PS3, Ex. 5: Luxembourg as a rogue state

Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican (V) and Denmark (D) are preparing an attack on Luxembourg's nuclear research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_v s_D,$$

where $s_i \in [0, 1]$ is the share of its military capacity that country i ($i \in \{V, D\}$) uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country i is

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost.

(a) Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium (NE) of this game.

(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

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(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \ \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

$$s_i^* = \frac{1}{3} \equiv s^{NE}$$

i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$FOC: \ \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{S} = 0$$

$$\bar{S} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

i.e. the SO is higher than the NE as the positive externality is not rewarded, leading to an incentive to free ride.