Microeconomics III: Problem Set 6[a]

Thor Donsby Noe (thor.noe@econ.ku.dk) & Christopher Borberg (christopher.borberg@econ.ku.dk)
October 30 2019

Department of Economics, University of Copenhagen

[a] Slides created for exercise class 3 and 4, with reservation for possible errors.

Outline

PS6, Ex. 1 (A): Sequential bargaining

PS6, Ex. 1 (A): Sequential bargaining

Consider the sequential bargaining game discussed in Lecture 6, but now with $K \geq 1$ stages (where K is some arbitrary but fixed integer). Suppose $\delta = 1$ and $K = 1, 2, 3$. Is there a first-mover advantage? Does your answer depend on the value of K?

Explain δ mathematically

delta is the discount factor which the payoff in the next game will be multiplied by, so if there player stand to gain 1 in the next round, and $\delta = 0.5$, it is only worth $1 * 0.5 = 0.5$ to the player in the current round.

Explain δ intuitively

Intuitively δ is the factor showing how patient the players are. The higher δ, thee less the players will mind waiting for the next round.

Explain the case $\delta = 0$

In the case $\delta = 0$, the players will have their payoff multiplied by 0 in the next round, so the game Rounds into an ultimatum game where the first mover can offer the other player anything and they will accept. There is a first mover advantage.

Explain the case $\delta = 1$

In the case $\delta = 1$, the players will have their payoff multiplied by 1 in the next round, so they won't care whether the game goes for another around. This will be the case for each round until the final round, which will then be an ultimatum game where the last mover can offer the other player anything and they will accept. There is no first mover advantage, but there is a last mover advantage.

Explain whether it depends on K

For δ=1, the last mover will get the whole price pool, no matter how many rounds (K) the game is. The only case with a first mover advantage is for $K = 1$, in which the first move is the same as the last.

PS6, Ex. 2 (A): Infinite-horizon bargaining

PS6, Ex. 2 (A): Infinite-horizon bargaining

Question 2.3 from Gibbons (p.131) looks at the infinite-horizon bargaining game where player 1 has discount factor δ_1 and player 2 has discount factor δ_2. It shows that the backward-induction outcome of this game is

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) \tag{1}$$

Discuss how these payoffs change as each player becomes more or less patient, i.e. as we vary δ_1 and δ_2. What is the intuition? Show that these payoffs simplify to those derived in Lecture 6

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \tag{2}$$

for the case where $\delta_1 = \delta_2$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step 1) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

Information so far:

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step 1) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

(Step 2) Use the fact that it's a zero sum game to look at the change in outcome for player 2

Information so far:

1. $\frac{\partial s_1*}{\partial \delta_1} = \frac{(1-\delta_2)\delta_2}{(1-\delta_1\delta_2)^2} > 0$
2. $\frac{\partial s_1*}{\partial \delta_2} = -\frac{1-\delta_1}{(1-\delta_1\delta_2)^2} < 0$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part one: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ Discuss how the payoff change as each player becomes more or less patient.

(Step 1) Write up partial derivatives for δ_2's and δ_1's effect on the outcome for player 1, are the partial derivatives positive or negative?

(Step 2) Use the fact that it's a zero sum game to look at the change in outcome for player 2

Answer Player 1s payoff is increasing in δ_1 and decreasing in δ_2, vice versa for Player 2. This intuitively makes sense, because player i's bargaining power in later rounds will increase when his patience increase relative to player j.

Information so far:

1 $\frac{\partial s_1*}{\partial\delta_1} = \frac{(1-\delta_2)\delta_2}{(1-\delta_1\delta_2)^2} > 0$

2 $\frac{\partial s_1*}{\partial\delta_2} = -\frac{1-\delta_1}{(1-\delta_1\delta_2)^2} < 0$

PS6, Ex. 2 (A): Infinite-horizon bargaining

Part two: For the payoffs: $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ show that for $\delta_2 = \delta_1$ the payoffs simplify to $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$

Write up the payoffs with $\delta = \delta_1 = \delta_2$ and use that: $1 - x^2 = (1+x)(1-x)$, to simplify

$$\left(\frac{1-\delta}{1-\delta^2}, \frac{\delta(1-\delta)}{1-\delta^2}\right) \Rightarrow \left(\frac{1-\delta}{(1-\delta)(1+\delta)}, \frac{\delta(1-\delta)}{(1-\delta)(1+\delta)}\right) \Rightarrow \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$$

