Microeconomics III: Problem Set 9[a]

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November 20 2019

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[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS8, Ex. 4: First-price sealed bid auctions with three bidders

PS8, Ex. 5: Winner's Curse

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim u(1, 3)$$

Thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$. Find c and d.

(b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

   i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
   ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
   iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

[PDF and CDF for the uniform distribution is written up on the next slide.]

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim u(1,3)$$

Thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

(b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for a uniform distribution $x \sim u(a,b)$:

PDF: Probability density function: $f(x) = \frac{1}{b-a}$

CDF: Cumulative distribution function: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim u(1,3)$, thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1st step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \frac{\frac{b_i-d}{c} - 1}{3-1}(v_i - b_i), \text{ using CDF} \\
&= \frac{b_i - d - c}{2c}(v_i - b_i)
\end{aligned}$$

2nd step: Take the FOC and SOC wrt. b_i.

3rd step: To find c^* and d^*, compare the best response function $b_i(v_i)$ to $(*)$.

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results:

2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d+c)] = 0$
SOC: $-\frac{1}{c} = 0$
i.e. expected utility is concave in b_i.

3rd: From the FOC, the BR is:

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d+c)}_{d^*}$$

Inserting the first term in the second term, $d^* = \frac{1}{2}(d^* + c^*) = \frac{1}{2}(d^* + \frac{1}{2})$, which solves for $c^* = d^* = \frac{1}{2}$.

PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i is

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\ &= \frac{cv_i - c}{2c}(cv_i + d),\ \text{using } (*), (**) \\ &= \frac{v_i - 1}{2}(cv_i + d) \\ &= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right),\ \text{using (a)} \\ &= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\ &= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

3rd step: The expected revenue to the seller is the ex-ante expected payment of both bidders:

$$\text{Seller's revenue} = \mathbb{E}[m_1(v_1)] + \mathbb{E}[m_2(v_2)] = \frac{5}{3}$$

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\ &= \int_1^3 \frac{v_i^2-1}{4} \cdot \frac{1}{3-1}dv_i \\ &= \frac{1}{8}\int_1^3 v_i^2 - 1dv_i \\ &= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\ &= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6} \end{aligned}$$

PS8, Ex. 3.c.ii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

PS8, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Player i's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j|v_j < v_i] \\ &= \frac{v_i - 1}{3 - 1} \cdot \frac{1 + v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

As this is the same as in (b), we know:

$$\text{Ex-ante expected payment} = \mathbb{E}[m_i(v_i)] = \frac{5}{6}$$

$$\text{Seller's revenue} = 2 \cdot \mathbb{E}[m_i(v_i)] = \frac{5}{3}$$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

PS8, Ex. 4: First-price sealed bid auctions with three bidders

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Consider the auction setting of the previous exercise. But now suppose that there are three identical bidders, $i = 1, 2, 3$, with values v_i where

$$v_i \sim u(1, 3)$$

and the values are independent, i.e. private. The auction is first-price sealed bid.

(a) Again, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

(c) (More difficult). Calculate the revenue to the seller.

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Step 1: Use that v_j and v_k are independent (private) to write i's' expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i\ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. 3.a}
\end{aligned}$$

$$FOC: \quad 0 = \frac{1}{2c}[2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2]$$

$$0 = 2(v_i - b_i) - (b_i - d - c), \qquad \text{assuming } b_i - d - c \neq 0$$

$$b_i^{**}(v_i) = \underbrace{\frac{2}{3}}_{c^* = \frac{2}{3}} v_i + \underbrace{\frac{1}{3}(c + d)}_{d^* = \frac{1}{3}\left(\frac{2}{3} + d^*\right) \Rightarrow d^* = \frac{1}{3}} = \underline{\underline{\frac{2}{3}v_i + \frac{1}{3}}} \qquad Q.E.D.$$

PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

Analytically, we can confirm this:

$$\begin{aligned} b_i^{**} &> b_i^* \Leftrightarrow \\ \frac{2}{3}v_i + \frac{1}{3} &> \frac{1}{2}v_i + \frac{1}{2} \Leftrightarrow \\ \frac{1}{6}v_i &> \frac{1}{6} \Leftrightarrow \\ v_i &> 1 \end{aligned}$$

I.e. except for the rare case where all players have the valuation $v = 1$, the seller's revenue is strictly higher with three players than with two players.

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: Calculate the expected payment of bidder i with valuation v_i:

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i)b_i(v_i)$$
$$= \left(\frac{cv_i - c}{2c}\right)^2 b_i(v_i),\ (*),(+)$$
$$= \left(\frac{v_i - 1}{2}\right)^2 (cv_i + d)$$
$$= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right),\ (4.a)$$
$$= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)$$

2[nd] step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

The expected payment from each bidder is lower due to the lower probability of winning.

3[rd] step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

(∗) $b_i(v_i) = cv_i + d$

(+) $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2 = \left(\frac{cv_i - c}{2c}\right)^2$

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2[nd]: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i)f_i(v_i)dv_i$$
$$= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i$$
$$= \frac{1}{24}\left[\frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i\right]_1^3$$
$$= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}$$

3[rd]: $Revenue = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$

However, the seller can expect a higher revenue as more players increases the chance of one having a high valuation.

PS8, Ex. 5: Winner's Curse

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Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

PS8, Ex. 5.a: Winner's Curse

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The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

Though the reports are private, the bidders' values are ***common*** since they are identical.

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Write up Bayes' rule.

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: Use Bayes' rule and the given probabilities to write up the probability that the value of the oil-field is high after having received the report $r_1 = H$.

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (*)$$

PS8, Ex. 5.b: Winner's Curse

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The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (*)$$

Step 3: Use $(*)$ to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high π_H or low π_L.

PS8, Ex. 5.b: Winner's Curse

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The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (*)$$

Step 3: Use $(*)$ to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high π_H or low π_L:

$$\mathbb{E}[\pi_i|r_1 = H] = \mathbb{P}[H|r_1 = H]\pi_H + \mathbb{P}[L|r_1 = L]\pi_L = p\pi_H + (1-p)\pi_L$$

PS8, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: Write up the probability that company 1 wins the auction after receiving the report $r_1 = H$.

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Step 1: The probability company 1 wins the auction after receiving the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L)$$

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$$= \frac{\mathbb{P}[r_1 = H \wedge r_2 = low|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = low|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = low|L] \times \mathbb{P}[L]}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (**)$$

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Step 2: Use $(**)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction.

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$$\begin{aligned}
\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L) \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = low|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = low|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = low|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (**)
\end{aligned}$$

Step 2: Use $(**)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}
\mathbb{E}[\pi_i|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]\pi_H + \mathbb{P}[L|r_1 = L \wedge win]\pi_L \\
= H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]\pi_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\,\pi_L \\
&= \frac{1}{2}\pi_H + \frac{1}{2}\pi_L < p\pi_H + (1-p)\pi_L \text{ since } p > \frac{1}{2}
\end{aligned}$$

Step 3: Explain the difference between the result in (b) and (c).

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$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (**)$$

Step 2: Use $(**)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\mathbb{E}[\pi_i|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]\pi_H + \mathbb{P}[L|r_1 = L \wedge win]\pi_L$$

$$= H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]\pi_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\,\pi_L$$

$$= \frac{1}{2}\pi_H + \frac{1}{2}\pi_L < p\pi_H + (1-p)\pi_L \text{ since } p > \frac{1}{2}$$

Step 3: This is an example of The Winner's Curse: The equally trustworthy reports of the two companies cancel each other out. Since the valuations of the auctioned object are correlated, you are likely to win the object when you overestimate the value.