Microeconomics III: Problem Set 10[a]

Thor Donsby Noe (thor.noe@econ.ku.dk) & Christopher Borberg (christopher.borberg@econ.ku.dk)
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Department of Economics, University of Copenhagen

[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS10, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

PS10, Ex. 2 (A): Crimea Through a Game-Theory Lens

PS10, Ex. 3 (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

PS10, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

PS10, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)

PS10, Ex. 6: Extensive form game (mixed-strategy Perfect Bayesian Equilibrium)

PS10, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

PS10, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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PS10, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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i. **Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.**

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i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

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Step 2: **How is this result affected by the distribution of the bidder's values?**

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(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

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i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

2: The result is independent of the distributions, thus it's still a best-response to bid one's value.

PS10, Ex. 2 (A): Crimea Through a Game-Theory Lens

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Read through the New York Times article Crimea Through a Game-Theory Lens by Tyler Cowen (co-author of the popular economics blog Marginal Revolution). Try to think about how you would set up models to describe the situations he writes about. *(This exercise is just for reflection, no answer will be provided).*

PS10, Ex. 3 (A): The ’Lemons’ model (Perfect Bayesian Equilibrium)

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Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

(a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.

(b) What would happen if buyers cannot do so, and know that half of all used cars are lemons? Draw this as a dynamic game of incomplete information, where nature chooses the type of the car, the seller observes this and sets a price (any positive real number) and the buyer decides whether to buy or not.

(c) Find a Perfect Bayesian Equilibrium of this model.

"In US English, a lemon is a vehicle (often new) that turns out to have several manufacturing defects affecting its safety, value or utility." (Source: Wikipedia)

PS10, Ex. 3.a (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

(a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.

If buyers can distinguish between beauties and lemons, they would be traded on two separate markets with prices within $[800, 1200]$ and $[2000, 2400]$ respectively.

PS10, Ex. 3.b (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality "beauties" and low-quality "lemons". Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

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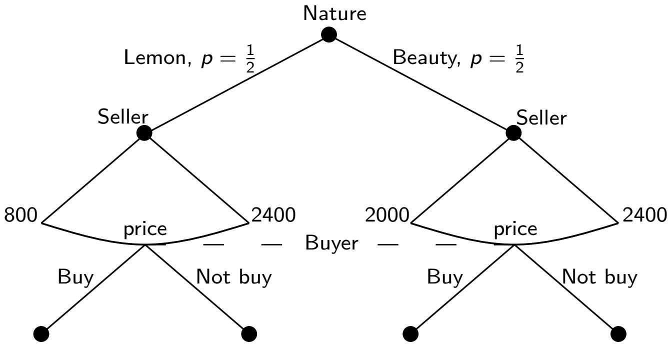
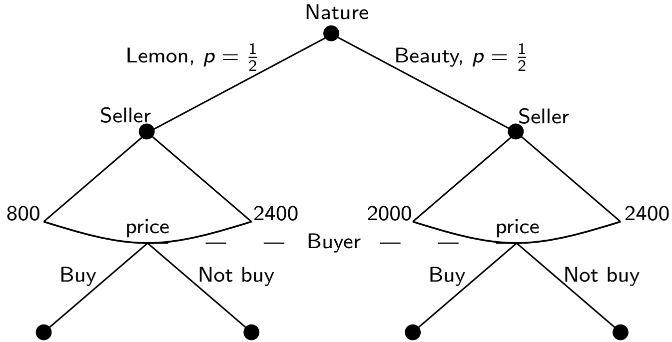


PS10, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



(c) Find a Perfect Bayesian Equilibrium of this model.

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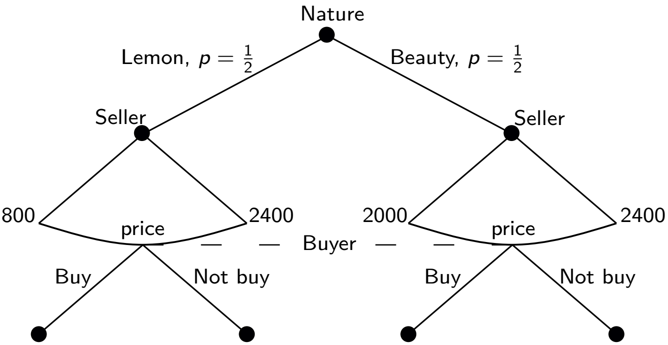

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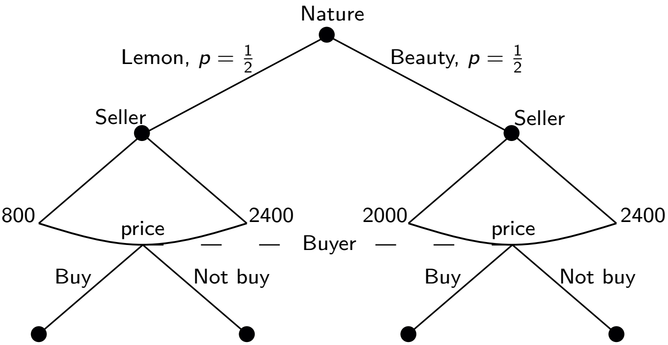
Step 1: **Write up buyer's expectation to the car's value given her beliefs regarding p**

PS10, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



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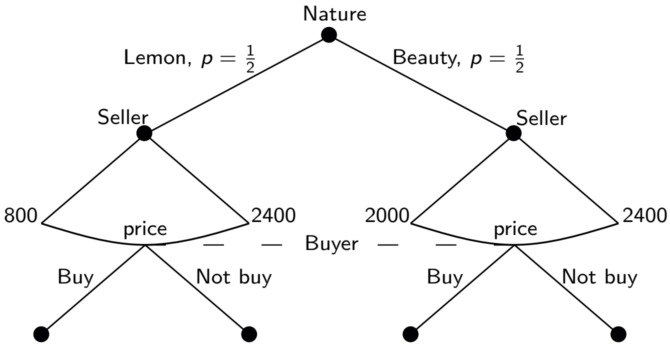

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Step 2: **As both the seller and the buyer know this expectation, what will the outcome be?**

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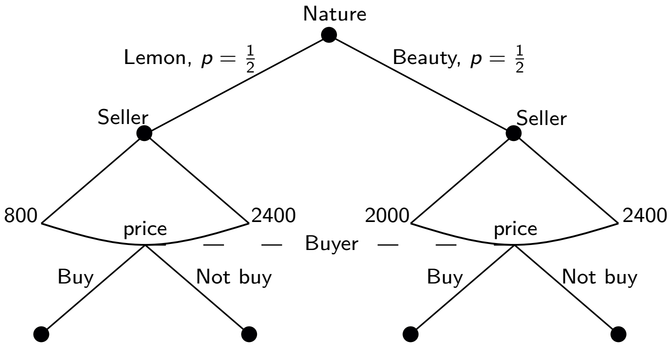

(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding p.

