



## Microeconomics III: Problem Set 11<sup>a</sup>

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<sup>a</sup>Slides created for exercise class 3 and 4, with reservation for possible errors.

PS11, Ex. 1 (A): Effect of the GED education as a signal

PS11, Ex. 2 (A): Asymmetric information (PBE)

Signaling games in general

PS11, Ex. 3: Signaling game (pooling and separating PBE)

PS11, Ex. 4: Signaling games (pooling and separating PBE)

PS11, Ex. 5: Signaling games (pooling PBE)

PS11, Ex. 6: Spence's education signaling model (PBE)

## **PS11, Ex. 1 (A): Effect of the GED education as a signal**

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Does signaling work? Read the article by Tyler, Murnane and Willett and think about their results. What is their hypothesis for why they do not find an effect for minority groups? Come up with an example of an education program that has mostly signaling value in your country.

*(This is a reflection question, no answer will be provided).*

**PS11, Ex. 2 (A): Asymmetric  
information (PBE)**

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Exercise 4.11 in Gibbons (p. 250). Difficult. A buyer and a seller have valuations  $v_b$  and  $v_s$ . It is common knowledge that there are gains from trade (i.e., that  $v_b > v_s$ ), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on  $[0,1]$ ; the buyer's valuation  $v_b = kv_s$ , where  $k > 1$  is common knowledge; the seller knows  $v_s$  (and hence  $v_b$ ) but the buyer does not know  $v_b$  (and hence  $v_s$ ). Suppose the buyer makes a single offer,  $p$ , which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when  $k < 2$ ? When  $k > 2$ ? (See Samuelson 1984.)

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Step 1: **Consider the uniform distribution  $x \sim U(a, b)$ . Use the cumulative distribution function (CDF) to write up the probability that a random draw of  $x$  is lower than a constant  $c$ . Use the mean to write up the expected value of a random draw of  $x$  where  $x$  is lower than a constant  $c \in [a, b]$ .**

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$$\max_p \mathbb{P}[v_s < p] \mathbb{E}[v_b - p | v_s < p]$$

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$$\max_p \mathbb{P}[v_s < p] \mathbb{E}[v_b - p | v_s < p]$$

$$= \max_p \frac{p-0}{1-0} \mathbb{E}[kv_s - p | v_s < p]$$

$$= \max_p p (k \mathbb{E}[v_s < p] - p)$$

$$= \max_p p \left( k \frac{0+p}{2} - p \right)$$

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$$\text{using } (\dagger) \quad 3. \max_p u_b(p, k) = \max_p p^2 \left( \frac{k}{2} - 1 \right)$$

using  $(\ddagger)$

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$$\begin{aligned}\frac{\delta u_b(p, k)}{\delta p} &= 0 \\ 2p \left( \frac{k}{2} - 1 \right) &= 0 \\ 2p \frac{k}{2} &= 2p \\ p \frac{k}{2} &= p\end{aligned}$$

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Step 5: Maximize buyer's utility for  $k < 2$ .

Step 6: Maximize buyer's utility for  $k > 2$ .

Step 7: **Looking at the seller's strategy, will trade occur when  $k > 2$ ?**

**What about  $k \in (1, 2)$ ? Have we seen something similar before?**

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$$6. \text{For } k > 2 : p \frac{k}{2} > p \Rightarrow p^{**} = 1$$

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Step 5: Maximize buyer's utility for  $k < 2$ .

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Step 7:  $k > 2$ : As  $v_s \in [0, 1] \leq 1 = p^{**}$ , seller will always accept this price.

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$$5. \text{For } k \in (1, 2) : p \frac{k}{2} < p \Rightarrow p^* = 0$$

$$6. \text{For } k > 2 : p \frac{k}{2} > p \Rightarrow p^{**} = 1$$

## PS11, Ex. 2 (A): Asymmetric information (PBE)

Exercise 4.11 in Gibbons (p. 250). Difficult. A buyer and a seller have valuations  $v_b$  and  $v_s$ . It is common knowledge that there are gains from trade (i.e., that  $v_b > v_s$ ), but the size of the gains is private information, as follows: the seller's valuation is uniformly distributed on  $[0,1]$ ; the buyer's valuation  $v_b = kv_s$ , where  $k > 1$  is common knowledge; the seller knows  $v_s$  (and hence  $v_b$ ) but the buyer does not know  $v_b$  (and hence  $v_s$ ). Suppose the buyer makes a single offer,  $p$ , which the seller either accepts or rejects. What is the perfect Bayesian equilibrium when  $k < 2$ ? When  $k > 2$ ? (See Samuelson 1984.)

Step 1: Use the CDF to write up  $\mathbb{P}(x < c)$ .

Use the mean to write up  $\mathbb{E}(x < c)$ .

Step 2: The buyer offers a price  $p$ . Write up the seller's strategy (best response).

Step 3: Write up the buyer's problem.

Step 4: Take the first-order condition.

Step 5: Maximize buyer's utility for  $k < 2$ .

Step 6: Maximize buyer's utility for  $k > 2$ .

Step 7:  **$k > 2$ :** As  $v_s \in [0, 1] \leq 1 = p^{**}$ , seller will always accept this price.  
 **$k \in (1, 2)$ :** Seller will not accept if  $v_s > 0$ , though trade would benefit both under perfect information.  
Similar to Akerlof's 'Lemons'.

1. Standard results for  $x \sim U(a, b)$  :

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(x < c) = \frac{c-a}{b-a} \quad (\dagger)$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(x < c) = \frac{a+c}{2} \quad (\ddagger)$$

$$2. S_s(p, v_s) = \begin{cases} \text{Sell} & \text{if } p \geq v_s \\ \text{Don't} & \text{if } p < v_s \end{cases}$$

$$3. \max_p u_b(p, k) = \max_p p^2 \left( \frac{k}{2} - 1 \right)$$

$$4. \text{FOC: } p \frac{k}{2} = p$$

$$5. \text{For } k \in (1, 2) : p \frac{k}{2} < p \Rightarrow p^* = 0$$

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## Signaling games in general

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## PS11: Signaling games in general

### Players:

- 2 players: Sender (S) and receiver (R). E.g. firm and consumer, or employer and employee (Spence).

### Timing:

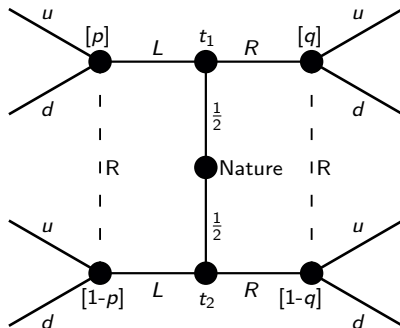
1. Nature chooses the sender's type from  $T = \{t_1, \dots\}$ .
2. S: The sender realizes her type and sends a signal from  $M = \{m_1, \dots\}$ , typically either  $L$  (left) or  $R$  (right).
3. R: The receiver observes  $m$  (but not the type  $t$ !) and forms his beliefs:

$$\mu(t_1|L) = p \text{ and } \mu(t_1|R) = q$$

Consequently, for  $S$  having two possible types:

$$\mu(t_2|L) = 1 - p \text{ and } \mu(t_2|R) = 1 - q$$

4. R: The receiver chooses an action from  $A = \{a_1, \dots\}$ , e.g. *up* or *down*.
5. Payoffs are realized.



## PS11: Signaling games in general

### Players:

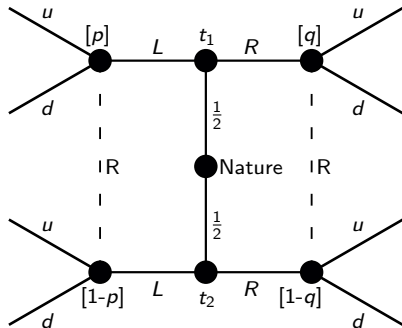
- 2 players: Sender (S) and receiver (R). E.g. firm and consumer, or employer and employee (Spence).

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 $p = \mu(t_1|L)$  and  $q = \mu(t_1|R)$   
Consequently, for  $S$  having two possible types:  
 $1 - p = \mu(t_2|L)$  and  $1 - q = \mu(t_2|R)$
4. R: The receiver chooses an action from  $A = \{a_1, \dots\}$ , e.g. *up* or *down*.
5. Payoffs are realized.

### Four possible equilibria for two types:

- Pooling on  $L$  or pooling on  $R$ .
- Separating:  $t_1$  plays  $L$  and  $t_2$  plays  $R$  or the other way around.



# PS11: Signaling games in general

## Players:

- 2 players: Sender (S) and receiver (R). E.g. firm and consumer, or employer and employee (Spence).

