Microeconomics III: Problem Set 9[a]

Thor Donsby Noe (thor.noe@econ.ku.dk) & Christopher Borberg (christopher.borberg@econ.ku.dk)
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Department of Economics, University of Copenhagen

[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS8, Ex. 4: First-price sealed bid auctions with three bidders

PS8, Ex. 5: Winner's Curse

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with

$$v_i \sim u(1,3)$$

Thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$.
Find c and d.

(b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

[Try to write up the PDF, CDF, and Mean for the uniform distribution $x \sim u(a,b)$, before going to the next slide.]

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

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(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

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i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for a uniform distribution $x \sim u(a, b)$:

PDF: Probability density function: $f(x) = \frac{1}{b-a}$

CDF: Cumulative distribution function: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations v_1 and v_2, respectively. These values are distributed independently uniformly with $v_i \sim u(1,3)$, thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1[st] step: **Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i.**

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

1st step: **Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:**

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \ wins|b_i)(v_i - b_i)$$

Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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1[st] step: Assuming bidder j follows the proposed strategy $b_j(v_j) = cv_j + d$, calculate bidder i's expected payoff from bidding b_i:

$$
\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins | b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\
&= \frac{b_i - d - c}{2c}(v_i - b_i)
\end{aligned}
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$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins | b_i)(v_i - b_i) \\
&= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\
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\end{aligned}$$

2nd step: **Take the FOC and SOC wrt. b_i.**

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Standard results for $x \sim u(a,b)$:

PDF: $f(x) = \frac{1}{b-a}$

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Results:

2nd: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$

2nd step: **Take the FOC and SOC wrt.** b_i**.**

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Results:

2[nd]: FOC: $\frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$

SOC: $-\frac{1}{c} = 0$

i.e. expected utility is concave in b_i.

2[nd] step: Take the FOC and SOC wrt. b_i.

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2nd step: Take the FOC and SOC wrt. b_i.

3rd step: **To find c^* and d^*, compare the best-response function $b_i(v_i)$ to $(*)$.**

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SOC: $-\frac{1}{c} = 0$
i.e. expected utility is concave in b_i.

3rd: From the FOC, the BR function is:

$$2b_i = v_i + d + c \Rightarrow$$

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

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\end{aligned}$$

2[nd] step: Take the FOC and SOC wrt. b_i.

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Standard results for $x \sim u(a,b)$:

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SOC: $-\frac{1}{c} = 0$
i.e. expected utility is concave in b_i.

3[rd]: From the FOC, the BR function is:

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

Inserting the first term in the second term, $d^* = \frac{1}{2}(d^* + c^*) = \frac{1}{2}(d^* + \frac{1}{2})$, which solves for $c^* = d^* = \frac{1}{2}$.

(b) Calculate the revenue to the seller.

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Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: **Calculate the expected payment of bidder i with valuation v_i.**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1^{st} step: **Calculate the expected payment of bidder i with valuation v_i:**

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i)b_i(v_i)$$

Standard results for $x \sim u(a,b)$:

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PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

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1st step: Calculate the expected payment of bidder i with valuation v_i:

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\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \ wins|v_i) b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d), \text{ cf. } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
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(3.a) $c^* = d^* = \frac{1}{2}$

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1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) b_i(v_i) \\ &= \frac{cv_i - c}{2c}(cv_i + d), \text{ cf. } (*), (**) \\ &= \frac{v_i - 1}{2}(cv_i + d) \\ &= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\ &= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\ &= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

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1st step: Calculate the expected payment of bidder i with valuation v_i:

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(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

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\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \; wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d), \text{ cf. } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \; wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\
&= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\
&= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\
&= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6}
\end{aligned}
$$

PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\
&= \frac{cv_i - c}{2c}(cv_i + d),\ \text{cf. } (*), (**) \\
&= \frac{v_i - 1}{2}(cv_i + d) \\
&= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right),\ \text{using (3.a)} \\
&= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\
&= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4}
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

3rd step: **Write up the expected revenue to the seller.**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i\ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\
&= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\
&= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\
&= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6}
\end{aligned}
$$

PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i)b_i(v_i) \\ &= \frac{cv_i - c}{2c}(cv_i + d), \text{ cf. } (*), (**) \\ &= \frac{v_i - 1}{2}(cv_i + d) \\ &= \frac{v_i - 1}{2}\left(\frac{v_i}{2} + \frac{1}{2}\right), \text{ using (3.a)} \\ &= \left(\frac{v_i}{2} - \frac{1}{2}\right)\left(\frac{v_i}{2} + \frac{1}{2}\right) \\ &= \left(\frac{v_i}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