Step 2: As both the seller and the buyer know this expectation, what will the outcome be?

1. $E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$
2. The seller will not sell *beauties* for a price below 2000. The buyer anticipates this, thus, there will only be a market for *lemons* being sold for *price* $\in [800, 1200]$ as 1200 is the highest amount that the buyer is willing to pay for a *lemon*.

PS10, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

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Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

The value to i of having his preferred policy implemented is v_i, where $v_i \sim U(0, 1)$ independently (private values). The lobbyists know their own valuation, but not that of the other lobbyist.

(a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c.

PS10, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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Step 1: **Write up the bidders, valuations, bids, and utilities.**

PS10, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

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(a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

Step 1: Write up the auction with bidders, valuations, bids, and utilities.

Step 2: **How is this different from the auctions we have seen so far?**

1. Two bidders, $i \in 1, 2$.

   Valuations are independently distributed $v_i \sim U(0, 1)$

   Bids $b_i \in [0, 1]$

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

PS10, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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2. Both bidders pay their bid b_i regardless of whether they win. This is known as an *all-pay auction*.

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists, $i = 1, 2$, are trying to persuade a policymaker to implement their preferred policy by making a costly effort $e_i \in [0, 1]$. The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c.

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

[Try to write up the cumulative distribution function (CDF) for a uniform distribution $x \sim U(a, b)$, and the probability that a constant c is higher than a random draw of x.]

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

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Step 1: **Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.**

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Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

$$\begin{aligned}
\mathbb{P}(i\ wins) &= \mathbb{P}(b_i > b_j(v_j)) \\
&= \mathbb{P}(b_i > cv_j^2), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i}{c} > v_j^2\right) \\
&= \mathbb{P}\left(\sqrt{\frac{b_i}{c}} > v_j\right) \\
&= \sqrt{\frac{b_i}{c}}, && \text{using CDF}
\end{aligned}$$

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1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$ (1)

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

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&= \mathbb{P}\left(\sqrt{\frac{b_i}{c}} > v_j\right) \\
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Step 2: **Write up bidder i's expected payoff from bidding b_i conditional on v_i.**

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Results so far:

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Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

$$\mathbb{E}[u_i(b_i)|v_i] = \mathbb{P}(i\ wins)v_i - b_i$$

$$= \sqrt{\frac{b_i}{c}}v_i - b_i, \qquad \text{using (1)}$$

Remember that the bid is always payed.

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

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1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$ (1)
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

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$$= \sqrt{\frac{b_i}{c}}v_i - b_i, \qquad \text{using (1)}$$

Remember that the bid is always payed.

Step 3: **Take the first-order condition and second-order condition with respect to b_i.**

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

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Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

$$
\begin{aligned}
\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} &= \frac{\delta}{\delta b_i}\left(\sqrt{\frac{b_i}{c}} v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(\frac{\sqrt{b_i}}{\sqrt{c}} v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(b_i^{\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - b_i\right) \\
&= \frac{1}{2} b_i^{-\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - 1 \qquad (**) \\
&= \frac{1}{2} \frac{1}{\sqrt{b_i}} \frac{1}{\sqrt{c}} v_i - 1 \\
&= \frac{1}{2\sqrt{b_i c}} v_i - 1
\end{aligned}
$$

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$
u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}
$$

1. $\mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$ (1)
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

$$\begin{aligned}
\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} &= \frac{\delta}{\delta b_i}\left(\sqrt{\frac{b_i}{c}}v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(\frac{\sqrt{b_i}}{\sqrt{c}}v_i - b_i\right) \\
&= \frac{\delta}{\delta b_i}\left(b_i^{\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - b_i\right) \\
&= \frac{1}{2}b_i^{-\frac{1}{2}}\frac{1}{\sqrt{c}}v_i - 1 \qquad (**) \\
&= \frac{1}{2}\frac{1}{\sqrt{b_i}}\frac{1}{\sqrt{c}}v_i - 1 \\
&= \frac{1}{2\sqrt{b_i c}}v_i - 1
\end{aligned}$$

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$ (1)
2. $\mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$
3. FOC: $\frac{1}{2\sqrt{b_i c}}v_i - 1 = 0$

   SOC: $-\frac{1}{4}b_i^{-\frac{3}{2}}\frac{1}{\sqrt{c}}v_i = 0$, using $(**)$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: **Solve to find $b_i(v_i)$.**

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

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PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

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Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

As the SOC is negative for all $b_i, v_i, c > 0$ bidder i maximizes expected utility for

$$0 = \frac{1}{2\sqrt{b_i(v_i)c}} v_i - 1 \Leftrightarrow$$

$$2\sqrt{b_i(v_i)c} = v_i \Leftrightarrow$$

$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i\ wins) = \sqrt{b_i/c}$ (1)
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4. $b_i(v_i) = \frac{1}{4c} v_i^2$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

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$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Step 5: **Set this equal to $(*)$ to find c^*:**

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2$$

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$ (1)
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3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0$, using $(**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

Step 4: Solve to find $b_i(v_i)$.

Step 5: Set this equal to $(*)$ to find c^*:

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$
$$c^* = \frac{1}{4c^*} \Leftrightarrow$$
$$2c^* = \frac{1}{4} \Leftrightarrow$$
$$c^* = \frac{1}{2}$$

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \; wins) = \sqrt{b_i/c}$ (1)
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   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0$, using $(**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

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Step 4: Solve to find $b_i(v_i)$.

Step 5: Set this equal to $(*)$ to find c^*:

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$
$$c^* = \frac{1}{4c^*} \Leftrightarrow$$
$$2c^* = \frac{1}{4} \Leftrightarrow$$
$$c^* = \frac{1}{2}$$

Step 6: **Write up the equilibrium bidding strategy.**

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

1. $\mathbb{P}(i \ wins) = \sqrt{b_i/c}$ (1)
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   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0$, using $(**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$

PS10, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

(b) Check that there is a symmetric Bayesian Nash Equilibrium of the type $b_i(v_i) = cv_i^2$ $(*)$, and find c. Values are independently distributed $v_i \sim U(0,1)$.

Step 1: Write up bidder i's probability of winning the auction if j sticks to the equilibrium strategy.

