Microeconomics III: Session 3

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Outline

Kahoot!

PS3, Ex. 1 (A): Dominance and best response

PS3, Ex. 2 (A): Equilibrium selection

PS3, Ex. 3 (A): NE proof using IEWDS

PS3, Ex. 4 (A): Mixed strategy price competition

PS4, Ex. 2: Entry deterrence in a game tree

PS3, Ex. 6: Cournot Oligopoly with three firms

PS3, Ex. 8: Mixed Strategy Nash Equilibria - best-response functions

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

Kahoot!

Kahoot: A exercises

Form a group for each table:

- Get prepared to answer the A exercises as a team (10 min).

PS3, Ex. 1 (A): Dominance and best response

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1. (A) Show that for each of the following two games, the only Nash equilibrium is in pure strategies. Describe the intuition for this result. What do these two games have in common?

		Player 2	
		L	R
Player 1	U	5, 5	1, 6
	D	6, 1	2, 2

(D, R) is a unique Pure Strategy Nash Equilibrium (PSNE). The game is a Prisoner's Dilemma as it fulfills:

$$T > R > P > S \Leftrightarrow 6 > 5 > 2 > 1$$

i.e. the **T**emptation to deviate (6) is greater than the **R**eward for cooperating on the socially optimal outcome (5) and the **P**unishment payoff (2) is greater than the "**S**ucker's" payoff (1).

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			Player 2	
		L	C	R
Player 1	U	1, 0	1, 2	0, 1
	D	0, 3	0, 1	2, 0

(U, C) is a unique Pure Strategy Nash Equilibrium (PSNE) as no other combination of (mixed or pure) strategies gives as high payoffs.

Iterated Elimination of Strictly Dominated Strategies (IESDS) leads to the same outcome as the best responses (eliminate R then D and lastly L).

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Iterated Elimination of Strictly Dominated Strategies (IESDS) leads to the same outcome as the best responses (eliminate R then D and lastly L).

In a mixed strategy NE both players must be indifferent between their respective pure strategies. This is impossible if one of the strategies are strictly dominated.

As both games can be solved by IESDS they both have a unique PSNE.

PS3, Ex. 2 (A): Equilibrium selection

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2. (A) Solve for all pure strategy Nash equilibria. Which equilibrium do you find most reasonable?

Player 2

		a	b	c
Player 1	A	2, 2	0, 0	-1, 2
	B	0, 0	0, 0	0, 0
	C	2, -1	0, 0	1, 1

$PSNE = \{(A, a), (B, b), (C, c)\}$.

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For **risk neutral** players (A, a) is the most reasonable as it maximizes payoff for both players.

For **risk averse** players avoiding A and a eliminates the risk of a negative payoff. (C, c) is more reasonable than (B, b) as the payoffs are positive.

PS3, Ex. 3 (A): NE proof using IEWDS

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3. (A) We have seen in the lectures that IESDS never eliminates a Nash Equilibrium. However, we saw in Problem Set 2 that this is not true if we do iterated elimination of weakly dominated strategies (IEWDS.) Go through the proof in the slides from lecture 2 and identify the step that is no longer true if we replace IESDS by IEWDS. That is, explain why the proof is no longer true when we replace ‘strict domination’ by ‘weak domination’.

PS3, Ex. 3 (A): NE proof using IEWDS

Informal proof: For the intuition, look at this example for now. At home, you can compare the two different formal proofs.

		Player 2	
		L	R
Player 1	U	3, 2	0, 0
	D	3, 0	1, 2

NE is any strategy where no player can be strictly better off by deviating:

$$NE = \{(U, L), (D, R)\}$$

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IESDS: If a strategy was strongly dominated there would be a clear incentive to deviate from it, thus, by definition the strategy could not be a NE.

IEWDS: In a NE where a player 1 is indifferent between the NE-payoff and her payoff from deviating, the NE-strategy can be weakly dominated if player 1's' alternative strategy would give a higher payoff in the case where player 2 deviates from his NE strategy as well.

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IESDS: If a strategy was strongly dominated there would be a clear incentive to deviate from it, thus, by definition the strategy could not be a NE.

IEWDS: In a NE where a player 1 is indifferent between the NE-payoff and her payoff from deviating, the NE-strategy can be weakly dominated if player 1's' alternative strategy would give a higher payoff in the case where player 2 deviates from his NE strategy as well.

E.g. Player 1 is indifferent between $u_1(U, L)$ and $u_1(D, L)$, however, $u_1(D, R) > u_1(U, R)$, i.e. D weakly dominates U and U can be eliminated. I.e. eliminating the NE (U, L), leaving behind the reduced form game:

		Player 2	
		L	R
Player 1	D	3, 0	1, 2

PS3, Ex. 3 (A): NE proof using IEWDS

Formal proof: The proof that all NE survive IESDS holds by contradiction. For the two-player case the IESDS proof is as follows:

- Let (s_1^*, s_2^*) be a NE.
- Say we carry out IESDS and s_1^* is the first NE strategy to be eliminated (in round n of elimination).
- Then there must be a strategy $s_1^{'} \neq s_1^*$ that strictly dominates s_1^*, i.e.

$$\forall s_2 \in S_2^n : \; u_1(s_1^*, s_2) < u_1(s_1^{'}, s_2) \quad (1)$$

where S_2^n is the set of player-2 strategies that have not been eliminated in rounds $1, ..., n-1$.

