



## Microeconomics III: Problem Set 10<sup>a</sup>

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<sup>a</sup>Slides created for exercise class 3 and 4, with reservation for possible errors.

PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

PS8, Ex. 2 (A): Crimea Through a Game-Theory

PS8, Ex. 3 (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

PS8, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

PS8, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)

PS8, Ex. 6: Extensive form game (mixed-strategy Perfect Bayesian Equilibrium)

PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

**PS8, Ex. 1 (A): Asymmetric values  
(second-price sealed bid auction)**

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## PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

Suppose there are two bidders who have private but asymmetric values. In particular,  $v_1 \sim U(0, 1)$  and  $v_2 \sim U(0, 2)$ . Suppose the auction format is second-price sealed bid. When the values are private and symmetric, it is a weakly dominant strategy to bid one's value. Is this still true when the values are asymmetric?

## PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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Step 1: Recall the argument in PS9, Ex. 3.c.

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Step 1: Recall the argument in PS9, Ex. 3.c.

- i. **Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .**

## PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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Step 1: Recall the argument in PS9, Ex. 3.c. (i) The expected payoffs of P1 given  $b_2$ :

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

## PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

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Step 1: Recall the argument in PS9, Ex. 3.c.

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- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.

Step 2: **How is this result affected by the distribution of the bidder's values?**

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of  $b_1$  unless  $b_1 < b_2$ , in which case P1 no longer wins, thus, gets zero payoff.

P1 loses: Payoff is independent of  $b_1$  unless  $b_1 > b_2$ , in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .



## PS8, Ex. 1 (A): Asymmetric values (second-price sealed bid auction)

Suppose there are two bidders who have private but asymmetric values. In particular,  $v_1 \sim U(0, 1)$  and  $v_2 \sim U(0, 2)$ . Suppose the auction format is second-price sealed bid. When the values are private and symmetric, it is a weakly dominant strategy to bid one's value. Is this still true when the values are asymmetric?

Step 1: Recall the argument in PS9, Ex. 3.c.

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Step 2: How is this result affected by the distribution of the bidder's values?

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

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i.e. there is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

2: The result is independent of the distributions, thus it's still a best-response to bid one's value.

## **PS8, Ex. 2 (A): Crimea Through a Game-Theory**

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## PS8, Ex. 2 (A):

Read through the New York Times article *Crimea Through a Game-Theory Lens* by Tyler Cowen (co-author of the popular economics blog *Marginal Revolution*). Try to think about how you would set up models to describe the situations he writes about. *(This exercise is just for reflection, no answer will be provided).*

**PS8, Ex. 3 (A): The 'Lemons' model  
(Perfect Bayesian Equilibrium)**

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## PS8, Ex. 3 (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality “beauties” and low-quality “lemons”. Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

- (a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.
- (b) What would happen if buyers cannot do so, and know that half of all used cars are lemons? Draw this as a dynamic game of incomplete information, where nature chooses the type of the car, the seller observes this and sets a price (any positive real number) and the buyer decides whether to buy or not.
- (c) Find a Perfect Bayesian Equilibrium of this model.

*"In US English, a lemon is a vehicle (often new) that turns out to have several manufacturing defects affecting its safety, value or utility." (Source: Wikipedia)*

## PS8, Ex. 3.a (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality “beauties” and low-quality “lemons”. Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

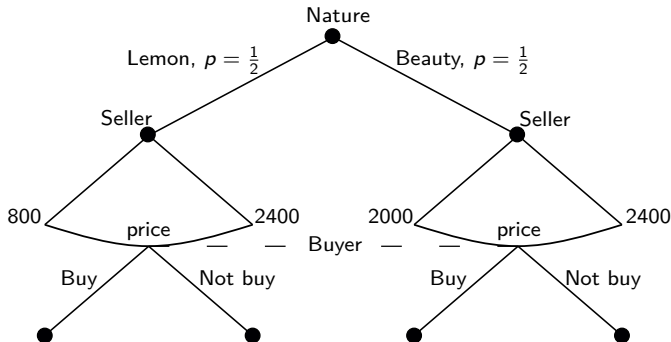
- (a) Describe what would happen in the used-car market if buyers can distinguish between beauties and lemons.

If buyers can distinguish between beauties and lemons they would be traded on two separate markets with prices within  $[800, 1200]$  and  $[2000, 2400]$  respectively.

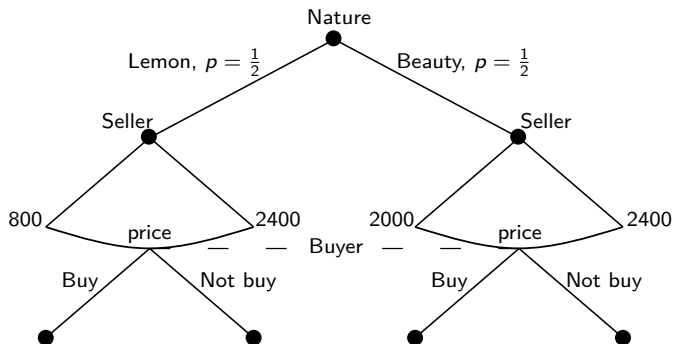
## PS8, Ex. 3.b (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

Consider the The 'Lemons' model of Akerlof. Suppose that used cars come in two types: high-quality “beauties” and low-quality “lemons”. Lemon-owners are willing to sell for \$800 but Beauty-owners will not sell for anything less than \$2000. Buyers will pay up to \$1200 for a lemon and up to \$2400 for a beauty.

- (b) What would happen if buyers cannot do so, and know that half of all used cars are lemons? Draw this as a dynamic game of incomplete information, where nature chooses the type of the car, the seller observes this and sets a price (any positive real number) and the buyer decides whether to buy or not.



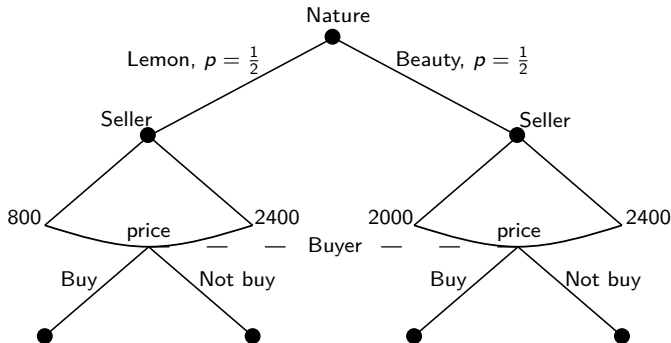
## PS8, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



(c) Find a Perfect Bayesian Equilibrium of this model.



