Microeconomics III: Problem Set 7[a]

Thor Donsby Noe (thor.noe@econ.ku.dk) & Christopher Borberg (christopher.borberg@econ.ku.dk)

November 6 2019

Department of Economics, University of Copenhagen

---

[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS7, Ex. 1 (A): Imperfect recall (imperfect information)

PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

PS7, Ex. 3 (A): A finite repeated game with one NE in the stagegame

PS7, Ex. 4:

PS7, Ex. 5:

PS7, Ex. 6:

PS7, Ex. 7:

PS7, Ex. 8: Trigger strategy (infinitely repeated game)

PS7, Ex. 9: Optimal Punishment Strategy (infinitely repeated game)

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Code examples

PS7, Ex. 1 (A): Imperfect recall (imperfect information)

PS7, Ex. 1 (A): Imperfect recall (imperfect information)

In this course we normally consider games in which there is 'perfect recall': players can always remember what they themselves have done in the past.

We have seen an example in class of a game with 'imperfect recall' where the player forgets his own actions. But what would a game where he forgets the opponent's actions look like? Construct a game with two players. The timing is as follows: Player 1 moves first, then Player 2, and then Player 2 again. Everytime they move, the players choose one of two actions: $\{L, R\}$.

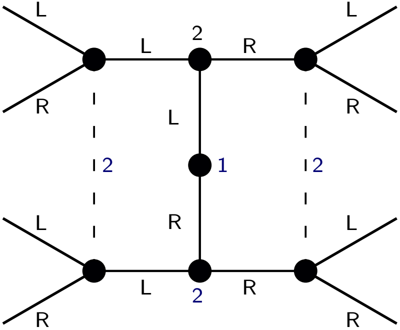
Draw the game tree and construct the information sets such that (a) Player 2 observes Player 1's action the first time he moves, but (b) when Player 2 moves the second time, he has forgotten what Player 1 chose. However, he recalls his own action.

PS7, Ex. 1 (A): Imperfect recall (imperfect information)

In this course we normally consider games in which there is 'perfect recall': players can always remember what they themselves have done in the past.

We have seen an example in class of a game with 'imperfect recall' where the player forgets his own actions. But what would a game where he forgets the opponent's actions look like? Construct a game with two players. The timing is as follows: Player 1 moves first, then Player 2, and then Player 2 again. Everytime they move, the players choose one of two actions: $\{L, R\}$.

Draw the game tree and construct the information sets such that (a) Player 2 observes Player 1's action the first time he moves, but (b) when Player 2 moves the second time, he has forgotten what Player 1 chose. However, he recalls his own action.



PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

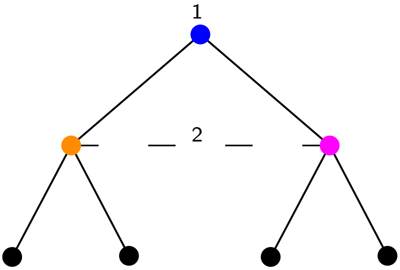
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.

Example of violation of condition 1:



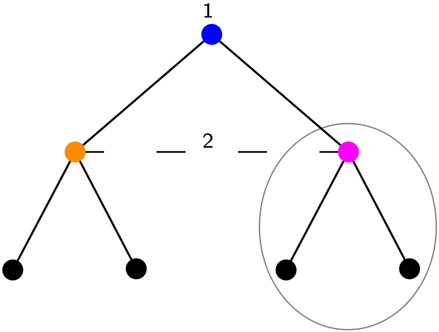
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.

Example of violation of condition 1:



The purple decision node to the right is not a singleton information set (nor is the orange decision node to the left).

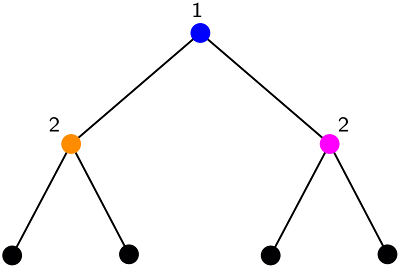
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. **It includes all following decision and terminal nodes following *n* in the game tree**, but no nodes that do not follow *n*.