## Timing:

- Nature chooses the sender's type from  $T = \{t_1, \dots\}$ .
- S: The sender realizes her type and sends a signal from  $M = \{m_1, \dots\}$ , typically either  $L$  (left) or  $R$  (right).
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$$p = \mu(t_1|L) \text{ and } q = \mu(t_1|R)$$

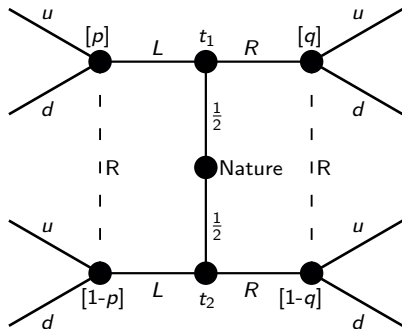
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- Pooling on  $L$  or pooling on  $R$ .
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**Cookbook: For each possible equilibrium go over signaling requirements 3 and 2:**

SR3: R: Find the beliefs  $p, q$  given  $S$ 's eq. strategy. (Only consider beliefs that are consistent with  $S$ 's eq. strategy.)

SR2R: R: Given beliefs, find  $a(m_j|\mu(t_1|m_j))$ .

SR2S: S: Does  $t_1$  or  $t_2$  want to deviate?

PBE: No deviation  $\rightarrow$  PBE. Pooling on  $L$ : Find off-eq.  $a(R|q) \rightarrow$  possibly two different PBE for different  $q$ .



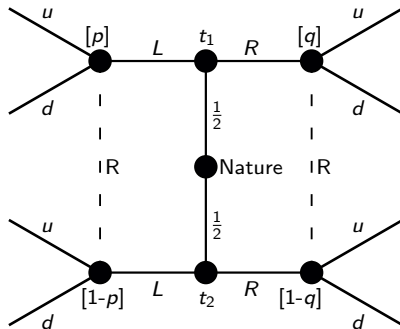
**PS11, Ex. 3: Signaling game  
(pooling and separating PBE)**

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## PS11, Ex. 3: Signaling game (pooling and separating PBE)

Consider the signaling game in Figure 1.

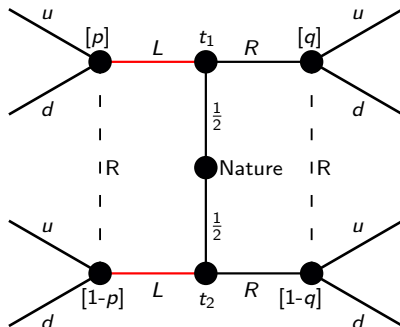
- (a) Suppose there is a pooling PBE where the Sender sends message  $L$  regardless of his type. What are the beliefs in this equilibrium?
- (b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?



## PS11, Ex. 3.a: Signaling game (pooling and separating PBE)

- (a) Suppose there is a pooling PBE where the Sender sends message  $L$  regardless of his type. What are the beliefs in this equilibrium?

SR3: **R:** Find the beliefs of R given S's equilibrium strategy. (In equilibrium, we only consider beliefs of R that are consistent with S's equilibrium strategy.)



## PS11, Ex. 3.a: Signaling game (pooling and separating PBE)

(a) Suppose there is a pooling PBE where the Sender sends message  $L$  regardless of his type. What are the beliefs in this equilibrium?

SR3: R: Find the beliefs of R given S's equilibrium strategy. (In equilibrium, we only consider beliefs of R that are consistent with S's eq. strategy.):

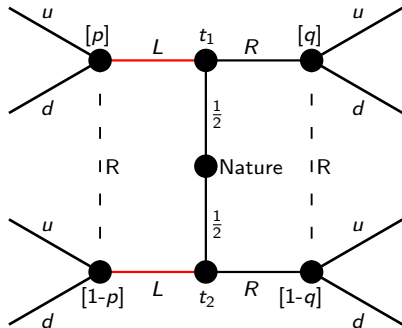
$$\mu(t_1|L) = \mu(t_2|L) = \frac{1}{2}$$

$$\Rightarrow p = 1 - p = \frac{1}{2}$$

$$q \in [0; 1]$$

I.e. in a pooling perfect Bayesian equilibrium where S always sends the message  $L$ , the receiver R believes that S can be type  $t_1$  or  $t_2$  with equal probability as the signal does not reveal anything.

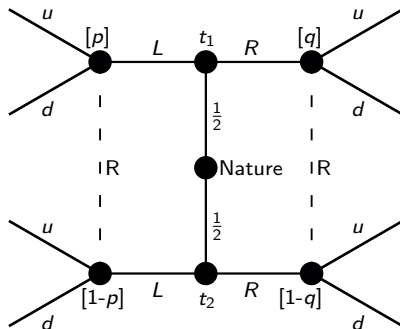
As the message  $R$  is not a part of S's equilibrium strategy, the receiver R has no beliefs about  $q$  other than  $q \in [0, 1]$  in the case where S would unexpectedly send the message  $R$  instead.



# PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

- (b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

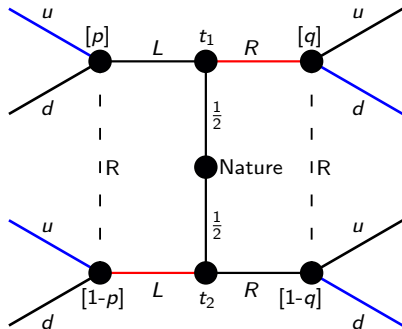
SR3:



## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

- (b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

SR3: R: Find the beliefs of R given S's equilibrium strategy. (In equilibrium, we only consider beliefs of R that are consistent with S's equilibrium strategy.)



## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

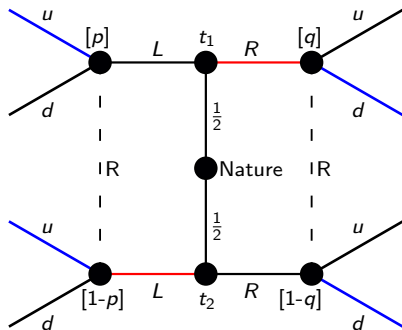
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SR2R:

SR2S:

PBE:



SR3: In the separating PBE, R has beliefs:

$$\mu(t_1|L) = p^* = 0$$

$$\mu(t_1|R) = q^* = 1$$

## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

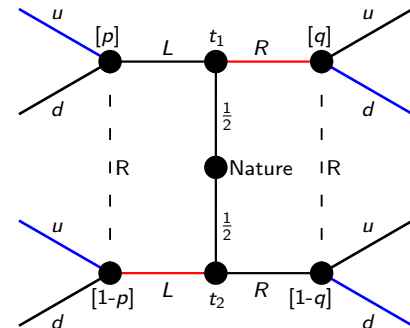
SR3: R: Find the beliefs of R given S's equilibrium strategy. (In equilibrium, we only consider beliefs of R that are consistent with S's eq. strategy.)

SR2R: R: Find R's optimal strategy given beliefs about S's strategy.

SR2S: S: Check whether S wants to deviate.

PBE: **Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:**

$$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0}_{\mu(t_1|L)}, \underbrace{q=1}_{\mu(t_1|R)}\}$$



SR3: In the separating PBE, R has beliefs:

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## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

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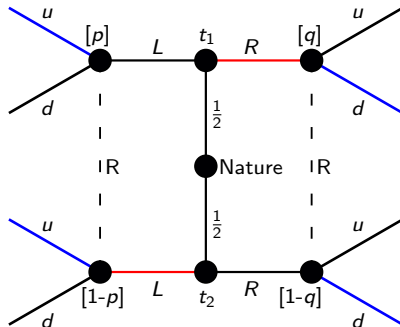
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$$\left\{ \underbrace{(R, L)}_{m(t_1), m(t_2)}, \underbrace{(u, d)}_{a(L), a(R)}, \underbrace{p=0, q=1}_{\mu(t_1|L), \mu(t_1|R)} \right\}$$



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$

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$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

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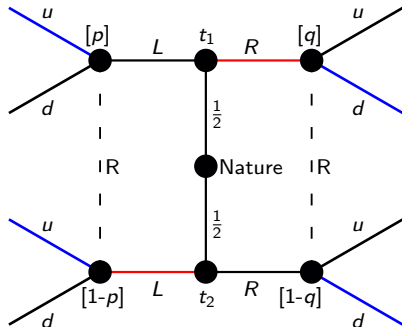
SR2R: R: Find R's optimal strategy given beliefs about S's strategy.

SR2S: S: Check whether S wants to deviate.

PBE: Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:

$$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0}_{\mu(t_1|L)}, \underbrace{q=1}_{\mu(t_1|R)}\}$$

→ **Construct payoffs that live up to these conditions.**



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$

$$\mu(t_1|R) = q^* = 1$$

$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

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## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

- (b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

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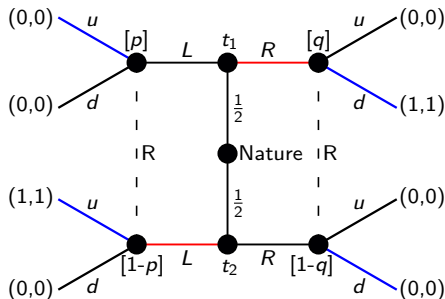
SR2S: S: Check whether S wants to deviate.