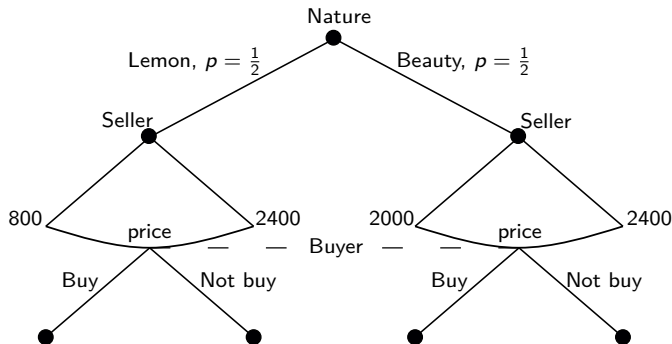
## PS8, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding  $p$

## PS8, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)

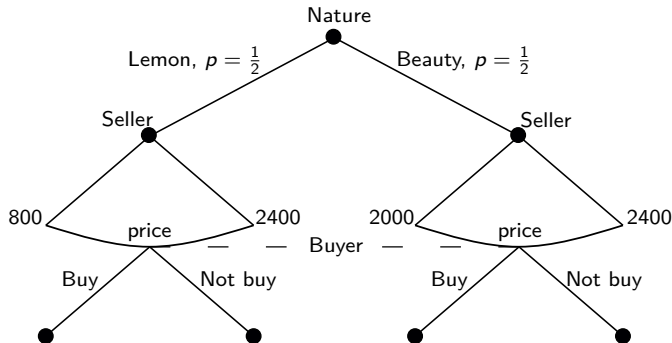


(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding  $p$ .

$$1. E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$$

## PS8, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



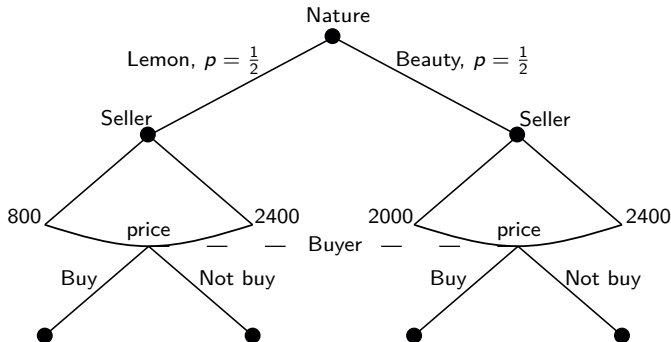
(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding  $p$ .

$$1. E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$$

Step 2: **As both the seller and the buyer know this expectation, what will the outcome be?**

## PS8, Ex. 3.c (A): The 'Lemons' model (Perfect Bayesian Equilibrium)



(c) Find a Perfect Bayesian Equilibrium of this model.

Step 1: Write up buyer's expectation to the car's value given her beliefs regarding  $p$ .

Step 2: As both the seller and the buyer know this expectation, what will the outcome be?

$$1. E[V] = \frac{1}{2}1200 + \frac{1}{2}2400 = 1800$$

2. The seller will not sell *beauties* for a price below 2000. The buyer anticipates this, thus, there will only be a market for *lemons* being sold for  $price \in [800, 1200]$  as 1200 is the highest amount that the buyer is willing to pay for a *lemon*.

**PS8, Ex. 4: A simple principal-agent  
model of corruption (all-pay auction)**

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## PS8, Ex. 4: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists,  $i = 1, 2$ , are trying to persuade a policymaker to implement their preferred policy by making a costly effort  $e_i \in [0, 1]$ . The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

The value to  $i$  of having his preferred policy implemented is  $v_i$ , where  $v_i \sim U(0, 1)$  independently (private values). The lobbyists know their own valuation, but not that of the other lobbyist.

- (a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?
- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ .

## PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists,  $i = 1, 2$ , are trying to persuade a policymaker to implement their preferred policy by making a costly effort  $e_i \in [0, 1]$ . The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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- (a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

Step 1: **Write up the bidders, valuations, bids, and utilities.**

## PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists,  $i = 1, 2$ , are trying to persuade a policymaker to implement their preferred policy by making a costly effort  $e_i \in [0, 1]$ . The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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(a) Rewrite this as an auction. What is the difference to the auctions we have seen so far?

Step 1: Write up the auction with bidders, valuations, bids, and utilities.

Step 2: **How is this different from the auctions we have seen so far?**

1. Two bidders,  $i \in 1, 2$ .

Valuations are independently distributed  $v_i \sim U(0, 1)$

Bids  $b_i \in [0, 1]$

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$



## PS8, Ex. 4.a: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists,  $i = 1, 2$ , are trying to persuade a policymaker to implement their preferred policy by making a costly effort  $e_i \in [0, 1]$ . The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

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Step 1: Write up the auction with bidders, valuations, bids, and utilities.

Step 2: How is this different from the auctions we have seen so far?

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$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

2. Both bidders pay their bid  $b_i$  regardless of whether they win. This is known as an *all-pay auction*.

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

Suppose two lobbyists,  $i = 1, 2$ , are trying to persuade a policymaker to implement their preferred policy by making a costly effort  $e_i \in [0, 1]$ . The policymaker can only implement one of the policies, and will implement the policy of the lobbyist who makes the most effort (you can also think of the policymaker as being corrupt, and the effort being a bribe.) The point is, that the lobbyist has to make the effort *before* he learns if his policy is implemented.

The value to  $i$  of having his preferred policy implemented is  $v_i$ , where  $v_i \sim U(0, 1)$  independently (private values). The lobbyists know their own valuation, but not that of the other lobbyist.

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ .

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_j & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_j & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

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Step 1: **Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.**

Standard results for  $x \sim u(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

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$$\begin{aligned}\mathbb{P}(i \text{ wins}) &= \mathbb{P}(b_i > b_j(v_j)) \\ &= \mathbb{P}(b_i > cv_j^2) \\ &= \mathbb{P}\left(\frac{b_i}{c} > v_j^2\right) \\ &= \mathbb{P}\left(\sqrt{\frac{b_i}{c}} > v_j\right) \\ &= \sqrt{\frac{b_i}{c}}\end{aligned}$$

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$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

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Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

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$$\begin{aligned}\mathbb{E}[u_i(b_i)|v_i] &= \mathbb{P}(i \text{ wins})v_i - b_i \\ &= \sqrt{\frac{b_i}{c}}v_i - b_i, \quad \text{cf. (1)}\end{aligned}$$

Remember that the bid is always paid.

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Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

$$\begin{aligned}\mathbb{E}[u_i(b_i)|v_i] &= \mathbb{P}(i \text{ wins})v_i - b_i \\ &= \sqrt{\frac{b_i}{c}}v_i - b_i, \quad \text{cf. (1)}\end{aligned}$$

Remember that the bid is always paid.

Step 3: **Take the first-order condition and second-order condition with respect to  $b_i$ .**

Standard results for  $x \sim u(a, b)$  :

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- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy. Standard results for  $x \sim u(a, b)$  :

PDF:  $f(x) = \frac{1}{b-a}$

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Results so far:

$$\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} = \frac{\delta}{\delta b_i} \left( \sqrt{\frac{b_i}{c}} v_i - b_i \right)$$

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$= \frac{\delta}{\delta b_i} \left( \frac{\sqrt{b_i}}{\sqrt{c}} v_i - b_i \right)$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$= \frac{\delta}{\delta b_i} \left( b_i^{\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - b_i \right)$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$= \frac{1}{2} b_i^{-\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - 1$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$$

$$= \frac{1}{2} \frac{1}{\sqrt{b_i}} \frac{1}{\sqrt{c}} v_i - 1$$

$$= \frac{1}{2\sqrt{b_i c}} v_i - 1$$



## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy. Standard results for  $x \sim u(a, b)$  :

PDF:  $f(x) = \frac{1}{b-a}$

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Results so far:

$$\frac{\delta \mathbb{E}[u_i(b_i)|v_i]}{\delta b_i} = \frac{\delta}{\delta b_i} \left( \sqrt{\frac{b_i}{c}} v_i - b_i \right)$$

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$= \frac{\delta}{\delta b_i} \left( \frac{\sqrt{b_i}}{\sqrt{c}} v_i - b_i \right)$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$= \frac{\delta}{\delta b_i} \left( b_i^{\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - b_i \right)$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$$

$$= \frac{1}{2} b_i^{-\frac{1}{2}} \frac{1}{\sqrt{c}} v_i - 1 \quad (**)$$

$$\text{SOC: } -\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad \text{cf. } (**)$$

$$= \frac{1}{2} \frac{1}{\sqrt{b_i}} \frac{1}{\sqrt{c}} v_i - 1$$

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## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Step 4: **Solve to find  $b_i(v_i)$ .**

Standard results for  $x \sim u(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$$

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## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Step 4: Solve to find  $b_i(v_i)$ .