Example of violation of the first part of condition 2:



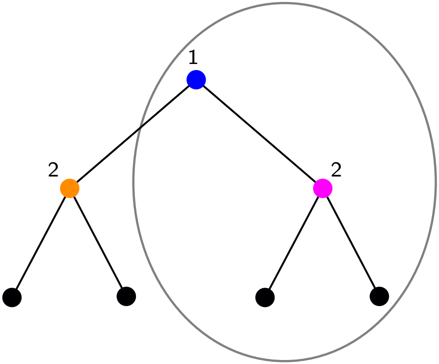
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. **It includes all following decision and terminal nodes following *n* in the game tree**, but no nodes that do not follow *n*.

Example of violation of the first part of condition 2:



For a subgame containing the blue decision node *n*, all following decision nodes must be included.

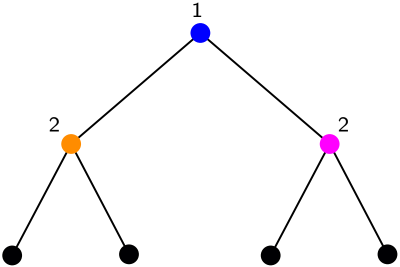
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, **but no nodes that do not follow *n*.**

Example of violation of the second part of condition 2:



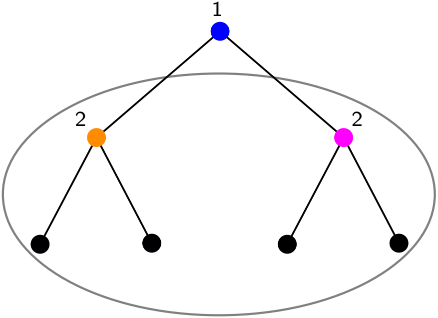
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, **but no nodes that do not follow *n*.**

Example of violation of the second part of condition 2:



Regardless of whether the orange or the purple node is chosen as the first decision node *n*, the other decision node does not follow *n*, and therefore cannot be part of the subgame.

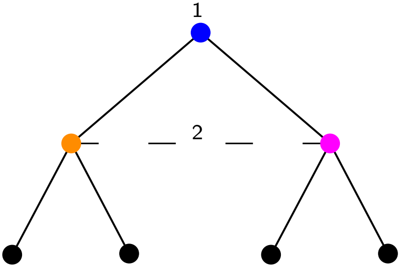
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node n that is a singleton information set.
2. It includes all following decision and terminal nodes following n in the game tree, but no nodes that do not follow n.
3. It does not "cut" any information set: if a decision node n' follows n in the game tree, then all other nodes in the information set including n' must also follow n (and so be included in the subgame).

Example of violation of condition 3:



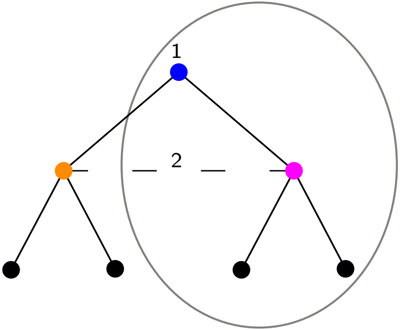
PS7, Ex. 2 (A): Three conditions for a subgame (imperfect information)

Recall that under imperfect information we have three conditions that define a subgame. Construct an example of a violation of each of the three conditions (pick different examples than those seen in the lectures).

Under imperfect information, a subgame must satisfy three properties:

1. It begins at a decision node *n* that is a singleton information set.
2. It includes all following decision and terminal nodes following *n* in the game tree, but no nodes that do not follow *n*.
3. It does not "cut" any information set: if a decision node n' follows *n* in the game tree, then all other nodes in the information set including n' must also follow *n* (and so be included in the subgame).

