Microeconomics III: Problem Set 8[a]

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[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

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PS8, Ex. 3 (A): Static Bayesian game (Bayesian Nash Equilibria)

PS8, Ex. 4: Static Bayesian game (Bayesian Nash Equilibria)

Recipe for a static Bayesian game (Bayesian Nash Equilibria)

PS8, Ex. 5: The dating game (Bayesian Nash Equilibria)

PS8, Ex. 6: Static public goods game (two-sided incomplete information)

PS8, Ex. 1 (A): Trigger strategy in infinitely repeated game:

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Consider the following game G:

		Player 2 X	Y	Z
Player 1	A	6, 6	0, 8	0, 0
	B	7, 1	2, 2	1, 1
	C	0, 0	1, 1	4, 5

Suppose that G is repeated infinitely many times, so that we have $G(1, \infty)$. Define trigger strategies such that the outcome of all stages is (A,X). Find the smallest value of δ such that these strategies constitute a SPNE.

PS8, Ex. 1 (A): Trigger strategy in infinitely repeated game - exam answer)

Suppose that G is repeated infinitely many times, so that we have G(1, ∞). Define trigger strategies such that the outcome of all stages is (A,X). Find the smallest value of δ such that these strategies constitute a SPNE.

Trigger strategies such that the outcome of all stages of the game is (A,X) are possible using respectively B,Y or C,Z as the threats. Since the threats B,Y will make the SPNE possible for the smallest δ, I will use B,Y in the trigger strategies i define:

1. Trigger strategy $P1$: In the 1st turn, play A. In every subsequent turn, if outcome from every previous turn was (A,X), play A, otherwise play B.
2. Trigger strategy $P2$: In the 1st turn, play X. In every subsequent turn, if outcome from every previous turn was (A,X), play X, otherwise play Y.

Player 2 has the highest incentive to deviate, so I only examine player 2's incentive to deviate. In order to find the lowest δ to secure cooperation, I set up the inequality for which the payoff for cooperation is higher than the payoff for deviating:

$$\frac{6}{1-\delta} \geq 8 + \frac{2\delta}{1-\delta} \Leftrightarrow$$

$$6 \geq 8 - 8\delta + 2\delta \Leftrightarrow$$

$$\delta \geq \frac{1}{3}$$

$\delta = \frac{1}{3}$ is the smallest value for which the strategies constitute a SPNE.

PS8, Ex. 2 (A): Hit-and-run cab (Bayes' Rule)

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Review the intuition from the 'Doctor' example in lecture 7 (slides 6-9), and then use Bayes' rule to solve the following problem:

> *A cab was involved in a hit and run accident at night.* 85% *of the cabs in the city are Green and* 15% *are Blue. A witness later recalls that the cab was Blue, and we know that this witness' memory is reliable* 80% *of the time. Given the statement from the witness, calculate the probability that the cab involved in the accident was actually Blue.*

First, try to write up Bayes' Rule on your own *(it is written on the next slide)*

PS8, Ex. 2 (A): Hit-and-run cab (Bayes' Rule)

Review the intuition from the 'Doctor' example in lecture 7 (slides 6-9), and then use Bayes' rule to solve the following problem:

> *A cab was involved in a hit and run accident at night.* 85% *of the cabs in the city are Green and* 15% *are Blue. A witness later recalls that the cab was Blue, and we know that this witness' memory is reliable* 80% *of the time. Given the statement from the witness, calculate the probability that the cab involved in the accident was actually Blue.*

Bayes' rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

PS8, Ex. 2 (A): Hit-and-run cab (Bayes' Rule)

Information so far:

1. P(B): The unconditional chance that a cab is green: $\frac{85}{100}$
2. P(B): The unconditional chance that a cab is blue: $\frac{15}{100}$
3. P(obs B|B): The chance of remembering a blue cab, given it was blue: $\frac{80}{100}$
4. P(obs B|G): The chance of remembering a blue cab, given it was green: $\frac{20}{100}$

P(obs B): The unconditional chance that the witness says the cab is blue, so the chance the witness would observe a blue cab and remember it as blue, plus the chance the witness would observe a green cab and remember it as blue:

PS8, Ex. 2 (A): Hit-and-run cab (Bayes' Rule)

Information so far:

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2. P(B): The unconditional chance that a cab is blue: $\frac{15}{100}$
3. P(obs B|B): The chance of remembering a blue cab, given it was blue: $\frac{80}{100}$
4. P(obs B|G): The chance of remembering a blue cab, given it was green: $\frac{20}{100}$

P(obs B): The unconditional chance that the witness says the cab is blue, so the chance the witness would observe a blue cab and remember it as blue, plus the chance the witness would observe a green cab and remember it as blue:

$$P(obs\ B) = P(obs\ B|B) \cdot P(B) + P(obs\ B|G) \cdot P(G) = \frac{80}{100} \cdot \frac{15}{100} + \frac{20}{100} \cdot \frac{85}{100} = \frac{29}{100}$$

We want to find the chance that the cab is blue, given that the witness says it's blue. Using Bayes' rule, this is the same as the odds that the cab will be blue and the witness says it's blue, divided by the unconditional chance the witness says it blue.