Step 2: Write up bidder i's expected payoff from bidding b_i conditional on v_i.

Step 3: Take the FOC and SOC wrt. b_i.

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$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$
$$c^* = \frac{1}{4c^*} \Leftrightarrow$$
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$$c^* = \frac{1}{2}$$

Step 6: Write up the equilibrium bidding strategy.

Standard result for $x \sim U(a,b)$:

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if} \quad b_i = b_j \\ -b_i & \text{if} \quad b_i < b_j \end{cases}$$

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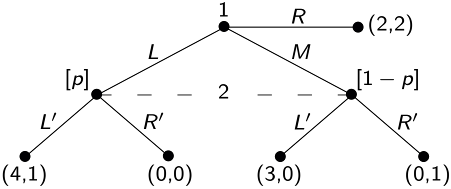
3. FOC: $\frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$

   SOC: $-\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0$, using $(**)$
4. $b_i(v_i) = \frac{1}{4c} v_i^2$
5. $c^* = \frac{1}{2}$
6. BNE: $b_i^*(v_i) = \frac{1}{2} v_i^2$

PS10, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

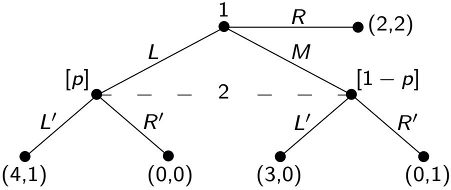


[Try to list the four requirements for a Perfect Bayesian Equilibrium (PBE).]

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

Derive the normal-form game.



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

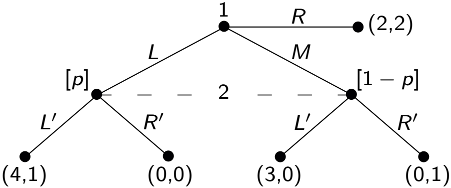
R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2



PSNE: **Find all PSNE in the bi-matrix.**

Requirements for a PBE:

- R2: In each information set, players have beliefs about where they are.
- R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.
- R3: Bayesian beliefs *on* equilibrium path.
- R4: Bayesian beliefs *off* equilibrium path.

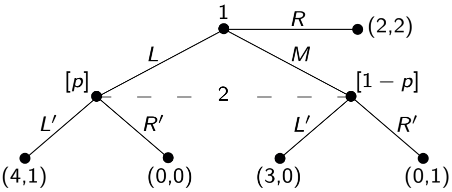
PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE: $\{(L, L'); (R, R')\}$

SPNE:



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
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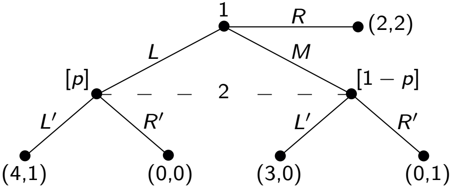
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE: $\{(L, L'); (R, R')\}$

SPNE: **How many proper subgames are there?**



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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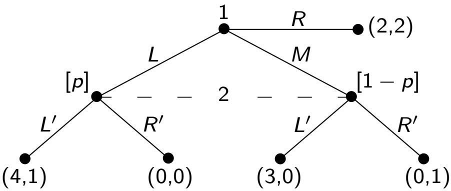

PSNE: $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE:

Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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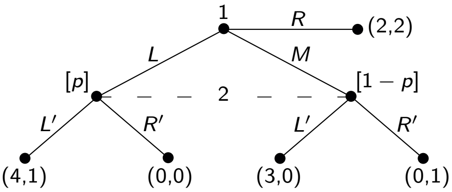

PSNE: $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: **Find the pure-strategy PBE including the beliefs of player 2 that secure the equilibrium/equilibria.**

Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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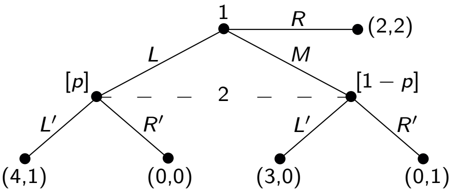

PSNE: $\{(L, L'); (R, R')\}$

SPNE $=$ PSNE, due to no proper subgames.

PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play R, P2 does not know whether the game has reached the information set following L or R. She assigns probabilities p and $[1 - p]$ to each.

Given her beliefs, find P2's expected utility of playing L' and R' respectively.

Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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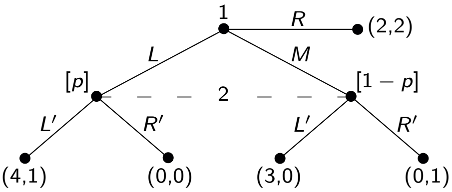

PSNE: $\{(L, L'); (R, R')\}$

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If P1 did not play R, P2 does not know whether the game has reached the information set following L or R. She assigns probabilities p and $[1-p]$ to each.

Given her beliefs, find P2's expected utility of playing L' and R' respectively:

$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1-p] = p$$

$$\mathbb{E}[u_2(R')|p] = 0 \cdot p + 1 \cdot [1-p] = 1-p$$

Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

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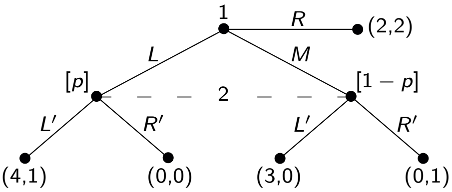

PSNE: $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play R, P2 does not know whether the game has reached the information set following L or R. She assigns probabilities p and $[1-p]$ to each.

Given her beliefs, find P2's expected utility of playing L' and R' respectively:

$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1-p] = p$$

$$\mathbb{E}[u_2(R')|p] = 0 \cdot p + 1 \cdot [1-p] = 1-p$$

P2 is indifferent between L' and R' if

$$\mathbb{E}[u_2(L')|p] = \mathbb{E}[u_2(R')|p] \Rightarrow p = 1-p \Rightarrow p = 1/2$$

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

Write up the best responses of player 2.

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
L [p]	4, 1	0, 0
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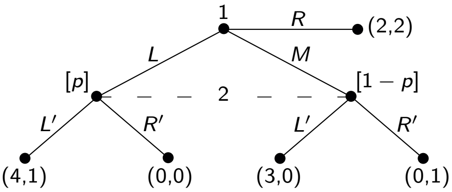

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R2: Players have beliefs.

R2: Sequential rationality.