Example of violation of condition 3:



The orange decision node to the left is part of the same information set as the purple node to the right, so it must be included in the same subgame.

PS7, Ex. 3 (A): A finite repeated game with one NE in the stagegame

PS7, Ex. 3 (A):

Let G be the following game:

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

Consider the repeated game G(T), where G is repeated T times and the outcomes of each round are observed by both players before the next round.

(a) If $T = 2$, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the first round?

(b) What if $T = 42$?

PS7, Ex. 3.a (A):

(a) If T = 2, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the first round?

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

No. Since there is only one NE (A,D) which is not (B,C), that NE will be played in both games.

Explanation:
In the last round, an NE from the stage game must be played. In this case there is only one NE, which is (A,D). Knowing that (A,D) will be played no matter what in the second round, no player has an incentive to cooperate in the first turn. Player A will play his dominant strategy A and player B will play her dominant strategy D.

PS7, Ex. 3.b (A):

(b) If T $=$ 42, is there a Subgame Perfect Nash Equilibrium such that (B,C) is played during the first round?

		Player 2	
		C	D
Player 1	A	27, -3	0, 0
	B	6, 6	-2, 7

No. Since there is only one NE (A,D) which is not (B,C), that NE will be played in every turn of any finite game G(T).

Explanation:
In the last round, an NE from the stage game must be played. In this case there is only one NE, which is (A,D). Knowing that (A,D) will be played no matter what in the last round, no player has an incentive to cooperate in the round before that. This keeps applying until the players reach the first stage of the game. Thus, the NE (A,D) will be played in every turn of any finite game G(T).

PS7, Ex. 4:

PS7, Ex. 4:

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (B,Y). Make sure to write down the full equilibrium.

(b) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (C,Z). Make sure to write down the full equilibrium.

(c) Can you find a subgame perfect Nash equilibrium such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

PS7, Ex. 4.a:

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(a) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (B,Y). Make sure to write down the full equilibrium.

The first step is to find the NE in the stage game, in this case there are two: $\{(B, Y), (C, Z)\}$.
Knowing this, it is possible for the outcome of any stage of the game to be (B,Y). We can therefore choose the strategies such that (B,Y) will be the outcome of the first stage, and then either of the NE's can be the outcome of the second stage.
Using this information, write up the SPNE, keeping in mind that you need to write up a second stage strategy for each of the possible outcomes of the first stage (3*3 matrix, so 9 possible outcomes).

SPNE=$\{(BBBBBBBBBB, YYYYYYYYYY)\}$
or SPNE=$\{(BCCCCCCCCC, YZZZZZZZZZ)\}$

PS7, Ex. 4.b:

Consider the two times repeated game where the stage game is:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(b) Find a subgame perfect Nash equilibrium such that the outcome of the first stage is (C,Z). Make sure to write down the full equilibrium.

The first step is to find the NE in the stage game, in this case there are two: $\{(B, Y), (C, Z)\}$.
Knowing this, it is possible for the outcome of any stage of the game to be (C,Z). We can therefore choose the strategies such that (C,Z) will be the outcome of the first stage, and then either of the NE's can be the outcome of the second stage.
Using this information, write up the SPNE, keeping in mind that you need to write up a second stage strategy for each of the possible outcomes of the first stage (3*3 matrix, so 9 possible outcomes).

SPNE=$\{(CCCCCCCCCC, ZZZZZZZZZZ)\}$
or SPNE=$\{(CBBBBBBBBB, ZYYYYYYYYY)\}$

PS7, Ex. 4.c:

		Player 2		
		X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

(c) Can you find a SPNE such that the total payoffs that the players receive are 10 for player 1 and 11 for player 2? If yes, write down the full equilibrium.