PS8, Ex. 2 (A): Hit-and-run cab (Bayes' Rule)

Information so far:

1. P(B): The unconditional chance that a cab is green: $\frac{85}{100}$
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4. P(obs B|G): The chance of remembering a blue cab, given it was green: $\frac{20}{100}$

P(obs B): The unconditional chance that the witness says the cab is blue, so the chance the witness would observe a blue cab and remember it as blue, plus the chance the witness would observe a green cab and remember it as blue:

$$P(obs\ B) = P(obs\ B|B) \cdot P(B) + P(obs\ B|G) \cdot P(G) = \frac{80}{100} \cdot \frac{15}{100} + \frac{20}{100} \cdot \frac{85}{100} = \frac{29}{100}$$

We want to find the chance that the cab is blue, given that the witness says it's blue. Using Bayes' rule, this is the same as the odds that the cab will be blue and the witness says it's blue, divided by the unconditional chance the witness says it blue.

$$P(B|obs\ B) = \frac{P(obs\ B|B) \cdot P(B)}{P(obs\ B)} = \frac{\frac{80}{100} * \frac{15}{100}}{\frac{29}{100}} = 0.414$$

PS8, Ex. 3 (A): Static Bayesian game (Bayesian Nash Equilibria)

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Consider the following static game, where a is a real number:

	L	R
U	2, 1	0, a
D	0, 1	1, a

(a) Suppose that $a = 2$. Does any player have a dominant strategy? What about when $a = -2$?

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(c) Find the Bayes-Nash equilibrium of the game described in (b).

PS8, Ex. 3.a (A): Static Bayesian game (Bayesian Nash Equilibria)

(a) Suppose that $a = 2$. Does any player have a dominant strategy? What about when $a = -2$?

(a) The value of a affects P2's payoff:

$a = 2:$

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

$a = -2:$

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

PS8, Ex. 3.a (A): Static Bayesian game (Bayesian Nash Equilibria)

(a) Suppose that $a = 2$. Does any player have a dominant strategy? What about when $a = -2$?

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

$a = 2$:

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

$a = -2$:

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

$a = 2$:

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

$a = -2$:

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (where he strongly prefers L and R respectively) and P1 has a belief about the distribution of these types (each happen $\frac{1}{2}$ of the time.).

$a = 2$:

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

$a = -2$:

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (where he strongly prefers L and R respectively) and P1 has a belief about the distribution of these types (each happen $\frac{1}{2}$ of the time.).

Players: P1, P2

Action sp.: $A_1 = (U, D), A_2 = (L, R)$

Type space: $T_1 = (t)$ [one type],

$T_2 = (t_1 : a = 2, t_2 : a = -2)$

Beliefs: $\mathbb{P}_1(a = 2) = \mathbb{P}_1(a = -2) = \frac{1}{2}$

$a = 2:$

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

$a = -2:$

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

Write as type-dependent payoff matrices

PS8, Ex. 3.b (A): Static Bayesian game (Bayesian Nash Equilibria)

(b) Now assume that player 2 knows the value of a, but player 1 only knows that $a = 2$ with probability 0.5 and $a = -2$ with probability 0.5. Explain how this situation can be modeled as a Bayesian game, describing the players, their action spaces, type spaces, beliefs and payoff functions.

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (where he strongly prefers L and R respectively) and P1 has a belief about the distribution of these types (each happen $\frac{1}{2}$ of the time.).

Players: P1, P2

Action sp.: $A_1 = (U, D), A_2 = (L, R)$

Type space: $T_1 = (t)$ [one type],

$T_2 = (t_1 : a = 2, t_2 : a = -2)$

Beliefs: $\mathbb{P}_1(a = 2) = \mathbb{P}_1(a = -2) = \frac{1}{2}$

Type-dependent payoff matrices:

Type $t_1 : a = 2$ ($p = \frac{1}{2}$)

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

Type $t_2 : a = -2$ ($p = \frac{1}{2}$)

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

PS8, Ex. 3.c (A): Static Bayesian game (Bayesian Nash Equilibria)

(c) Find the Bayes-Nash equilibrium of the game described in (b).

