# Microeconomics III: Problem Set 11[a]

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[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

# Outline

PS11, Ex. 1 (A): Effect of the GED education as a signal

PS11, Ex. 2 (A): Asymmetric information (PBE)

Signaling games in general

PS11, Ex. 3: Signaling game (pooling and separating PBE)

PS11, Ex. 4: Signaling games (pooling and separating PBE)

PS11, Ex. 5: Signaling games (pooling PBE)

PS11, Ex. 6: Spence's education signaling model (PBE)

## PS11, Ex. 1 (A): Effect of the GED education as a signal

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Does signaling work? Read the article by Tyler, Murnane and Willett and think about their results. What is their hypothesis for why they do not find an effect for minority groups? Come up with an example of an education program that has mostly signaling value in your country.

*(This is a reflection question, no answer will be provided).*

## PS11, Ex. 2 (A): Asymmetric information (PBE)

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Step 1: **Consider the uniform distribution $x \sim U(a, b)$. Use the cumulative distribution function (CDF) to write up the probability that a random draw of $x$ is lower than a constant $c$. Use the mean to write up the expected value of a random draw of $x$ where $x$ is lower than a constant $c \in [a, b]$.**

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1. Standard results for $x \sim U(a, b)$ :

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(x < c) = \frac{c-a}{b-a}$ (†)

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(x < c) = \frac{a+c}{2}$ (‡)

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$$\max_p \mathbb{P}[v_s < p]\mathbb{E}[v_b - p | v_s < p]$$

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$$\max_p \mathbb{P}[v_s < p]\mathbb{E}[v_b - p | v_s < p]$$

$$= \max_p \frac{p-0}{1-0}\mathbb{E}[kv_s - p | v_s < p] \qquad \text{using } (\dagger)$$

$$= \max_p p\left(k\mathbb{E}[v_s < p] - p\right)$$

$$= \max_p p\left(k\frac{0+p}{2} - p\right) \qquad \text{using } (\ddagger)$$

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2. $S_s(p, v_s) = \begin{cases} Sell & \text{if } p \geq v_s \\ Don't & \text{if } p < v_s \end{cases}$

3. $\max_p u_b(p,k) = \max_p p^2\left(\frac{k}{2} - 1\right)$

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$$\frac{\delta u_b(p,k)}{\delta p} = 0$$

$$2p\left(\frac{k}{2} - 1\right) = 0$$

$$2p\frac{k}{2} = 2p$$

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Step 4: Take the first-order condition.

Step 5: Maximize buyer's utility for $k < 2$.

Step 6: Maximize buyer's utility for $k > 2$.

Step 7: **Looking at the seller's strategy, will trade occur when $k > 2$? What about $k \in (1, 2)$? Have we seen something similar before?**

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Step 4: Take the first-order condition.

Step 5: Maximize buyer's utility for $k < 2$.

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Step 7: $\boldsymbol{k > 2}$: As $v_s \in [0,1] \leq 1 = p^{**}$, seller will always accept this price. **What about $\boldsymbol{k \in (1,2)}$? Have we seen something similar before?**

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Step 7: $\boldsymbol{k > 2}$: As $v_s \in [0,1] \leq 1 = p^{**}$, seller will always accept this price. $\boldsymbol{k \in (1,2)}$: Seller will not accept if $v_s > 0$, though trade would benefit both under perfect information. Similar to Akerlof's *'Lemons'*.

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6. For $k > 2$ : $p\frac{k}{2} > p \Rightarrow p^{**} = 1$

# Signaling games in general

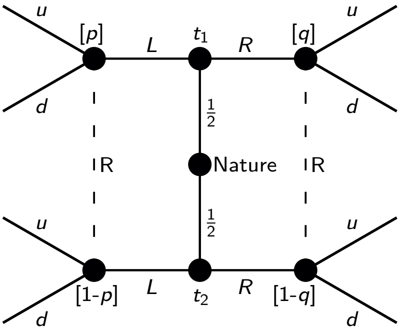
# PS11: Signaling games in general

**Players:**

- 2 players: Sender (S) and receiver (R). E.g. firm and consumer, or employer and employee (Spence).