PS6, Ex. 3: Dynamic games (imperfect information)

PS6, Ex. 3: Dynamic games (imperfect information)

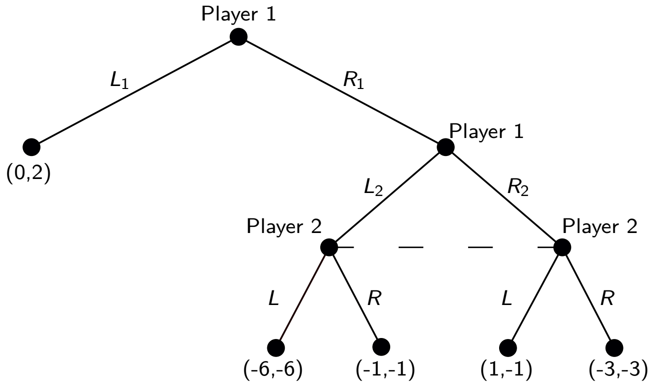
Find the SPNE in the four games.

Hints:

1. It becomes much easier to grasp dynamic games with imperfect information if you write the part with imperfect information in normal form (bi-matrix).
2. Be careful to cover all of the strategy profile (in every subgame!) when writing up the subgame perfect Nash Equilibria (SPNE).

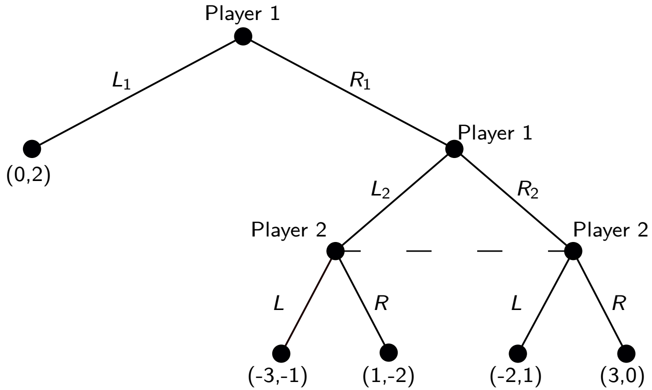
PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:



PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:

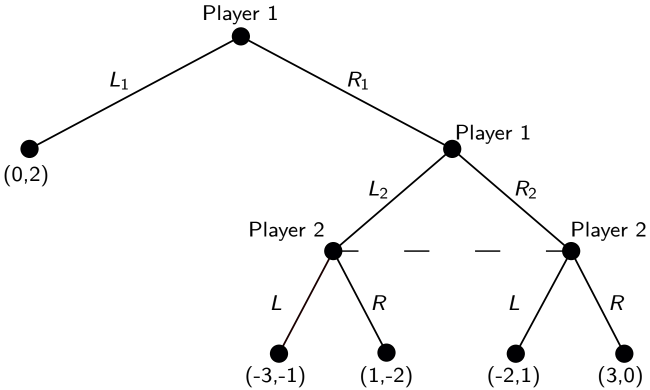


2[nd] and 3[rd] stage in normal form:

		Player 2	
		L	R
Player 1	L_2	-3, -1	1, -2
	R_2	-2, 1	3, 0

PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:

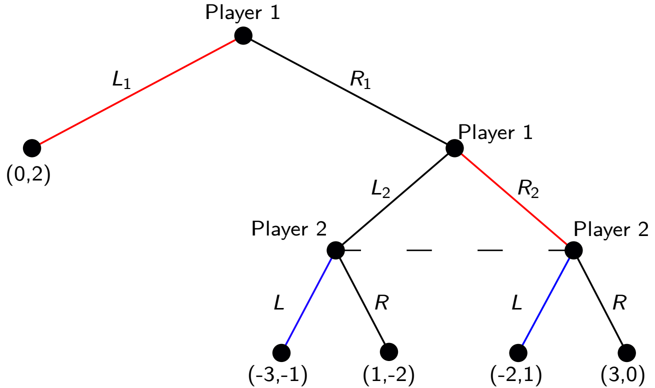


2[nd] and 3[rd] stage in normal form:

Player 1 \ Player 2	L	R
L_2	-3, -1	1, -2
R_2	-2, 1	3, 0

PS6, Ex. 3.a: Dynamic games (imperfect information)

(a) Find the SPNE in the following game:

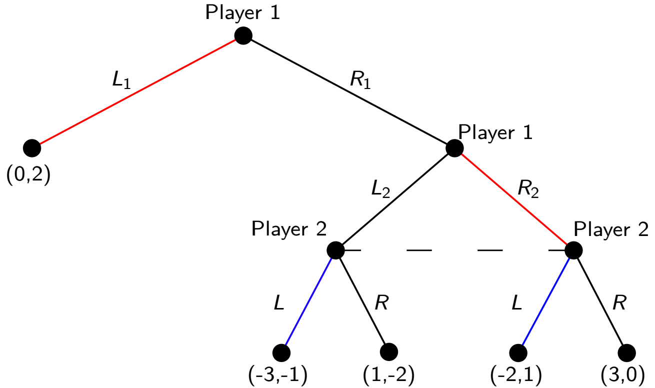


2nd and 3rd stage in normal form:

Player 1 \ Player 2	L	R
L_2	-3, -1	1, -2
R_2	-2, 1	3, 0

Write up the SPNE!

(a) Find the SPNE in the following game:



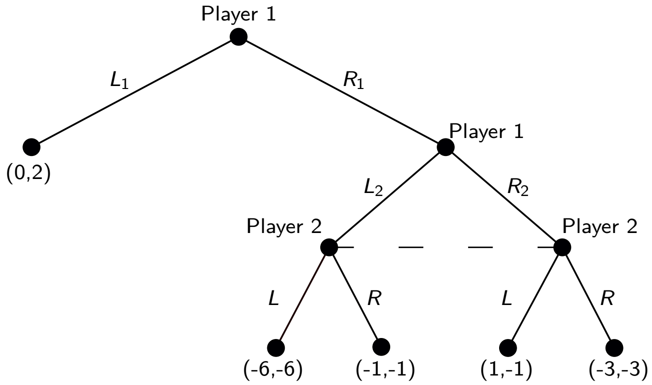
2nd and 3rd stage in normal form:

Player 1 \ Player 2	L	R
L_2	-3, -1	1, -2
R_2	-2, 1	3, 0

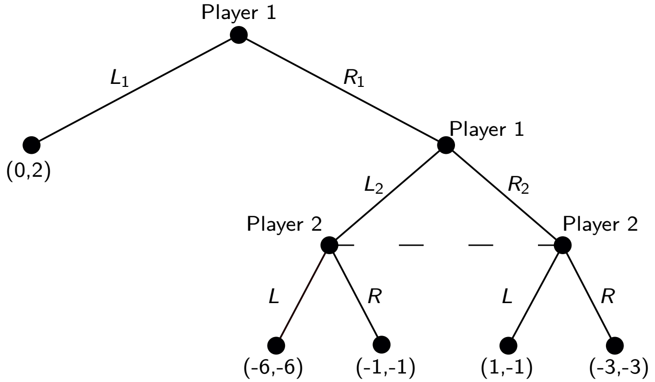
$SPNE = \{s_1^*, s_2^*\} = \{(L_1, R_2), L\}$ with outcome (0,2).

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:



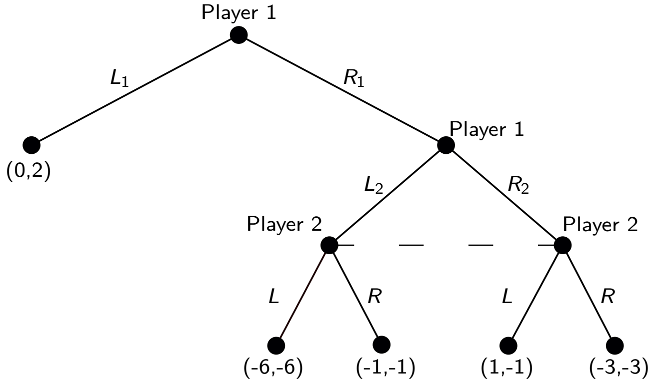
(b) Find the SPNE in the following game:



2$^{\text{nd}}$ and 3$^{\text{rd}}$ stage in normal form:

		Player 2: L	Player 2: R
Player 1	L_2	-6, -6	-1, -1
	R_2	1, -1	-3, -3

(b) Find the SPNE in the following game:



2[nd] and 3[rd] stage in normal form:

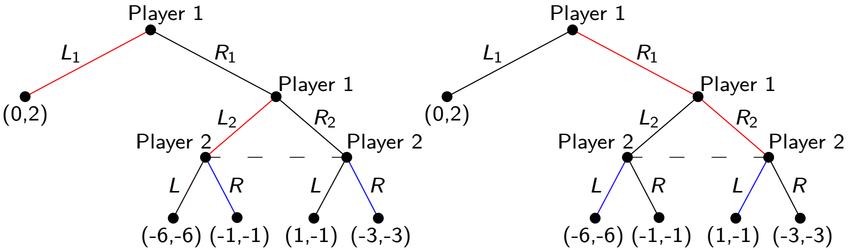
Player 2

Player 1	L	R
L_2	-6, -6	-1, -1
R_2	1, -1	-3, -3

Two different pure strategy NE (PSNE) in the subgame. What now?

(b) Find the SPNE in the following game:

R_1 is strictly dominated by L_1 and we have two subgame perfect solutions:

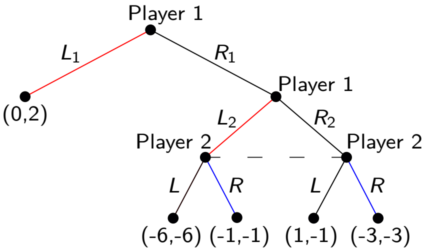


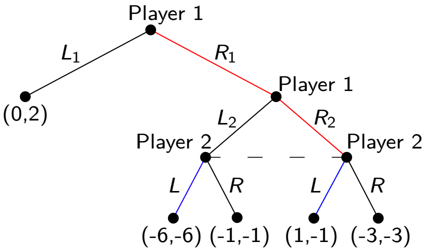
Write up the SPNE!

PS6, Ex. 3.b: Dynamic games (imperfect information)

(b) Find the SPNE in the following game:

R_1 is strictly dominated by L_1 and we have two subgame perfect solutions:

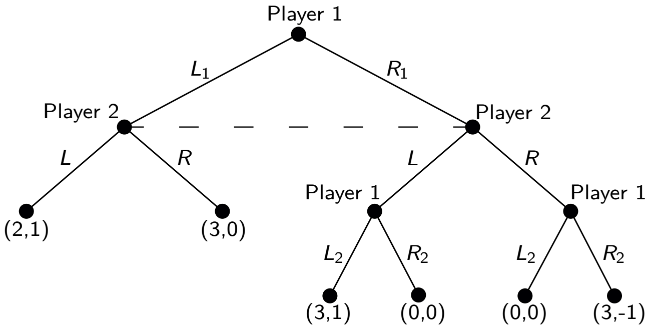




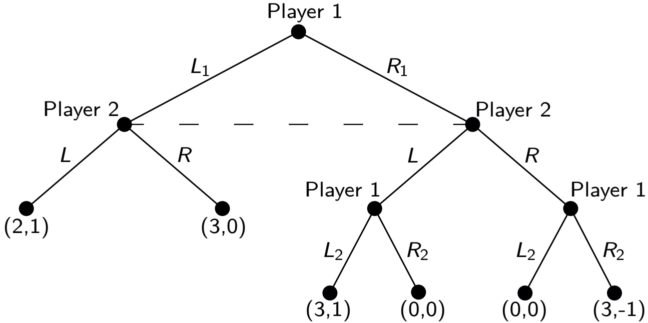
$SPNE = \{s_1^*, s_2^*\} = \{(L_1, L_2), R; (L_1, R_2), L\}$ with outcomes {(0,2);(1,-1)}.

PS6, Ex. 3.c: Dynamic games (imperfect information)

(c) Find the SPNE in the following game:

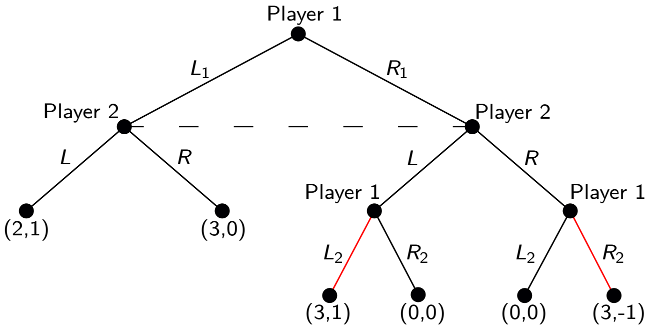


(c) Find the SPNE in the following game:



Backwards Induction: First solve the 3rd stage.