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (where he strongly prefers L and R respectively) and P1 has a belief about the distribution of these types (each happen $\frac{1}{2}$ of the time.).

Players: P1, P2

Action sp.: $A_1 = (U, D), A_2 = (L, R)$

Type space: $T_1 = (t)$ [one type],

$T_2 = (t_1 : a = 2, t_2 : a = -2)$

Beliefs: $\mathbb{P}_1(a = 2) = \mathbb{P}_1(a = -2) = \frac{1}{2}$

Type-dependent payoff matrices:

Type $t_1 : a = 2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

Type $t_2 : a = -2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

PS8, Ex. 3.c (A): Static Bayesian game (Bayesian Nash Equilibria)

(c) Find the Bayes-Nash equilibrium of the game described in (b).

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (he either has L or R as a dominant strategy) and P1 has a belief about the distribution of these types (Each happen $\frac{1}{2}$ the time.).

Players: P1, P2

Action sp.: $A_1 = (U, D), A_2 = (L, R)$

Type space: $T_1 = (t)$ [one type],
$T_2 = (t_1 : a = 2, t_2 : a = -2)$

Beliefs: $\mathbb{P}_1(a = 2) = \mathbb{P}_1(a = -2) = \frac{1}{2}$

Type-dependent payoff matrices:

Type $t_1 : a = 2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

Type $t_2 : a = -2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

(c) The strategy for P1 is just one action, whereas the strategy for P2 is an action for each of the two types. Using this, we can write up the expected payoff matrix:

	LL	LR	RL	RR
U	2, 1	1, $-\frac{1}{2}$	1, $\frac{3}{2}$	0, 0
D	0, 1	$\frac{1}{2}$, $-\frac{1}{2}$	$\frac{1}{2}$, $\frac{3}{2}$	1, 0

Find the Bayesian Nash Equilibria.

PS8, Ex. 3.c (A): Static Bayesian game (Bayesian Nash Equilibria)

(c) Find the Bayes-Nash equilibrium of the game described in (b).

(a) The value of a not only affects P2's payoff, but also P2's strategy. For $a = 2$, P2 will have R as a dominant strategy; and for $a = -2$, P2 will have L as a dominant strategy.

(b) This can be modelled as a Bayesian game since P2 has two types (he either has L or R as a dominant strategy) and P1 has a belief about the distribution of these types (Each happen $\frac{1}{2}$ the time.).

Players: P1, P2

Action sp.: $A_1 = (U, D), A_2 = (L, R)$

Type space: $T_1 = (t)$ [one type],
$T_2 = (t_1 : a = 2, t_2 : a = -2)$

Beliefs: $\mathbb{P}_1(a = 2) = \mathbb{P}_1(a = -2) = \frac{1}{2}$

Type-dependent payoff matrices:

Type $t_1 : a = 2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, 2
D	0, 1	1, 2

Type $t_2 : a = -2$ $(p = \frac{1}{2})$

	L	R
U	2, 1	0, -2
D	0, 1	1, -2

(c) The strategy for P1 is just one action, whereas the strategy for P2 is an action for each of the two types. Using this, we can write up the expected payoff matrix:

	LL	LR	RL	RR
U	2,1	1,$-\frac{1}{2}$	1,$\frac{3}{2}$	0, 0
D	0, 1	$\frac{1}{2}$, $-\frac{1}{2}$	$\frac{1}{2}$, $\frac{3}{2}$	1, 0

The BNE is: (U,RL)

PS8, Ex. 4: Static Bayesian game (Bayesian Nash Equilibria)

PS8, Ex. 4: Static Bayesian game (Bayesian Nash Equilibria)

Exercise 3.4 in Gibbons (p. 169). Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

a. Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.
b. Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.
c. Player 1 chooses either T or B; player 2 simultaneously chooses either L or R.
d. Payoffs are given by the game drawn by nature.

Game 1 $\left(t_1, p = \frac{1}{2}\right)$:

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game 2 $\left(t_2, p = \frac{1}{2}\right)$:

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

PS8, Ex. 4.a: Static Bayesian game (Bayesian Nash Equilibria)

Find all the Bayesian Nash equilibria in the following static Bayesian game:

Game 1 $\left(t_1, p = \frac{1}{2}\right)$:

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game 2 $\left(t_2, p = \frac{1}{2}\right)$:

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

Use the fact that each type of game happens half the time to write up the expected payoff matrix for the following possibilities:

- P2 plays L and P1 plays T if game is type 1 and T if game is type 2
- P2 plays L and P1 plays T if game is type 1 and B if game is type 2
- P2 plays L and P1 plays B if game is type 1 and T if game is type 2
- P2 plays L and P1 plays B if game is type 1 and B if game is type 2
- P2 plays R and P1 plays T if game is type 1 and T if game is type 2
- P2 plays R and P1 plays T if game is type 1 and B if game is type 2
- P2 plays R and P1 plays B if game is type 1 and T if game is type 2
- P2 plays R and P1 plays B if game is type 1 and B if game is type 2

PS8, Ex. 4.a: Static Bayesian game (Bayesian Nash Equilibria)

Find all the Bayesian Nash equilibria in the following static Bayesian game:

Game 1 $\left(t_1, p = \frac{1}{2}\right)$:

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game 2 $\left(t_2, p = \frac{1}{2}\right)$:

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

- P2 plays L and P1 plays T if game is type 1 and T if game is type 2
- P2 plays L and P1 plays T if game is type 1 and B if game is type 2
- P2 plays L and P1 plays B if game is type 1 and T if game is type 2
- P2 plays L and P1 plays B if game is type 1 and B if game is type 2
- P2 plays R and P1 plays T if game is type 1 and T if game is type 2
- P2 plays R and P1 plays T if game is type 1 and B if game is type 2
- P2 plays R and P1 plays B if game is type 1 and T if game is type 2
- P2 plays R and P1 plays B if game is type 1 and B if game is type 2

The expected payoff matrix:

		Player 2	
		L	R
Player 1	TT	$\frac{1}{2}, \frac{1}{2}$	0, 0
	TB	$\frac{1}{2}, \frac{1}{2}$	1, 1
	BT	0, 0	0, 0
	BB	0, 0	1, 1

Find all Bayesian Nash Equilibria.

PS8, Ex. 4.a: Static Bayesian game (Bayesian Nash Equilibria)

Find all the Bayesian Nash equilibria in the following static Bayesian game:

Game 1 $\left(t_1, p = \frac{1}{2}\right)$:

	L	R
T	1, 1	0, 0
B	0, 0	0, 0

Game 2 $\left(t_2, p = \frac{1}{2}\right)$:

	L	R
T	0, 0	0, 0
B	0, 0	2, 2

- P2 plays L and P1 plays T if game is type 1 and T if game is type 2
- P2 plays L and P1 plays T if game is type 1 and B if game is type 2
- P2 plays L and P1 plays B if game is type 1 and T if game is type 2
- P2 plays L and P1 plays B if game is type 1 and B if game is type 2
- P2 plays R and P1 plays T if game is type 1 and T if game is type 2
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- P2 plays R and P1 plays B if game is type 1 and B if game is type 2

The expected payoff matrix:

Player 2

Player 1	L	R
TT	$\frac{1}{2}$, $\frac{1}{2}$	0, 0
TB	$\frac{1}{2}$, $\frac{1}{2}$	1, 1
BT	0, 0	0, 0
BB	0, 0	1, 1

This gives the following BNE:

$$BNE = \{(TT, L), (TB, R), (BB, R)\}$$

Recipe for a static Bayesian game (Bayesian Nash Equilibria)

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1. The timing is as follows where p is a commonly known distribution:
    1.1 Nature draws all players' type according to p.
    1.2 Each player i learns her own type t_i.
    1.3 Players form their beliefs about the type profile.
    1.4 Players simultaneously choose actions and payoffs are realized.

Recipe for a static Bayesian game (Bayesian Nash Equilibria)

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   1.1 Nature draws all players' type according to p.
   1.2 Each player i learns her own type t_i.
   1.3 Players form their beliefs about the type profile.
   1.4 Players simultaneously choose actions and payoffs are realized.
2. The static Bayesian game consists of:
   2.1 Players: $Player\ 1, ..., Player\ n$
   2.2 Type spaces: $T_1 = \{t_{11}, ..., t_{1K}\}, ...$
   2.3 Beliefs: $\mathbb{P}_1[t_2 = t_{21}] = \cdot, ...$
   2.4 Action spaces: $A_1 = \{\cdot\}, ...$
   2.5 Strategy spaces: $S_1 = \{s_1(t_1), \cdot\} = \{(s_1|t_{11}, ..., s_1|t_{1K}), \cdot\}, ...$
   2.6 Type-dependent payoff matrices.

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   2.5 Strategy spaces: $S_1 = \{s_1(t_1), \cdot\} = \{(s_1|t_{11}, ..., s_1|t_{1K}), \cdot\}, ...$
   2.6 Type-dependent payoff matrices.
3. Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for a player i (the player with the smallest strategy space). For each strategy $s_i(t_i)$:
   3.1 Write up the best-response of the other player(s): $s_j^*(t_j) \equiv BR_j\left(s_i(t_i)|t_j\right)$.
   3.2 If $s_i(t_i) = BR_i\left(s_j(t_j)|t_i\right) \equiv s_i^*(t_i)$ then $\left(s_i^*(t_i), s_j^*(t_j)\right)$ is a BNE.