PBE: Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:

$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0}_{\mu(t_1|L)}, \underbrace{q=1}_{\mu(t_1|R)}\}$

→ Construct payoffs that live up to these conditions.

i: *Simplest possible example.*



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$   
 $\mu(t_1|R) = q^* = 1$

SR2R:  $\mathbb{E}[u_R(L, u|p=0)] \geq \mathbb{E}[u_R(L, d|p=0)]$   
 $\mathbb{E}[u_R(R, d|q=1)] \geq \mathbb{E}[u_R(R, u|q=1)]$

SR2S:  $u_S(R, d|t_1) \geq u_S(L, d|t_1)$   
 $u_S(L, u|t_2) \geq u_S(R, u|t_2)$

## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

SR3: R: Find the beliefs of R given S's equilibrium strategy. (In equilibrium, we only consider beliefs of R that are consistent with S's eq. strategy.)

SR2R: R: Find R's optimal strategy given beliefs about S's strategy.

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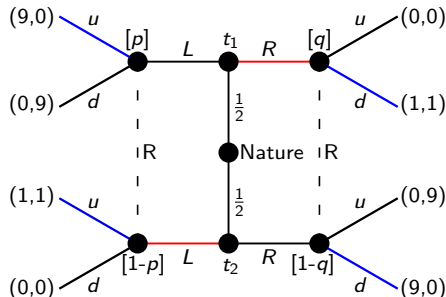
PBE: Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:

$$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0}_{\mu(t_1|L)}, \underbrace{q=1}_{\mu(t_1|R)}\}$$

→ Construct payoffs that live up to these conditions.

i: Simplest possible example.

ii: **Does the PBE still hold for this example?**



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$

$$\mu(t_1|R) = q^* = 1$$

$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

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# PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

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PBE: Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:

$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0, q=1}_{\mu(t_1|L) \mu(t_1|R)}\}$  SR3: In the separating PBE, R has beliefs:

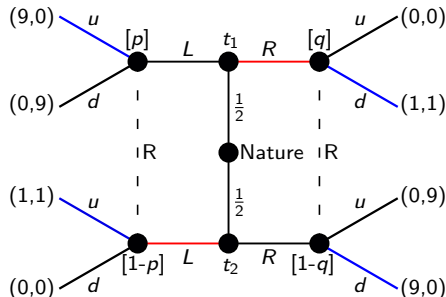
$$\mu(t_1|L) = p^* = 0$$

$$\mu(t_1|R) = q^* = 1$$

→ Construct payoffs that live up to these conditions.

i: Simplest possible example.

ii: Yes, all conditions still hold.



$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

$$\begin{aligned} \text{SR2S: } u_5(R, d|t_1) &\geq u_5(L, d|t_1) \\ u_5(L, u|t_2) &\geq u_5(R, u|t_2) \end{aligned}$$

## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

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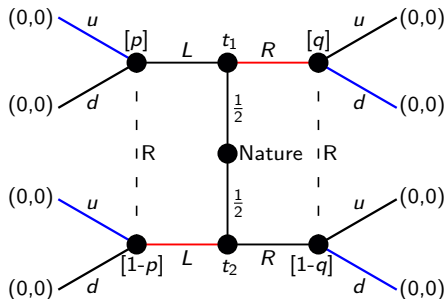
$$\{(\underbrace{R}_{m(t_1)}, \underbrace{L}_{m(t_2)}), (\underbrace{u}_{a(L)}, \underbrace{d}_{a(R)}), \underbrace{p=0}_{\mu(t_1|L)}, \underbrace{q=1}_{\mu(t_1|R)}\}$$

→ Construct payoffs that live up to these conditions.

i: Simplest possible example.

ii: Yes, all conditions still hold.

iii: **What about zero payoffs all over?**



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$

$$\mu(t_1|R) = q^* = 1$$

$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

$$\begin{aligned} \text{SR2S: } u_5(R, d|t_1) &\geq u_5(L, d|t_1) \\ u_5(L, u|t_2) &\geq u_5(R, u|t_2) \end{aligned}$$

## PS11, Ex. 3.b: Signaling game (pooling and separating PBE)

(b) Consider a possible separating PBE where  $t_1$  sends message  $R$ ,  $t_2$  sends message  $L$ , and where the receiver chooses  $u$  if and only if he receives message  $L$ . Can you write down payoffs for this game such that nobody has an incentive to deviate?

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PBE: Write up the conditions such that SR2R and SR2S hold (no incentive to deviate) for the following PBE:

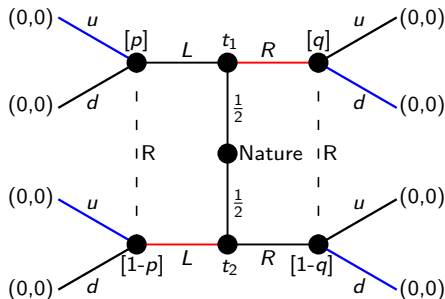
$$\left\{ \underbrace{(R, L)}_{m(t_1)}, \underbrace{(u, d)}_{a(L)}, \underbrace{(p=0, q=1)}_{\mu(t_1|L) \mu(t_1|R)} \right\}$$

→ Construct payoffs that live up to these conditions.

i: Simplest possible example.

ii: Yes, all conditions still hold.

iii: *All conditions hold with equality.*



SR3: In the separating PBE, R has beliefs:  
 $\mu(t_1|L) = p^* = 0$

$$\mu(t_1|R) = q^* = 1$$

$$\begin{aligned} \text{SR2R: } \mathbb{E}[u_R(L, u|p=0)] &\geq \mathbb{E}[u_R(L, d|p=0)] \\ \mathbb{E}[u_R(R, d|q=1)] &\geq \mathbb{E}[u_R(R, u|q=1)] \end{aligned}$$

$$\begin{aligned} \text{SR2S: } u_5(R, d|t_1) &\geq u_5(L, d|t_1) \\ u_5(L, u|t_2) &\geq u_5(R, u|t_2) \end{aligned}$$

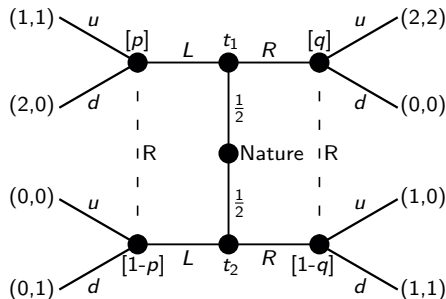
**PS11, Ex. 4: Signaling games  
(pooling and separating PBE)**

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## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

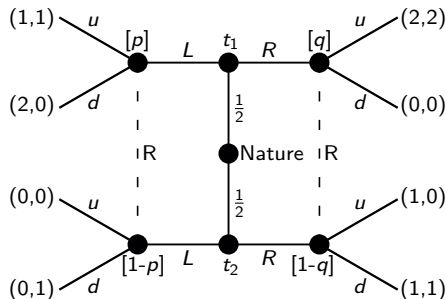
Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.



## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

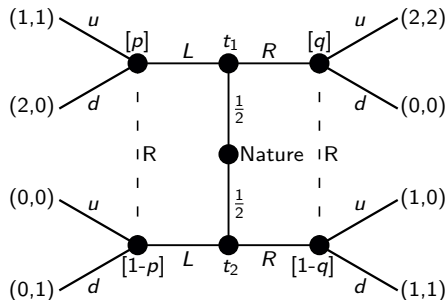
Step 1: Write up S's possible strategies.



# PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.



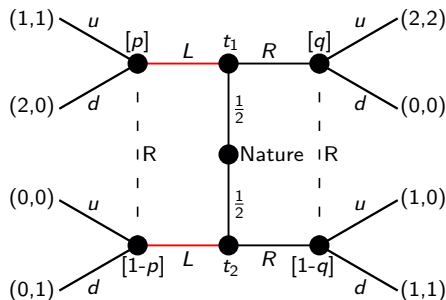
$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.

Step 2: **For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.**



$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.

Step 2: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = \frac{1}{2} \text{ and } \mu(t_1|R) = q \in [0, 1]$$

SR2R: R: Indifferent between  $u$  and  $d$ :

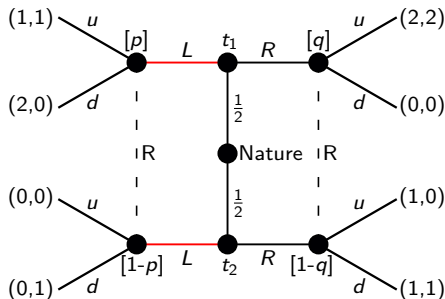
$$\mathbb{E}[u_R(L, u|p)] = \mathbb{E}[u_R(L, d|p)]$$

$$1p + 0[1 - p] = 0p + 1[1 - p]$$

$$\frac{1}{2} = \frac{1}{2}$$

SR2S: S:  $t_2$  wants to deviate as  $L|t_2$  is strictly dominated by  $R|t_2$ .