- Since $s_2^* \in S_2^n$, inequality (1) also means

$$u_1(s_1^*, s_2^*) < u_1(s_1^{'}, s_2^*)$$

- But (s_1^*, s_2^*) is a NE, so by definition

$$\forall s_1 \in S_1 : \; u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*)$$

- Contradiction! We can do the same for player 2. It follows that s_i^* survives IESDS for $i = 1, 2$.

PS3, Ex. 3 (A): NE proof using IEWDS

Formal proof: The proof that all NE survive IESDS holds by contradiction. We **highlight** where the contradiction breaks down using IEWDS instead:

- Let (s_1^*, s_2^*) be a NE.
- Say we carry out **IEWDS** and s_1^* is the first NE strategy to be eliminated (in round n of elimination).
- Then there must be a strategy $s_1^{'} \neq s_1^*$ that **weakly** dominates s_1^*, i.e.

$$\forall s_2 \in S_2^n : \; u_1(s_1^*, s_2) \underbrace{\leq}_{\textbf{Weak}} u_1(s_1^{'}, s_2) \quad (2)$$

and the inequality holds strictly for at least one strategy $s_2^{'} \in S_2^n$ where S_2^n is the set of player-2 strategies that have not been eliminated in rounds 1, ..., $n-1$.

- Since $s_2^* \in S_2^n$, inequality (2) also means

$$u_1(s_1^*, s_2^*) \underbrace{\leq}_{\textbf{Weak}} u_1(s_1^{'}, s_2^*)$$

- ~~But~~ (s_1^*, s_2^*) is a NE, so by definition

$$\forall s_1 \in S_1 : \; u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*)$$

- **No** contradiction!

Conclusion: for a NE (s_1^*, s_2^*) IEWDS can eliminate s_1^* if $s_1^{'}, s_2^{'}$ exist such that:

$$\text{for } s_1^{'} \in S_1^n : \; u_1(s_1^*, s_2^*) = u_1(s_1^{'}, s_2^*)$$

and

$$\text{for } s_2^{'} \in S_2^n : \; u_1(s_1^*, s_2^{'}) < u_1(s_1^{'}, s_2^{'})$$

PS3, Ex. 4 (A): Mixed strategy price competition

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4. (A). Consider price competition between two firms when some consumers are informed about prices and others are not. Firms have zero marginal cost and they set price simultaneously; for the sake of this example, assume each price can only take one of the following values: 80, 54, 38. The market consists of two consumers. The uninformed consumer will visit a firm at random (probabilities $\frac{1}{2}, \frac{1}{2}$) and buy from it, regardless of the price. The informed consumer will visit the firm with the lowest price and buy from it. If both firms set the same price, assume that the informed consumer picks a firm at random (probabilities $\frac{1}{2}, \frac{1}{2}$).

(a) Argue that this game can be represented by the following bimatrix.

	p_2=80	p_2=54	p_2=38
p_1=80	80, 80	40, 81	40, 57
p_1=54	81, 40	54, 54	27, 57
p_1=38	57, 40	57, 27	38, 38

(b) Show that there is no Nash equilibrium in pure strategies.

(c) Confirm that the following strategy profile is a Nash equilibrium: each firm plays price 80 with probability 0.232, price 54 with probability 0.361, and price 38 with probability 0.407.

(d) Why do you think the equilibrium is so different from the standard Bertrand pricing game (i.e. where competition drives price down to marginal cost)?

PS3, Ex. 4 (A): Mixed strategy price competition

(a) The game in normal form and bimatrix:

Players: *Firm 1, Firm 2*. Strategies: $p_i \in S_i = S = \{80, 54, 38\}$

Payoffs consist of *payoff from the informed consumer* + *payoff from the uninformed*.
I.e. payoffs for player $i \neq j$:

$$u_i(p_i, p_j) = \begin{cases} p_i + \frac{1}{2} p_i & \text{if} \quad p_i < p_j \\ \frac{1}{2} p_i + \frac{1}{2} p_i & \text{if} \quad p_i = p_j \\ 0 + \frac{1}{2} p_i & \text{if} \quad p_i > p_j \end{cases}$$

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Which can be represented as:

	p_j=80	p_j=54	p_j=38
p_i=80	80, -	$\frac{1}{2}80=40$, -	$\frac{1}{2}80=40$, -
p_i=54	$\frac{3}{2}54=81$, -	54, -	$\frac{1}{2}54=27$, -
p_i=38	$\frac{3}{2}80=57$, -	$\frac{3}{2}38=57$, -	38, -

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p_i=54	$\frac{3}{2}$54=81, -	54, -	$\frac{1}{2}$54=27, -
p_i=38	$\frac{3}{2}$80=57, -	$\frac{3}{2}$38=57, -	38, -

(b) Show that there is no Nash equilibrium in pure strategies:

		Firm 2		
		p_2=80	p_2=54	p_2=38
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PS3, Ex. 4 (A): Mixed strategy price competition

(c) Confirm that the following strategy profile is a Nash equilibrium: each firm plays price 80 with probability 0.232, price 54 with probability 0.361, and price 38 with probability 0.407.