As the SOC is negative for all  $b_i, v_i, c > 0$  bidder  $i$  maximizes expected utility for

$$0 = \frac{1}{2\sqrt{b_i(v_i)c}} v_i - 1 \Leftrightarrow$$

$$2\sqrt{b_i(v_i)c} = v_i \Leftrightarrow$$

$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Standard results for  $x \sim u(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$$

$$\text{SOC: } -\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad \text{cf. (**)}$$

$$4. b_i(v_i) = \frac{1}{4c} v_i^2$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Step 4: Solve to find  $b_i(v_i)$ .

As the SOC is negative for all  $b_i, v_i, c > 0$  bidder  $i$  maximizes expected utility for

$$0 = \frac{1}{2\sqrt{b_i(v_i)c}} v_i - 1 \Leftrightarrow$$

$$2\sqrt{b_i(v_i)c} = v_i \Leftrightarrow$$

$$2^2 b_i(v_i)c = v_i^2 \Leftrightarrow$$

$$b_i(v_i) = \frac{1}{4c} v_i^2$$

Step 5: Set this equal to (\*) to find  $c^*$ .

Standard results for  $x \sim u(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i c}} v_i - 1 = 0$$

$$\text{SOC: } -\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad \text{cf. (**)}$$

$$4. b_i(v_i) = \frac{1}{4c} v_i^2$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Step 4: Solve to find  $b_i(v_i)$ .

Step 5: Set this equal to (\*) to find  $c^*$ .

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$

$$c^* = \frac{1}{4c^*} \Leftrightarrow$$

$$2c^* = \frac{1}{4} \Leftrightarrow$$

$$c^* = \frac{1}{2}$$

Standard results for  $x \sim u(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

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Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i/c}} v_i - 1 = 0$$

$$\text{SOC: } -\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad \text{cf. (**)}$$

$$4. b_i(v_i) = \frac{1}{4c} v_i^2$$

$$5. c^* = \frac{1}{2}$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Step 4: Solve to find  $b_i(v_i)$ .

Step 5: Set this equal to (\*) to find  $c^*$ .

$$c^* v_i^2 = \frac{1}{4c^*} v_i^2 \Leftrightarrow$$

$$c^* = \frac{1}{4c^*} \Leftrightarrow$$

$$2c^* = \frac{1}{4} \Leftrightarrow$$

$$c^* = \frac{1}{2}$$

Step 6: **Write up the equilibrium bidding strategy.**

Standard results for  $x \sim u(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

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Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

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$$4. b_i(v_i) = \frac{1}{4c} v_i^2$$

$$5. c^* = \frac{1}{2}$$

## PS8, Ex. 4.b: A simple principal-agent model of corruption (all-pay auction)

- (b) Check that there is a symmetric Bayesian Nash Equilibrium of the type  $b_i(v_i) = cv_i^2$  (\*), and find  $c$ . Values are independently distributed  $v_i \sim U(0, 1)$ .

Step 1: Write up bidder  $i$ 's probability of winning the auction if  $j$  sticks to the equilibrium strategy.

Standard results for  $x \sim u(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

Step 2: Write up bidder  $i$ 's expected payoff from bidding  $b_i$  conditional on  $v_i$ .

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Step 3: Take the FOC and SOC wrt.  $b_i$ .

Results so far:

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{v_i}{2} - b_i & \text{if } b_i = b_j \\ -b_i & \text{if } b_i < b_j \end{cases}$$

Step 4: Solve to find  $b_i(v_i)$ .

Step 5: Set this equal to (\*) to find  $c^*$ .

Step 6: Write up the equilibrium bidding strategy.

$$1. \mathbb{P}(i \text{ wins}) = \sqrt{b_i/c}$$

$$2. \mathbb{E}[u_i(b_i)|v_i] = \sqrt{b_i/c} \cdot v_i - b_i$$

$$3. \text{FOC: } \frac{1}{2\sqrt{b_i/c}} v_i - 1 = 0$$

$$\text{SOC: } -\frac{1}{4} b_i^{-\frac{3}{2}} \frac{1}{\sqrt{c}} v_i = 0, \quad \text{cf. (**)}$$

$$4. b_i(v_i) = \frac{1}{4c} v_i^2$$

$$5. c^* = \frac{1}{2}$$

$$6. \text{BNE: } b_i^*(v_i) = \frac{1}{2} v_i^2$$

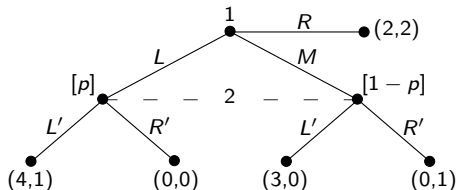
## **PS8, Ex. 5: Extensive form games (Perfect Bayesian Equilibria)**

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## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, **derive the normal-form game** and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

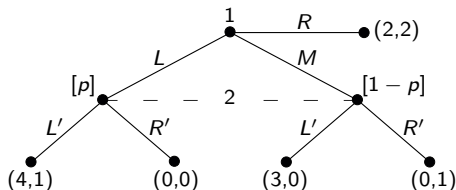


## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE: Find all PSNE in the bi-matrix.



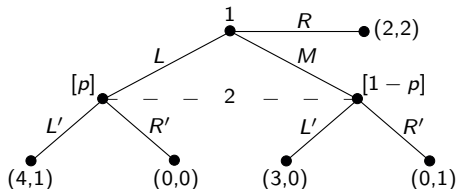
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M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE:  $\{(L, L'); (R, R')\}$

SPNE:



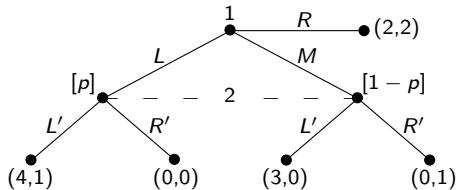
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	L'	R'
L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE:  $\{(L, L'); (R, R')\}$

SPNE: **How many proper subgames are there?**



## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

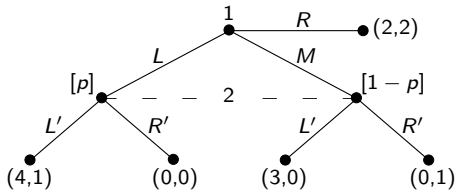
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L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE:



## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

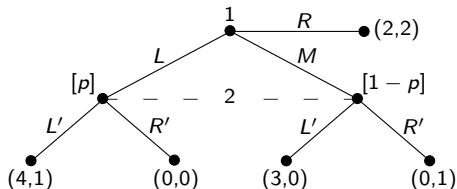
Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
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M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: **Find the pure-strategy PBE including the beliefs of player 2 that secure the equilibrium/equilibria.**



Requirements for a PBE:

- R2: In each information set, players have beliefs about where they are.
- R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.
- R3: Bayesian beliefs *on* equilibrium path.
- R4: Bayesian beliefs *off* equilibrium path.

## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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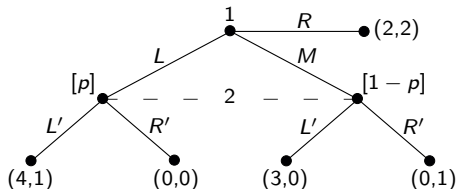
PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play  $R$ , P2 does not know whether the game has reached the information set following  $L$  or  $R$ . She assigns probabilities  $p$  and  $[1 - p]$  to each.

**Given her beliefs, find P2's expected utility of playing  $L'$  and  $R'$  respectively.**



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
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PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

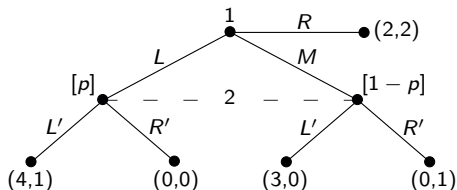
PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play  $R$ , P2 does not know whether the game has reached the information set following  $L$  or  $R$ . She assigns probabilities  $p$  and  $[1 - p]$  to each.

Given her beliefs, find P2's expected utility of playing  $L'$  and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1 - p] = p$$

$$\mathbb{E}[u_2(R')|p] = 0 \cdot p + 1 \cdot [1 - p] = 1 - p$$



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.



## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
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PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play  $R$ , P2 does not know whether the game has reached the information set following  $L$  or  $R$ . She assigns probabilities  $p$  and  $[1 - p]$  to each.

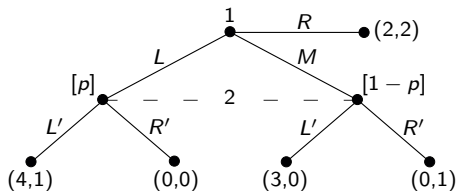
Given her beliefs, find P2's expected utility of playing  $L'$  and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1 - p] = p$$

$$\mathbb{E}[u_2(R')|p] = 0 \cdot p + 1 \cdot [1 - p] = 1 - p$$

P2 is indifferent between  $L'$  and  $R'$  if

$$\mathbb{E}[u_2(L')|p] = \mathbb{E}[u_2(R')|p] \Rightarrow p = 1 - p \Rightarrow p = \frac{1}{2}$$



R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

**Write up the best responses of player 2.**

## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	R'
L [p]	4, 1	0, 0
M [1-p]	3, 0	0, 1
R	2, 2	2, 2

PSNE:  $\{(L, L'); (R, R')\}$

SPNE = PSNE, due to no proper subgames.

PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play  $R$ , P2 does not know whether the game has reached the information set following  $L$  or  $R$ . She assigns probabilities  $p$  and  $[1 - p]$  to each.

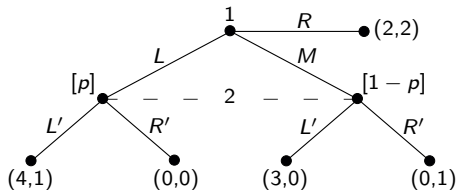
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$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1 - p] = p$$

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P2 is indifferent between  $L'$  and  $R'$  if

$$\mathbb{E}[u_2(L')|p] = \mathbb{E}[u_2(R')|p] \Rightarrow p = 1 - p \Rightarrow p = \frac{1}{2}$$



R2: Players have beliefs.

R2: Sequential rationality.

R3: Bayesian beliefs *on* equilibrium path.

R4: Bayesian beliefs *off* equilibrium path.

Write up the best responses of player 1:

$$BR_1(L') = L, \text{ with } u_1(L|L') = 4$$

$$BR_1(R') = R, \text{ with } u_1(R|R') = 2$$

Find the PBE s.t. requirements 1-4.

## PS8, Ex. 5.a: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.a in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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PBE: Find the pure-strategy PBE including the beliefs of player 2.

If P1 did not play  $R$ , P2 does not know whether the game has reached the information set following  $L$  or  $R$ . She assigns probabilities  $p$  and  $[1 - p]$  to each.

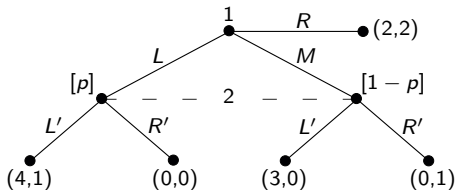
Given her beliefs, find P2's expected utility of playing  $L'$  and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 1 \cdot p + 0 \cdot [1 - p] = p$$

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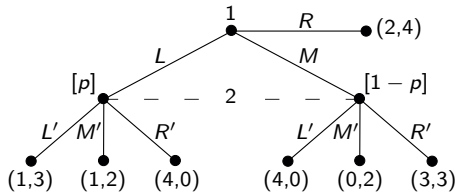
$$BR_1(R') = R, \text{ with } u_1(R|R') = 2$$

P2 knows the  $BR_1$ 's, e.g. if P2 plays  $L'$  she expects P1 to play  $L$  with  $p = 1$ :

$$PBE = \left\{ (L, L'), p = 1 ; (R, R'), p < \frac{1}{2} \right\}$$

## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, **derive the normal-form game** and find all the pure-strategy Nash, subgame-perfect, and perfect Bayesian equilibria.

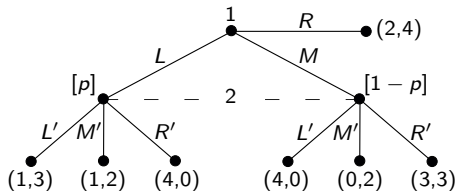


## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:



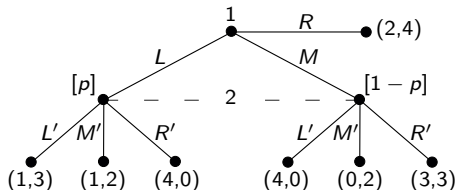
## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
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M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE:



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

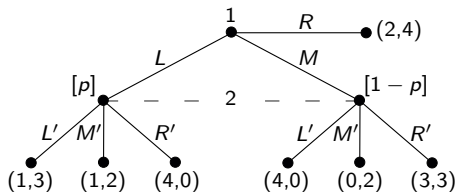
Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

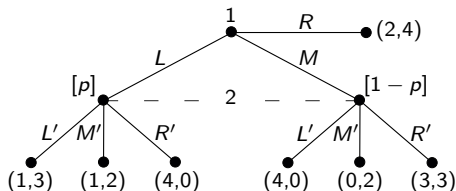
Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
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M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:



Requirements for a PBE:

R2: In each information set, players have beliefs about where they are.

R2: Sequential rationality: At each information set the action taken is optimal given the player's belief at the information set and the other player's subsequent strategies.

R3: Bayesian beliefs *on* equilibrium path.