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Write up the best responses of player 1:

$$BR_1(L') = L, \text{ with } u_1(L, L') = 4$$

$$BR_1(R') = R, \text{ with } u_1(R, R') = 2$$

Find the PBE s.t. requirements 1-4 by analyzing the intervals $p \in \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]$

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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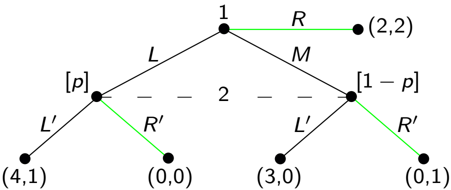

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$p \in \left[0, \frac{1}{2}\right]$: P2 plays R' if she expects P1 to play L with probability $p \leq \frac{1}{2}$. This is a PBE if P1 plays R.

$\boldsymbol{p \in \left[\frac{1}{2}, 1\right]}$**: ?**

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

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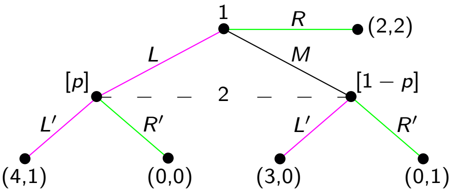

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$p \in \left[\frac{1}{2}, 1\right]$: P2 plays L' if she expects P1 to play L with probability $p \geq \frac{1}{2}$. This is a PBE if P1 plays L and P2 expects this with beliefs $p = 1$.

Write up the set of PBE.

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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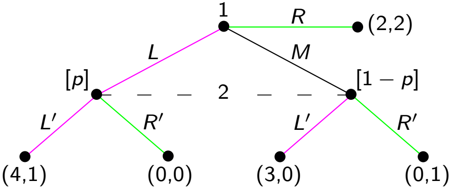

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$p \in \left[0, \frac{1}{2}\right]$: P2 plays R' if she expects P1 to play L with probability $p \leq \frac{1}{2}$. This is a PBE if P1 plays R.

$p \in \left[\frac{1}{2}, 1\right]$: P2 plays L' if she expects P1 to play L with probability $p \geq \frac{1}{2}$. This is a PBE if P1 plays L and P2 expects this with beliefs $p = 1$.

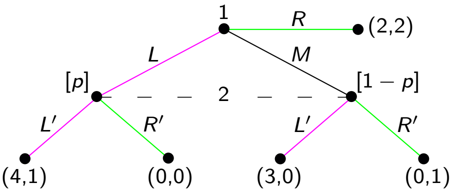
$$\boldsymbol{PBE = \left\{(L, L'), p = 1 \; ; \; (R, R'), p \leq 1/2\right\}}$$

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

If P1 did not play R, P2 does not know whether the game has reached the information set following L or R. She assigns probabilities p and [1-p] to each.

Bonus: **Explain why $\{(L, L'), p \geq \frac{1}{2}\}$ isn't a PBE but $\{(R, R'), p \leq 1/2\}$ is?**



Write up the best responses of player 1:

$$BR_1(L') = L, \text{ with } u_1(L, L') = 4$$

$$BR_1(R') = R, \text{ with } u_1(R, R') = 2$$

$p \in \left[0, \frac{1}{2}\right]$: P2 plays R' if she expects P1 to play L with probability $p \leq \frac{1}{2}$. This is a PBE if P1 plays R.

$p \in \left[\frac{1}{2}, 1\right]$: P2 plays L' if she expects P1 to play L with probability $p \geq \frac{1}{2}$. This is a PBE if P1 plays L and P2 expects this with beliefs $p = 1$.

$$PBE = \left\{(L, L'), p = 1\ ;\ (R, R'), p \leq 1/2\right\}$$

PS10, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

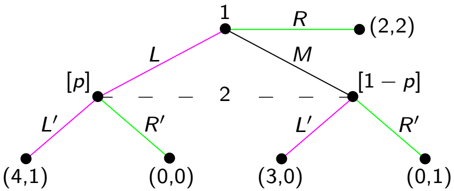
Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

If P1 did not play R, P2 does not know whether the game has reached the information set following L or R. She assigns probabilities p and [1-p] to each.

Bonus: Explain why $\{(L, L'), p \geq 1/2\}$ isn't a PBE but $\{(R, R'), p \leq 1/2\}$ is?

(L, L') is a **NE** for $p \geq \frac{1}{2}$. However, to be a PBE it is required that beliefs are Bayesian, i.e. consistent with the equilibrium. In the equilibrium L is played with probability $p = 1$ which should be the believed as well (updating beliefs).

On the contrary, if P2 plays R', P1 would want to play R. Off the equilibrium path, P1 is indifferent between L and M when P2 plays R', i.e. any off-equilibrium path beliefs of $p \in [0, 1]$ would be rational, but only $p \leq \frac{1}{2}$ is consistent with P2 preferring R'.



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$$BR_1(L') = L, \text{ with } u_1(L, L') = 4$$

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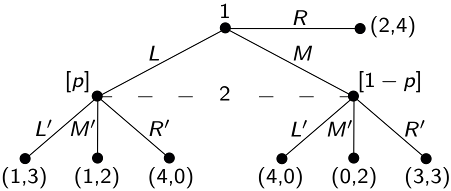
$$PBE = \left\{(L, L'), p = 1 \; ; \; (R, R'), p \leq 1/2\right\}$$

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

Derive the normal-form game.



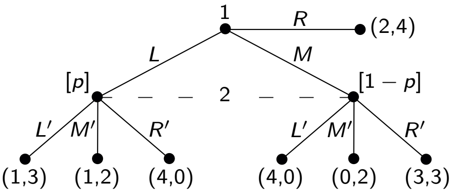
Requirements for a PBE:

- R2: In each information set, players have beliefs about where they are.
- R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.
- R3: Bayesian beliefs *on* equilibrium path.
- R4: Bayesian beliefs *off* equilibrium path.

