE as the positive externality is not rewarded, leading to an incentive to free ride.