PBE: Not a PBE as  $t_2$  would deviate.



1.  $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$
2. No PBE that includes  $(L, L)$ .

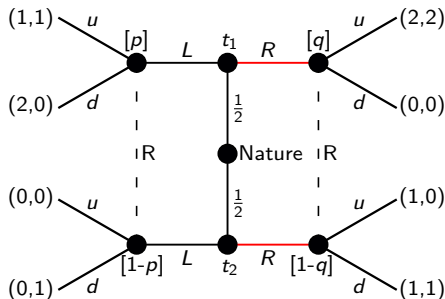
## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

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Step 3: **For the pooling strategy  $(R,R)$ , go over SR3, SR2R, and SR2S.**



1.  $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$
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# PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.

Step 2: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 3: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p \in [0, 1] \text{ and } \mu(t_1|R) = q = \frac{1}{2}$$

SR2R: R: Best response is to play  $u$  as

$$\mathbb{E}[u_R(R, u|q=\frac{1}{2})] = 2\frac{1}{2} + 0\frac{1}{2} = 1$$

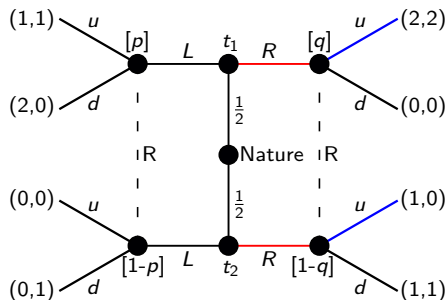
$$\mathbb{E}[u_R(R, d|q=\frac{1}{2})] = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$$

SR2S:  $t_1$  will not deviate even if  $a(L) = d$ :

$$u_S(R, u|t_1) = 2 \geq 2 = \max u_S(L, a(L)|t_1)$$

$t_2$  will not deviate as  $R|t_2$  strictly dominates  $L|t_2$ .

PBE: Find the off-equilibrium beliefs  $p$  to identify  $a(L|p)$  (possibly 2 for different  $p$ .)



$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

2. No PBE that includes  $(L, L)$ .

3.

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

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SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p \in [0, 1] \text{ and } \mu(t_1|R) = q = \frac{1}{2}$$

SR2R: R: Best response is to play  $u$  as

$$\mathbb{E}[u_R(R, u|q=\frac{1}{2})] = 2\frac{1}{2} + 0\frac{1}{2} = 1$$

$$\mathbb{E}[u_R(R, d|q=\frac{1}{2})] = 0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$$

SR2S:  $t_1$  will not deviate even if  $a(L) = d$ :

$$u_S(R, u|t_1) = 2 \geq 2 = \max u_S(L, a(L)|t_1)$$

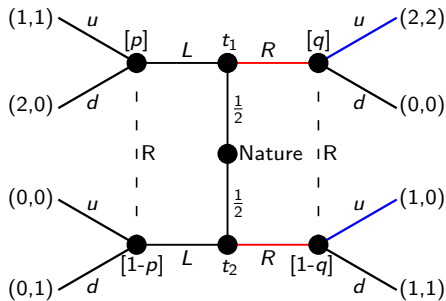
$t_2$  will not deviate as  $R|t_2$  strictly dominates  $L|t_2$ .

PBE: Find the off-equilibrium beliefs  $p$  to identify (two different)  $a(L|p)$ :

$$\mathbb{E}[u_R(L, u|p) \geq \mathbb{E}[u_R(L, d|p)$$

$$1p + 0[1 - p] \geq 0p + 1[1 - p]$$

$$p \geq \frac{1}{2}$$



1.  $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$
2. No PBE that includes  $(L, L)$ .
3. **Write up all PBE including  $(R, R)$ .**



## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.

Step 2: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 3: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:  
 $\mu(t_1|L) = p \in [0, 1]$  and  $\mu(t_1|R) = q = \frac{1}{2}$

SR2R: R: Best response is to play  $u$  as  
 $\mathbb{E}[u_R(R, u|q=\frac{1}{2})] = 2\frac{1}{2} + 0\frac{1}{2} = 1$   
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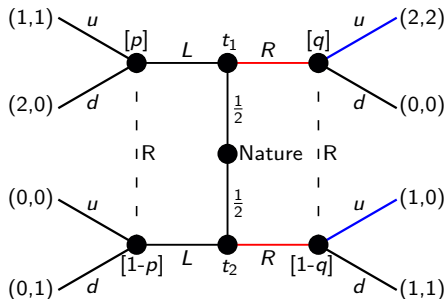
SR2S:  $t_1$  will not deviate even if  $a(L) = d$ :  
 $u_S(R, u|t_1) = 2 \geq 2 = \max u_S(L, a(L)|t_1)$   
 $t_2$  will not deviate as  $R|t_2$  strictly dominates  $L|t_2$ .

PBE: Find the off-equilibrium beliefs  $p$  to identify (two different)  $a(L|p)$ :

$$\mathbb{E}[u_R(L, u|p) \geq \mathbb{E}[u_R(L, d|p)]$$

$$1p + 0[1 - p] \geq 0p + 1[1 - p]$$

$$p \geq \frac{1}{2}$$



$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

2. No PBE that includes  $(L, L)$ .

$$3. \left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$$

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

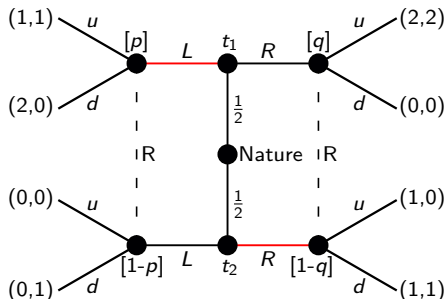
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Step 3: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 4: **For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.**



- $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$
- No PBE that includes  $(L, L)$ .
- $\left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

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Step 4: For the separating strategy  $(L,R)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = 1 \text{ and } \mu(t_1|R) = q = 0$$

SR2R: R: Best response is to play  $u|L, d|R$ .

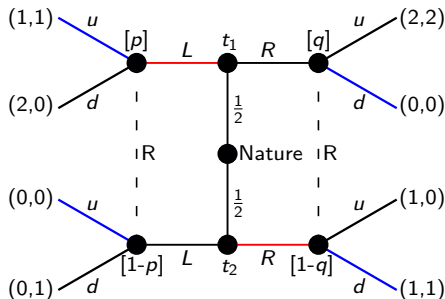
SR2S:  $t_1$  will not deviate as

$$u_S(L, u|t_1) = 1 > 0 = u_S(R, d|t_1)$$

$t_2$  will not deviate as

$$u_S(R, d|t_2) = 1 > 0 = u_S(L, u|t_2)$$

PBE: No deviation, thus, it's a PBE.



$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

2. No PBE that includes  $(L, L)$ .

$$3. \left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$$

$$4. \left\{ (L, R), (u, d), p = 1, q = 0 \right\}$$

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

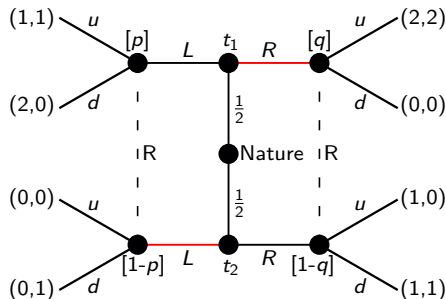
Step 1: Write up S's possible strategies.

Step 2: For the pooling strategy  $(L,L)$ , go over SR3, SR2R, and SR2S.

Step 3: For the pooling strategy  $(R,R)$ , go over SR3, SR2R, and SR2S.

Step 4: For the separating strategy  $(L,R)$ , go over SR3, SR2R, and SR2S.

Step 5: **For the separating strategy  $(R,L)$ , go over SR3, SR2R, and SR2S.**



$$1. S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

2. No PBE that includes  $(L, L)$ .

$$3. \left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$$

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## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

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Step 4: For the separating strategy  $(L,R)$ , go over SR3, SR2R, and SR2S.

Step 5: For the separating strategy  $(R,L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = 0 \text{ and } \mu(t_1|R) = q = 1$$

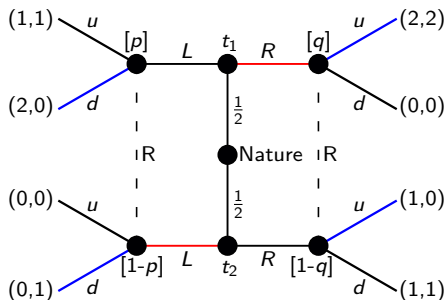
SR2R: R: Best response is to play  $d|L, u|R$ .