Remember: In an equilibrium in mixed strategies, a player is indifferent between all pure strategies that she is choosing with positive probability.

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Check that firm i is indifferent between all pure strategies when the opposing firm's strategy is given by the probability distribution $\widehat{p_j} = (0.232, 0.361)$:

$$u_i(p_i = 80, \widehat{p_j}) = 0.232 \cdot 80 + 0.361 \cdot 40 + (1 - 0.232 - 0.361) \cdot 40 = 49.280 \approx 49.3$$

$$u_i(p_i = 54, \widehat{p_j}) = 0.232 \cdot 81 + 0.361 \cdot 54 + (1 - 0.232 - 0.361) \cdot 27 = 49.275 \approx 49.3$$

$$u_i(p_i = 38, \widehat{p_j}) = 0.232 \cdot 57 + 0.361 \cdot 57 + (1 - 0.232 - 0.361) \cdot 38 = 49.267 \approx 49.3$$

There are rounding errors as the exact mixed strategy profile is $\widehat{p_j} = \left(\frac{193}{833}, \frac{8127}{22491}\right)$.

PS3, Ex. 4 (A): Mixed strategy price competition

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In the standard Bertrand Oligopoly price competition would lead to the perfectly competitive outcome (price = marginal cost), here:

$$p_1^* = p_2^* = c = 0$$

Introduction of an uninformed consumer dampens the effect of price competition as a firm i can expect a revenue of at least $\frac{1}{2}p_i$ no matter what price p_i it sets.

Price competition could be increased by lowering the probability that an uninformed customer randomly picks the firm, i.e. through:

1. A higher share of informed customers.
2. More competing firms (moreover, increasing the pure price competition).

PS4, Ex. 2: Entry deterrence in a game tree

PS4, Ex. 2: Entry deterrence in a game tree

Consider the following dynamic game: firm 1 owns a shop in town A. Firm 2 decides whether to enter the market in town A. If firm 2 enters, firm 1 chooses whether to fight or accommodate the entrant. If firm 2 does not enter, firm 1 receives a profit of 2 and firm 2 gets 0. If firm 2 enters and firm 1 accommodates, they share the market and each of them receives a profit of 1. If firm 2 enters and firm 1 decides to fight, firm 2 suffers a loss of 1 (so that the payoff is -1), but fighting is costly for firm 1, lowering its payoff to 0.

(a) Draw the game tree.
(b) Solve the game by backwards induction.

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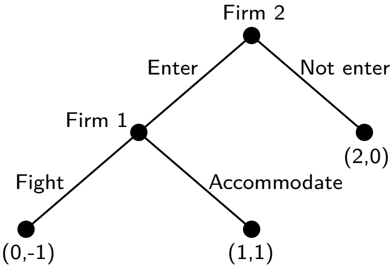

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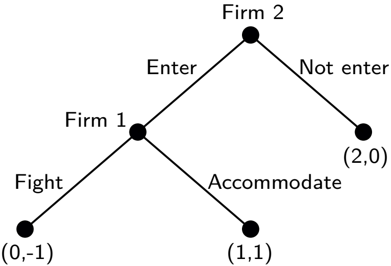
Starting from the bottom: If Firm 2 has entered the market in the 1st round, then Firm 1 can choose to either fight or accommodate in the 2nd round.



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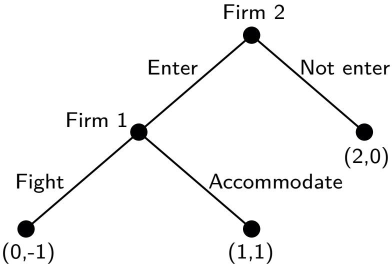
Firm 1 will always accommodate, as it is more costly to fight ($1 > 0$).