PS10, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4



PSNE:

Requirements for a PBE:

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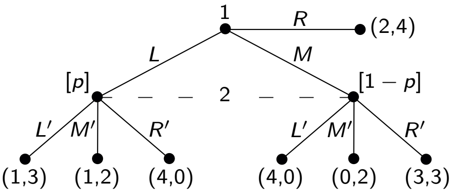
PSNE: $\{(R, M')\}$

SPNE:



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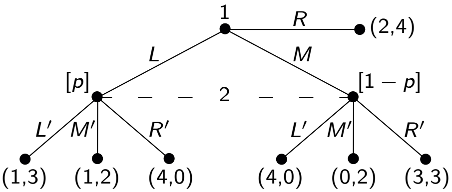

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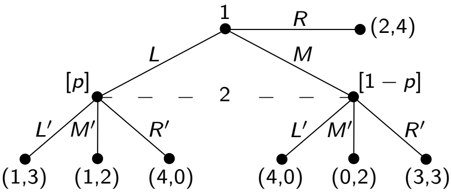

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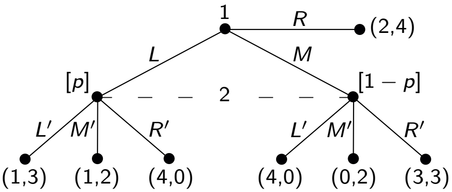

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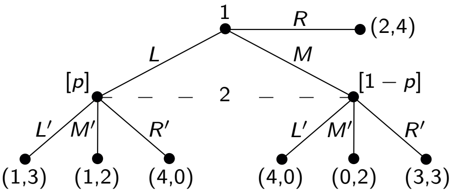

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Draw the expected utility of each choice as functions of *p*.

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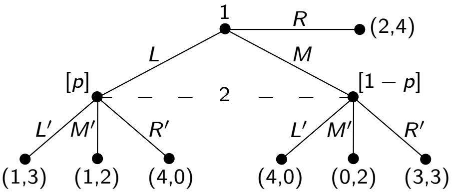

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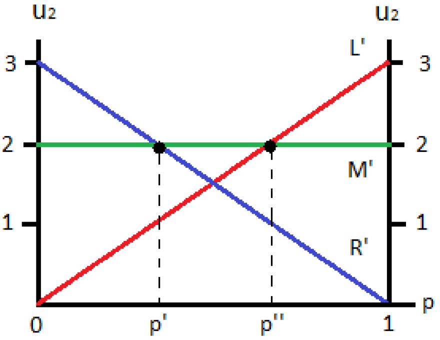

Using the diagram and the expected utility functions, find the intersections p' and p''.

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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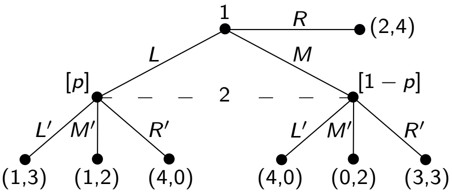

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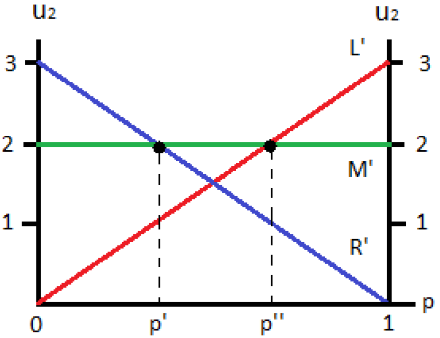
$$\mathbb{E}[u_2(R')|p] = 0p + 3[1-p] = 3 - 3p$$

P2 is indifferent at p' and p'':

$\mathbb{E}[u_2(M')] = \mathbb{E}[u_2(R')] \Rightarrow 2 = 3\text{-}3p \Rightarrow p' = 1/3$

$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(M')] \Rightarrow 3p = 2 \Rightarrow p'' = 2/3$



PS10, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

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M [1-p]	4, 0	0, 2	3, 3
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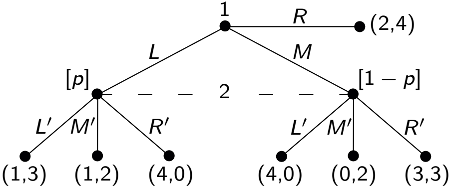

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Given her beliefs, find P2's expected utility of L', M', and R' respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1-p] = 3p$$

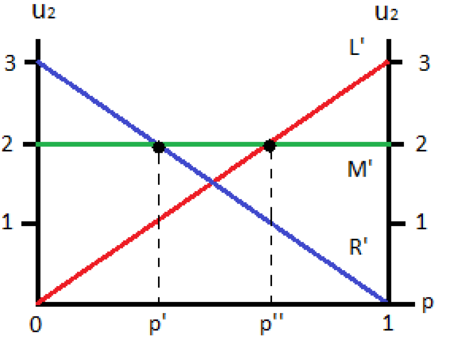
$$\mathbb{E}[u_2(M')|p] = 2p + 2[1-p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1-p] = 3 - 3p$$

P2 is indifferent at p' and p'':

$\mathbb{E}[u_2(M')]$=$\mathbb{E}[u_2(R')]$ $\Rightarrow$ 2=3-3p $\Rightarrow$ p'=1/3

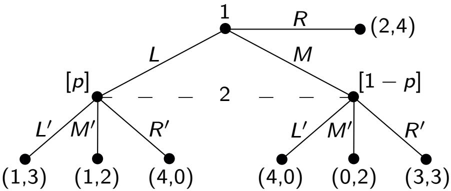
$\mathbb{E}[u_2(L')]$=$\mathbb{E}[u_2(M')]$ $\Rightarrow$ 3p=2 $\Rightarrow$ p''=2/3



Writing up the BR_1's, are they consistent with the beliefs of P2 for the relevant interval $[0, p'], [p', p''], [p'', 1]$?

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4



PSNE: $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE: ***Now, write up the PBE.***

Given her beliefs, find P2's expected utility of L', M', and R' respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1-p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1-p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1-p] = 3 - 3p$$

P2 is indifferent at p' and p'':

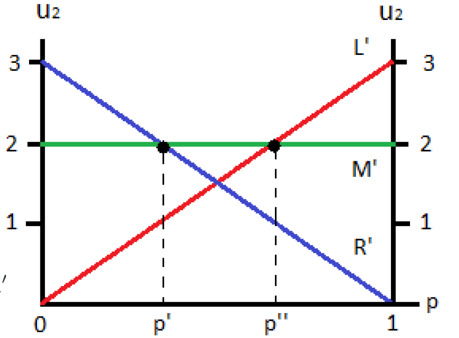
$\mathbb{E}[u_2(M')]{=}\mathbb{E}[u_2(R')] \Rightarrow 2{=}3\text{-}3p \Rightarrow p'{=}1/3$

$\mathbb{E}[u_2(L')]{=}\mathbb{E}[u_2(M')] \Rightarrow 3p{=}2 \Rightarrow p''{=}2/3$

$p \leq 1/3$: $BR_1(R') = L \rightarrow$ P2 deviates to L'

$p \in \left[\frac{1}{3}, \frac{2}{3}\right]$: $BR_1(M') = R \rightarrow$ no deviation