First, find out which combination of outcomes would yield the payoff (10,11), under the restriction that the last stage must be be an NE:

Stage1:(A,X), Stage2:(C,Z), of which stage 2 is an NE so it doesn't need examination.
Now, look for a threatening strategy which if abode by will lead to the combination. Threatening that the players will go for the (Y,B) NE rather than (C,Z) in the second stage if (A,X) is not the outcome of stage 1.

Using the threat, we get that player 1 can choose between going along (6+4=10) and playing B in the first round (7+2=9). Player 2 can choose between going along (6+5=11) and playing Y in the first round (8+2=10). For both players going along yield a strictly higher payoff.
Now write up the strategy:

SPNE=$\{(ACBBBBBBBB, XZYYYYYYYY)\}$

PS7, Ex. 5:

PS7, Ex. 5:

Consider the situation of two flatmates. They both prefer having a clean kitchen, but cleaning is a tedious task, so that it is individually rational not to clean regardless of what the other does. This results in the following game G:

Player 1		Player 2: Cl	Player 2: DCl
	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

Now consider the situation where the two flatmates have to decide every day whether to clean or not, i.e. consider the infinitely repeated game $G(\infty, \delta)$

- (a) Define trigger strategies such that the outcome of all stages will be (Clean,Clean).
- (b) Find the lowest value of δ such that the trigger strategies from (b) constitute a SPNE in $G(\infty, \delta)$. Recall: you have to check for deviations both on and off the equilibrium path.

PS7, Ex. 5.a:

PS7, Ex. 6:

PS7, Ex. 6:

Consider again the the infinitely repeated game $G(\infty, \delta)$ with the stage game:

		Player 2	
		Cl	DCl
Player 1	Cl	4, 4	0, 6
	Dcl	5, 0	1, 1

- (a) Define a tit-for-tat strategy such that the outcome of all stages will be (Clean, Clean).
- (b) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Always play 'Do not clean''
- (c) Check for which δ tit-for-tat is optimal on the equilibrium path against the following strategy: 'Start by playing 'Do not clean', then play 'tit-for-tat' forever after that'.
- (d) Argue informally that 'tit-for-tat' is a NE for the appropriate values of δ. In particular, think about whether there are other deviations that would be better for the players.

When we say "against", it doesn't mean that the other player is playing the "against" strategy. It means to compare the two strategies, in this case "on the equilibrium path", so if the other player is playing "tit-for-tat"

PS7, Ex. 6.a:

PS7, Ex. 7:

PS7, Ex. 7:

PS7, Ex. 7.a:

PS7, Ex. 8: Trigger strategy (infinitely repeated game)

PS7, Ex. 8: Trigger strategy (infinitely repeated game)

The next exercises use the following game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

Suppose that the Players play the infinitely repeated game $G(\infty)$ and that they would like to support as a SPNE the 'collusive' outcome in which (L, L) is played every round.

(a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

PS7, Ex. 8.a: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

PS7, Ex. 8.a: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

(Step a) Write up the trigger strategy.

PS7, Ex. 8.a: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(a) Define a trigger strategy which delivers the collusive outcome in every period where no deviation has been made, and gives (x_1, x_2) forever after a deviation.

(Step a) Write up the trigger strategy.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."

PS7, Ex. 8.b: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."

PS7, Ex. 8.b: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(Step a) Find the PSNE in the stage game G.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."

PS7, Ex. 8.b: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(Step a) Find the PSNE in the stage game G.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."
2. Stage game NE: (M, M).

PS7, Ex. 8.b: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(Step a) Find the PSNE in the stage game G.

(Step b) Explain.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."
2. Stage game NE: (M, M).

PS7, Ex. 8.b: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(b) A necessary (but not sufficient) condition for a SPNE is $x_1 = x_2 = M$. Explain why.

(Step a) Find the PSNE in the stage game G.

(Step b) Explain.

For a trigger strategy to constitute a SPNE, the threat of (eternal and unchangeable) punishment must be credible, i.e. must be a stage game NE.

Thus, $x_1 = x_2 = M$ is a necessary (but not sufficient) condition for the trigger strategies to constitute a SPNE.