**Timing:**

1. Nature chooses the sender's type from $T = \{t_1, ...\}$.
2. S: The sender realizes her type and sends a signal from $M = \{m_1, ...\}$, typically either $L$ (left) or $R$ (right).
3. R: The receiver observes $m$ (but not the type $t$!) and forms his beliefs:
   $\mu(t_1|L) = p$ and $\mu(t_1|R) = q$
   Consequently, for $S$ having two possible types:
   $\mu(t_2|L) = 1 - p$ and $\mu(t_2|R) = 1 - q$
4. R: The receiver chooses an action from $A = \{a_1, ...\}$, e.g. *up* or *down*.
5. Payoffs are realized.



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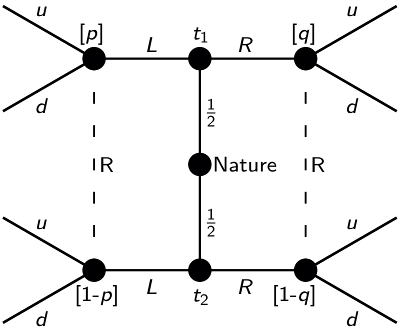
   $p = \mu(t_1|L)$ and $q = \mu(t_1|R)$
   Consequently, for $S$ having two possible types:
   $1 - p = \mu(t_2|L)$ and $1 - q = \mu(t_2|R)$
4. R: The receiver chooses an action from $A = \{a_1, ...\}$, e.g. *up* or *down*.
5. Payoffs are realized.

**Four possible equilibria for two types:**

- Pooling on $L$ or pooling on $R$.
- Separating: $t_1$ plays $L$ and $t_2$ plays $R$ or the other way around.



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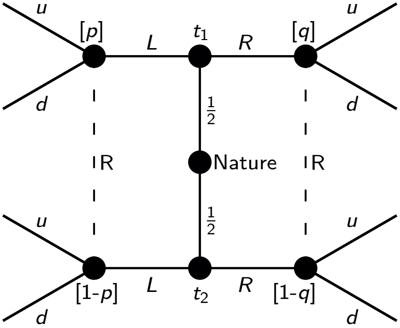



**Cookbook: For each possible equilibrium go over signaling requirements 3 and 2:**

- SR3: R: Find the beliefs $p, q$ given S's eq. strategy. (Only consider beliefs that are consistent with S's eq. strategy.)
- SR2R: R: Given beliefs, find $a(m_j|\mu(t_1|m_j))$.
- SR2S: S: Does $t_1$ or $t_2$ want to deviate?
- PBE: No deviation $\rightarrow$ PBE. Pooling on $L$: Find off-eq. $a(R|q) \rightarrow$ possibly two different PBE for different $q$.

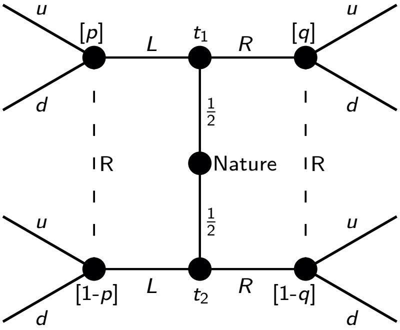
# PS11, Ex. 3: Signaling game (pooling and separating PBE)

# PS11, Ex. 3: Signaling game (pooling and separating PBE)

Consider the signaling game in Figure 1.

(a) Suppose there is a pooling PBE where the Sender sends message $L$ regardless of his type. What are the beliefs in this equilibrium?

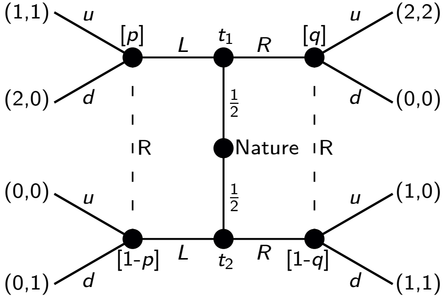
(b) Consider a possible separating PBE where $t_1$ sends message $R$, $t_2$ sends message $L$, and where the receiver chooses $u$ if and only if he receives message $L$. Can you write down payoffs for this game such that nobody has an incentive to deviate?