SR2S:  $t_2$  wants to deviate as

$$u_S(L, d|t_2) = 0 < 1 = u_S(R, u|t_2)$$

PBE: No PBE as  $t_2$  will want to deviate.

Step 6: **Write up the full set of PBE.**



1.  $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$
2. No PBE that includes  $(L, L)$ .
3.  $\left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$
4.  $\left\{ (L, R), (u, d), p = 1, q = 0 \right\}$
5. No PBE that includes  $(R, L)$ .

## PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

Step 1: Write up S's possible strategies.

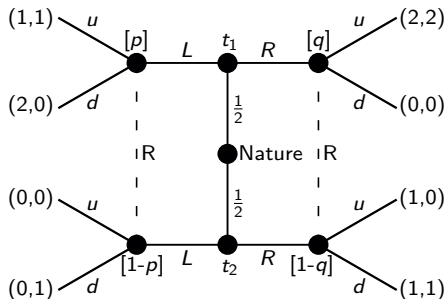
Step 2: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 3: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 4: For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.

Step 5: For the separating strategy  $(R, L)$ , go over SR3, SR2R, and SR2S:

Step 6: Write up the full set of PBE.



1.  $S_S = \{(L, L); (R, R); (L, R); (R, L)\}$

2. No PBE that includes  $(L, L)$ .

3.  $\left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \end{array} \right\}$

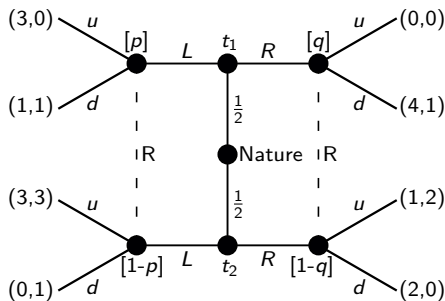
4.  $\left\{ (L, R), (u, d), p = 1, q = 0 \right\}$

5. No PBE that includes  $(R, L)$ .

6.  $\left\{ \begin{array}{l} (R, R), (u, u), p \geq \frac{1}{2}, q = \frac{1}{2} \\ (R, R), (d, u), p \leq \frac{1}{2}, q = \frac{1}{2} \\ (L, R), (u, d), p = 1, q = 0 \end{array} \right\}$

## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

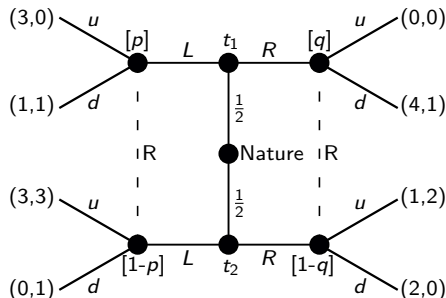


# PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider  $S$ 's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$





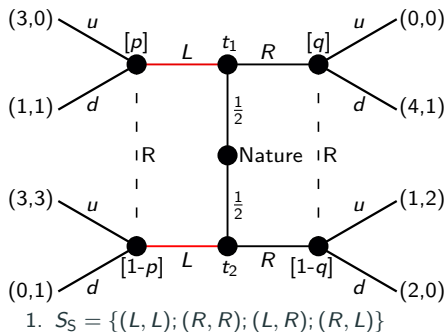
## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: **For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.**



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = \frac{1}{2} \text{ and } \mu(t_1|R) = q \in [0, 1]$$

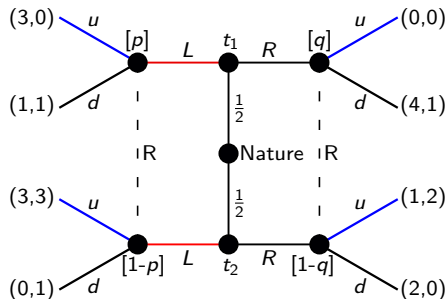
SR2R: R: Best response is to play  $u|L$  as

$$\mathbb{E}[u_R(L, u|p=\frac{1}{2})] = 0\frac{1}{2} + 3\frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}[u_R(L, d|p=\frac{1}{2})] = 1\frac{1}{2} + 1\frac{1}{2} = 1$$

SR2S:  $t_1, t_2$  will not deviate if R plays  $u|R$ .

PBE: **So, now what?**



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = \frac{1}{2} \text{ and } \mu(t_1|R) = q \in [0, 1]$$

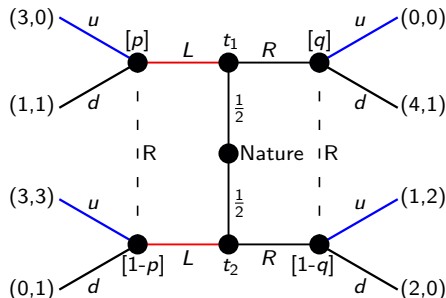
SR2R: R: Best response is to play  $u|L$  as

$$\mathbb{E}[u_R(L, u|p=\frac{1}{2})] = 0\frac{1}{2} + 3\frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}[u_R(L, d|p=\frac{1}{2})] = 1\frac{1}{2} + 1\frac{1}{2} = 1$$

SR2S:  $t_1, t_2$  will not deviate if R plays  $u|R$ .

PBE: Find values of  $q$  such that the receiver plays  $u|R$ .



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = \frac{1}{2} \text{ and } \mu(t_1|R) = q \in [0, 1]$$

SR2R: R: Best response is to play  $u|L$  as

$$\mathbb{E}[u_R(L, u|p=\frac{1}{2})] = 0\frac{1}{2} + 3\frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}[u_R(L, d|p=\frac{1}{2})] = 1\frac{1}{2} + 1\frac{1}{2} = 1$$

SR2S:  $t_1, t_2$  will not deviate if R plays  $u|R$ .

PBE: Find values of  $q$  such that the receiver plays  $u|R$ :

$$\mathbb{E}[u_R(R, u|q) \geq \mathbb{E}[u_R(R, d|q)]$$

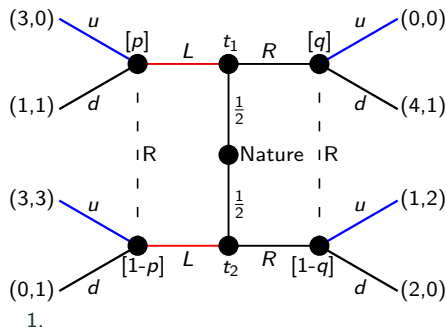
$$0q + 2[1 - q] \geq 1q + 0[1 - q]$$

$$2 - 2q \geq q$$

$$2 \geq 3q$$

$$\frac{2}{3} \geq q$$

Write up the PBE including beliefs.



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = \frac{1}{2} \text{ and } \mu(t_1|R) = q \in [0, 1]$$

SR2R: R: Best response is to play  $u|L$  as

$$\mathbb{E}[u_R(L, u|p=\frac{1}{2})] = 0\frac{1}{2} + 3\frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}[u_R(L, d|p=\frac{1}{2})] = 1\frac{1}{2} + 1\frac{1}{2} = 1$$

SR2S:  $t_1, t_2$  will not deviate if R plays  $u|R$ .

PBE: Find values of  $q$  such that the receiver plays  $u|R$ :

$$\mathbb{E}[u_R(R, u|q) \geq \mathbb{E}[u_R(R, d|q)]$$

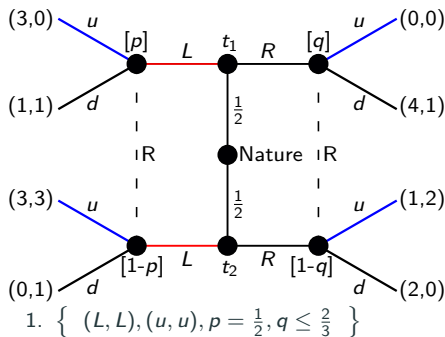
$$0q + 2[1 - q] \geq 1q + 0[1 - q]$$

$$2 - 2q \geq q$$

$$2 \geq 3q$$

$$\frac{2}{3} \geq q$$

Write up the PBE including beliefs.



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

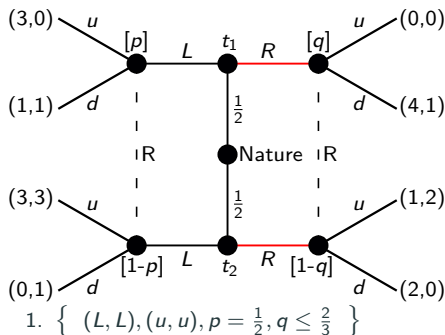
Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: **For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.**



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

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Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p \in [0, 1] \text{ and } \mu(t_1|R) = q = \frac{1}{2}$$

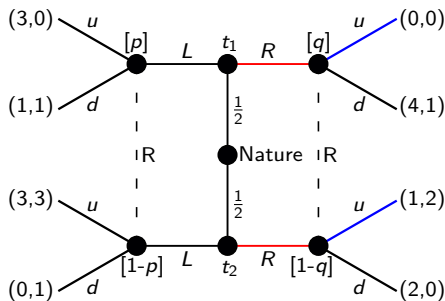
SR2R: R: Best response is to play  $u$  as

$$\mathbb{E}[u_R(R, u|q=\frac{1}{2})] = 0\frac{1}{2} + 2\frac{1}{2} = 1$$

$$\mathbb{E}[u_R(R, d|q=\frac{1}{2})] = 1\frac{1}{2} + 0\frac{1}{2} = \frac{1}{2}$$

SR2S:  $t_1$  will deviate as the payoff from  $(L, a(L)|t_1)$  is strictly higher than  $(R, u|t_1) = 0$ .

