Microeconomics III: Session 4

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Outline

PS4, Ex. 1 (A): MSNE and best-response functions

Form a group for each table:

- Get prepared to answer exercise 1 in problem set 4 as a team (5 min).

Find all equilibria (pure and mixed) in the following games, first analytically and then
through plotting the best-response functions.

(a)   (b)

<table>
<tr><td></td><td></td><td colspan="2" align="center">Player 2</td></tr>
<tr><td></td><td></td><td align="center">L (q)</td><td align="center">L (1-q)</td></tr>
<tr><td rowspan="2">Player 1</td><td>T (p)</td><td align="center">3, 3</td><td align="center">0, 0</td></tr>
<tr><td>B (1-p)</td><td align="center">0, 0</td><td align="center">4, 4</td></tr>
</table>

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<tr><td>B (1-p)</td><td align="center">1, 0</td><td align="center">2, 1</td></tr>
</table>

(a) Find all equilibria (pure and mixed), first analytically and then through plotting the BR functions.

	Player 2	
	L (q)	L (1-q)
T (p)	3, 3	0, 0
B (1-p)	0, 0	4, 4

Player 1 (left label, rows T (p), B (1-p))

Find q such that Player 1 expect to have equal payoffs from playing T and B:

$$E[u_1 | T] = E[u_1 | B]$$
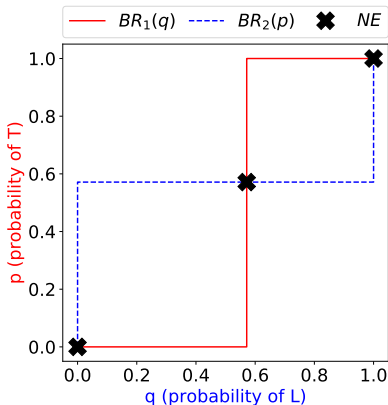
$$3q = 4(1 - q) \Rightarrow q = \frac{4}{7}$$

The players have symmetric payoffs, thus:

$$NE = (p^*, q^*) = \left\{ (0,0); (1,1); \left(\frac{4}{7}, \frac{4}{7} \right) \right\}$$

Plot the BR functions:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 4/7 \\ p \in [0,1] & \text{if} \quad q = 4/7 \\ p = 1 & \text{if} \quad q > 4/7 \end{cases}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 4/7 \\ q \in [0,1] & \text{if} \quad p = 4/7 \\ q = 1 & \text{if} \quad p > 4/7 \end{cases}$$

(b) Find all NE, first analytically:

	Player 2	
	L (q)	L (1-q)
T (p)	1, 1	0, 0
B (1-p)	1, 0	2, 1

Player 1 is indifferent for:

$$E[u_1|T] = E[u_1|B]$$
$$q = q + 2(1 - q) \Rightarrow q = 1$$

Player 2 is indifferent for:

$$E[u_2|L] = E[u_2|R]$$
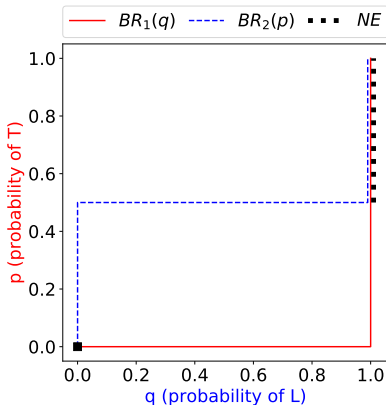$$p = 1 - p \Rightarrow p = \frac{1}{2}$$

The pure and mixed strategy NE are:

$$NE : \left\{ (0,0); (1,1); \left(p \in \left[\frac{1}{2}, 1 \right], q = 1 \right) \right\}$$

Then through plotting the BR functions:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0, 1] & \text{if} \quad q = 1 \end{cases}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < 1/2 \\ q \in [0, 1] & \text{if} \quad p = 1/2 \\ q = 1 & \text{if} \quad p > 1/2 \end{cases}$$

PS4, Ex. 3: The Focal Point

Thomas and Alice want to meet on a Friday night. There are two bars in their home town: "The Focal Point" and "The Other Place". They have to decide independently where they go. If they meet in the same bar, they both get utility of 1. If they end up in different bars, they get utility of 0.