Information so far:

1. Trigger strategy for Player $i \in 1, 2$: "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. Otherwise, play x_i."
2. Unique stage game NE: (M, M).

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

Information so far:

1. Trigger Strategy (TS): ”If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M.”

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

(Step b) Given Player j plays TS, write up Player i's respective payoffs from playing TS and ODS.

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

(Step b) Given Player j plays TS, write up Player i's respective payoffs from playing TS and ODS.

Player i's payoff from playing TS:

$$10 + 10\delta + 10\delta^2 + ... = \sum_{t=1}^{\infty} 10\delta^{t-1} = \frac{10}{1-\delta}$$

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."
3. $U_i(TS, TS) = \frac{10}{1-\delta}$.

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

(Step b) Given Player j plays TS, write up Player i's respective payoffs from playing TS and ODS.

Player i's payoff from playing TS:

$$10 + 10\delta + 10\delta^2 + ... = \sum_{t=1}^{\infty} 10\delta^{t-1} = \frac{10}{1-\delta}$$

Player i's payoff from playing ODS:

$$15 + 7\delta + 7\delta^2 + ... = 15 + \sum_{t=2}^{\infty} 7\delta^{t-1} = 15 + \frac{7\delta}{1-\delta}$$

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."
3. $U_i(TS, TS) = \frac{10}{1-\delta}$
4. $U_i(ODS, TS) = 15 + \frac{7\delta}{1-\delta}$

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

(Step b) Given Player j plays TS, write up Player i's respective payoffs from playing TS and ODS.

(Step c) Show that the deviation is preferred for $\delta = 4/7$.

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."
3. $U_i(TS, TS) = \frac{10}{1-\delta}$
4. $U_i(ODS, TS) = 15 + \frac{7\delta}{1-\delta}$

PS7, Ex. 8.c: Trigger strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(c) Suppose $\delta = 4/7$. Show by finding a profitable deviation that the above trigger strategy is not a SPNE.

(Step a) Given Player j plays the Trigger Strategy (TS), write up Player i's Optimal Deviation Strategy (ODS).

(Step b) Given Player j plays TS, write up Player i's respective payoffs from playing TS and ODS.

(Step c) Show that the deviation is preferred for $\delta = 4/7$:

$$U_i\left(ODS, TS\right) > U_i\left(TS, TS\right)$$

$$\Rightarrow 15 + \frac{7\frac{4}{7}}{1-\frac{4}{7}} > \frac{10}{1-\frac{4}{7}}, \qquad \text{for } \delta = \frac{4}{7}$$

$$\Rightarrow \frac{73}{3} > \frac{70}{3} \qquad Q.E.D.$$

Information so far:

1. Trigger Strategy (TS): "If $t = 1$ or if the outcome in all previous stages was (L, L), play L. If not, play M."
2. Optimal Deviation Strategy (ODS): "Always play M."
3. $U_i(TS, TS) = \frac{10}{1-\delta}$
4. $U_i(ODS, TS) = 15 + \frac{7\delta}{1-\delta}$

PS7, Ex. 9: Optimal Punishment Strategy (infinitely repeated game)

PS7, Ex. 9: Optimal Punishment Strategy (infinitely repeated game)

We continue analyzing $G(\infty)$. As in ~~Lecture 8~~ (Lecture 6, slides 50-68), consider the strategy profile (OP, OP), where OP stands for optimal punishment...

[See the lecture slides and the full description of the exercise in the problem set.]

Stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

PS7, Ex. 9.a: Optimal Punishment Strategy (infinitely repeated game)

Consider $G(\infty)$, i.e. the infinitely repeated game with stage game G:

	L	M	H
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(a) Discuss how this increased leniency over time may give player 1 an incentive to accept his punishment (and actually play according to Q_D, rather than deviate again).