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## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

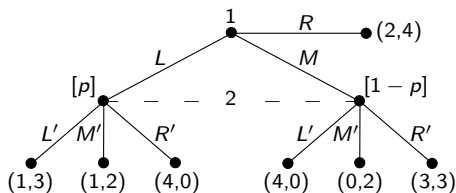
	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:

**Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively.**



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
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PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

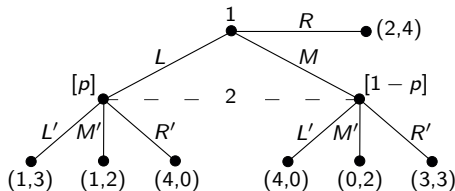
PBE:

Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:

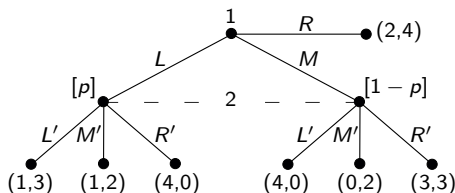
Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$

Draw the expected utility of each choice as functions of  $p$ .



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

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L [p]	1, 3	1, 2	4, 0
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R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

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PBE:

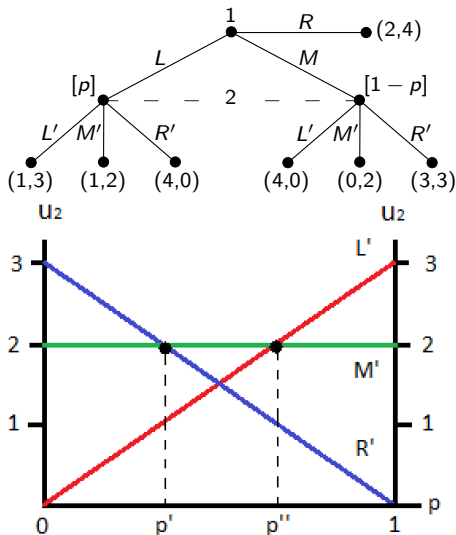
Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$

Using the diagram and the expected utility functions, find the intersections  $p'$  and  $p''$ .



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
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R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

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PBE:

Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

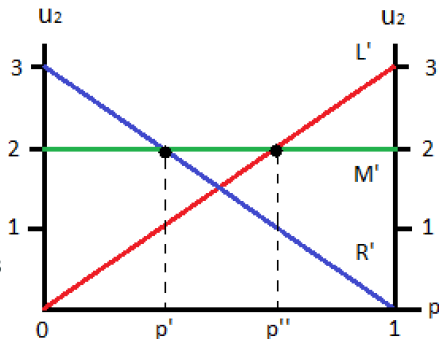
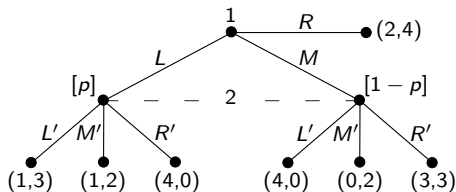
$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$

P2 is indifferent at  $p'$  and  $p''$ :

$$\mathbb{E}[u_2(M')] = \mathbb{E}[u_2(R')] \Rightarrow 2 = 3 - 3p \Rightarrow p' = 1/3$$

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(M')] \Rightarrow 3p = 2 \Rightarrow p'' = 2/3$$



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:

Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1-p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1-p] = 2$$

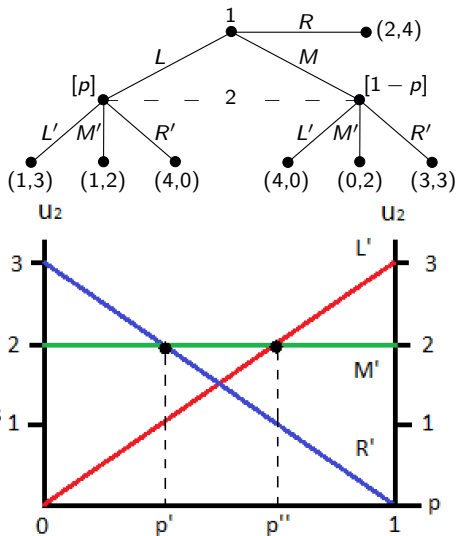
$$\mathbb{E}[u_2(R')|p] = 0p + 3[1-p] = 3-3p$$

P2 is indifferent at  $p'$  and  $p''$ :

$$\mathbb{E}[u_2(M')] = \mathbb{E}[u_2(R')] \Rightarrow 2 = 3-3p \Rightarrow p' = 1/3$$

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(M')] \Rightarrow 3p = 2 \Rightarrow p'' = 2/3$$

Writing up the  $BR_1$ 's, are they consistent with the beliefs of P2 for the relevant interval  $[0, p']$ ,  $[p', p'']$ ,  $[p'', 1]$ ?



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE: **Now, write up the PBE.**

Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1-p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1-p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1-p] = 3 - 3p$$

P2 is indifferent at  $p'$  and  $p''$ :

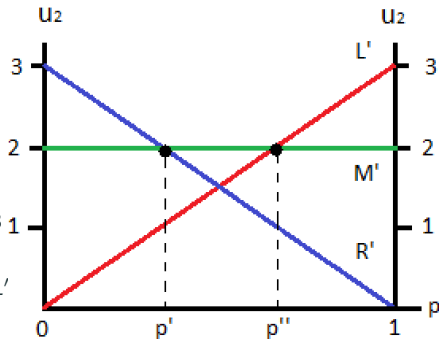
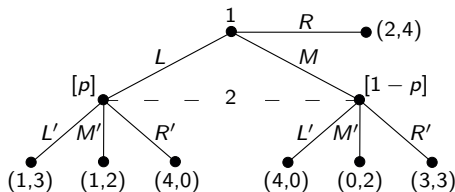
$$\mathbb{E}[u_2(M')] = \mathbb{E}[u_2(R')] \Rightarrow 2 = 3 - 3p \Rightarrow p' = 1/3$$

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(M')] \Rightarrow 3p = 2 \Rightarrow p'' = 2/3$$

$$p \leq 1/3: BR_1(R') = L \rightarrow P2 \text{ deviates to } L'$$

$$p \in \left[\frac{1}{3}, \frac{2}{3}\right]: BR_1(M') = R \rightarrow \text{no deviation}$$

$$p \geq 2/3: BR_1(L') = M \rightarrow P2 \text{ deviates to } R'$$



## PS8, Ex. 5.b: Extensive form games (Perfect Bayesian Equilibria)

Exercise 4.1.b in Gibbons (p. 245). In the following extensive-form games, derive the normal-form game and find all the pure-strategy NE, SPNE, and PBE.

	L'	M'	R'
L [p]	1, 3	1, 2	4, 0
M [1-p]	4, 0	0, 2	3, 3
R	2, 4	2, 4	2, 4

PSNE:  $\{(R, M')\}$

SPNE = PSNE, due to no proper subgames.