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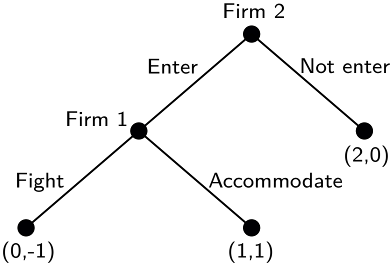
Knowing that Firm 1 is rational and will accommodate in the 2nd round, **Firm 2** (first mover), will always chose to enter in the 1st round ($1 > 0$), i.e. the backwards induction solution is the strategy profile:

$$(s_1, s_2) = (Accommodate, Enter)$$



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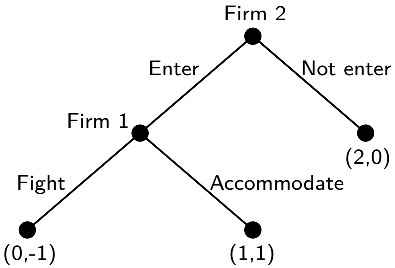
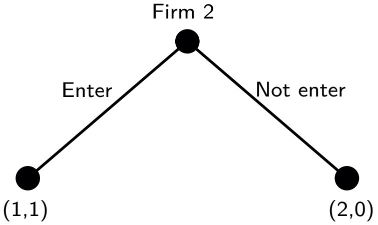
Intuition: Firm 2 has *first mover advantage*, thus, to "Fight" would not be a credible threat given Firm 1 is rational. I.e. Firm 2's decision can be reduced to the upper part of the game tree.





PS3, Ex. 6: Cournot Oligopoly with three firms

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There are three identical firms in an industry. Their production quantities are denoted q_1, q_2, and q_3. The inverse demand function is

$$p = 1 - Q, \text{ where } Q = q_1 + q_2 + q_3.$$

The marginal cost is zero.

(a) Compute the quantities in the Cournot equilibrium, i.e., the Nash Equilibrium of the game where the firms simultaneously choose quantities.
(b) What is the price in the Cournot-equilibrium?
(c) Show that if two of the three firms merge (transforming the industry into a duopoly), the profits of the merging firms decrease. Explain.
(d) What happens if all three firms merge?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

$$\pi_i = (1 - q_i - q_j - q_k)q_i$$

Best-Response (BR) function for firm i:

$$\frac{\delta \pi_i}{\delta q_i} = 1 - 2q_i - q_j - q_k = 0$$

$$q_i = \frac{1 - q_j - q_k}{2}$$

Due to symmetry $q_i^* = q_j^* = q_k^* = q^{NE}$:

$$q_i^* = \frac{1 - 2q_i^*}{2}$$

$$q_i^* = \frac{1}{4} \equiv q^{NE}$$

PS3, Ex. 6: Cournot Oligopoly with three firms

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(b) What is the price in the Cournot-equilibrium?

PS3, Ex. 6: Cournot Oligopoly with three firms

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(b) Price in the Cournot-equilibrium:

$$p^* = 1 - q_i^* - q_j^* - q_k^* = \frac{1}{4} \Rightarrow \pi_i^* = \frac{1}{16}$$

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(c) Show that if two of the three firms merge (transforming the industry into a duopoly), the profits of the merging firms decrease. Explain.

PS3, Ex. 6: Cournot Oligopoly with three firms

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(c) Firm *1* and *2* merge to firm m.

The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j) q_i$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

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The payoff function for firm $i \in \{\boldsymbol{m}, 3\}$:

$$\pi_i = (1 - q_i - q_j)q_i$$

BR function for firm i in the duopoly:

$$q_i = \frac{1 - q_j}{2}$$

$$q_i^* = \frac{1 - 2q_i^*}{2}, \quad q_i^* = q_j^*$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

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$$q_i^* = \frac{1 - 2q_i^*}{2}, \quad q_i^* = q_j^*$$

$$q_i^* = \frac{1}{3} \equiv q^{NE}$$

By merging the rivalry is internalized by reducing joint output which increase market price and the profit margin:

$$p^* = 1 - q_m^* - q_3^* = \frac{1}{3} \Rightarrow \pi_m^* = \pi_3^* = \frac{1}{9}$$

Are Firm *1* and *2* better or worse off? Why?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

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However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

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However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

(d) What happens if all three firms merge?

PS3, Ex. 6: Cournot Oligopoly with three firms

a) Quantities in the Cournot equilibrium

The payoff function for firm $i \in \{1, 2, 3\}$:

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However, Firm *1* and *2* each get $\frac{1}{18} < \frac{1}{16}$ and are worse off as the third firm reacts to the higher price by increasing output.

(d) A full merger maximizes joint profits:

$$q_{\text{monopoly}}^* = p_{\text{monopoly}}^* = \frac{1}{2} \Rightarrow \pi_{\text{monopoly}}^* = \frac{1}{4} > \frac{2}{9}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - best-response functions

PS3, Ex. 8: Mixed Strategy Nash Equilibria - best-response functions

Plot the mixed best responses of each player (in a "(p,q)-diagram" - see the textbook). And find all Nash equilibria (pure and mixed) in the games below

(a)

Player 2

		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

(b)

Player 2

		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

(c)

Player 2

		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

(d)

Player 2

		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (p_2)	1, 2	4, 3
	s_3 (1-p_1-p_2)	0, 1	0, 3

PS3, Ex. 8: Mixed Strategy Nash Equilibria - best-response functions

Plot the mixed best responses of each player (in a "(p,q)-diagram" - see the textbook). And find all Nash equilibria (pure and mixed) in the games below

(a)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
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(b)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

(c)

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

(d)

		Player 2	
		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (p_2)	1, 2	4, 3
	s_3 (1-p_1-p_2)	0, 1	0, 3

Hint: Find the probabilities q for which Player 1 is indifferent, e.g. $u_1(T, q) = u_1(B, q)$. and the probabilities p for which Player 2 is indifferent, e.g. $u_2(L, p) = u_2(R, p)$.