Recipe for a static Bayesian game (Bayesian Nash Equilibria)

1. The timing is as follows where p is a commonly known distribution:
   1.1 Nature draws all players' type according to p.
   1.2 Each player i learns her own type t_i.
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2. The static Bayesian game consists of:
   2.1 Players: $\mathit{Player}\ 1, ..., \mathit{Player}\ n$
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   2.4 Action spaces: $A_1 = \{\cdot\}, ...$
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3. Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for a player i (the player with the smallest strategy space). For each strategy $s_i(t_i)$:
   3.1 Write up the best-response of the other player(s): $s_j^*(t_j) \equiv BR_j\left(s_i(t_i)|t_j\right)$.
   3.2 If $s_i(t_i) = BR_i\left(s_j(t_j)|t_i\right) \equiv s_i^*(t_i)$ then $\left(s_i^*(t_i), s_j^*(t_j)\right)$ is a BNE.
4. In BNE, a strategy must maximize expected utility given the strategy of the other player(s) and the probability of them being each type, i.e. no type of any player has an incentive to deviate as in equilibrium player i's strategy is a best response to player j's strategy given player i's beliefs:

$$\max_{s_i} \sum_{j \neq i} \sum_{t_{jk} \in T_j} \mathbb{P}_i[t_j = t_{jk}] \cdot u_i\left(s_i(t_i), s_j^*(t_j)|t_i\right)$$

PS8, Ex. 5: The dating game (Bayesian Nash Equilibria)

PS8, Ex. 5: The dating game (Bayesian Nash Equilibria)

Consider the 'dating game'. In this game, player 1 wants to meet with player 2, even if player 2 is of type t_2, who does not want to meet with player 1. Player 2's types have equal probability.

Now suppose we construct an alternative version of the game, where player 1 has self respect, and only wants to meet up with player 2 if player 2 also wants to meet up (i.e. if player 2 is of type t_1). Otherwise, player 1 prefers to *miscoordinate* (i.e. not to meet up) as well. Player 1 still has the same preference for going to football versus the opera.

(a) Write up this new game. Use the same payoffs as before (0,1, and 2) such that the payoff matrix when player 2's type is t_1 is the same as in the slides (Lecture 7, slides 22-26).

(b) Find the set of (pure strategy) Bayesian Nash Equilibria of this game.

[Hints on the next slide.]

PS8, Ex. 5: The dating game (Bayesian Nash Equilibria)

Consider the 'dating game'. In this game, player 1 wants to meet with player 2, even if player 2 is of type t_2, who does not want to meet with player 1. Player 2's types have equal probability.

Now suppose we construct an alternative version of the game, where player 1 has self respect, and only wants to meet up with player 2 if player 2 also wants to meet up (i.e. if player 2 is of type t_1). Otherwise, player 1 prefers to *miscoordinate* (i.e. not to meet up) as well. Player 1 still has the same preference for going to football versus the opera.

(a) Write up this new game. Use the same payoffs as before (0,1, and 2) such that the payoff matrix when player 2's type is t_1 is the same as in the slides (Lecture 7, slides 22-26).

Hint: Write up the Bayesian game (players, type spaces, beliefs, action spaces, strategy spaces, and the type-dependent payoff matrices.)

(b) Find the set of (pure strategy) Bayesian Nash Equilibria of this game.

Hints:

1. Check for equilibria where player 1 plays *Football* and *Opera* respectively.
2. In equilibrium, a strategy should maximize expected payoff given the strategy of the other player and the probability of each type.

PS8, Ex. 5.a: The dating game (Bayesian Nash Equilibria)

(a) Write up this new game. Use the same payoffs as before (0,1, and 2) such that the payoff matrix when player 2's type is t_1 is the same as in the slides (Lecture 7, slides 22-26).

1. Players: P1, P2.
2. Type spaces: $T_1 = \{t\},\ T_2 = \{t_1, t_2\}$
3. Beliefs: $\mathbb{P}_1(T_2 = t_1) = \mathbb{P}_1(T_2 = t_2) = \frac{1}{2},\ \mathbb{P}_2(T_1 = t) = 1$
4. Action space: $A_i = \{Football, Opera\},$ for $i \in 1, 2$
5. Strategy spaces: $S_1 = \{F, O\},\ S_2 = \{FF, FO, OF, OO\}$
6. Type-dependent payoff matrices:

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*.