(c) Find the SPNE in the following game:

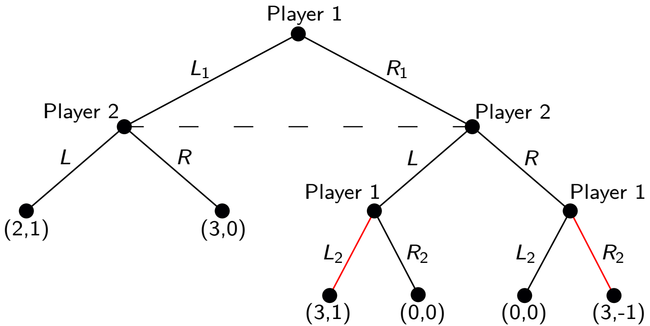


Backwards Induction: First solve the 3rd stage.

1st and 2nd stage in normal form (taking the 3rd stage as given):

Player 1 \ Player 2	L	R
L_1	2, 1	3, 0
R_1	3, 1	3, -1

(c) Find the SPNE in the following game:



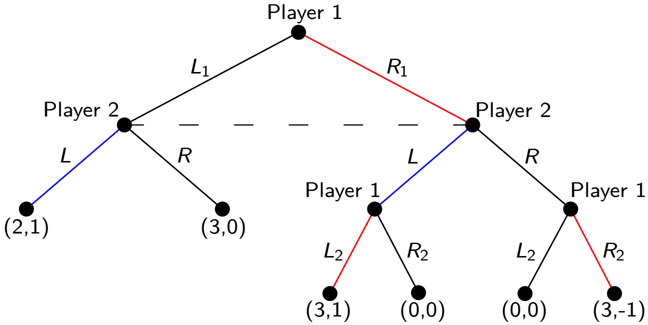
Backwards Induction: First solve the 3rd stage.

1st and 2nd stage in normal form (taking the 3rd stage as given):

Player 2

Player 1	L	R
L_1	2, 1	3, 0
R_1	3, 1	3, -1

(c) Find the SPNE in the following game:



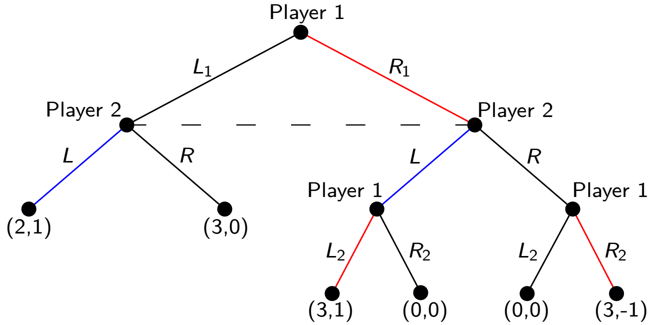
Backwards Induction: First solve the 3rd stage.

1st and 2nd stage in normal form (taking the 3rd stage as given):

Player 1 \ Player 2	L	R
L_1	2, 1	3, 0
R_1	3, 1	3, -1

Consider how many subgames there are and write up the SPNE.

(c) Find the SPNE in the following game:



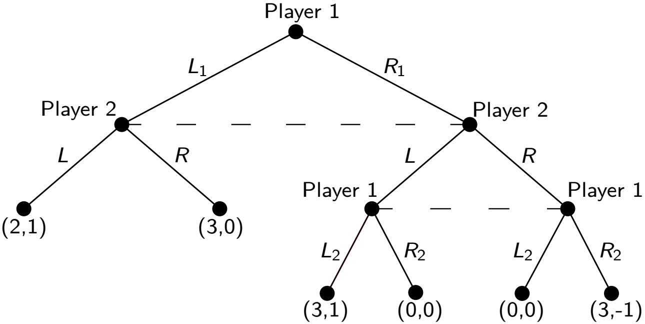
Backwards Induction: First solve the 3rd stage.