(a) Find all equilibria (pure and mixed). Which equilibrium do you consider the most realistic? Where would you go if you were one of them?

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Thomas gets a payoff of 1 if he meets Alice, and -1 otherwise. Alice gets a payoff of -1 for meeting Thomas, and 1 otherwise. Find all equilibria (pure and mixed).

(c) Now assume again that Thomas and Alice both want to meet (so that payoffs are as in part (a)), but now there are N bars in town, where N can be very large. Show that there are $2^N - 1$ equilibria (pure and mixed). Say that the bars have names: "The First Bar in Town", "The Second Bar in Town", and so on. Which equilibrium is the most realistic?

LANSING, IOWA

Thomas and Alice want to meet on a Friday night. There are two bars in their home town: "The Focal Point" and "The Other Place". They have to decide independently where they go. If they meet in the same bar, they both get utility of 1. If they end up in different bars, they get utility of 0.

(a) Find all equilibria (pure and mixed). Which equilibrium do you consider the most realistic? Where would you go if you were one of them?

		Thomas	
		F (q)	O (1-q)
Alice	F (p)	1, 1	0, 0
	O (1-p)	0, 0	1, 1

Alice is indifferent for:

$$E[u_A|Focal] = E[u_A|Other]$$
$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Taking advantage of symmetry:

$$NE = (p^*, q^*) = \left\{ (0,0); \left(\frac{1}{2}, \frac{1}{2} \right) ; (1,1) \right\}$$

Which is the most realistic?

$\left(\frac{1}{2}, \frac{1}{2} \right)$ seems unlikely as expected payoff is $\frac{1}{2}$ while it is 1 for $(0,0)$ and $(1,1)$.

Where would you go?

I would go to the "The Focal Point" - it sounds like the place to meet.

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Thomas gets a payoff of 1 if he meets Alice, and -1 otherwise. Alice gets a payoff of -1 for meeting Thomas, and 1 otherwise. Find all equilibria (pure and mixed).

		Thomas	
		F (q)	O (1-q)
Alice	F (p)	-1, 1	1, -1
	O (1-p)	1, -1	-1, 1

No PSNE. The BR functions are:

$$BR_A(q) = \begin{cases} p = 1 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 0 & \text{if } q > 1/2 \end{cases}$$

$$BR_T(p) = \begin{cases} q = 0 & \text{if } p < 1/2 \\ q \in [0,1] & \text{if } p = 1/2 \\ q = 1 & \text{if } p > 1/2 \end{cases}$$
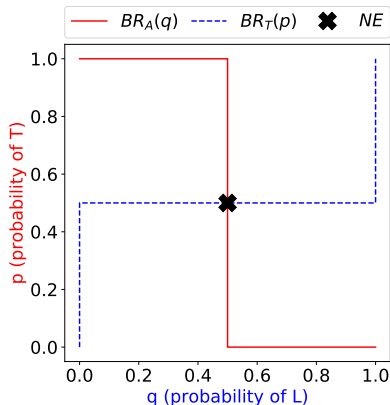
The only NE is the Mixed Strategy NE:

$$(p^*, q^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

(c) Now assume again that Thomas and Alice both want to meet (so that payoffs are as in part (a)), but now there are N bars in town, where N can be very large. Show that there are $2^N - 1$ equilibria (pure and mixed). Say that the bars have names: "The First Bar in Town", "The Second Bar in Town", and so on. Which equilibrium is the most realistic?

See "Problem Set 3 Solutions", ex. 8

PS4, Ex. 4: Generalized Battle of the Sexes

Consider the following Generalized Battle of the Sexes game, with $N > 1$:

	Player 2	
	C1 (q)	C2 (1-q)
C1 (p)	N, 1	0, 0
C2 (1-p)	0, 0	1, N

Player 1

(a) How can you interpret the parameter N?

(b) Solve for the mixed strategy Nash equilibrium (MSNE). When N becomes very large, what happens to the probability of successful coordination?