PBE: No PBE, as  $t_1$  wants to deviate.



- $\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \}$
- No PBE that includes  $(R, R)$ .

## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

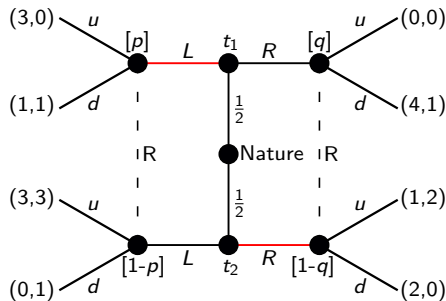
- Consider  $S$ 's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 3: **For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.**



- $\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \}$
- No PBE that includes  $(R, R)$ .



## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 3: For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = 1 \text{ and } \mu(t_1|R) = q = 0$$

SR2R: R: Best response is to play  $d|L, u|R$ .

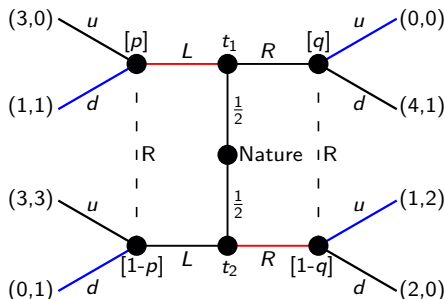
SR2S:  $t_1$  will not deviate as

$$u_S(L, d|t_1) = 1 > 0 = u_S(R, u|t_1)$$

$t_2$  will not deviate as

$$u_S(R, u|t_2) = 1 > 0 = u_S(L, d|t_2)$$

PBE: No deviation, thus, it's a PBE.



- $\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \}$
- No PBE that includes  $(R, R)$ .
- $\{ (L, R), (d, u), p = 1, q = 0 \}$

## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

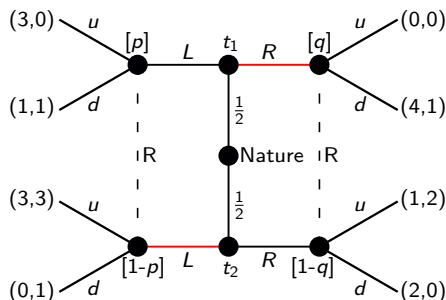
$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 3: For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.

Step 4: **For the separating strategy  $(R, L)$ , go over SR3, SR2R, and SR2S.**



- $\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \}$
- No PBE that includes  $(R, R)$ .
- $\{ (L, R), (d, u), p = 1, q = 0 \}$

## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 3: For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.

Step 4: For the separating strategy  $(R, L)$ , go over SR3, SR2R, and SR2S:

SR3: R: Beliefs given S's eq. strategy:

$$\mu(t_1|L) = p = 0 \text{ and } \mu(t_1|R) = q = 1$$

SR2R: R: Best response is to play  $u|L, d|R$ .

SR2S:  $t_1$  will not deviate as

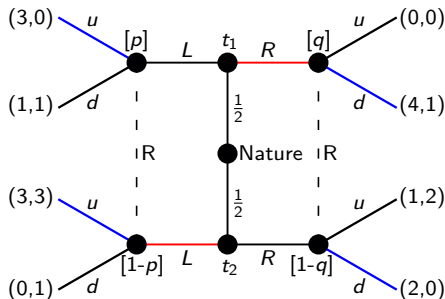
$$u_S(R, d|t_1) = 4 > 3 = u_S(L, u|t_1)$$

$t_2$  will not deviate as

$$u_S(L, u|t_2) = 3 > 2 = u_S(R, d|t_2)$$

PBE: No deviation, thus, it's a PBE.

Step 5: **Write up the full set of PBE.**



- $\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \}$
- No PBE that includes  $(R, R)$ .
- $\{ (L, R), (d, u), p = 1, q = 0 \}$
- $\{ (R, L), (u, d), p = 0, q = 1 \}$

## PS11, Ex. 4.b: Signaling games (pooling and separating PBE)

Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.

- Consider S's possible strategies:

$$S_S = \{(L, L); (R, R); (L, R); (R, L)\}$$

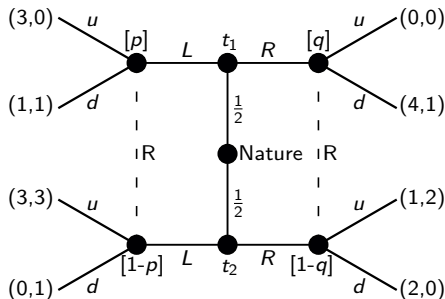
Step 1: For the pooling strategy  $(L, L)$ , go over SR3, SR2R, and SR2S.

Step 2: For the pooling strategy  $(R, R)$ , go over SR3, SR2R, and SR2S.

Step 3: For the separating strategy  $(L, R)$ , go over SR3, SR2R, and SR2S.

Step 4: For the separating strategy  $(R, L)$ , go over SR3, SR2R, and SR2S:

Step 5: Write up the full set of PBE.



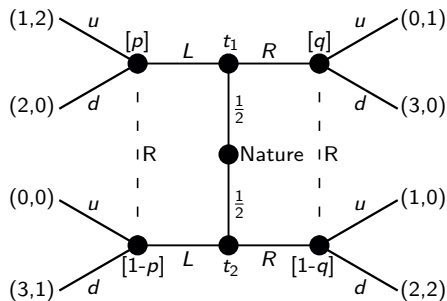
- $\left\{ (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \right\}$
- No PBE that includes  $(R, R)$ .
- $\left\{ (L, R), (d, u), p = 1, q = 0 \right\}$
- $\left\{ (R, L), (u, d), p = 0, q = 1 \right\}$
- $\left\{ \begin{array}{l} (L, L), (u, u), p = \frac{1}{2}, q \leq \frac{2}{3} \\ (L, R), (d, u), p = 1, q = 0 \\ (R, L), (u, d), p = 0, q = 1 \end{array} \right\}$

**PS11, Ex. 5: Signaling games  
(pooling PBE)**

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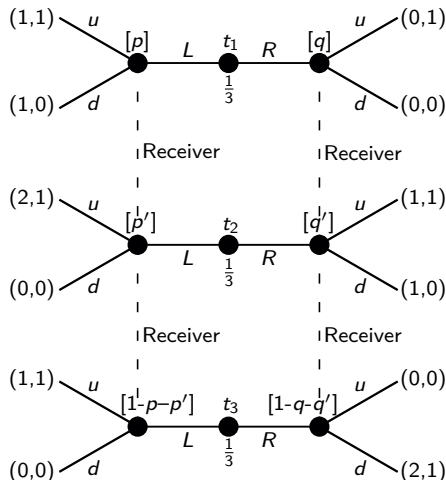
## PS11, Ex. 5.a: Signaling games (pooling PBE)

Exercise 4.3.a in Gibbons (p. 246). Specify a pooling perfect Bayesian equilibria in which both Sender types play  $R$  in the following signaling game.



## PS11, Ex. 5.b: Signaling games (pooling PBE)

Exercise 4.3.b in Gibbons (p. 246). The following three-type signaling game begins with a move by nature, not shown in the tree, that yields one of the three types with equal probability. Specify a pooling perfect Bayesian equilibria in which all three Sender types play  $L$ .



**PS11, Ex. 6: Spence's education  
signaling model (PBE)**

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## PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers is characterized by their type  $\theta$ , which measures their ability. There are two worker types:  $\theta \in \{\theta_L, \theta_H\}$ . Nature chooses the worker's type, with  $p_H = \mathbb{P}[\theta = \theta_H]$  and  $p_H = \mathbb{P}[\theta = \theta_H] = 1 - p_H$ .

The worker observes his own type, but the firm does not. The worker can choose his level of education:  $e \in \mathbb{R}^+$ . The cost to him of acquiring this education is  $c_\theta(e) = e/\theta$ . Education is observed by the firm, who then forms beliefs about the workers type:  $\mu(\theta|e)$ . We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity:  $w(e) = \mathbb{E}[\theta|e]$ . Thus, the payoff to a worker conditional on his type and education is  $u_\theta(e) = w(e) - c_\theta(e)$ . Suppose for this exercise that  $\theta_H = 3$  and  $\theta_L = 1$ .