PS7, Ex. 9.a: Optimal Punishment Strategy (infinitely repeated game)

Consider $G(\infty)$ with stage game G, underlining (Q^D, Q^P) in the 'tough' stage:

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

(a) Discuss how this increased leniency over time may give player 1 an incentive to accept his punishment (and actually play according to Q^D, rather than deviate again).

The 'tough' stage (the 1st round of punishment):

- P1 earns -4 from (M, H) and 3 from (L, M) in all later rounds.
- He can deviate by playing L and earn 0, but then the punishment will start over again and P1 will therefore stay in the 'tough' stage.
- Both -4 and 0 is less than 3, which is why this punishment structure leaves P1 without an incentive to deviate from the 'punishment path' (Q^D, Q^P).

PS7, Ex. 9.a: Optimal Punishment Strategy (infinitely repeated game)

Consider $G(\infty)$ with stage game G, underlining (Q^D, Q^P) in the 'mild' stage:

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

(a) Discuss how this increased leniency over time may give player 1 an incentive to accept his punishment (and actually play according to Q^D, rather than deviate again).

The 'tough' stage (the 1st round of punishment):

- P1 earns -4 from (M, H) and 3 from (L, M) in all later rounds.
- He can deviate by playing L and earn 0, but then the punishment will start over again and P1 will therefore stay in the 'tough' stage.
- Both -4 and 0 is less than 3, which is why this punishment structure leaves P1 without an incentive to deviate from the 'punishment path' (Q^D, Q^P).

The 'mild' stage (from the 2nd round of punishment):

- P1 earns 3 from (L, M) in this and all subsequent rounds.
- He can deviate by playing M and earn 7, but then the punishment will start over again and P1 will earn -4 in the 'tough' stage (or alternatively 0 by deviating again).
- Both -4 and 0 is less than 3, which is why this punishment structure provides P1 with an incentive to stay in the 'mild stage'.

PS7, Ex. 9.b: Optimal Punishment Strategy (infinitely repeated game)

The 'tough' stage (the 1st round of punishment):

- P1 earns -4 from (M, H) and 3 from (L, M) in all later rounds.
- He can deviate by playing L and earn 0, but then the punishment will start over again and P1 will therefore stay in the 'tough' stage.
- Both -4 and 0 is less than 3, which is why this punishment structure leaves P1 without an incentive to deviate from the punishment path.

The 'mild' stage (from the 2nd round of punishment):

- P1 earns 3 from (L, M) in this and all subsequent rounds.
- He can deviate by playing M and earn 7, but then the punishment will start over again and P1 will earn -4 in the 'tough' stage (or alternatively 0 by deviating again).
- Both -4 and 0 is less than 3, which is why this punishment structure provides P1 with an incentive to stay in the 'mild stage'.

(b) How does your answer relate to the following quote from Wikipedia?
The "carrot and stick" approach is an idiom that refers to a policy of offering a combination of rewards and punishment to induce behavior. It is named in reference to a cart driver dangling a carrot in front of a mule and holding a stick behind it. The mule would move towards the carrot because it wants the reward of food, while also moving away from the stick behind it, since it does not want the punishment of pain, thus drawing the cart.

PS7, Ex. 9.b: Optimal Punishment Strategy (infinitely repeated game)

(b) How does your answer relate to the following quote from Wikipedia?

The "carrot and stick" approach is an idiom that refers to a policy of offering a combination of rewards and punishment to induce behavior. It is named in reference to a cart driver dangling a carrot in front of a mule and holding a stick behind it. The mule would move towards the carrot because it wants the reward of food, while also moving away from the stick behind it, since it does not want the punishment of pain, thus drawing the cart.

The 'tough' stage (the 1^{st} round of punishment):

- The promise of a future 'mild punishment' should be regarded as a reward for accepting the punishment without deviating from the 'punishment path' and is therefore the carrot.

PS7, Ex. 9.b: Optimal Punishment Strategy (infinitely repeated game)

(b) How does your answer relate to the following quote from Wikipedia?