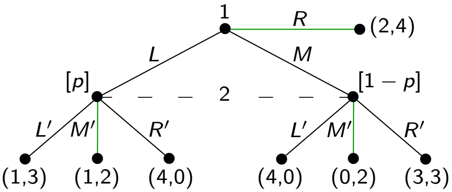
$p \geq 2/3$: $BR_1(L') = M \rightarrow$ P2 deviates to R'



PS10, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4



PSNE: $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE: $\left\{ (\boldsymbol{R}, \boldsymbol{M'}), \boldsymbol{p} \in \left[\frac{1}{3}, \frac{2}{3}\right] \right\}$

Given her beliefs, find P2's expected utility of L', M', and R' respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$

P2 is indifferent at p' and p'':

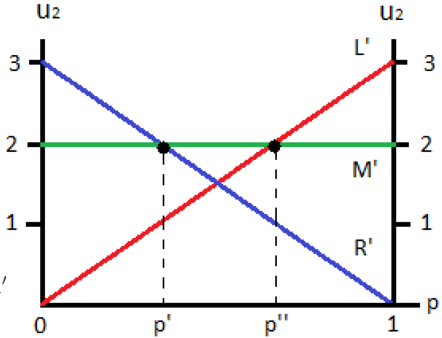
$\mathbb{E}[u_2(M')]=\mathbb{E}[u_2(R')] \Rightarrow 2=3\text{-}3p \Rightarrow p'=1/3$

$\mathbb{E}[u_2(L')]=\mathbb{E}[u_2(M')] \Rightarrow 3p=2 \Rightarrow p''=2/3$

$p \leq 1/3$: $BR_1(R') = L \rightarrow$ P2 deviates to L'

$p \in \left[\frac{1}{3}, \frac{2}{3}\right]$: $BR_1(M') = R \rightarrow$ no deviation

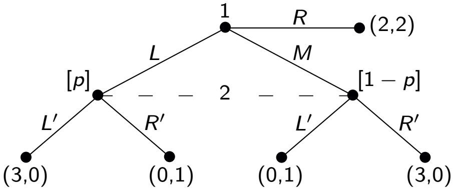
$p \geq 2/3$: $BR_1(L') = M \rightarrow$ P2 deviates to R'



PS10, Ex. 6: Extensive form game (mixed-strategy Perfect Bayesian Equilibrium)

PS10, Ex. 6: Extensive form game (mixed-strategy PBE)

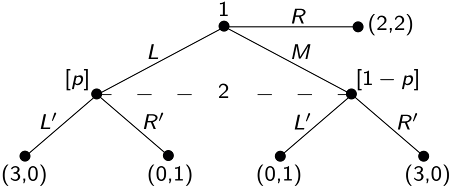
Exercise 4.2 in Gibbons (p. 245). (i) Show that there does not exist a pure-strategy PBE in the following extensive-form game. (ii) What is the mixed-strategy PBE?



PS10, Ex. 6 (i): Extensive form game (mixed-strategy PBE)

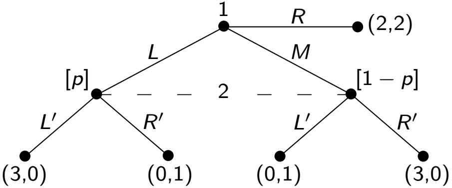
(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

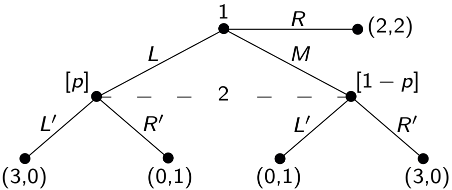
	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

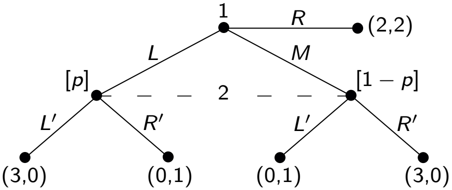
From the bi-matrix it is clear there is no equilibrium in pure strategies as one of the players would always want to deviate.



PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

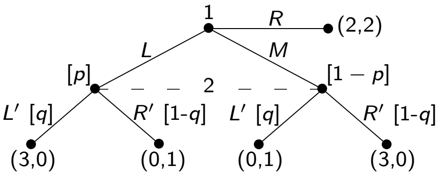


(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

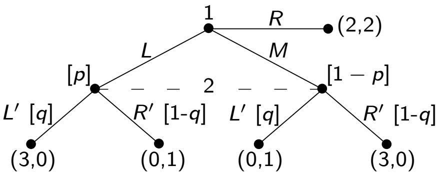
Given his beliefs, find P1's expected utility of L, M, and R respectively.



PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

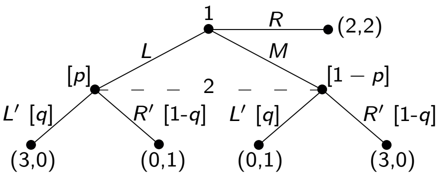
$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

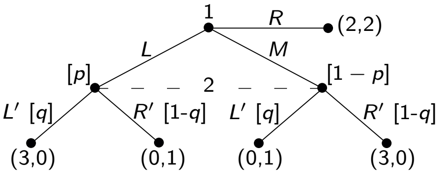
$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

Draw the expected utility of each choice as functions of q.

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



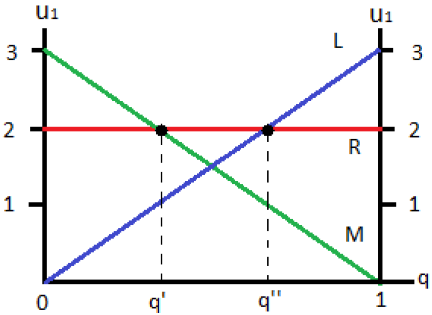
Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

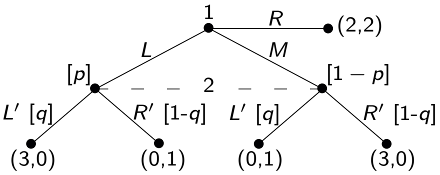
$$\mathbb{E}[u_1(R)] = 2$$



PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



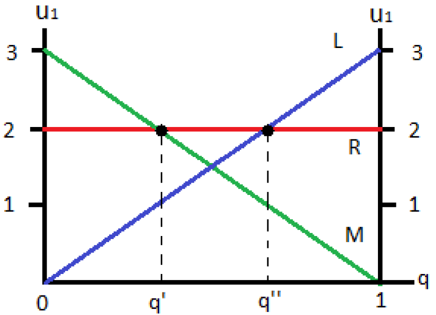
Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$



