Microeconomics III: Problem Set 10[a]

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[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

PS8, Ex. 2 (A): Crimea Through a Game-Theory

PS8, Ex. 3 (A): The 'Lemons' model (asymmetric information)

PS8, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

PS8, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)

PS8, Ex. 6: Extensive form game (Mixed-strategy Perfect Bayesian Equilibrium)

PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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i. **Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.**

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i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

Step 2: **How is this result affected by the distribution of the bidder's values?**

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(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

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i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

2: The result is independent of the distributions, thus it's still a best-response to bid one's value.

PS8, Ex. 2 (A): Crimea Through a Game-Theory

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Read through the New York Times article Crimea Through a Game-Theory Lens by Tyler Cowen (co-author of the popular economics blog Marginal Revolution). Try to think about how you would set up models to describe the situations he writes about. *(This exercise is just for reflection, no answer will be provided).*

PS8, Ex. 3 (A): The 'Lemons' model (asymmetric information)

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Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

(a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.

(b) What would happen if buyers cannot do so, and know that half of all used cars are lemons? Draw this as a dynamic game of incomplete information, where nature chooses the type of the car, the seller observes this and sets a price (any positive real number) and the buyer decides whether to buy or not.

(c) Find a Perfect Bayesian Equilibrium of this model.

"In US English, a lemon is a vehicle (often new) that turns out to have several manufacturing defects affecting its safety, value or utility." (Source: Wikipedia)

PS8, Ex. 3.a (A): The 'Lemons' model (asymmetric information)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for $800 but Beauty-owners will not sell for anything less than $2000. Buyers will pay up to $1200 for a lemon and up to $2400 for a beauty.

(a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.

If buyers can distinguish between beauties and lemons they would be traded on two separate markets with prices within $[800, 1200]$ and $[2000, 2400]$ respectively.

PS8, Ex. 3.b (A): The 'Lemons' model (asymmetric information)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

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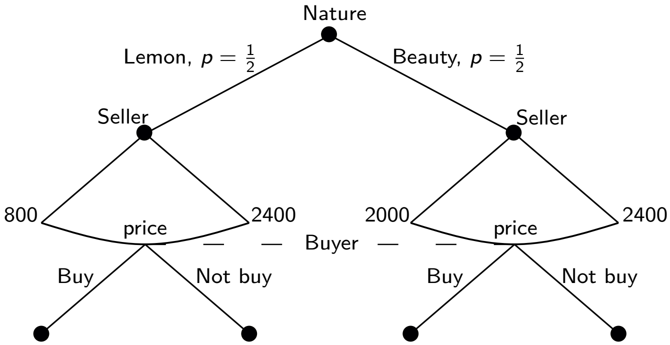
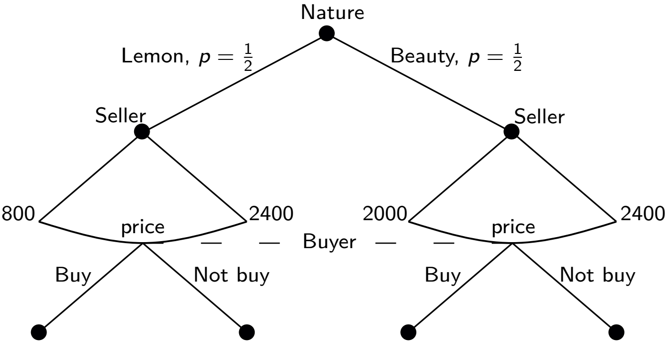
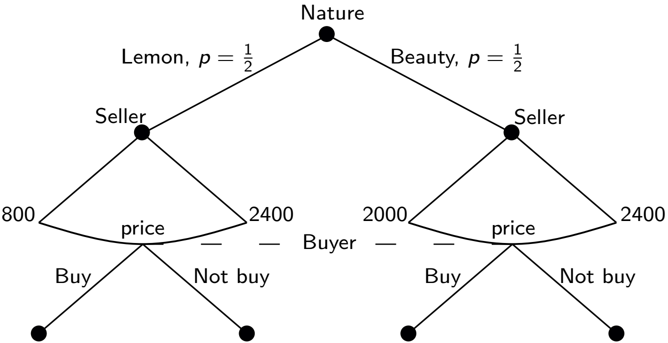


PS8, Ex. 3.c (A): The 'Lemons' model (asymmetric information)



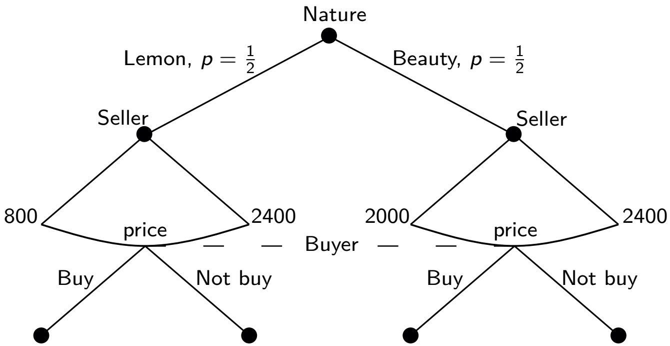
(c) Find a Perfect Bayesian Equilibrium of this model.