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria

(a) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria

(a) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
- Prefers B if $q < 1 \Rightarrow p = 0$.

Player 2:

- Indifferent if $p = 1 \Rightarrow q \in [0, 1]$
- Prefers R if $p < 1 \Rightarrow q = 0$.

i.e. two Pure Strategy Nash Equilibria:

$$PSNE = \{(T, L), (B, R)\}$$

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1

	L (q)	R (1-q)
T (p)	0, 0	0, 0
B (1-p)	0, 0	1, 1

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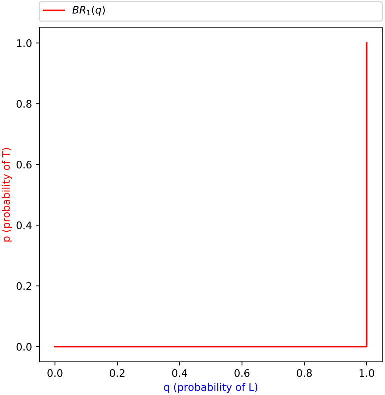

PS3, Ex. 7.a: Mixed Strategy Nash Equilibria

(a) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1		L (q)	R (1-q)
	T (p)	0, 0	0, 0
	B (1-p)	0, 0	1, 1

Player 1:

- Indifferent if $q = 1 \Rightarrow p \in [0, 1]$
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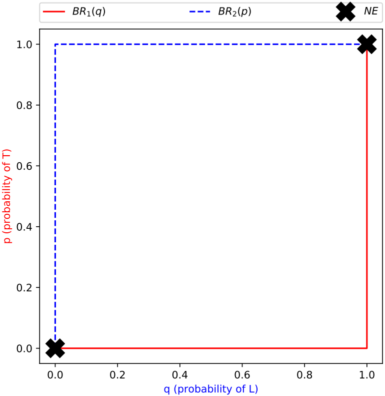

We only find the two Mixed Strategy NE (MSNE). Both coincide with the PSNE:

$$(p^*, q^*) = \{(1, 1), (0, 0)\}$$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$\begin{aligned} 1 &= 1q + 5(1-q) \\ 5q &= 4 \\ q &= 1 \end{aligned}$$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$
\begin{aligned}
1 &= 1q + 5(1-q) \\
5q &= 4 \\
q &= 1
\end{aligned}
$$

Player 2 is indifferent if:

$$
\begin{aligned}
3p + 1(1-p) &= 0p + 5(1-p) \\
7p &= 4 \\
p &= \frac{4}{7}
\end{aligned}
$$

i.e. two Pure Strategy Nash Equilibria:

$$PSNE = \{(T, L), (B, R)\}$$

PS3, Ex. 7.b: Mixed Strategy Nash Equilibria

(b) Plot the mixed best responses and find all NE (pure and mixed):

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

$$1 = 1q + 5(1-q)$$
$$5q = 4$$
$$q = 1$$

Player 2 is indifferent if:

$$3p + 1(1-p) = 0p + 5(1-p)$$
$$7p = 4$$
$$p = \frac{4}{7}$$

i.e. two Pure Strategy Nash Equilibria:

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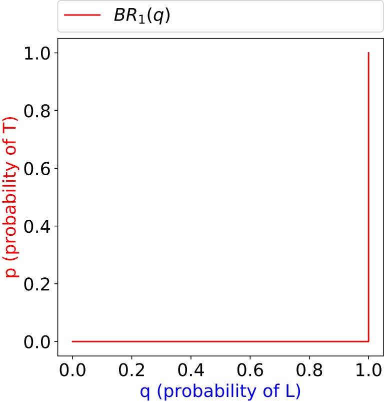


PS3, Ex. 7.b: Mixed Strategy Nash Equilibria

(b) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

		L (q)	R (1-q)
Player 1	T (p)	1, 3	1, 0
	B (1-p)	1, 1	5, 5

Player 1 is indifferent if:

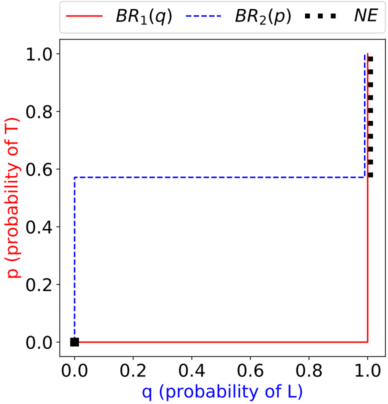
$$\begin{aligned} 1 &= 1q + 5(1-q) \\ 5q &= 4 \\ q &= 1 \end{aligned}$$