PBE:  $\{(R, M'), p \in [\frac{1}{3}, \frac{2}{3}]\}$

Given her beliefs, find P2's expected utility of  $L'$ ,  $M'$ , and  $R'$  respectively:

$$\mathbb{E}[u_2(L')|p] = 3p + 0[1 - p] = 3p$$

$$\mathbb{E}[u_2(M')|p] = 2p + 2[1 - p] = 2$$

$$\mathbb{E}[u_2(R')|p] = 0p + 3[1 - p] = 3 - 3p$$

P2 is indifferent at  $p'$  and  $p''$ :

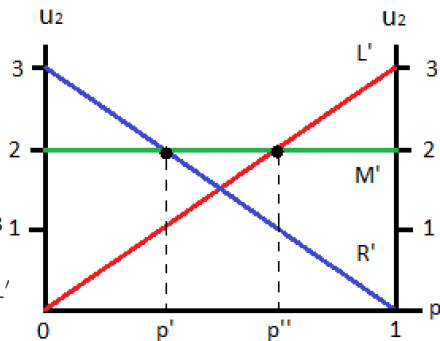
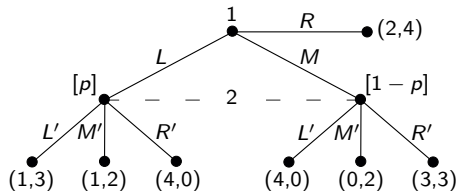
$$\mathbb{E}[u_2(M')] = \mathbb{E}[u_2(R')] \Rightarrow 2 = 3 - 3p \Rightarrow p' = 1/3$$

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(M')] \Rightarrow 3p = 2 \Rightarrow p'' = 2/3$$

$p \leq 1/3$ :  $BR_1(R') = L \rightarrow$  P2 deviates to  $L'$

$p \in [\frac{1}{3}, \frac{2}{3}]$ :  $BR_1(M') = R \rightarrow$  no deviation

$p \geq 2/3$ :  $BR_1(L') = M \rightarrow$  P2 deviates to  $R'$



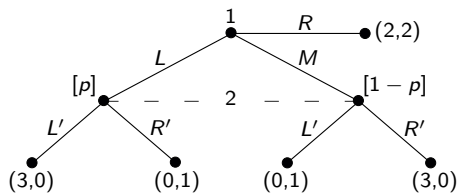


**PS8, Ex. 6: Extensive form game  
(mixed-strategy Perfect Bayesian  
Equilibrium)**

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## PS8, Ex. 6: Extensive form game (mixed-strategy PBE)

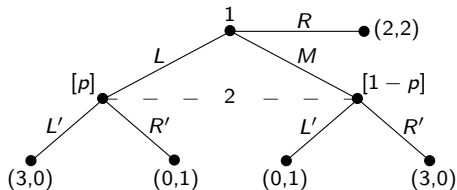
Exercise 4.2 in Gibbons (p. 245). (i) Show that there does not exist a pure-strategy PBE in the following extensive-form game. (ii) What is the mixed-strategy PBE?



# PS8, Ex. 6 (i): Extensive form game (mixed-strategy PBE)

(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

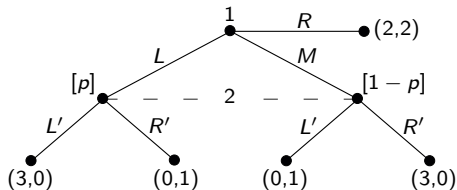
	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



# PS8, Ex. 6 (i): Extensive form game (mixed-strategy PBE)

(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

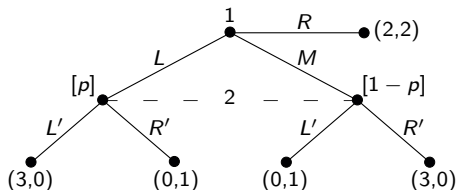


# PS8, Ex. 6 (i): Extensive form game (mixed-strategy PBE)

(i) Show that there does not exist a pure-strategy perfect Bayesian equilibrium.

	L'	R'
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

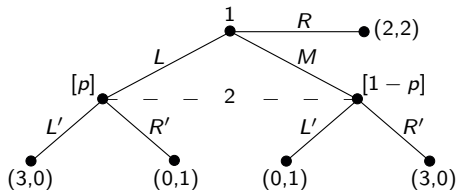
From the bi-matrix it is clear there is no equilibrium in pure strategies as one of the players would always want to deviate.



## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2



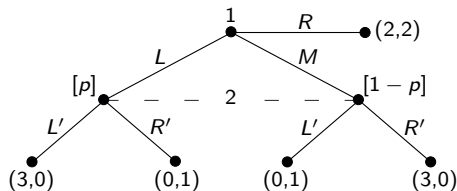
## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

Assume that P2 plays  $L'$  and  $R'$  with probability  $q$  and  $1 - q$  respectively.

**Given his beliefs, find P1's expected utility of  $L$ ,  $M$ , and  $R$  respectively.**



## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

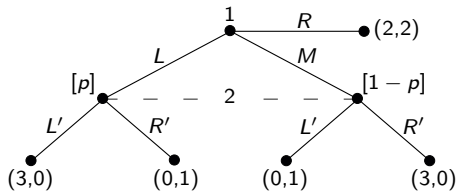
Assume that P2 plays  $L'$  and  $R'$  with probability  $q$  and  $1 - q$  respectively.

Given his beliefs, find P1's expected utility of  $L$ ,  $M$ , and  $R$  respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1 - q] = 3q$$

$$\mathbb{E}[u_1(M)|q] = 0q + 3[1 - q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)|q] = 2$$





## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
M [1-p]	0, 1	3, 0
R	2, 2	2, 2

Assume that P2 plays  $L'$  and  $R'$  with probability  $q$  and  $1 - q$  respectively.

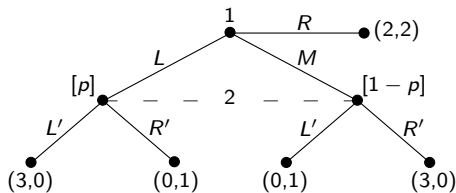
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Draw the expected utility of each choice as functions of  $q$ .



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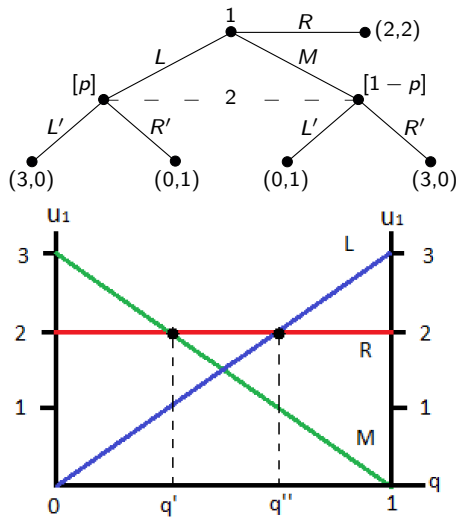
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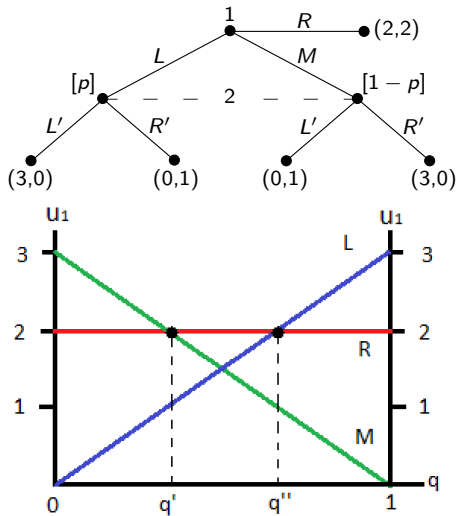
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Using the diagram and the expected utility functions, find the intersections  $q'$  and  $q''$ .



## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

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Assume that P2 plays  $L'$  and  $R'$  with probability  $q$  and  $1 - q$  respectively.

Given his beliefs, find P1's expected utility of  $L$ ,  $M$ , and  $R$  respectively:

$$\mathbb{E}[u_1(L)|q] = 3q + 0[1 - q] = 3q$$

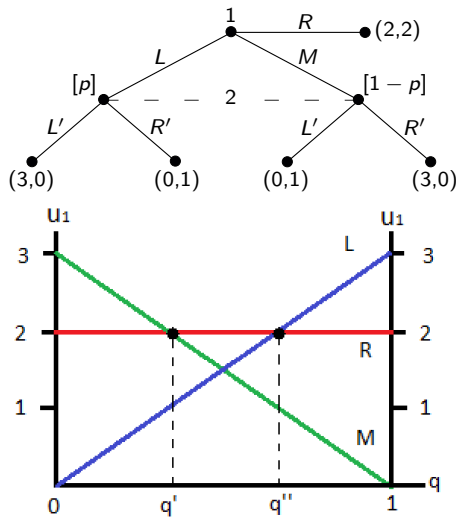
$$\mathbb{E}[u_1(M)|q] = 0q + 3[1 - q] = 3 - 3q$$

$$\mathbb{E}[u_1(R)|q] = 2$$

P1 is indifferent at  $q'$  and  $q''$ :

$$\mathbb{E}[u_1(M)] = \mathbb{E}[u_1(R)] \Rightarrow 3 - 3q = 2 \Rightarrow q' = 1/3$$

$$\mathbb{E}[u_1(L)] = \mathbb{E}[u_1(R)] \Rightarrow 3q = 2 \Rightarrow q'' = 2/3$$



## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

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$$\mathbb{E}[u_1(L)|q] = 3q + 0[1 - q] = 3q$$

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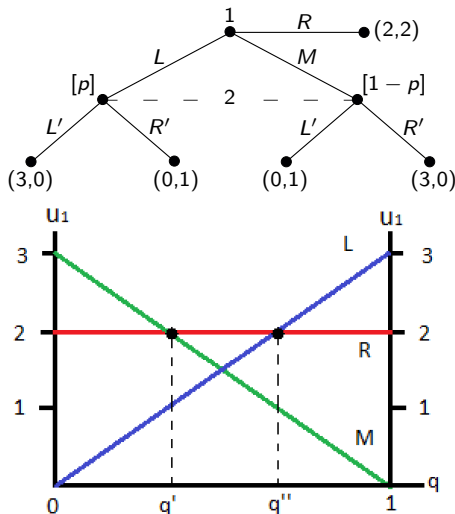
$$\mathbb{E}[u_1(R)|q] = 2$$

P1 is indifferent at  $q'$  and  $q''$ :

$$\mathbb{E}[u_1(M)] = \mathbb{E}[u_1(R)] \Rightarrow 3 - 3q = 2 \Rightarrow q' = 1/3$$

$$\mathbb{E}[u_1(L)] = \mathbb{E}[u_1(R)] \Rightarrow 3q = 2 \Rightarrow q'' = 2/3$$

Writing up the  $BR_2$ 's, are they consistent with the beliefs of P1 for the relevant interval  $[0, q']$ ,  $[q', q'']$ ,  $[q'', 1]$ ?



# PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

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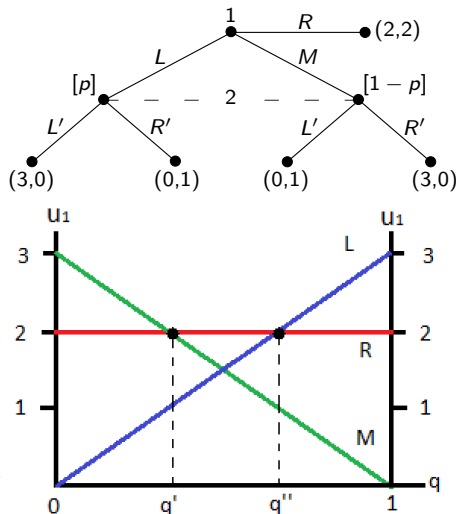
$$\mathbb{E}[u_1(L)] = \mathbb{E}[u_1(R)] \Rightarrow 3q = 2 \Rightarrow q'' = 2/3$$

$q \leq 1/3$ :  $BR_2(M) = L' \rightarrow$  P1 deviates to L

$q \in [1/3, 2/3]$ : P1 plays R  $\rightarrow$  P2 is indifferent

$q \geq 2/3$ :  $BR_2(L) = R' \rightarrow$  P2 deviates to M

Find the beliefs  $p$  such that P2 is indifferent between  $L'$  and  $R'$ .



## PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

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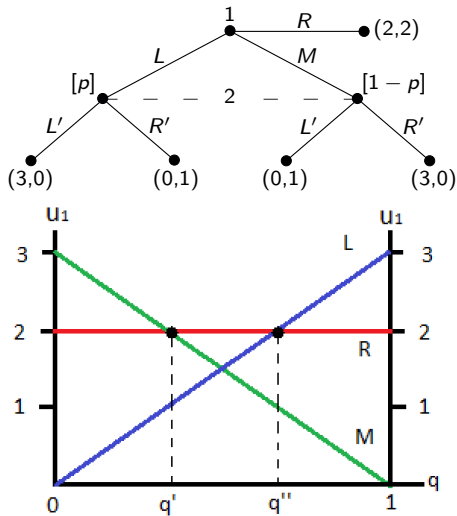
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$q \geq 2/3$ :  $BR_2(L) = R' \rightarrow$  P2 deviates to M

P2 is indifferent if she believes that:

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(R')] \Rightarrow 1 - p = p \Rightarrow p^* = 1/2$$



# PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

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$q \leq 1/3$ :  $BR_2(M) = L' \rightarrow$  P1 deviates to L

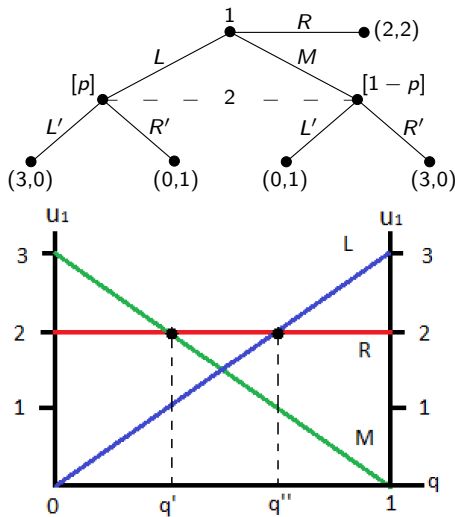
$q \in [\frac{1}{3}, \frac{2}{3}]$ : P1 plays R  $\rightarrow$  P2 is indifferent

$q \geq 2/3$ :  $BR_2(L) = R' \rightarrow$  P2 deviates to M

P2 is indifferent if she believes that:

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(R')] \Rightarrow 1 - p = p \Rightarrow p^* = 1/2$$

Is  $p = \frac{1}{2}$  compatible with  $q \in [\frac{1}{3}, \frac{2}{3}]$ ?





# PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

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L [p]	3, 0	0, 1
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P1's expected utility of L, M, and R:

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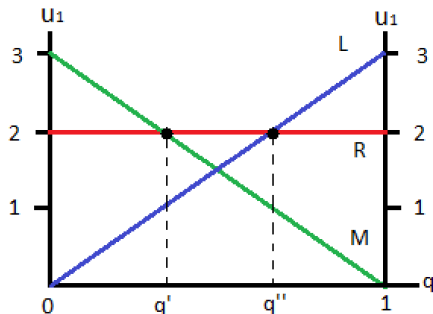
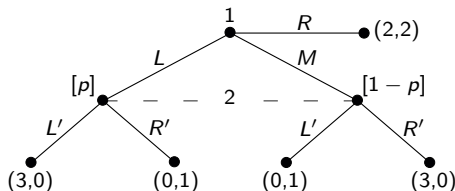
$q \in [1/3, 2/3]$ : P1 plays R  $\rightarrow$  P2 is indifferent

$q \geq 2/3$ :  $BR_2(L) = R' \rightarrow$  P2 deviates to M

P2 is indifferent if she believes that:

$$\mathbb{E}[u_2(L')] = \mathbb{E}[u_2(R')] \Rightarrow 1 - p = p \Rightarrow p^* = 1/2$$

$q \in [1/3, 2/3]$ : P1 plays R.  $p = 1/2$ : P2 mixes.



If compatible, write up the mixed-strategy PBE.

# PS8, Ex. 6 (ii): Extensive form game (mixed-strategy PBE)

(ii) What is the mixed-strategy PBE?

	L' [q]	R' [1-q]
L [p]	3, 0	0, 1
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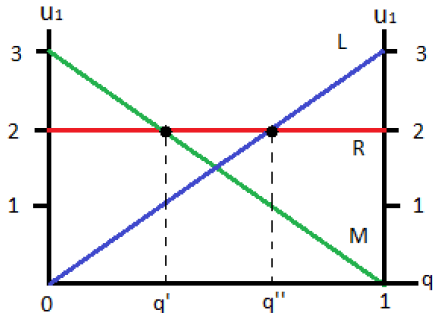
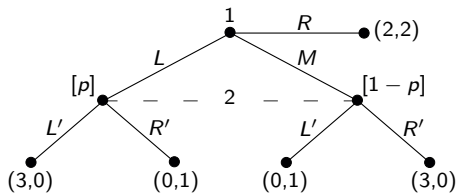
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$q \in [1/3, 2/3]$ : P1 plays R.  $p = 1/2$ : P2 mixes.



$$PBE = \left\{ \left( R, q \in \left[ \frac{1}{3}, \frac{2}{3} \right] \right), p = \frac{1}{2} \right\}$$

**PS8, Ex. 7: Dissolving a partnership  
(Perfect Bayesian Equilibria)**

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## PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

Difficult. Exercise 4.10 in Gibbons (p. 250). Two partners must dissolve their partnership. Partner 1 currently owns share  $s$  of the partnership, partner 2 owns share  $1 - s$ . The partners agree to play the following game: partner 1 names a price,  $p$ , for the whole partnership, and partner 2 then chooses either to buy 1's share for  $ps$  or to sell his or her share to 1 for  $p(1 - s)$ . Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on  $[0, 1]$ , but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?

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**Step 1: Write up the PDF, CDF, and Mean for a uniform distribution.**

## PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

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Step 1: Write up the probability density function (PDF), cumulative distribution function (CDF), and mean for a uniform distribution.

1. Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

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Step 2: **Write up P1's expected utility from naming the price  $p$ .**

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Step 2: **Write up P1's expected utility from naming the price  $p$ .** Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

$$\mathbb{E}[u_1(v_1, p)] = \mathbb{P}[\text{P2 buys}](p - v_1)s + \mathbb{P}[\text{P2 sells}](v_1 - p)(1 - s)$$



## PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

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$$\begin{aligned} \mathbb{E}[u_1(v_1, p)] &= \mathbb{P}[\text{P2 buys}](p - v_1)s + \mathbb{P}[\text{P2 sells}](v_1 - p)(1 - s) \\ &= \mathbb{P}[v_2 > p](p - v_1)s + \mathbb{P}[v_2 < p](v_1 - p)(1 - s) \\ &= (1 - \mathbb{P}[v_2 < p])(p - v_1)s + \mathbb{P}[v_2 < p](v_1 - p)(1 - s) \\ &= \left(1 - \frac{p-0}{1-0}\right)(p - v_1)s + \frac{p-0}{1-0}(v_1 - p)(1 - s), && \text{using the CDF} \\ &= (1 - p)(p - v_1)s + p(v_1 - p)(1 - s) \\ &= ps - v_1s - p^2s + pv_1s + pv_1 - pv_1s - p^2 + p^2s \\ &= ps - v_1s + pv_1 - p^2 \end{aligned} \quad (*)$$

## PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

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Step 1: Write up the probability density function (PDF), cumulative distribution function (CDF), and mean for a uniform distribution.

Step 2: P1's expected utility from price  $p$ .

Step 3: **Maximize P1's expected utility (\*) with respect to the price  $p$ .**

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Step 1: Write up the probability density function (PDF), cumulative distribution function (CDF), and mean for a uniform distribution.

Step 2: P1's expected utility from price  $p$ .

Step 3:  $\max \mathbb{E}[u_1(v_1, p)]$  wrt.  $p$ :

$$\frac{\delta \mathbb{E}[u_1(v_1, p)]}{\delta p} = 0 \Rightarrow$$

$$s + v_1 - 2p = 0 \Leftrightarrow$$

$$s + v_1 = 2p \Leftrightarrow$$

$$p^* = \frac{s + v_1}{2}$$

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$$2. \mathbb{E}[u_1(v_1, p)] = ps - v_1s + pv_1 - p^2 \quad (*)$$

**What is the PBE?**

## PS8, Ex. 7: Dissolving a partnership (Perfect Bayesian Equilibria)

Difficult. Exercise 4.10 in Gibbons (p. 250). Two partners must dissolve their partnership. Partner 1 currently owns share  $s$  of the partnership, partner 2 owns share  $1 - s$ . The partners agree to play the following game: partner 1 names a price,  $p$ , for the whole partnership, and partner 2 then chooses either to buy 1's share for  $ps$  or to sell his or her share to 1 for  $p(1 - s)$ . Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on  $[0, 1]$ , but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?

Step 1: Write up the probability density function (PDF), cumulative distribution function (CDF), and mean for a uniform distribution.

Step 2: P1's expected utility from price  $p$ .

Step 3:  $\max \mathbb{E}[u_1(v_1, p)]$  wrt.  $p$ :

$$\frac{\delta \mathbb{E}[u_1(v_1, p)]}{\delta p} = 0 \Rightarrow$$

$$s + v_1 - 2p = 0 \Leftrightarrow$$

$$s + v_1 = 2p \Leftrightarrow$$

$$p^* = \frac{s + v_1}{2}$$

1. Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(x < c) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

$$2. \mathbb{E}[u_1(v_1, p)] = ps - v_1s + pv_1 - p^2 \quad (*)$$

$$3. \text{ In the PBE, the price is } p^* = \frac{s+v_1}{2}$$