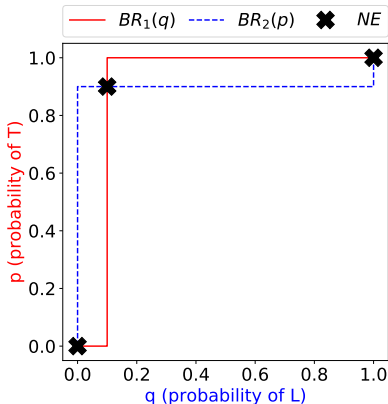|	Player 2	
	C1 (q)	C2 (1-q)
Player 1 C1 (p)	N, 1	0, 0
Player 1 C2 (1-p)	0, 0	1, N

(a) How can you interpret the N?

(b) Solve for the MSNE.

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < \frac{1}{1+N} \\ p \in [0, 1] & \text{if} \quad q = \frac{1}{1+N} \\ p = 1 & \text{if} \quad q > \frac{1}{1+N} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if} \quad p < \frac{N}{1+N} \\ q \in [0, 1] & \text{if} \quad p = \frac{N}{1+N} \\ q = 0 & \text{if} \quad p > \frac{N}{1+N} \end{cases}$$

$$NE : (p^*, q^*) = \left\{ (0, 0); (1, 1); \left(\frac{1}{2}, \frac{1}{2} \right) \right\}$$
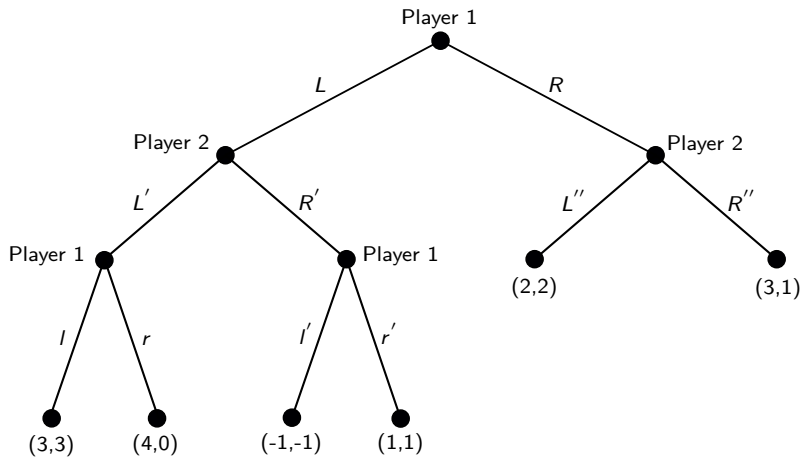
(example plot for $N = 1$)

When $N \to \infty$, what happens to the probability of successful coordination?
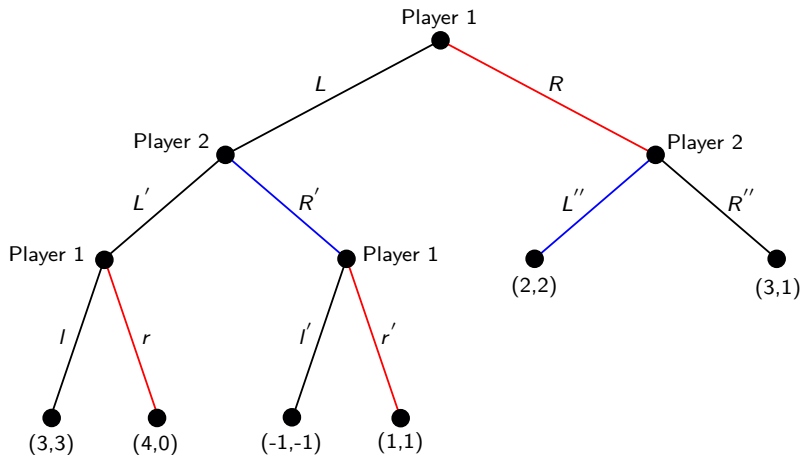
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Take Home Assignment 1: Common issues

(3a) Backwards Induction (BI)

(3a) Backwards Induction (BI)

The BI solution is the complete strategy profile - on and off the equilibrium path:

(*preferred strategies for player* 1, *preferred strategies for player* 2) $= (Rrr', R'L'')$