Player 2 is indifferent if:

$$\begin{aligned} 3p + 1(1-p) &= 0p + 5(1-p) \\ 7p &= 4 \\ p &= \frac{4}{7} \end{aligned}$$

i.e. two Pure Strategy Nash Equilibria:

$$PSNE = \{(T, L), (B, R)\}$$



From drawing, the two PSNE are contained in two Mixed Strategy Nash Equilibria (MSNE):

$$(p^*, q^*) = \{(0,0)\} \cup \left\{(p, 1) : p \in \left[\frac{4}{7}, 1\right]\right\}$$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 1		Player 2: L (q)	Player 2: R (1-q)
	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$

$$q = 0$$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 1		Player 2: L (q)	Player 2: R (1-q)
	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

$$3q + (1 - q) = (1 - q)$$
$$q = 0$$

Player 2 is indifferent if:

$$2p + (1 - p) = 2$$
$$p + 1 = 2$$
$$p = 1$$

i.e. three PSNE exist:

$$PSNE = \{(T, L), (T, R), (B, R)\}$$

PS3, Ex. 7.c: Mixed Strategy Nash Equilibria

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 2

Player 1	L (q)	R (1-q)
T (p)	3, 2	1, 2
B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

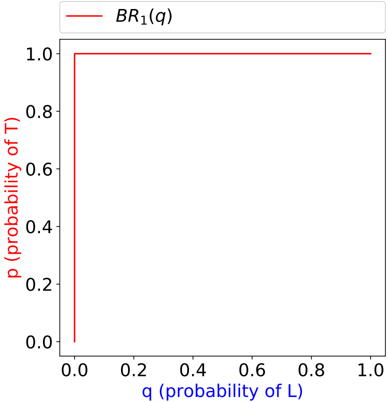
$$3q + (1 - q) = (1 - q)$$
$$q = 0$$

Player 2 is indifferent if:

$$2p + (1 - p) = 2$$
$$p + 1 = 2$$
$$p = 1$$

i.e. three PSNE exist:

$$PSNE = \{(T, L), (T, R), (B, R)\}$$



PS3, Ex. 7.c: Mixed Strategy Nash Equilibria

(c) Plot the mixed best responses and find all NE (pure and mixed):

Player 1		Player 2 L (q)	Player 2 R (1-q)
	T (p)	3, 2	1, 2
	B (1-p)	0, 1	1, 2

Player 1 is indifferent if:

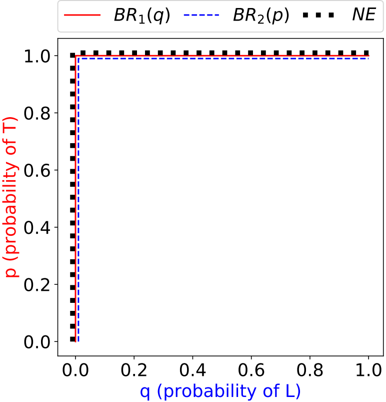
$$3q + (1-q) = (1-q)$$
$$q = 0$$

Player 2 is indifferent if:

$$2p + (1-p) = 2$$
$$p + 1 = 2$$
$$p = 1$$

i.e. three PSNE exist:

$$PSNE = \{(T, L), (T, R), (B, R)\}$$



From drawing, we find that the three PSNE are contained in just two MSNE:

$$(p^*, q^*) = \{(1, q) : q \in (0, 1]\} \cup \{(p, 0) : p \in [0, 1]\}$$

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated
and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

		Player 2	
		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (1-p_1)	1, 2	4, 3

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated
and 1-p_1-p_2 $= 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 (1-q)
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 (1-p_1)	1, 2	4, 3

Player 1 is indifferent if:

$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 $(1\text{-}q)$
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 $(1\text{-}p_1\text{-}p_2)$	0, 1	0, 3

IESDS: $s_2 > s_3$, thus s_3 can be eliminated and $1\text{-}p_1\text{-}p_2 = 0 \Rightarrow p_2 = 1 - p_1$

Player 2

		t_1 (q)	t_2 $(1\text{-}q)$
Player 1	s_1 (p_1)	2, 1	3, 0
	s_2 $(1\text{-}p_1)$	1, 2	4, 3

Player 1 is indifferent if:

$$2q + 3(1-q) = q + 4(1-q)$$

$$q = 1 - q \Rightarrow q = \frac{1}{2}$$

Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

$$p_1 = 1 - p_1 \Rightarrow p_1 = \frac{1}{2}$$

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 $(1\text{-}q)$
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		t_1 (q)	t_2 $(1\text{-}q)$
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Player 2 is indifferent if:

$$p_1 + 2(1-p_1) = 3(1-p_1)$$

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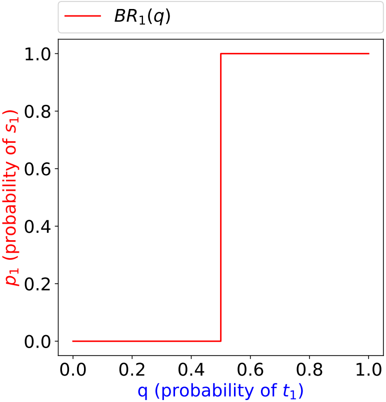

