The expected highest and second-highest draw from a uniform distribution

To find *seller's expected revenue* from a sealed bid auction (e.g. bidders simultaneously submit their bids in sealed envelopes without knowing the bids of others) with symmetric bidders with valuation drawn from a uniform distribution, there are two different approaches:

1. One approach is to derive each *bidder's expected payment* as a function of her valuation and then integrate this expression using the PDF to get the *ex-ante expected payment* of each bidder which can then be added together to find seller's expected revenue.
2. However, a more simple approach is to for N number of bidders to calculate the expected highest value (first-price sealed bid auction) or the expected second-highest value (second-price sealed bid auction). Plugging the value into the bid-function gives the seller's expected revenue.

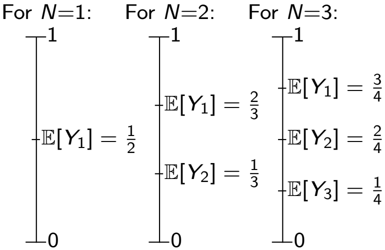
Deriving the optimal bid-function is a prerequisite for both approaches.

The expected highest and second-highest draw from a uniform distribution

First, let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from the ***standard* uniform distribution** $\boldsymbol{x \sim U(0,1)}$. The highest draw Y_1 and the second-highest draw Y_2 of all N draws are expected to be:

$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \quad \text{where } Y_1 = max(X)$$

$$\mathbb{E}[Y_2] = \frac{N-1}{N+1}, \quad \text{where } Y_2 = max(X \neq Y_1)$$



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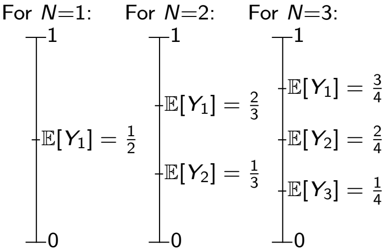

Generalized, let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from a uniform distribution $x \sim U(a,b)$. The highest draw Y_1 and the second-highest draw Y_2 of all N draws are expected to be:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}, \qquad \text{where } Y_1 = max(X)$$

$$\mathbb{E}[Y_2] = a + (b-a)\frac{N-1}{N+1}, \qquad \text{where } Y_2 = max(X \neq Y_1)$$

E.g. for $N = 1$, $\mathbb{E}[Y_1]$ simply collapses to the expression for the expected mean, μ:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1} = a + (b-a)\frac{1}{1+1} = \frac{2a}{2} + \frac{b-a}{2} = \frac{a+b}{2} \equiv \mu$$

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Applied to auctions: Consider N number of bidders where each bidder i has the value v_i that is independently drawn from the same uniform distribution $v_i \sim U(a, b)$.

1st step: The highest value Y_1 and the second-highest value Y_2 for all N bidders are expected to be:

$$\mathbb{E}[Y_1] = a + (b-a)\frac{N}{N+1}, \qquad \text{where } Y_1 = max(V),\ V = v_1, v_2, ..., v_N$$

$$\mathbb{E}[Y_2] = a + (b-a)\frac{N-1}{N+1}, \qquad \text{where } Y_2 = max(V \neq Y_1)$$

2nd step: To calculate the seller's expected revenue, insert the expected highest value $\mathbb{E}[Y_1]$ (first-price sealed bid auction) or the expected second-highest value $\mathbb{E}[Y_2]$ (second-price sealed bid auction) in the derived bid-function.

The expected highest and second-highest draw from a uniform distribution

Proof: [only for those interested] Let $X = x_1, x_2, ..., x_N$ be N independent and identically distributed (i.i.d.) draws from a uniform distribution $x \sim U(a, b)$. Denote the highest draw $Y_1 = max(X)$. The cumulative distribution function (CDF) of Y_1 is:
$G(x) = \mathbb{P}[Y_1 \leq x]$

$$
\begin{aligned}
&= \mathbb{P}[x_1 \leq x, x_1 \leq x, ..., x_N \leq x], && \text{since } Y_1 \text{ is the max of } X \\
&= \mathbb{P}[x_1 \leq x] \times \mathbb{P}[x_1 \leq x] \times ... \times \mathbb{P}[x_N \leq x], && \text{since draws are independent} \\
&= F(x) \times F(x) \times ... \times F(x) = (F(x))^N, && F(x) \text{ is the CDF of } x : F(x) = \frac{x-a}{b-a} \ (*)
\end{aligned}
$$

The first-derivative of the CDF gives the probability density function (PDF) of Y_1:

$$
\begin{aligned}
g(x) &= \frac{\delta G(x)}{\delta x} \\
&= F'(x) N (F(x))^{N-1} \\
&= f(x) N (F(x))^{N-1}, \qquad f(x) \text{ is the PDF of } x : f(x) = \frac{1}{b-a} \ (**)
\end{aligned}
$$

The expectation to Y_1 is found by integrating x times the PDF of Y_1, $g(x)$:

$$
\begin{aligned}
\mathbb{E}[Y_1] &= \int_a^b x \cdot g(x) \ dx \\
&= \int_a^b x \cdot f(x) N (F(x))^{N-1} \ dx \\
&= \int_a^b x \cdot \tfrac{1}{b-a} N \left(\tfrac{x-a}{b-a}\right)^{N-1} \ dx, \qquad \text{using } (**) \text{ and } (*)
\end{aligned}
$$

While the general solution isn't too obvious, the integral solves easily for $x \sim (0, 1)$:
$\mathbb{E}[Y_1] = \int_0^1 x \frac{1}{1-0} N \left(\frac{x-0}{1-0}\right)^{N-1} dx = \int_0^1 x N x^{N-1} dx = \int_0^1 N x^N dx = \left[\frac{N}{N+1} x^{N+1}\right]_0^1 = \frac{N}{N+1}$