3rd step: The expected revenue to the seller is the ex-ante expected payment of both bidders:

$$\text{Seller's revenue} = \mathbb{E}[m_1(v_1)] + \mathbb{E}[m_2(v_2)] = \frac{5}{3}$$

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(**)$ $\mathbb{P}(i \ wins|v_i) = \frac{b_i(v_i)-d-c}{2c} = \frac{cv_i - c}{2c}$

(3.a) $c^* = d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i)f_i(v_i)dv_i \\ &= \int_1^3 \frac{v_i^2-1}{4} \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{8}\int_1^3 v_i^2 - 1 dv_i \\ &= \frac{1}{8}\left[\frac{1}{3}v_i^3 - v_i\right]_1^3 \\ &= \frac{1}{8}\left(\frac{3^3}{3} - 3 - \frac{1^3}{3} + 1\right) = \frac{5}{6} \end{aligned}$$

PS8, Ex. 3.c: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.
ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

PS8, Ex. 3.c.i: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

PS8, Ex. 3.c.ii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of b_1 unless $b_1 < b_2$, in which case P1 no longer wins, thus, gets zero payoff.

P1 looses: Payoff is independent of b_1 unless $b_1 > b_2$, in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) **Calculate player *i*'s expected payment in the BNE.**

PS8, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) **Calculate player *i*'s expected payment in the BNE:**

$$m_i(v_i) = \mathbb{P}(i \ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)]$$

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Player i's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j|v_j < v_i] \\ &= \frac{v_i - 1}{3-1} \cdot \frac{1+v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

PS8, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

i. Suppose player 2 bids his valuation: $b_2(v_2) = v_2$. Write down the expected payoffs to player 1 from bidding b_1.

ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for $x \sim u(a, b)$:

PDF: $f(x) = \frac{1}{b-a}$

CDF: $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean: $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

(i) The expected payoffs of P1 given b_2:

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if} \quad b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if} \quad b_1 = b_2 \\ 0 & \text{if} \quad b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$.

(iii) Player i's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i) \cdot \mathbb{E}[b_j^*(v_j)|b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j|v_j < v_i] \\ &= \frac{v_i - 1}{3 - 1} \cdot \frac{1 + v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

As this is the same as in (3.b), we know:

$$\text{Ex-ante expected payment} = \mathbb{E}[m_i(v_i)] = \frac{5}{6}$$

$$\text{Seller's revenue} = 2 \cdot \mathbb{E}[m_i(v_i)] = \frac{5}{3}$$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

PS8, Ex. 4: First-price sealed bid auctions with three bidders

PS8, Ex. 4: First-price sealed bid auctions with three bidders

Consider the auction setting of the previous exercise. But now suppose that there are three identical bidders, $i = 1, 2, 3$, with values v_i where

$$v_i \sim u(1, 3)$$

and the values are independent, i.e. private. The auction is first-price sealed bid.

(a) Again, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

(c) (More difficult). Calculate the revenue to the seller.

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write bidder i's expected payoff in the proposed equilibrium.

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write bidder i's expected payoff in the proposed equilibrium:

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \ wins | b_i)(v_i - b_i)$$

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$
\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i\ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}
$$

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}$$

Take the FOC and isolate $b_i^{**}(v_i)$.

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins | b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}$$

$$\begin{aligned}
FOC: \quad 0 &= \frac{1}{2c}[2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2] \\
0 &= 2(v_i - b_i) - (b_i - d - c), && \text{assuming } b_i - d - c \neq 0 \\
b_i^{**}(v_i) &= \frac{2}{3}v_i + \frac{1}{3}(c + d)
\end{aligned}$$

PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

(a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies: $b_i(v_i) = cv_i + d$ $(*)$. Find c and d.

Hint: Use that v_j and v_k are independent (private) to write i's expected payoff in eq.:

$$
\begin{aligned}
\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \ wins|b_i)(v_i - b_i) \\
&= \mathbb{P}\left(b_i > b_j(v_j), b_i > b_k(v_k)\right)(v_i - b_i) \\
&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using } (*) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
\end{aligned}
$$

$$
\begin{aligned}
FOC: \quad 0 &= \frac{1}{2c}[2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2] \\
0 &= 2(v_i - b_i) - (b_i - d - c), && \text{assuming } b_i - d - c \neq 0 \\
b_i^{**}(v_i) &= \underbrace{\frac{2}{3}}_{c^* = \frac{2}{3}} v_i + \underbrace{\frac{1}{3}(c + d)}_{d^* = \frac{1}{3}\left(\frac{2}{3} + d^*\right) \Rightarrow d^* = \frac{1}{3}} = \underline{\underline{\frac{2}{3}v_i + \frac{1}{3}}} && Q.E.D.
\end{aligned}
$$

PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

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PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading $\left(\frac{2}{3} > \frac{1}{2}\right)$ and therefore a *higher* revenue for the seller.