PS4, Ex. 5: North-Atlantic, 1943

North-Atlantic, 1943. An allied convoy, counting 100 ships, is heading east and it can choose between a northern route where icebergs are known to be numerous or a more southern route. The northern route is dangerous - because of the icebergs - and it is estimated that 6 ships will get lost due to icebergs. Below the surface, the wolf-pack lures. If the u-boats catch the convoy on the southern route, it is a field day, and 40 ships from the convoy are estimated to get lost. If the u-boats catch the convoy on the northern route, they do not have as much time hunting down the convoy - due to petrol shortages - and they are only expected to be able to sink 20 ships from the convoy. The wolf-pack does not have time to check both locations, north and south. Each headquarter (allied or nazi) has to decide whether to go north or south. Unfortunately, there is no radar etc, so one cannot observe the move of the enemy before taking a decision. Each headquarter has a simple payoff function. For the allied headquarter it equals the number of ships making it across the Atlantic. For the nazi headquarter payoff equals the number of ships lost by the allies.

(a) Write down this strategic situation in a bi-matrix.
(b) Find the Nash Equilibrium (equilibria?)
(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

PS4, Ex. 5: North-Atlantic, 1943

(a) It's a zero-sum (100-sum) type game:

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

(b) There are no PSNE, find the MSNE:

The Allied are indifferent for:

$$E[u_A|North] = E[u_A|South]$$

$$74q + 94(1-q) = 100q + 60(1-q) \Rightarrow q = \frac{17}{30}$$

The Nazis are indifferent for:

$$E[u_N|North] = E[u_N|South]$$

$$26p = 6p + 40(1-p) \Rightarrow q = \frac{2}{3}$$

$$\Rightarrow NE = (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30} \right)$$

(c) In equilibrium, the expected number of ships that make it across the Atlantic:

$$u_A(p^*, q^*) = p^* u_A(North) + (1 - p^*)u_A(South)$$
$$= \frac{2}{3}(74q + 94(1 - q)) + \left(1 - \frac{2}{3}\right)(100q + 60(1 - q))$$
$$= \vdots$$
$$\approx 82.67$$

PS4, Ex. 6: Stopping the bike thief

As in Problem Set 2, there are $N \geq 2$ people observing someone trying to steal a parked bike. Each of the witnesses would like the thief to be stopped, but prefers not to do it him/herself (because it is unpleasant and perhaps even dangerous). More precisely, if the thief is stopped by someone else, each of the witnesses gets a utility of $v > 0$. Every person who stops the thief gets a utility of $v - c > 0$, where c is the cost of interaction with the thief. Finally, if nobody stops the thief and the bike gets stolen, every witness gets a utility of 0. The witnesses decide whether or not to stop the thief simultaneously and independently.

a) Solve for a symmetric mixed strategy equilibrium of this game, where each witness stops the thief with probability $p \in (0, 1)$.

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?
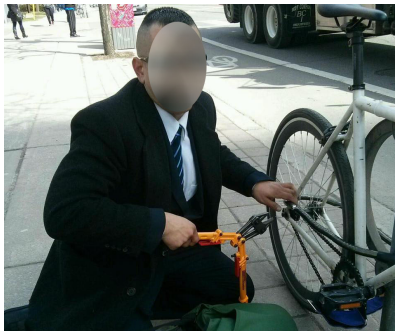
a) Solve for a symmetric mixed strategy equilibrium of this game, where each
   witness stops the thief with probability $p \in (0, 1)$.

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?

PS4, Ex. 7: To keep or split (backwards induction)

Consider the following 2 × 2 game where payoffs are monetary:

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

Before this game is played, Player 1 can choose whether, after the game is played, players should keep their own payoffs or split the aggregate payoff evenly between them.

(a) Draw the game tree of this two-stage game (assuming that Players 1's choice of whether to split payoffs is revealed to Player 2 before the second stage).

(b) Solve by backwards induction.

(a) Draw the game tree:

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

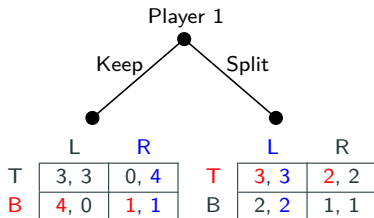2nd stage: They play the static game and payoffs are realized.