1st and 2nd stage in normal form (taking the 3rd stage as given):

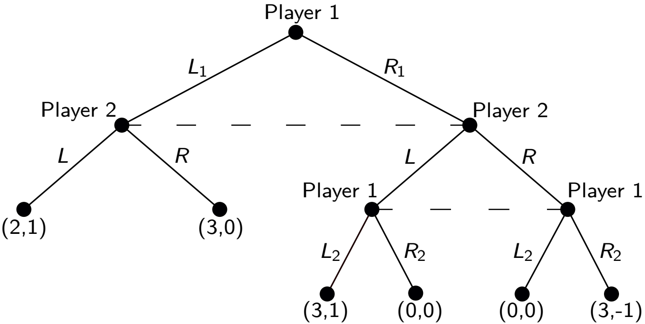
Player 1 \ Player 2	L	R
L_1	2, 1	3, 0
R_1	3, 1	3, -1

$SPNE = \{s_1^*, s_2^*\} = \{(R_1, L_2, R_2), L\}$ with outcome (3,1).

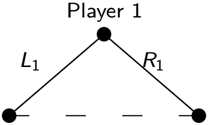
(d) Find the SPNE in the following game:



(d) Find the SPNE in the following game:



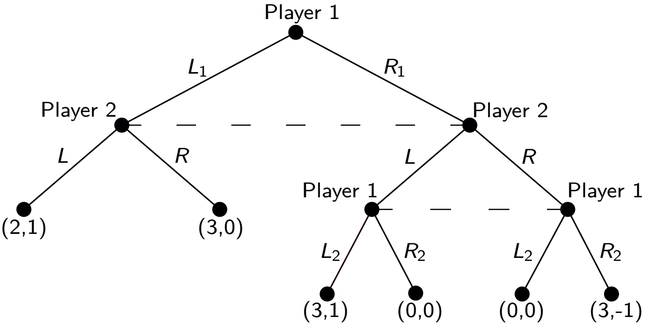
2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



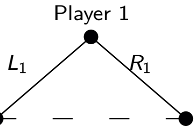
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



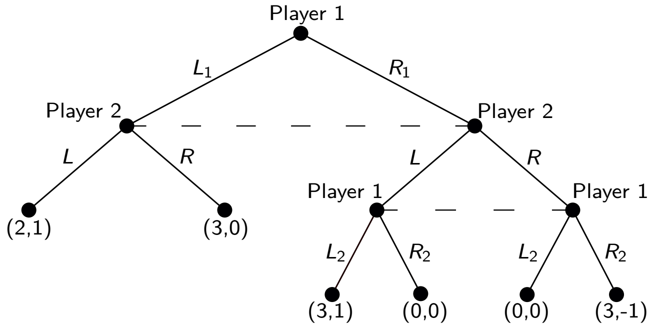
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

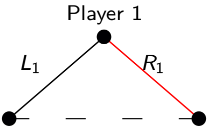
Which equilibria are subgame perfect?

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):

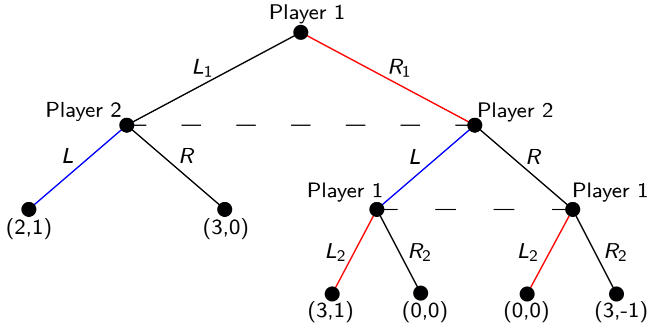


	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

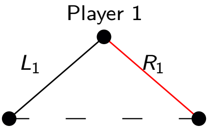
	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

Player 2: R is strictly dominated by L. Player 1: Expecting L, she plays (R_1, L_2).