(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: **Write up buyer's expectation to the car's value given her beliefs regarding p**

PS8, Ex. 3.c (A): The 'Lemons' model (asymmetric information)



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1. $E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$

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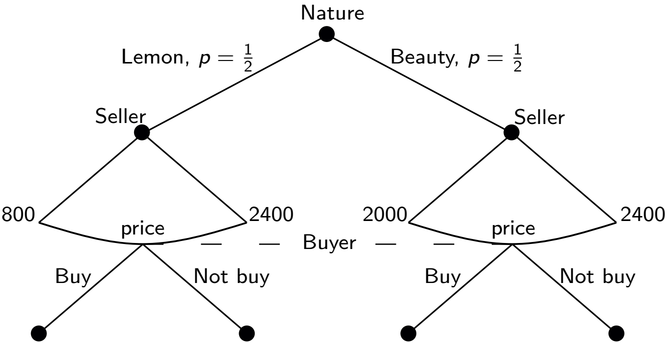

(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding p.

Step 2: **As both the seller and the buyer know this expectation, what will the outcome be?**

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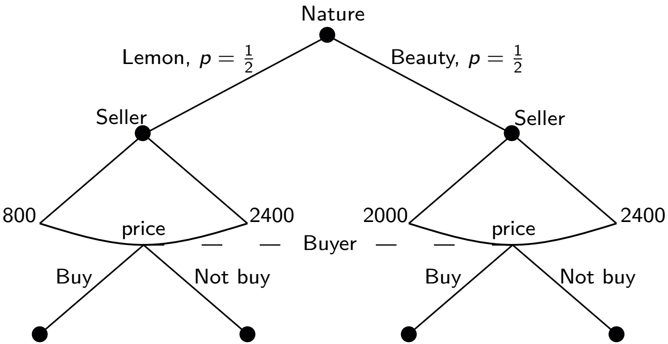

(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding p.

Step 2: As both the seller and the buyer know this expectation, what will the outcome be?

1. $E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$
2. The seller will not sell *beauties* for a price below 2000. The buyer anticipates this, thus, there will only be a market for *lemons* being sold for *price* $\in [800, 1200]$ as 1200 is the highest amount that the buyer is willing to pay for a *lemon*.

PS8, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

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Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

The value to i of having his preferred policy implemented is v_i, where $v_i \sim U(0, 1)$ independently (private values). The lobbyists know their own valuation, but not that of the other lobbyist.

(a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c.

PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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Step 1: **Write up the bidders, valuations, bids, and utilities.**

PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

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(a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

Step 1: Write up the auction with bidders, valuations, bids, and utilities.

Step 2: **How is this different from the auctions we have seen so far?**

1. Two bidders, $i \in 1, 2$.

   Valuations are independently distributed $v_i \sim U(0, 1)$

   Bids $b_i \in [0, 1]$

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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   $$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

2. Both bidders pay their bid b_i regardless of whether they win. This is known as an *all-pay auction*.

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c.

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: **Write up bidder *i*'s probability of winning the auction if *j* sticks to the equilibrium strategy.**

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

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(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

$$\begin{aligned}
\mathbb{P}(i\ wins) &= \mathbb{P}(b_i > b_j(v_j)) \\
&= \mathbb{P}(b_i > cv_j^2) \\
&= \mathbb{P}\left(\frac{b_i}{c} > v_j^2\right) \\
&= \mathbb{P}\left(\sqrt{\frac{b_i}{c}} > v_j\right) \\
&= \sqrt{\frac{b_i}{c}}
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Standard results for $x \sim u(a,b)$:

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1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$

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1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$

Step 2: **Write up bidder i's expected payoff from bidding b_i conditional on v_i.**

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

$$\mathbb{E}[u_i(b_i)|v_i] = \mathbb{P}(i\ wins)v_i - b_i$$
$$= \sqrt{\frac{b_i}{c}}v_i - b_i, \qquad cf.\ (1)$$

Remember that the bid is always payed.

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1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0, 1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

$$\mathbb{E}[u_i(b_i)|v_i] = \mathbb{P}(i\ wins)v_i - b_i$$

$$= \sqrt{\frac{b_i}{c}}v_i - b_i, \qquad cf.\ (1)$$

Remember that the bid is always payed.