PS4, Ex. 7: To keep or split (backwards induction)

(a) Draw the game tree:

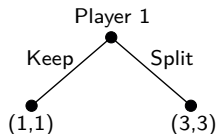1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

2nd stage: They play the static game and payoffs are realized.

Player 1

Keep / Split

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L	R
T	3, 3	2, 2
B	2, 2	1, 1
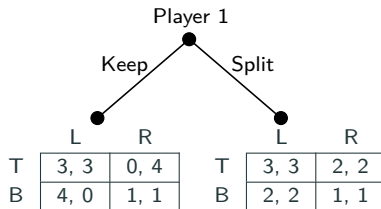
(b) Solve by backwards induction:

2nd stage: Each bi-matrix has a unique NE that can be founds using IESDS.

1st stage: Player 1's choice can be reduced to choosing between the subgame NE in each bi-matrix:

Player 1

Keep / Split

(1,1)   (3,3)

BI gives the subgame perfect NE:
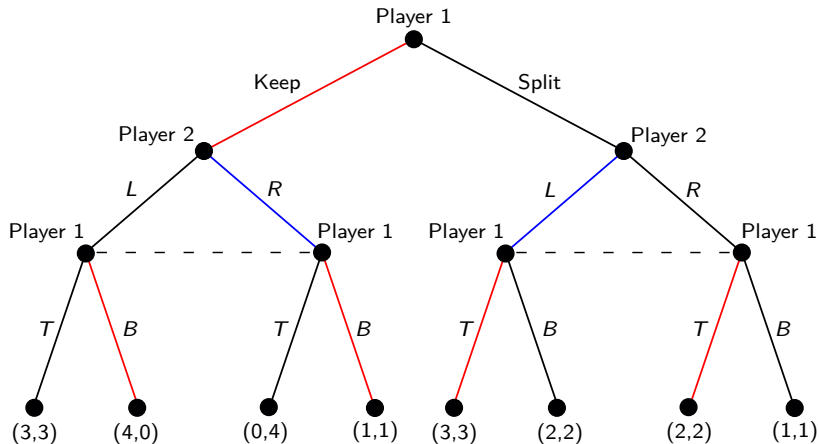
$$SPNE = (Keep\ B\ T,\ R\ L)$$

PS4, Ex. 7: To keep or split (backwards induction)

Alternatively, draw the game tree in extensive form to find $SPNE = (Keep\ B\ T,\ R\ L)$.

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

2nd stage: They play the static game and payoffs are realized (doesn't matter who "moves first" as one's actions are private information till the other has moved).

PS3, Ex. 5: Luxembourg as a rogue state

Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican (V) and Denmark (D) are preparing an attack on Luxembourg's nuclear research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_V s_D,$$

where $s_i \in [0, 1]$ is the share of its military capacity that country i ($i \in \{V, D\}$) uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country i is

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost.

(a) Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium (NE) of this game.

(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

PS3, Ex. 5: Luxembourg as a rogue state

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \quad \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

$$s_i^* = \frac{1}{3} \equiv s^{NE}$$

i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

PS3, Ex. 5: Luxembourg as a rogue state

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Expected payoff for player $i \neq j$:

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Taking advantage of symmetry $s_i^* = s_j^*$:

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i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

(b) Find the SO given shares are equal:

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \quad \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

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i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

PS3, Ex. 5: Luxembourg as a rogue state

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \quad \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

$$s_i^* = \frac{1}{3} \equiv s^{NE}$$

i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

PS3, Ex. 5: Luxembourg as a rogue state

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

$$FOC: \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

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i.e. $NE = \left\{ (s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3}) \right\}$

(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$FOC: \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{s} = 0$$

$$\bar{s} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

i.e. the SO is higher than the NE as the positive externality is not rewarded, leading to an incentive to free ride.

PS4, Ex. 8: Building a playground (Stackelberg)

Two neighbors are building a common playground for their children. The time spent on the project by neighbor i is $x_i \geq 0$, $i = 1, 2$. The resulting quality of the playground is

$$q(x_1, x_2) = x_1 + x_2 - x_1 x_2$$

Spending time on the project is costly. More precisely, the cost function of the neighbors are:

$$C_i(x_i) = x_i^2, \quad i = 1, 2$$

The payoff of neighbor i, U_i, is equal to the quality of the playground minus his cost.

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.