PS3, Ex. 7.d: Mixed Strategy Nash Equilibria

(d) $PSNE = \{(s_1, t_1), (s_2, t_2)\}$

	t_1 (q)	t_2 (1-q)
s_1 (p_1)	2, 1	3, 0
s_2 (p_2)	1, 2	4, 3
s_3 (1-p_1-p_2)	0, 1	0, 3

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Player 2

Player 1

	t_1 (q)	t_2 (1-q)
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Player 2 is indifferent if:

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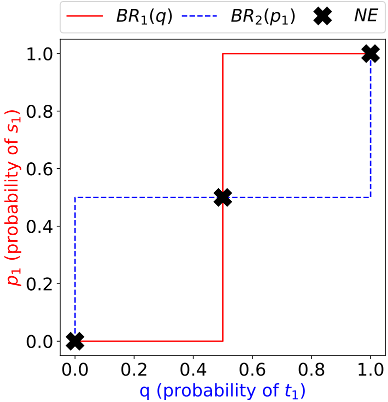

In the reduced game, three MSNE exist:

$$(p_1^*, q^*) = \{(0,0), (1/2, 1/2), (1,1)\}$$

And in the full game: $\left[(p_1^*, p_2^*), (q^*)\right] =$

$$\left\{[(0,1),(0)]\,;\left[\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2}\right)\right];[(1,0),(1)]\right\}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

Find all (pure and mixed) Nash equilibria in the following game:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

Find all (pure and mixed) Nash equilibria in the following game:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
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Hints:

1. Highlight the best responses in the matrix.
2. Find the relationship between q_1 and q_2 for which **Player 1 is indifferent**.
3. Write up the **best responses for Player 1**: $p^*(q_1, q_2)$, i.e. $BR_1(q_1, q_2)$.
4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.
5. Write up the **best responses for Player 2**:

$$BR_2(p) = (q_1^*(p), q_2^*(p)) = \begin{cases} \vdots & \vdots \\ \{(0, x) : x \in [0, 1]\} & p = 2/3 \\ (0, 0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a Mixed Strategy Nash Equilibriumm (MSNE) both players must be indifferent between their respective pure strategies.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

1. Highlight the best responses in the matrix:

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

No Pure Strategy Nash Equilibrium (PSNE) exist.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

2. Find the relationship between q_1 and q_2 for which **Player 1 is indifferent**:

Player 1 is indifferent if:

$$4q_1 + 2q_2 = 2q_1 + q_2 + 5(1 - q_1 - q_2)$$

$$7q_1 + 6q_2 = 5$$

$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
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$$7q_1 + 6q_2 = 5$$

$$q_1 + \frac{6}{7}q_2 = \frac{5}{7}$$

3. Write up the **best responses for Player 1**: $p^*(q_1, q_2) =$, i.e.

$$BR_1(q_1, q_2) = \begin{cases} 1 & q_1 + \frac{6}{7}q_2 > \frac{5}{7} \\ [0,1] & q_1 + \frac{6}{7}q_2 = \frac{5}{7} \\ 0 & q_1 + \frac{6}{7}q_2 < \frac{5}{7} \end{cases}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
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Player 1's best responses: $p^*(q_1, q_2)$, i.e.

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4. Pairwise find the probabilities p for which **Player 2 is indifferent**, e.g. between L and C, then L and R, and finally between C and R.

Player 2 is indifferent between L and C if:

$$p + 3(1-p) = 3p + 2(1-p)$$

$$1 - p = 2p$$

$$p = \frac{1}{3}$$

If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

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$$\begin{aligned} p + 3(1-p) &= 3p + 2(1-p) \\ 1 - p &= 2p \\ p &= \frac{1}{3} \end{aligned}$$

If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

Player 2 is indifferent between L and R if:

$$\begin{aligned} p + 3(1-p) &= 4p \\ 3 &= 6p \\ p &= \frac{1}{2} \end{aligned}$$

If $p < 1/2$ prefer L; if $p > 1/2$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
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Player 2 is indifferent between C and R if:

$$3p + 2(1-p) = 4p \Leftrightarrow 2 = 3p \Leftrightarrow p = \frac{2}{3}$$

If $p < 2/3$ prefer C; if $p > 2/3$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

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Player 2: $\boldsymbol{BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =}$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ \vdots & \vdots \end{cases}$$

Player 2 is indifferent between L and C if:

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	L (q_1)	C (q_2)	R (1-q_1-q_2)
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Note: if $p = \frac{1}{2} : u_2(C) > u_2(L) = u_2(R)$

$$\Rightarrow \text{For } p = \frac{1}{2} : \frac{3+2}{2} > \frac{1+3}{2} = \frac{4+0}{2}$$

$$\Rightarrow \quad \frac{5}{2} > \frac{4}{2} = \frac{4}{2}$$

Player 2 is indifferent between L and C if:

$$p + 3(1-p) = 3p + 2(1-p)$$

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PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
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Note: if $p = \frac{1}{2}$: $u_2(C) > u_2(L) = u_2(R)$

Player 2 is indifferent between L and C if:

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If $p < 1/3$ prefer L; if $p > 1/3$ prefer C.