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.a: $BR_2(F) = (FO)$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.a: $BR_2(F) = (FO)$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2\ plays\ FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$

$u_1(O|P2\ plays\ FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$

$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 $(p = \frac{1}{2})$

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 $(p = \frac{1}{2})$

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

- 1.a: Write up player 2's best-response.
- 1.b: Does player 1 have an incentive to deviate and play *Opera*?
- 1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

- 2.a: Write up player 2's best-response.
- 2.b: Does player 1 have an incentive to deviate and play *Football*?

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$

$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2 \text{ plays } OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$

$u_1(F|P2 \text{ plays } OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

2.c: If no, it's a BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2 \text{ plays } OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$
$u_1(F|P2 \text{ plays } OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

2.c: If no, it's a BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2\ plays\ FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2\ plays\ FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e.
$BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2\ plays\ OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$
$u_1(F|P2\ plays\ OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

2.c: No incentive to deviate, i.e.
$BNE_2 = \{O, OF\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

2.c: If no, it's a BNE.

Step 3: Write up the set of all BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2\ plays\ FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$

$u_1(O|P2\ plays\ FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2\ plays\ OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$

$u_1(F|P2\ plays\ OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

2.c: No incentive to deviate, i.e. $BNE_2 = \{O, OF\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

2.c: If no, it's a BNE.

Step 3: Write up the set of all BNE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2\ plays\ FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2\ plays\ FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e.
$BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2\ plays\ OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$
$u_1(F|P2\ plays\ OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

2.c: No incentive to deviate, i.e.
$BNE_2 = \{O, OF\}$

3: $BNE = \{(F, FO), (O, OF)\}$

PS8, Ex. 5.b: The dating game (Bayesian Nash Equilibria)

Type t_1 ($p = \frac{1}{2}$)

	F	O
F	2, 1	0, 0
O	0, 0	1, 2

Type t_2 ($p = \frac{1}{2}$)

	F	O
F	0, 0	2, 2
O	1, 1	0, 0

(b) Find the set of (pure strategy) Bayesian Nash Equilibria (BNE) of this game.

Step 1: Check for a BNE where player 1 plays *Football*:

1.a: Write up player 2's best-response.

1.b: Does player 1 have an incentive to deviate and play *Opera*?

1.c: If no, it's a BNE.

Step 2: Check for a BNE where player 1 plays *Opera*:

2.a: Write up player 2's best-response.

2.b: Does player 1 have an incentive to deviate and play *Football*?

2.c: If no, it's a BNE.

Step 3: Write up the set of all BNE.

Alternative: Instead, do as in ex. 3 and 4 and write up the full expected payoff matrix and look for NE.

1.a: $BR_2(F) = (FO)$

1.b: $u_1(F|P2 \text{ plays } FO) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
$u_1(O|P2 \text{ plays } FO) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

1.c: No incentive to deviate, i.e. $BNE_1 = \{F, FO\}$

2.a: $BR_2(O) = (OF)$

2.b: $u_1(O|P2 \text{ plays } OF) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$
$u_1(F|P2 \text{ plays } OF) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

2.c: No incentive to deviate, i.e. $BNE_2 = \{O, OF\}$

3: $BNE = \{(F, FO), (O, OF)\}$

PS8, Ex. 6: Static public goods game (two-sided incomplete information)

PS8, Ex. 6: Static public goods game (two-sided incomplete information)

Difficult. Consider the public goods game from lecture 7 (slides 34-40):

	W	D
W	$1-c_1, 1-c_2$	$1-c_1, 1$
D	$1, 1-c_2$	$0, 0$

Suppose now instead that there is two-sided incomplete information. In particular, the cost of writing the reference is uniformly distributed between 0 and 2:

$$c_i \sim u(0,2) \text{ for } i = 1,2$$

In this setting, we can show that the players optimally follow a 'cutoff' strategy. Thus, the equilibrium strategies take the form

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases}$$

$$s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(a) Let $z_{-i}^* = \mathbb{P}(s_{-i}^* = Write)$, i.e. the probability that the other player plays 'Write' in equilibrium. Argue that
$$1 - c_i^* = z_{-i}^* \qquad (1)$$
Hint: Calculate i's expected payoff from writing the reference and from not writing the reference, conditional on z_{-i}^*.

(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\mathbb{P}(x < a) = \frac{a}{2}$. Use this to find z_i^*. *Hint* : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

(c) Use the result from (b) together with equation (1) to find (c_1^*, c_2^*).