The "carrot and stick" approach is an idiom that refers to a policy of offering a combination of rewards and punishment to induce behavior. It is named in reference to a cart driver dangling a carrot in front of a mule and holding a stick behind it. The mule would move towards the carrot because it wants the reward of food, while also moving away from the stick behind it, since it does not want the punishment of pain, thus drawing the cart.

The 'tough' stage (the 1st round of punishment):

- The promise of a future 'mild punishment' should be regarded as a reward for accepting the punishment without deviating from the 'punishment path' and is therefore the carrot.

The 'mild' stage (from the 2nd round of punishment):

- The threat of a future 'tough punishment' should be regarded as further punishment for deviating in the first place and is therefore the stick.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

We continue analyzing $G(\infty)$. Complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$. We checked the first three points of the road map in Lecture 6. The last two points consist of checking that it is optimal for Player 2 to punish Player 1 after a deviation. In particular, you need to check that Player 2 has no profitable deviation when he is in the first and in the second round of punishing Player 1.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

We continue analyzing $G(\infty)$. Complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$. We checked the first three points of the road map in Lecture 6. The last two points consist of checking that it is optimal for Player 2 to punish Player 1 after a deviation. In particular, you need to check that Player 2 has no profitable deviation when he is in the first and in the second round of punishing Player 1.

In the lecture it was checked that Player 1 will not deviate from (OP, OP) in:

1. Round 1, or if (L, L) was played in all previous rounds.
2. The 1st round of being punished.
3. Subsequent rounds of being punished.

Underlining (Q^D, Q^P) in the 1st round of punishment (the 'tough' stage):

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

Underlining (Q^D, Q^P) in subsequent rounds of punishment (the 'mild' stage):

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

We continue analyzing $G(\infty)$. Complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$. We checked the first three points of the road map in Lecture 6. The last two points consist of checking that it is optimal for Player 2 to punish Player 1 after a deviation. In particular, you need to check that Player 2 has no profitable deviation when he is in the first and in the second round of punishing Player 1.

In the lecture it was checked that Player 1 will not deviate from (OP, OP) in:

1. Round 1, or if (L, L) was played in all previous rounds.
2. The 1st round of being punished.
3. Subsequent rounds of being punished.

Check that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1.
5. When he is in subsequent rounds of punishing Player 1.

Underlining (Q^D, Q^P) in the 1st round of punishment (the 'tough' stage):

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

Underlining (Q^D, Q^P) in subsequent rounds of punishment (the 'mild' stage):

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

4. When he is in the 1^{st} round of punishing Player 1:

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1:

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

If Player 2 does not deviate from Q^P, his expected utility is:

$$U_2(Q^P; Q^D) = 5 + 15\delta + 15\delta^2 + ... = 5 + \sum_{t=2}^{\infty} 15\delta^{t-1} = 5 + \frac{15\delta}{1-\delta} = 25, \text{ for } \delta = \frac{4}{7}$$

If Player 2 deviates from Q^P to play M, from the next round Player 1 will instead force Player 2 to take the punishment and play according to Q^D forever:

$$U_2(M, Q^D; M, Q^P) = 7 + \delta U_2(Q^D; Q^P) = 7 + \delta\left(-4 + \sum_{t=3}^{\infty} 3\delta^{t-1}\right) = 7 + \delta\left(-4 + \frac{3\delta}{1-\delta}\right)$$

$$= 7, \text{ for } \delta = 4/7$$

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1:

	L	M	**H**
L	10, 10	3, 15	0, 7
M	15, 3	7, 7	**-4, 5**
H	7, 0	5, -4	-15, -15

If Player 2 does not deviate from Q^P, his expected utility is:

$$U_2(Q^P; Q^D) = 5 + 15\delta + 15\delta^2 + \ldots = 5 + \sum_{t=2}^{\infty} 15\delta^{t-1} = 5 + \frac{15\delta}{1-\delta} = 25, \text{ for } \delta = \frac{4}{7}$$