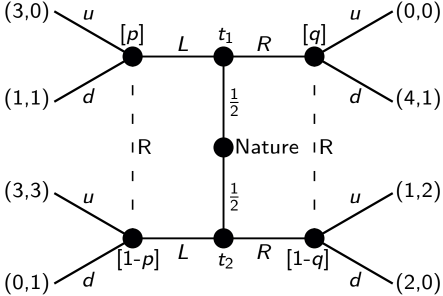
## PS11, Ex. 4: Signaling games (pooling and separating PBE)

# PS11, Ex. 4.a: Signaling games (pooling and separating PBE)

Exercise 4.4.a in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.



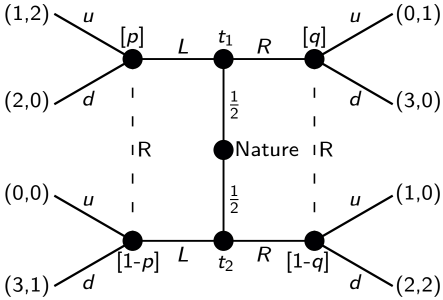
Exercise 4.4.b in Gibbons (p. 248). Describe all the pure-strategy pooling and separating perfect Bayesian equilibria in the following signaling game.



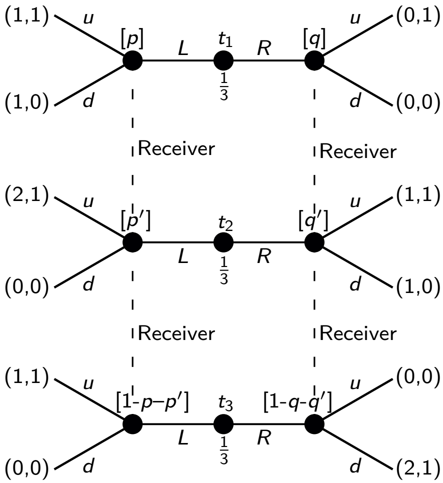
# PS11, Ex. 5: Signaling games (pooling PBE)

## PS11, Ex. 5.a: Signaling games (pooling PBE)

Exercise 4.3.a in Gibbons (p. 246). Specify a pooling perfect Bayesian equilibria in which both Sender types play $R$ in the following signaling game.



Exercise 4.3.b in Gibbons (p. 246). The following three-type signaling game begins with a move by nature, not shown in the tree, that yields one of the three types with equal probability Specify a pooling perfect Bayesian equilibria in which all three Sender types play $L$.



# PS11, Ex. 6: Spence's education signaling model (PBE)

# PS11, Ex. 6: Spence's education signaling model (PBE)

Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type $\theta$, which measures their ability. There are two worker types: $\theta \in \{\theta_L, \theta_H\}$. Nature chooses the worker's type, with $p_H = \mathbb{P}[\theta = \theta_H]$ and $p_H = \mathbb{P}[\theta = \theta_H] = 1 - p_H$.

The worker observes his own type, but the firm does not. The worker can choose his level of education: $e \in \mathbb{R}^+$. The cost to him of acquiring this education is $c_\theta(e) = e/\theta$. Education is observed by the firm, who then forms beliefs about the workers type: $\mu(\theta|e)$. We assume that the marginal productivity of a worker is equal to his ability and that the company is in competition such it pays the marginal productivity: $w(e) = \mathbb{E}[\theta|e]$. Thus, the payoff to a worker conditional on his type and education is $u_\theta(e) = w(e) - c_\theta(e)$. Suppose for this exercise that $\theta_H = 3$ and $\theta_L = 1$.

(a) Find a separating pure strategy Perfect Bayesian Equilibrium.

(b) Find a pooling pure strategy Perfect Bayesian Equilibrium.