Using the diagram and the expected utility functions, find the intersections q' **and** q''**.**

PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

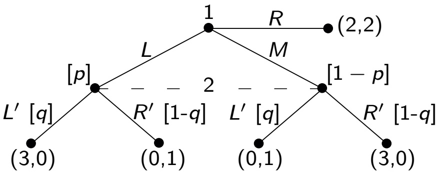
$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

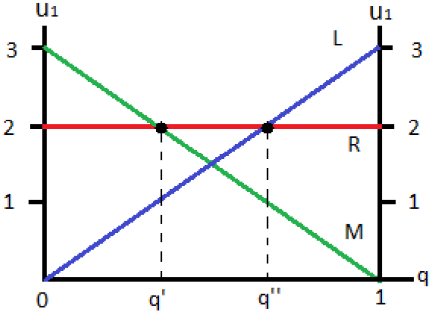
$$\mathbb{E}[u_1(R)] = 2$$

P1 is indifferent at q' and q'':

$\mathbb{E}[u_1(M)]=\mathbb{E}[u_1(R)] \Rightarrow$ 3-3q=2 $\Rightarrow q'$=1/3

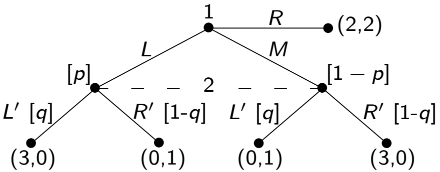
$\mathbb{E}[u_1(L)]=\mathbb{E}[u_1(R)] \Rightarrow 3q$=2 $\Rightarrow q''$=2/3





(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



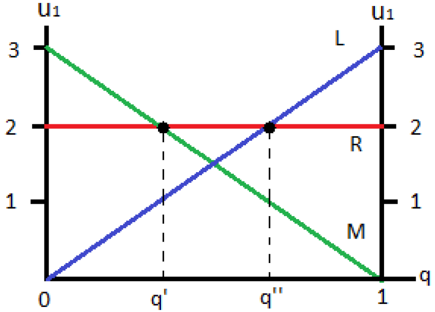
Assume that P2 plays L' and R' with probability q and $1-q$ respectively.

Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3-3q$$

$$\mathbb{E}[u_1(R)] = 2$$



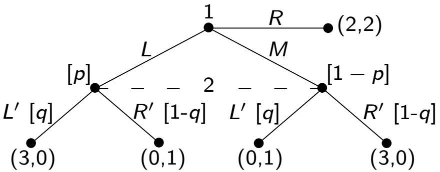
P1 is indifferent at q' and q'':
$\mathbb{E}[u_1(M)]$=$\mathbb{E}[u_1(R)] \Rightarrow$ 3-3q=2 $\Rightarrow q'$=1/3

$\mathbb{E}[u_1(L)]$=$\mathbb{E}[u_1(R)] \Rightarrow$ 3q=2 $\Rightarrow q''$=2/3

Writing up the BR_2's, are they consistent with the beliefs of P1 for the relevant interval $[0, q']$, $[q', q'']$, $[q'', 1]$?

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

P1 is indifferent at q' and q'':

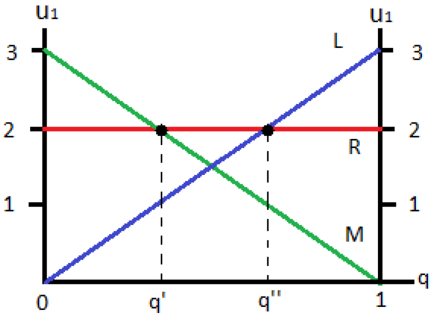
$\mathbb{E}[u_1(M)]=\mathbb{E}[u_1(R)] \Rightarrow$ 3-3q=2 $\Rightarrow q'$=1/3

$\mathbb{E}[u_1(L)]=\mathbb{E}[u_1(R)] \Rightarrow$ 3q=2 $\Rightarrow q''$=2/3

$q \leq 1/3$: $BR_2(M)=L'$ → P1 deviates to L

$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R → does P2 mix?

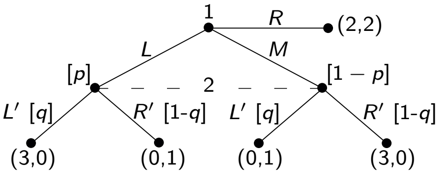
$q \geq 2/3$: $BR_2(L)=R'$ → P2 deviates to M



Find the beliefs p such that P2 is indifferent between L' and R'.

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

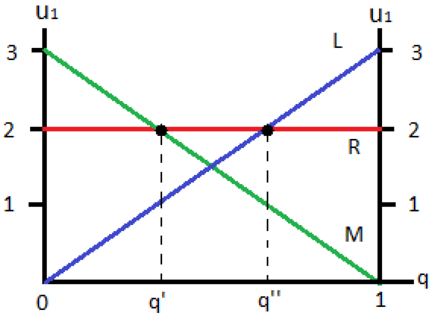


Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$



P1 is indifferent at q' and q'':

$\mathbb{E}[u_1(M)]$=$\mathbb{E}[u_1(R)] \Rightarrow$ 3-3q=2 $\Rightarrow q'$=1/3

$\mathbb{E}[u_1(L)]$=$\mathbb{E}[u_1(R)] \Rightarrow$ 3q=2 $\Rightarrow q''$=2/3

$q \leq 1/3$: $BR_2(M)$=L' $\rightarrow$ P1 deviates to L

$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R $\rightarrow$ does P2 mix?

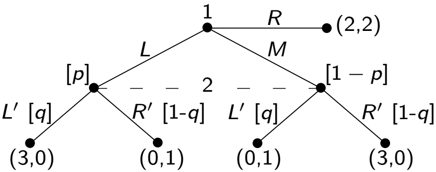
$q \geq 2/3$: $BR_2(L)$=R' $\rightarrow$ P2 deviates to M

P2 is indifferent if she believes that:

$\mathbb{E}[u_2(L')]$=$\mathbb{E}[u_1(R')] \Rightarrow$ 0p+1[1-p]=1p+0[1-p] $\Rightarrow$ 1-p=p $\Rightarrow p^*$=1/2

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



Given his beliefs, find P1's expected utility of L, M, and R respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

P1 is indifferent at q' and q'':

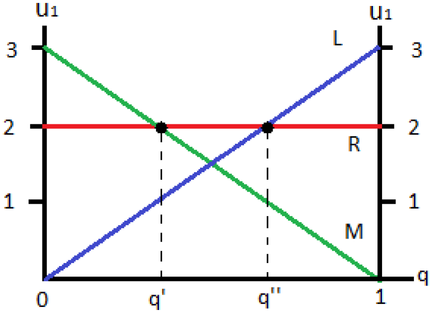
$\mathbb{E}[u_1(M)]=\mathbb{E}[u_1(R)] \Rightarrow$ 3-3q=2 $\Rightarrow q'$=1/3

$\mathbb{E}[u_1(L)]=\mathbb{E}[u_1(R)] \Rightarrow$ 3q=2 $\Rightarrow q''$=2/3

$q \leq 1/3$: $BR_2(M)$=L' $\rightarrow$ P1 deviates to L

$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R $\rightarrow$ does P2 mix?