Analytically, we can confirm this:

$$b_i^{**} > b_i^* \Leftrightarrow$$
$$\frac{2}{3}v_i + \frac{1}{3} > \frac{1}{2}v_i + \frac{1}{2} \Leftrightarrow$$
$$\frac{1}{6}v_i > \frac{1}{6} \Leftrightarrow$$
$$v_i > 1$$

I.e. except for the rare case where all players have the valuation $v = 1$, the seller's revenue is strictly higher with three players than with two players.

BNE found:

(3.a) $b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2}$ for $i \in 1, 2, 3$

(4.a) $b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}$ for $i \in 1, 2, 3$

(c) (More difficult). Calculate the revenue to the seller.

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i \; wins | v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3}, \; d^* = \frac{1}{2}$

PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: **Calculate the expected payment of bidder *i* with valuation v_i.**

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i \; wins | v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

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PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: **Calculate the expected payment of bidder *i* with valuation v_i:**

$$m_i(v_i) = \mathbb{P}(i \ wins | v_i) b_i(v_i)$$

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i \ wins | v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3}, \ d^* = \frac{1}{2}$

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i),\ \text{cf. } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d),\ \text{cf. } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3} v_i + \frac{1}{3}\right),\ \text{cf. } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
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Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

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(3.a) $c^* = d^* = \frac{1}{2}$

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PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \ wins|v_i) b_i(v_i) \\ &= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ cf. } (\dagger) \\ &= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ cf. } (*) \\ &= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3} v_i + \frac{1}{3}\right), \text{ cf. } (4.a) \\ &= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \end{aligned}$$

2[nd] step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i \ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

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(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\ &= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ cf. } (\dagger) \\ &= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ cf. } (*) \\ &= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ cf. } (4.a) \\ &= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \end{aligned}$$

2nd step: **Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.**

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2nd: Ex-ante payment of bidder i:

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1[st] step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i),\ \text{cf. } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d),\ \text{cf. } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right),\ \text{cf. } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2[nd] step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Why is the ex-ante expected payment lower than in exercise 3.b?

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

(4.a) $c^* = \frac{2}{3},\ d^* = \frac{1}{2}$

2[nd]: Ex-ante payment of bidder i:

$$
\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\
&= \int_1^3 \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right) \cdot \frac{1}{3-1} dv_i \\
&= \frac{1}{24}\left[\frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i\right]_1^3 \\
&= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i\ wins|v_i)b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ cf. } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ cf. } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ cf. } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

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&= \frac{1}{24}\left[\frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i\right]_1^3 \\
&= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$m_i(v_i) = \mathbb{P}(i\ wins|v_i) b_i(v_i)$$
$$= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i),\ \text{cf. } (\dagger)$$
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$$= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3rd step: **Calculate the seller's revenue and compare to exercise (3.b).**

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

$(\dagger)$ $\mathbb{P}(i\ wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

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PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1^{st} step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \ wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ cf. } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ cf. } (*) \\
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&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2^{nd} step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3^{rd} step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

$(*)$ $b_i(v_i) = cv_i + d$

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2^{nd}: Ex-ante payment of bidder i:

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&= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

3^{rd}: $Revenue = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$

Why is seller's revenue higher than in exercise 3.b?

(c) (More difficult). Calculate the revenue to the seller.

1st step: Calculate the expected payment of bidder i with valuation v_i:

$$
\begin{aligned}
m_i(v_i) &= \mathbb{P}(i \; wins|v_i) b_i(v_i) \\
&= \left(\frac{b_i - d - c}{2c}\right)^2 b_i(v_i), \text{ cf. } (\dagger) \\
&= \left(\frac{cv_i - c}{2c}\right)^2 (cv_i + d), \text{ cf. } (*) \\
&= \left(\frac{v_i - 1}{2}\right)^2 \left(\frac{2}{3}v_i + \frac{1}{3}\right), \text{ cf. } (4.a) \\
&= \left(\frac{2v_i^3 - 3v_i^2 + 1}{12}\right)
\end{aligned}
$$

2nd step: Find the ex-ante expected payment by integrating $m_i(v_i)$ using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3rd step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

($*$) $b_i(v_i) = cv_i + d$

($\dagger$) $\mathbb{P}(i \; wins|v_i) = \left(\frac{b_i - d - c}{2c}\right)^2$ from (4.a)

(3.a) $c^* = d^* = \frac{1}{2}$

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2nd: Ex-ante payment of bidder i:

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\begin{aligned}
\mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\
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&= \frac{1}{24}\left(\frac{33}{2} - \frac{1}{2}\right) = \frac{2}{3} < \frac{5}{6}
\end{aligned}
$$

3rd: $Revenue = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$

The seller can expect higher revenue as more players increases competition and the chance of one having high valuation.