(d) What's the probability of under-investment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

PS8, Ex. 6.a: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

(a) Let $z_{-i}^* = \mathbb{P}(s_{-i}^* = \textit{Write})$, i.e. the probability that the other player plays 'Write' in equilibrium. Argue that $1 - c_i^* = z_{-i}^* \quad (1)$

Hint: Calculate i's expected payoff from writing the reference and from not writing the reference, conditional on z_{-i}^*.

Step 1: Write up the expected payoffs from playing W and D respectively.

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Hint: Calculate i's expected payoff from writing the reference and from not writing the reference, conditional on z_{-i}^*.

Step 1: Write up the expected payoffs from playing W and D respectively:

$$E[s_i = W] = \mathbb{P}\left[s_{-i}^* = W\right] \cdot (1 - c_i) + \mathbb{P}\left[s_{-i}^* = D\right] \cdot (1 - c_i) = 1 - c_i$$
$$E[s_i = D] = \mathbb{P}\left[s_{-i}^* = W\right] \cdot 1 + \mathbb{P}\left[s_{-i}^* = D\right] \cdot 0 = \mathbb{P}\left[s_{-i}^* = W\right] = z_{-i}^*$$

PS8, Ex. 6.a: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \le c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \le c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(a) Let $z_{-i}^* = \mathbb{P}(s_{-i}^* = Write)$, i.e. the probability that the other player plays 'Write' in equilibrium. Argue that $1 - c_i^* = z_{-i}^* \quad (1)$

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Step 2: Use this to argue that equation (1) holds.

Hint: the 'cutoff' value c_i^* is where player i is indifferent between W and D.

PS8, Ex. 6.a: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(a) Let $z_{-i}^* = \mathbb{P}(s_{-i}^* = Write)$, i.e. the probability that the other player plays 'Write' in equilibrium. Argue that $1 - c_i^* = z_{-i}^* \quad (1)$

Hint: Calculate i's expected payoff from writing the reference and from not writing the reference, conditional on z_{-i}^*.

Step 1: Write up the expected payoffs from playing W and D respectively:

$$E[u_i|s_i = W] = \mathbb{P}\left[s_{-i}^* = W\right] \cdot (1 - c_i) + \mathbb{P}\left[s_{-i}^* = D\right] \cdot (1 - c_i) = 1 - c_i$$

$$E[u_i|s_i = D] = \mathbb{P}\left[s_{-i}^* = W\right] \cdot 1 + \mathbb{P}\left[s_{-i}^* = D\right] \cdot 0 = \mathbb{P}\left[s_{-i}^* = W\right] = z_{-i}^*$$

Step 2: Use this to argue that equation (1) holds.

Hint: the 'cutoff' value c_i^* is where player i is indifferent between W and D.

For $c_i = c_i^*$ player i's expected payoff is the same regardless of strategy, i.e.

$$E[u_i|s_i = W] = E[u_i|s_i = D] \Rightarrow$$

$$1 - c_i^* = z_{-i}^*$$

Q.E.D.

PS8, Ex. 6.b: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\mathbb{P}(x < a) = \frac{a}{2}$. Use this to find z_i^*.

Hint : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

PS8, Ex. 6.b: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\mathbb{P}(x < a) = \frac{a}{2}$. Use this to find z_i^*.

Hint : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

Step 1: Knowing that z_i^* is the probability that the other player plays W in equilibrium, write up z_i as the probability player i herself plays W in equilibrium.

PS8, Ex. 6.b: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\mathbb{P}(x < a) = \frac{a}{2}$. Use this to find z_i^*.

Hint : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

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1. $z_i^* = \mathbb{P}[s_i^* = \textit{write}] = \mathbb{P}[c_i \leq c_i^*]$

PS8, Ex. 6.b: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

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Hint : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

Step 1: Knowing that z_i^* is the probability that the other player plays W in equilibrium, write up z_i as the probability player i herself plays W in equilibrium.

Step 2: Use that c_i is uniformly distributed $c_i \sim u(0,2)$.

1. $z_i^* = \mathbb{P}[s_i^* = \textit{write}] = \mathbb{P}[c_i \leq c_i^*]$

PS8, Ex. 6.b: Static public goods game (two-sided incomplete information)

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(b) A standard result on uniform distributions gives the following: if $x \sim u(0,2)$, then $\mathbb{P}(x < a) = \frac{a}{2}$. Use this to find z_i^*.

Hint : Use the equilibrium strategy and your knowledge of the distribution of c_{-i}.

Step 1: Knowing that z_i^* is the probability that the other player plays W in equilibrium, write up z_i as the probability player i herself plays W in equilibrium.

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1. $z_i^* = \mathbb{P}[s_i^* = write] = \mathbb{P}[c_i \leq c_i^*]$
2. $z_i^* = \mathbb{P}[c_i \leq c_i^*] = \frac{c_i^*}{2}$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

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(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = \textit{write}] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = \textit{write}] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

Step 2: Substitute in z_i^* from (b).