$q \geq 2/3$: $BR_2(L)$=R' $\rightarrow$ P2 deviates to M



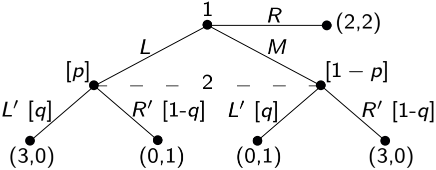
P2 is indifferent if she believes that:

$\mathbb{E}[u_2(L')]=\mathbb{E}[u_1(R')] \Rightarrow$ 1-p=p $\Rightarrow p^*$=1/2

Is $p = \frac{1}{2}$ compatible with $q \in \left[\frac{1}{3}, \frac{2}{3}\right]$?

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



P1's expected utility of L, M, and R:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

P1 is indifferent at q' and q'':

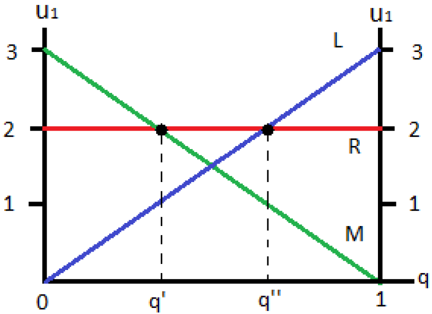
$\mathbb{E}[u_1(M)]$=$\mathbb{E}[u_1(R)]$ $\Rightarrow$ 3-3q=2 $\Rightarrow$ q'=1/3

$\mathbb{E}[u_1(L)]$=$\mathbb{E}[u_1(R)]$ $\Rightarrow$ 3q=2 $\Rightarrow$ q''=2/3

$q \leq 1/3$: $BR_2(M)$=L' $\rightarrow$ P1 deviates to L

$q\in\left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R $\rightarrow$ does P2 mix?

$q \geq 2/3$: $BR_2(L)$=R' $\rightarrow$ P2 deviates to M



P2 is indifferent if she believes that:

$\mathbb{E}[u_2(L')]$=$\mathbb{E}[u_1(R')]$ $\Rightarrow$ 1-p=p $\Rightarrow$ p^*=1/2

$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R. $p = \frac{1}{2}$: P2 mixes.

If compatible, write up the mixed-strategy PBE.

PS10, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

P1's expected utility of L, M, and R:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1-q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1-q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)] = 2$$

P1 is indifferent at q' and q'':

$\mathbb{E}[u_1(M)]$=$\mathbb{E}[u_1(R)]$ $\Rightarrow$ 3-3q=2 $\Rightarrow$ q'=1/3

$\mathbb{E}[u_1(L)]$=$\mathbb{E}[u_1(R)]$ $\Rightarrow$ 3q=2 $\Rightarrow$ q''=2/3

$q \leq 1/3$: $BR_2(M)$=L' $\rightarrow$ P1 deviates to L

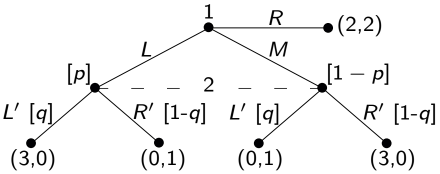
$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R $\rightarrow$ does P2 mix?

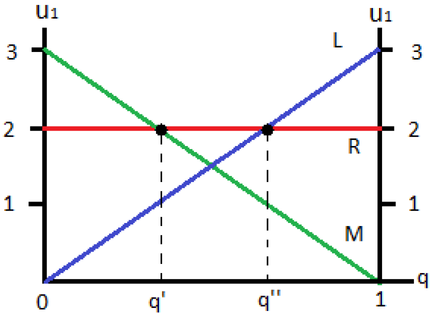
$q \geq 2/3$: $BR_2(L)$=R' $\rightarrow$ P2 deviates to M

P2 is indifferent if she believes that:

$\mathbb{E}[u_2(L')]$=$\mathbb{E}[u_1(R')]$ $\Rightarrow$ 1-p=p $\Rightarrow$ p^*=1/2

$q \in \left[\frac{1}{3}, \frac{2}{3}\right]$: P1 plays R. $p = \frac{1}{2}$: P2 mixes.





$$\boldsymbol{PBE} = \left\{ \left(\boldsymbol{R}, \boldsymbol{q} \in \left[\frac{1}{3}, \frac{2}{3}\right] \right), \boldsymbol{p} = \frac{1}{2} \right\}$$

PS10, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

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$$\begin{aligned}
\mathbb{E}[u_1(v_1,p)] &= \mathbb{P}[\text{P2 buys}](p-v_1)s + \mathbb{P}[\text{P2 sells}](v_1-p)(1-s) \\
&= \mathbb{P}[v_2 > p](p-v_1)s + \mathbb{P}[v_2 < p](v_1-p)(1-s) \\
&= (1-\mathbb{P}[v_2 < p])(p-v_1)s + \mathbb{P}[v_2 < p](v_1-p)(1-s) \\
&= \left(1-\frac{p-0}{1-0}\right)(p-v_1)s + \frac{p-0}{1-0}(v_1-p)(1-s), \qquad \text{using the CDF} \\
&= (1-p)(p-v_1)s + p(v_1-p)(1-s) \\
&= ps - v_1 s - p^2 s + pv_1 s + pv_1 - pv_1 s - p^2 + p^2 s
\end{aligned}$$

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3. As the quadratic term is negative, $(*)$ is a parabola which opens downward from the vertex p^*.
   I.e. p^* is the maximum of $(*)$, thus:

$$\frac{\delta \mathbb{E}[u_1(v_1, p)]}{\delta p} = 0 \Rightarrow$$

$$s + v_1 - 2p = 0 \Leftrightarrow$$

$$s + v_1 = 2p \Leftrightarrow$$

$$p^* = \frac{s + v_1}{2}$$