Step 3: **Take the first-order condition and second-order condition with respect to b_i.**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

$$\begin{aligned}
\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} &= \frac{\delta}{\delta b_i}\left(\sqrt{\frac{b_i}{c}}v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(\frac{\sqrt{b_i}}{\sqrt{c}}v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(b_i^{\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - b_i\right) \\
&= \frac{1}{2}b_i^{-\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - 1 \\
&= \frac{1}{2}\frac{1}{\sqrt{b_i}}\frac{1}{\sqrt{c}}v_i - 1 \\
&= \frac{1}{2\sqrt{b_i c}}v_i - 1
\end{aligned}$$

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}}v_i - 1 = 0$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

$$\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} = \frac{\delta}{\delta b_i}\left(\sqrt{\frac{b_i}{c}}v_i - b_i\right)$$

$$= \frac{\delta}{\delta b_i}\left(\frac{\sqrt{b_i}}{\sqrt{c}}v_i - b_i\right)$$

$$= \frac{\delta}{\delta b_i}\left(b_i^{\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - b_i\right)$$

$$= \frac{1}{2}b_i^{-\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - 1 \qquad (**)$$

$$= \frac{1}{2}\frac{1}{\sqrt{b_i}}\frac{1}{\sqrt{c}}v_i - 1$$

$$= \frac{1}{2\sqrt{b_i c}}v_i - 1$$

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c}\cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}}v_i - 1 = 0$

   SOC: $-\frac{1}{4}b_i^{-\frac{3}{2}}\frac{1}{\sqrt{c}}v_i = 0, \quad cf.\ (**)$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: **Solve to find $b_i(v_i)$.**

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

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Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

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   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf.\ (**)$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

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Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

As the SOC is negative for all $b_i, v_i, c > 0$ bidder i maximizes expected utility for

$$0 = \frac{1}{2\sqrt{b_i(v_i)c}} v_i - 1 \Leftrightarrow$$

$$2\sqrt{b_i(v_i)c} = v_i \Leftrightarrow$$

$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \; wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf. \; (**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

As the SOC is negative for all $b_i, v_i, c > 0$ bidder i maximizes expected utility for

$$0 = \frac{1}{2\sqrt{b_i(v_i)c}} v_i - 1 \Leftrightarrow$$

$$2\sqrt{b_i(v_i)c} = v_i \Leftrightarrow$$

$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Step 5: **Set this equal to $(*)$ to find c^*.**

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf.\ (**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

Step 5: Set this equal to $(*)$ to find c^*.

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$
$$c^* = \frac{1}{4c^*} \Leftrightarrow$$
$$2c^* = \frac{1}{4} \Leftrightarrow$$
$$c^* = \frac{1}{2}$$

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$
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3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf.\ (**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

Step 5: Set this equal to $(*)$ to find c^*.

$$\begin{aligned} c^* v_i^2 &= \frac{1}{4c^*} v_i^2 \Leftrightarrow \\ c^* &= \frac{1}{4c^*} \Leftrightarrow \\ 2c^* &= \frac{1}{4} \Leftrightarrow \\ c^* &= \frac{1}{2} \end{aligned}$$

Step 6: **Write up the equilibrium bidding strategy.**

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf. \ (**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$

PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

Step 5: Set this equal to $(*)$ to find c^*.

Step 6: Write up the equilibrium bidding strategy.

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

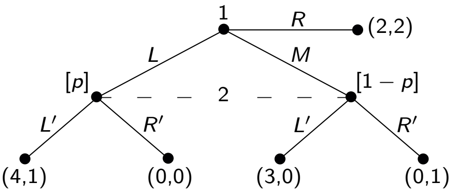
1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad cf. \ (**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$
6. BNE: $b_i^*(v_i) = \frac{1}{2} v_i^2$

PS8, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)

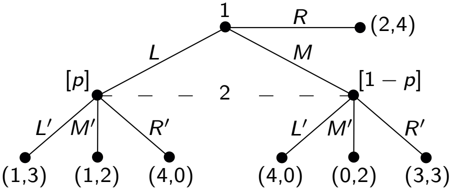
PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.



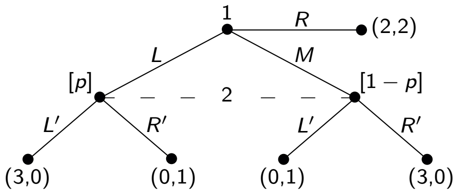
PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.



PS8, Ex. 6: Extensive form game (Mixed-strategy Perfect Bayesian Equilibrium)

Exercise 4.2 in Gibbons (pp. 245-246). Show that there does not exist a pure-strategy perfect Bayesian equilibrium in the following extensive-form game. What is the mixed-strategy perfect Bayesian equilibrium?



PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

Difficult. Exercise 4.10 in Gibbons (p. 250). Two partners must dissolve their partnership. Partner 1 currently owns share s of the partnership, partner 2 owns share $1 - s$. The partners agree to play the following game: partner 1 names a price, p, for the whole partnership, and partner 2 then chooses either to buy l's share for ps or to sell his or her share to 1 for $p(1 - s)$. Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on $[0, 1]$, but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?