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

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(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

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Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$
2. $\frac{c_i^*}{2} = 1 - c_{-i}^*$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

Step 2: Substitute in z_i^* from (b).

Step 3: Use symmetry in the distribution of the costs to find the cutoff value c_i^*.

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$
2. $\frac{c_i^*}{2} = 1 - c_{-i}^*$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

Step 2: Substitute in z_i^* from (b).

Step 3: Use symmetry in the distribution of the costs to find the cutoff value c_i^*.

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$
2. $\frac{c_i^*}{2} = 1 - c_{-i}^*$
3. Due to symmetry, they must have the same 'cutoff' value c_i^*:

$$\frac{c_i^*}{2} = 1 - c_i^*$$
$$c_i^* = 2 - 2c_i^*$$
$$c_i^* = \frac{2}{3}$$

PS8, Ex. 6.c: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(c) Use the result from question (b) together with equation (1) to find (c_1^*, c_2^*).

Step 1: Though the drawn costs might differ, take advantage of the symmetry in the distribution of the costs to write up equation (1) for player i instead of $-i$.

Step 2: Substitute in z_i^* from (b).

Step 3: Use symmetry in the distribution of the costs to find the cutoff value c_i^*.

Information so far:

(a) $z_{-i}^* = \mathbb{P}[s_{-i}^* = write] = 1 - c_i^* \quad (1)$

(b) $z_i^* = \mathbb{P}[s_i^* = write] = \frac{c_i^*}{2}$

1. $z_i^* = 1 - c_{-i}^*$
2. $\frac{c_i^*}{2} = 1 - c_{-i}^*$
3. Due to symmetry, they must have the same 'cutoff' value c_i^*:
$$\frac{c_i^*}{2} = 1 - c_i^*$$
$$c_i^* = 2 - 2c_i^*$$
$$c_i^* = \frac{2}{3}$$
4. Hence, $(c_1^*, c_2^*) = \left(\frac{2}{3}, \frac{2}{3}\right)$

PS8, Ex. 6.d: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

PS8, Ex. 6.d: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} \textit{Write} & \text{if} \quad c_1 \leq c_1^* \\ \textit{Don't} & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} \textit{Write} & \text{if} \quad c_2 \leq c_2^* \\ \textit{Don't} & \text{if} \quad c_2 > c_2^* \end{cases}$$

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

Hint : As seen in question (c) we can drop the subscripts due to symmetry.

PS8, Ex. 6.d: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

Hint : As seen in question (c) we can drop the subscripts due to symmetry.

$$\mathbb{P}[Nobody\ writes] = (1 - \mathbb{P}[s_i^* = write])(1 - \mathbb{P}[s_{-i}^* = write]) = (1 - z_i^*)(1 - z_{-i}^*)$$

PS8, Ex. 6.d: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

Hint : As seen in question (c) we can drop the subscripts due to symmetry.

$$\mathbb{P}[Nobody\ writes] = (1 - \mathbb{P}[s_i^* = write])(1 - \mathbb{P}[s_{-i}^* = write]) = (1 - z_i^*)(1 - z_{-i}^*)$$
$$= (1 - z^*)^2 = (1 - (1 - c^*))^2 = \left(1 - \left(1 - \frac{2}{3}\right)\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

PS8, Ex. 6.d: Static public goods game (two-sided incomplete information)

$$s_1^*(c_1) = \begin{cases} Write & \text{if} \quad c_1 \leq c_1^* \\ Don't & \text{if} \quad c_1 > c_1^* \end{cases} \qquad s_2^*(c_2) = \begin{cases} Write & \text{if} \quad c_2 \leq c_2^* \\ Don't & \text{if} \quad c_2 > c_2^* \end{cases}$$

(d) What's the probability of underinvestment (i.e. that nobody writes the reference)? What's the probability of overinvestment (i.e. that both write the reference)?

Hint : As seen in question (c) we can drop the subscripts due to symmetry.

$$\begin{aligned}\mathbb{P}[Nobody\ writes] &= (1 - \mathbb{P}[s_i^* = write])(1 - \mathbb{P}[s_{-i}^* = write]) = (1 - z_i^*)(1 - z_{-i}^*) \\ &= (1 - z^*)^2 = (1 - (1 - c^*))^2 = \left(1 - \left(1 - \frac{2}{3}\right)\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}\end{aligned}$$

$$\mathbb{P}[Both\ write] = (z^*)^2 = (1 - c^*)^2 = \left(1 - \frac{2}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$