- (a) Find a separating pure strategy Perfect Bayesian Equilibrium.
- (b) Find a pooling pure strategy Perfect Bayesian Equilibrium.

## PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type  $\theta$ , which measures their ability. There are two worker types:  $\theta \in \{\theta_L, \theta_H\}$ . Nature chooses the worker's type, with  $p_H = \mathbb{P}[\theta = \theta_H]$  and  $p_L = \mathbb{P}[\theta = \theta_L] = 1 - p_H$ .

The worker observes his own type, but the firm does not. The worker can choose his level of education:  $e \in \mathbb{R}^+$ . The cost to him of acquiring this education is  $c_\theta(e) = e/\theta$ . Education is observed by the firm, who then forms beliefs about the worker's type:  $\mu(\theta|e)$ . We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity:  $w(e) = \mathbb{E}[\theta|e]$ . Thus, the payoff to a worker conditional on his type and education is  $u_\theta(e) = w(e) - c_\theta(e)$ . Suppose for this exercise that  $\theta_H = 3$  and  $\theta_L = 1$ .

- (a) Find a separating pure strategy PBE.
- (b) Find a pooling pure strategy PBE.

**Given the firm expects a worker to be type  $\theta_L$  and pays  $\theta_L = 1$ , find the utility maximizing education level for each type:**

$$\text{Type } \theta_L: \max_{e_L} u_{\theta_L}(e_L)$$

$$\text{Type } \theta_H: \max_{e_L} u_{\theta_H}(e_L)$$

## PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type  $\theta$ , which measures their ability. There are two worker types:  $\theta \in \{\theta_L, \theta_H\}$ . Nature chooses the worker's type, with  $p_H = \mathbb{P}[\theta = \theta_H]$  and  $p_L = \mathbb{P}[\theta = \theta_L] = 1 - p_H$ .

The worker observes his own type, but the firm does not. The worker can choose his level of education:  $e \in \mathbb{R}^+$ . The cost to him of acquiring this education is  $c_\theta(e) = e/\theta$ . Education is observed by the firm, who then forms beliefs about the worker's type:  $\mu(\theta|e)$ . We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity:  $w(e) = \mathbb{E}[\theta|e]$ . Thus, the payoff to a worker conditional on his type and education is  $u_\theta(e) = w(e) - c_\theta(e)$ . Suppose for this exercise that  $\theta_H = 3$  and  $\theta_L = 1$ .

(a) Find a separating pure strategy PBE.

(b) Find a pooling pure strategy PBE.

Given the firm expects a worker to be type  $\theta_L$  and pays  $\theta_L = 1$ , find the utility maximizing education level for each type:

$$\text{Type } \theta_L: \max_{e_L} u_{\theta_L}(e_L) = \max_{e_L} 1 - e_L \Rightarrow e_L = 0$$

$$\text{Type } \theta_H: \max_{e_L} u_{\theta_H}(e_L) = \max_{e_L} 1 - \frac{e_L}{3} \Rightarrow e_L = 0$$

**Write up the firm's profit from  $h$  (hiring) or  $n$  (not hiring) for each type sending a signal of either  $e_H$  or  $e_L$ .**

## PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type  $\theta$ , which measures their ability. There are two worker types:  $\theta \in \{\theta_L, \theta_H\}$ . Nature chooses the worker's type, with  $p_H = \mathbb{P}[\theta = \theta_H]$  and  $p_L = \mathbb{P}[\theta = \theta_L] = 1 - p_H$ .

The worker observes his own type, but the firm does not. The worker can choose his level of education:  $e \in \mathbb{R}^+$ . The cost to him of acquiring this education is  $c_\theta(e) = e/\theta$ . Education is observed by the firm, who then forms beliefs about the worker's type:  $\mu(\theta|e)$ . We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity:  $w(e) = \mathbb{E}[\theta|e]$ . Thus, the payoff to a worker conditional on his type and education is  $u_\theta(e) = w(e) - c_\theta(e)$ . Suppose for this exercise that  $\theta_H = 3$  and  $\theta_L = 1$ .

(a) Find a separating pure strategy PBE.

(b) Find a pooling pure strategy PBE.

Given the firm expects a worker to be type  $\theta_L$  and pays  $\theta_L = 1$ , find the utility maximizing education level for each type:

$$\text{Type } \theta_L: \max_{e_L} u_{\theta_L}(e_L) = \max_{e_L} 1 - e_L \Rightarrow e_L = 0$$

$$\text{Type } \theta_H: \max_{e_L} u_{\theta_H}(e_L) = \max_{e_L} 1 - \frac{e_L}{3} \Rightarrow e_L = 0$$

$$\pi_F = \begin{cases} -2 & \text{for } h|e_H(\theta_L) \\ 0 & \text{for } n \vee h|e_H(\theta_H) \vee h|e_L(\theta_L) \\ 2 & \text{for } h|e_L(\theta_H) \end{cases}$$

**Draw the extensive form of this**

**signaling game where either type of worker sends the signal of taking an education of level  $e_L = 0$  or  $e_H > 0$ .**

**Observing the signal, the firm  $F$  forms their beliefs and choose  $h$  (hire and pay the wage according to the education level) or  $n$  (not hire).**

## PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type  $\theta$ , which measures their ability. There are two worker types:  $\theta \in \{\theta_L, \theta_H\}$ . Nature chooses the worker's type, with  $p_H = \mathbb{P}[\theta = \theta_H]$  and  $p_L = \mathbb{P}[\theta = \theta_L] = 1 - p_H$ .

The worker observes his own type, but the firm does not. The worker can choose his level of education:  $e \in \mathbb{R}^+$ . The cost to him of acquiring this education is  $c_\theta(e) = e/\theta$ . Education is observed by the firm, who then forms beliefs about the worker's type:  $\mu(\theta|e)$ . We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity:  $w(e) = \mathbb{E}[\theta|e]$ . Thus, the payoff to a worker conditional on his type and education is  $u_\theta(e) = w(e) - c_\theta(e)$ . Suppose for this exercise that  $\theta_H = 3$  and  $\theta_L = 1$ .

(a) Find a separating pure strategy PBE.

(b) Find a pooling pure strategy PBE.

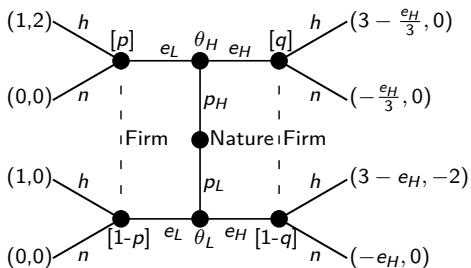
Given the firm expects a worker to be type  $\theta_L$  and pays  $\theta_L = 1$ , find the utility maximizing education level for each type:

$$\text{Type } \theta_L: \max_{e_L} u_{\theta_L}(e_L) = \max_{e_L} 1 - e_L \Rightarrow e_L = 0$$

$$\text{Type } \theta_H: \max_{e_L} u_{\theta_H}(e_L) = \max_{e_L} 1 - \frac{e_L}{3} \Rightarrow e_L = 0$$

$$\pi_F = \begin{cases} -2 & \text{for } h|e_H(\theta_L) \\ 0 & \text{for } n \vee h|e_H(\theta_H) \vee h|e_L(\theta_L) \\ 2 & \text{for } h|e_L(\theta_H) \end{cases}$$

Draw the extensive form:

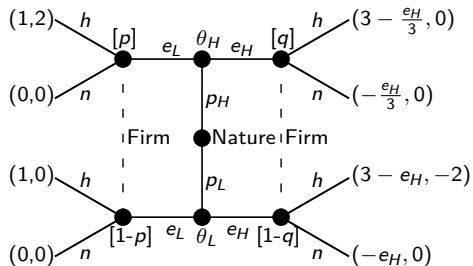


Now, solve question (a) and (b).

# PS11, Ex. 6.a: Spence's education signaling model (PBE)

(a) Find a separating pure strategy PBE.

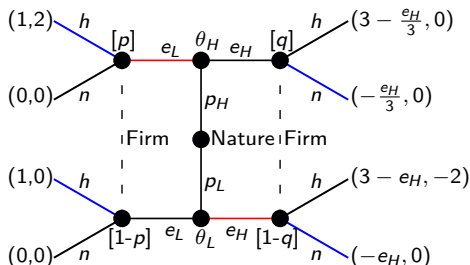
Step 1: **Looking at the game tree, which separating PBE are not viable?**



## PS11, Ex. 6.a: Spence's education signaling model (PBE)

(a) Find a separating pure strategy PBE.

Step 1: Looking at the game tree, which separating PBE are not viable?