PS8, Ex. 5: Winner's Curse

PS8, Ex. 5: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

PS8, Ex. 5.a: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

Though the reports investigating the values are private, the bidders' actual values are ***common*** since they are identical, i.e. $v_1 = v_2 = v$.

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Write up Bayes' rule.

PS8, Ex. 5.b: Winner's Curse

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: Use Bayes' rule and the given probabilities to write up the probability that the value of the oil-field is high after having received the report $r_1 = H$.

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use ($\star$) to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L.

PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use $(\star)$ to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L:

$$\mathbb{E}[v|r_1 = H] = \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L$$

PS8, Ex. 5.b: Winner's Curse

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(b) Assume that company 1 receives a report of H. Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report $r_1 = H$:

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p\frac{1}{2}}{p\frac{1}{2} + (1-p)\frac{1}{2}} = \frac{p\frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use ($\star$) to write up the expected value of the oil-field after receiving the report $r_1 = H$ where the profits can be either high v_H or low v_L:

$$\begin{aligned}\mathbb{E}[v|r_1 = H] &= \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L \\ &= \mathbb{P}[H|r_1 = H]v_H + (1 - \mathbb{P}[H|r_1 = H])\, v_L, \text{ cf. } (\star) \\ &= pv_H + (1-p)v_L\end{aligned}$$

PS8, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either H or L, and is correct with probability p, where $\frac{1}{2} < p < 1$. The probability of a mistake is independent across the two reports.

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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Step 1: Write up the probability that the value of the oil-field is H given company 1 receives a report $r_1 = H$ and wins the auction.

PS8, Ex. 5.c: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high (H) or low (L) with equal probability.

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Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L)$$

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\begin{aligned}
\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L) \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)
\end{aligned}$$

PS8, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

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\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L) \\
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&= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)
\end{aligned}$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction.

PS8, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

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\mathbb{P}[H|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L) \\
&= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
&= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)
\end{aligned}$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\mathbb{E}[v|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = L \wedge win]v_L$$

PS8, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = L \wedge win]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H],\ \text{cf. (b)}} \qquad \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c).

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

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Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\begin{aligned}
\mathbb{E}[v|r_1 = H \wedge win] &= \mathbb{P}[H|r_1 = H \wedge win] v_H + \mathbb{P}[L|r_1 = L \wedge win] v_L \\
&= \mathbb{P}[H|r_1 = H \wedge win] v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L \\
&= \frac{1}{2} v_H + \frac{1}{2} v_L < \underbrace{p v_H + (1-p) v_L}_{\mathbb{E}[v|r_1 = H],\ \text{cf. (b)}} \qquad \text{since } p > \frac{1}{2}
\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c).
In other words, why is company 1 less certain that the value is H after they actually win the auction?

PS8, Ex. 5.c: Winner's Curse

(c) Continue to assume that company 1 receives a report of H, and suppose that this company bids b_H in the auction. Assume that company 2 will bid $b_L < b_H$ if its own report is L and b_H if it is H. Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is H given company 1 receives a report $r_1 = H$ and wins:

$$\mathbb{P}[H|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \qquad \text{(company 1 only wins if } r_2 = L\text{)}$$

$$= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} \qquad (\star\star)$$

Step 2: Use $(\star\star)$ to write up the expected value of the oil-field conditional on the report being $r_1 = H$ and company 1 winning the auction:

$$\mathbb{E}[v|r_1 = H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]v_H + \mathbb{P}[L|r_1 = L \wedge win]v_L$$

$$= H \wedge win] = \mathbb{P}[H|r_1 = H \wedge win]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge win])\, v_L$$

$$= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H],\ \text{cf. (b)}} \qquad \text{since } p > \frac{1}{2}$$

Step 3: This is an example of The Winner's Curse: The equally trustworthy reports of the two companies cancel each other out. Since the valuations of the auctioned object are correlated, you are likely to win the object when you overestimate the value.