If Player 2 deviates from Q^P to play M, from the next round Player 1 will instead force Player 2 to take the punishment and play according to Q^D forever:

$$U_2(M, Q^D; M, Q^P) = 7 + \delta U_2(Q^D; Q^P) = 7 + \delta\left(-4 + \sum_{t=3}^{\infty} 3\delta^{t-1}\right) = 7 + \delta\left(-4 + \frac{3\delta}{1-\delta}\right)$$

$$= 7, \text{ for } \delta = 4/7$$

In the 1st round of punishing Player 1, Player 2 expects higher utility from playing according to Q^P (25) than from deviating (7), i.e. Player 2 has no incentive to deviate.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

5. When he is in subsequent rounds of punishing Player 1:

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

5. When he is in subsequent rounds of punishing Player 1:

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

In each subsequent round, the outcome from (Q^D, Q^P) is (L, M) with payoffs $(3, 15)$.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

Check that Player 2 will not deviate:

5. When he is in subsequent rounds of punishing Player 1:

	L	**M**	H
L	10, 10	**3, 15**	0, 7
M	15, 3	7, 7	-4, 5
H	7, 0	5, -4	-15, -15

In each subsequent round, the outcome from (Q^D, Q^P) is (L, M) with payoffs $(3, 15)$.

From the 2nd round of punishing Player 1, Player 2 expects to earn 15 in every round, i.e. he has no incentive to deviate.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

In the lecture it was checked that Player 1 will not deviate from (OP, OP) in:

1. Round 1, or if (L, L) was played in all previous rounds.
2. The 1st round of being punished.
3. Subsequent rounds of being punished.

In this exercise we have checked that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1.
5. When he is in subsequent rounds of punishing Player 1.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

In the lecture it was checked that Player 1 will not deviate from (OP, OP) in:

1. Round 1, or if (L, L) was played in all previous rounds.
2. The 1st round of being punished.
3. Subsequent rounds of being punished.

In this exercise we have checked that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1.
5. When he is in subsequent rounds of punishing Player 1.

Condition 1 secures that (OP, OP) is optimal ***on*** the equilibrium path.

Condition 2-5 secure that (OP, OP) is optimal ***off*** the equilibrium path.

PS7, Ex. 10: Is the punishment credible? (infinitely repeated game)

Use the roadmap to complete the proof that (OP, OP) is a SPNE when $\delta = 4/7$.

In the lecture it was checked that Player 1 will not deviate from (OP, OP) in:

1. Round 1, or if (L, L) was played in all previous rounds.
2. The 1st round of being punished.
3. Subsequent rounds of being punished.

In this exercise we have checked that Player 2 will not deviate:

4. When he is in the 1st round of punishing Player 1.
5. When he is in subsequent rounds of punishing Player 1.

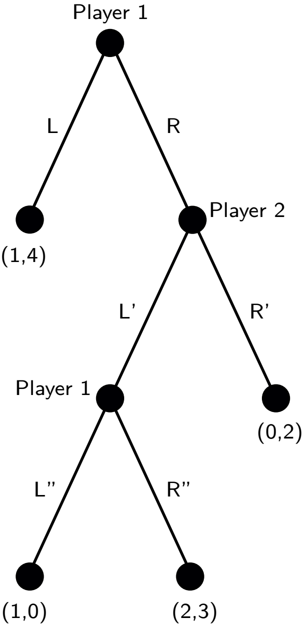
Condition 1 secures that (OP, OP) is optimal ***on*** the equilibrium path.

Condition 2-5 secure that (OP, OP) is optimal ***off*** the equilibrium path.

Therefore, we can conclude that (OP, OP) is a SPNE for $\delta = 4/7$.

Code examples

Code examples



Matrix, no player names:

	L (q)	R (1-q)
T (p)		
B (1-p)		

Matrix, no colors:

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)		
	B (1-p)		

Matrix, with colors:

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 1	
	B (1-p)		