~~Player 2 is indifferent between L and R if:~~

$$\cancel{pp + 3(1-p)} = \cancel{4p}$$
$$\cancel{p3} = \cancel{6p}$$
$$\cancel{p} = \frac{\cancel{1}}{\cancel{2}}$$

~~If $p < 1/2$ prefer L; if $p > 1/2$ prefer R.~~

Player 2 is indifferent between C and R if:

$$3p + 2(1-p) = 4p \Leftrightarrow 2 = 3p \Leftrightarrow p = \frac{2}{3}$$

If $p < 2/3$ prefer C; if $p > 2/3$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
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$$\cancel{pp + 3(1-p)} = \cancel{4p}$$

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$$\cancel{p} = \cancel{\frac{1}{2}}$$

~~If $p < 1/2$ prefer L; if $p > 1/2$ prefer R.~~

Player 2 is indifferent between C and R if:

$$3p + 2(1-p) = 4p \Leftrightarrow 2 = 3p \Leftrightarrow p = \frac{2}{3}$$

If $p < 2/3$ prefer C; if $p > 2/3$ prefer R.

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

Player 1's best responses: $p^*(q_1, q_2)$, i.e.

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6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 1: $p = 1/3$:

$$BR_2\left(\frac{1}{3}\right) = \{(x, 1-x) : x \in [0,1]\} \Rightarrow$$

$$\underbrace{x}_{q_1} + \frac{6}{7}\underbrace{1-x}_{q_2} > \frac{5}{7} \Rightarrow BR_1\left(BR_2(\frac{1}{3})\right) = 1 \neq \frac{1}{3}$$

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
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Player 1's best responses: $p^*(q_1, q_2)$, i.e.

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Player 2: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 1: $p = 1/3$:

$$\underbrace{x}_{q_1} + \frac{6}{7}\underbrace{(1-x)}_{q_2} > \frac{5}{7}$$

$$\Rightarrow BR_1\left(BR_2\left(\frac{1}{3}\right)\right) = 1 \neq \frac{1}{3}$$

- MSNE, Case 2: $p = 2/3$:

$$\underbrace{0}_{q_1} + \frac{6}{7}\underbrace{x}_{q_2} = \frac{5}{7} \Leftrightarrow x = \frac{5}{6}$$

$$\Rightarrow BR_1\left(0, \frac{5}{6}\right) = [0,1] \ni \frac{2}{3}$$

How many NE are there in total?

PS3, Ex. 8: Mixed Strategy Nash Equilibria - analytical solution

	L (q_1)	C (q_2)	R (1-q_1-q_2)
T (p)	4, 1	2, 3	0, 4
B (1-p)	2, 3	1, 2	5, 0

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Player 2: $BR_2(p) = \left(q_1^*(p), q_2^*(p)\right) =$

$$\begin{cases} (1,0) & p < 1/3 \\ \{(x, 1-x) : x \in [0,1]\} & p = 1/3 \\ (0,1) & p \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \{(0,x) : x \in [0,1]\} & p = 2/3 \\ (0,0) & p > 2/3 \end{cases}$$

6. **Find the NE** (pure and mixed). In a MSNE both players must be indifferent between their respective pure strategies.

- MSNE, Case 1: $p = 1/3$:

$$\underbrace{x}_{q_1} + \frac{6}{7}\underbrace{(1-x)}_{q_2} > \frac{5}{7}$$

$$\Rightarrow BR_1\left(BR_2\left(\frac{1}{3}\right)\right) = 1 \neq \frac{1}{3}$$

- MSNE, Case 2: $p = 2/3$:

$$\underbrace{0}_{q_1} + \frac{6}{7}\underbrace{x}_{q_2} = \frac{5}{7} \Leftrightarrow x = \frac{5}{6}$$

$$\Rightarrow BR_1\left(0, \frac{5}{6}\right) = [0,1] \ni \frac{2}{3}$$

$\Rightarrow BR_2\left(\frac{2}{3}\right) = \left(0, \frac{5}{6}\right)$ is a unique MSNE:

$$[(p^*), (q_1^*, q_2^*)] = \left\{\left[\left(\frac{2}{3}\right), \left(0, \frac{5}{6}\right)\right]\right\}$$