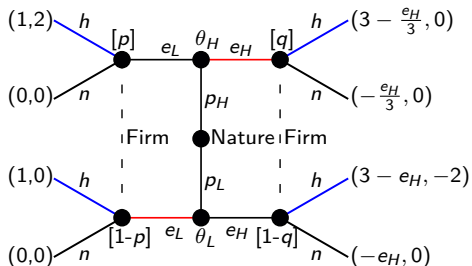
1.  $(e_L, e_H)$  is not viable, as  $F$  would not hire  $\theta_L$  despite his high education level  $e_H$ . Nor is any other PBE where either type of worker is not hired as he would then want to deviate.

## PS11, Ex. 6.a: Spence's education signaling model (PBE)

(a) Find a separating pure strategy PBE.

Step 1: Looking at the game tree, which separating PBE are not viable?

Step 2: **Instead, go over SR3, SR2R and SR2S for the PBE candidate  $((e_H, e_L), (h, h))$ .**



1.  $(e_L, e_H)$  is not viable, as  $F$  would not hire  $\theta_L$  despite his high education level  $e_H$ . Nor is any other PBE where either type of worker is not hired as he would then want to deviate.



# PS11, Ex. 6.a: Spence's education signaling model (PBE)

(a) Find a separating pure strategy PBE.

Step 1: Looking at the game tree, which separating PBE are not viable?

Step 2: Go over SR3, SR2R and SR2S for the PBE candidate  $((e_H, e_L), (h, h))$ :

SR3: F: Beliefs given worker's strategy:

$$\mu(\theta_H|e_L) = p = 0 \text{ and } \mu(\theta_H|e_H) = q = 1$$

SR2R: F: Is indifferent between  $h$  and  $n$ .

SR2S: Type  $\theta_H$  will not deviate when

$$u_{\theta_H}(e_H, h) \geq u_{\theta_H}(e_L, h)$$

$$3 - \frac{e_H}{3} \geq 1$$

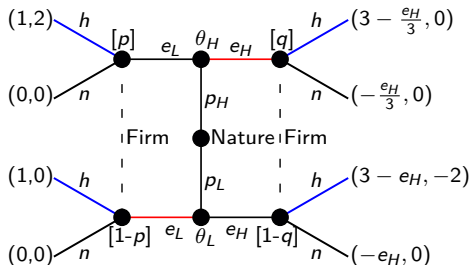
$$e_H \leq 6$$

Type  $\theta_L$  will not deviate when

$$u_{\theta_L}(e_L, h) \geq u_{\theta_L}(e_H, h)$$

$$1 \geq 3 - e_H$$

$$e_H \geq 2$$



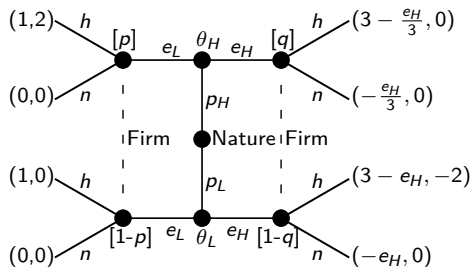
1.  $(e_L, e_H)$  is not viable, as  $F$  would not hire  $\theta_L$  despite his high education level  $e_H$ . Nor is any other PBE where either type of worker is not hired as he would then want to deviate.

2. No deviation for  $e_H \in [2, 6]$ . It's optimal for  $\theta_H$  to choose  $e_H = 2$  as it's sufficient for credibly signaling his type. Above 2, worker's marginal effect of education is negative.

$$PBE = \{(e_H = 2, e_L = 0), (h, h), p = 0, q = 1\}_{73}$$

(b) Find a pooling pure strategy PBE.

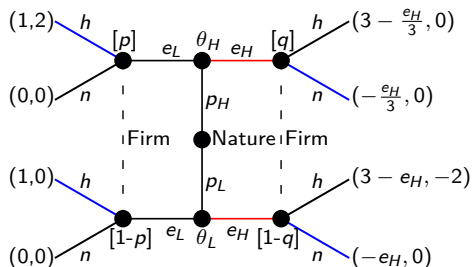
Step 1: **Looking at the game tree, which pooling PBE is not viable?**



# PS11, Ex. 6.b: Spence's education signaling model (PBE)

(b) Find a pooling pure strategy PBE.

Step 1: Looking at the game tree, which pooling PBE is not viable?



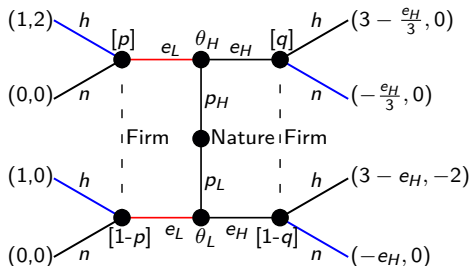
1.  $(e_H, e_H)$  is not viable, as  $F$  would not hire unless the probability of type  $\theta_L$  is  $p_L = 0$ .

## PS11, Ex. 6.b: Spence's education signaling model (PBE)

(b) Find a pooling pure strategy PBE.

Step 1: Looking at the game tree, which pooling PBE is not viable?

Step 2: **Instead, go over SR3, SR2R and SR2S for the PBE candidate  $((e_L, e_L), (h, n))$ .**



1.  $(e_H, e_H)$  is not viable, as  $F$  would not hire unless the probability of type  $\theta_L$  is  $p_L = 0$ .

## PS11, Ex. 6.b: Spence's education signaling model (PBE)

(b) Find a pooling pure strategy PBE.

Step 1: Looking at the game tree, which pooling PBE is not viable?

Step 2: Go over SR3, SR2R and SR2S for the PBE candidate  $((e_L, e_L), (h, n))$ :

SR3: F: Beliefs given worker's strategy:

$$\mu(\theta_H|e_L) = p_H \text{ and } \mu(\theta_H|e_H) = q \in [0, 1]$$

SR2R: F:  $(h, n)$  is strictly dominant except for  $p_H = 0, q = 1$  where it's weakly dominant.

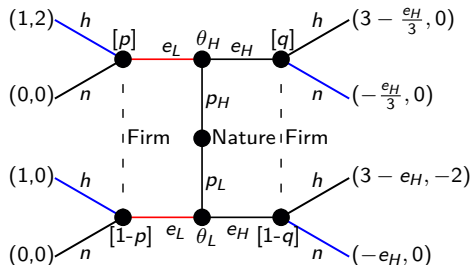
SR2S: Type  $\theta_H$  will not deviate as

$$u_{\theta_H}(e_L, h) = 1 > -\frac{e_H}{3} = u_{\theta_H}(e_H, n)$$

Type  $\theta_L$  will not deviate when

$$u_{\theta_L}(e_L, h) = 1 > -e_H = u_{\theta_L}(e_H, n)$$

Step 3: **Explain: Which 2 assumptions make it possible to have an equilibrium where both high-ability and low-ability workers take zero education?**



1.  $(e_H, e_H)$  is not viable, as  $F$  would not hire unless the probability of type  $\theta_L$  is  $p_L = 0$ .

2. No deviation, thus, we have a PBE:

$$\{(e_L = 0, e_L = 0), (h, n), p = p_H, q \in [0, 1]\}$$

## PS11, Ex. 6.b: Spence's education signaling model (PBE)

(b) Find a pooling pure strategy PBE.

Step 1: Looking at the game tree, which pooling PBE is not viable?

Step 2: Go over SR3, SR2R and SR2S for the PBE candidate  $((e_L, e_H), (h, n))$ :

SR3: F: Beliefs given worker's strategy:

$$\mu(\theta_H|e_L) = p_H \text{ and } \mu(\theta_H|e_H) = q \in [0, 1]$$

SR2R: F:  $(h, n)$  is strictly dominant except for  $p_H = 0, q = 1$  where it's weakly dominant.

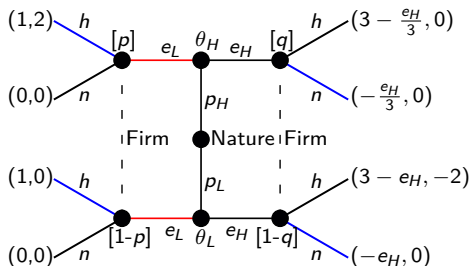
SR2S: Type  $\theta_H$  will not deviate as

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Type  $\theta_L$  will not deviate when

$$u_{\theta_L}(e_L, h) = 1 > -e_H = u_{\theta_L}(e_H, n)$$

Step 3: Explain: Which 2 assumptions make it possible to have an eq. where both high-ability and low-ability workers take zero education?



1.  $(e_H, e_H)$  is not viable, as  $F$  would not hire unless the probability of type  $\theta_L$  is  $p_L = 0$ .

2. No deviation, thus, we have a PBE:

$$\{(e_L = 0, e_L = 0), (h, n), p = p_H, q \in [0, 1]\}$$

3.i Education is unproductive; it only serves as a signal of one's ability.

3.ii Under competition,  $F$  pays marginal productivity and is indifferent between  $n$ ,  $h(e_L)|\theta_L$ , and  $h(e_H)|\theta_H$ . Thus,  $F$  has no reason to run the risk of overpaying  $\theta_L$  imitating  $\theta_H$ .