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



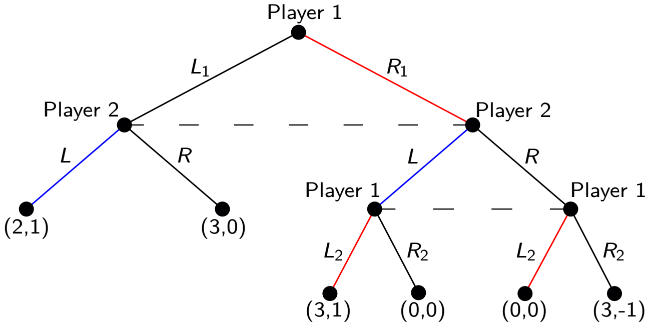
	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

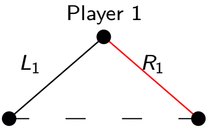
Write up the SPNE!

PS6, Ex. 3.d: Dynamic games (imperfect information)

(d) Find the SPNE in the following game:



2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



	L	R
L_2	2, 1	3, 0
R_2	2, 1	3, 0

	L	R
L_2	3, 1	0, 0
R_2	0, 0	3, -1

$SPNE = \{s_1^*, s_2^*\} = \{(R_1, L_2), L\}$ with outcome (3,1).

PS6, Ex. 4: The Mutated Seabass (imperfect information)

PS6, Ex. 4:

Go back to exercise 4 in problem set 5. Write up the game tree for the situation in part (c), where the choice to acquire the weapon is not observed. Find the SPNE. What has changed?

Last class we actually solved and discussed this part as an extension...

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement

$$(s^*, 1 - s^*) = \left(\frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \tag{3}$$

which player 2 accepts.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement

$$(s^*, 1 - s^*) = \left(\frac{1 - \delta_2}{1 - \delta_1 \delta_2}, \frac{\delta_2 (1 - \delta_1)}{1 - \delta_1 \delta_2} \right) \tag{4}$$

which player 2 accepts.

Hints:

1. Start with a three stage game where Player 1 gets a payoff s_3 in round 3.
2. Use this to find a stationary solution where $s_1 = s_3$.
3. Remember that the outcome in period $\boldsymbol{t}$ is always denoted $\boldsymbol{s_t}$ for Player 1 and $\mathbf{1} - \boldsymbol{s_t}$ for Player 2.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 1) Start with a three stage game where Player 1's payoff in round 3 is denoted s_3. Write up the outcome for Player 1 in round 3. Then use the potential outcome of round 3 to find the outcome in round 2. Do the same for round 1 with respects to round 2.

Round 3 What does Player 1 propose? Does Player 2 accept? What does Player 1 get himself?

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1 - s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 1) Start with a three stage game where Player 1's payoff in round 3 is denoted s_3. Write up the outcome for Player 1 in round 3. Then use the potential outcome of round 3 to find the outcome in round 2. Do the same for round 1 with respects to round 2.

Round 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0; 1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Round 2

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 1) Start with a three stage game where Player 1's payoff in round 3 is denoted s_3. Write up the outcome for Player 1 in round 3. Then use the potential outcome of round 3 to find the outcome in round 2. Do the same for round 1 with respects to round 2.

Round 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0; 1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Round 2 P1 will choose to accept or decline an offer $s_2 \in [0; 1]$: He will accept if $s_2 \geq s_3\delta_1$. P2 proposes $s_2 = s_3\delta_1$ which P1 accepts. P2 gets $1 - s_2 = 1 - s_3\delta_1$.

Round 1

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 1) Start with a three stage game where Player 1's payoff in round 3 is denoted s_3. Write up the outcome for Player 1 in round 3. Then use the potential outcome of round 3 to find the outcome in round 2. Do the same for round 1 with respects to round 2.

Round 3 P2 will choose to accept or decline an offer $1 - s_3 \in [0; 1]$: She will accept anything. P1 proposes $1 - s_3$ which P2 accepts. P1 gets s_3 for himself.

Round 2 P1 will choose to accept or decline an offer $s_2 \in [0; 1]$: He will accept if $s_2 \geq s_3\delta_1$. P2 proposes $s_2 = s_3\delta_1$ which P1 accepts. P2 gets $1 - s_2 = 1 - s_3\delta_1$.

Round 1 P2 will choose to accept or decline an offer $1 - s_1 \in [0; 1]$: She will accept if $1 - s_1 \geq (1 - s_3\delta_1)\delta_2$. P1 proposes $1 - s_1 = (1 - s_3\delta_1)\delta_2$ which P2 accepts. P1 gets $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 2) Since the game is infinite, the players are playing the same game in Round 3 as in Round 1, thus, the outcome of Round 1 should be the same as in Round 3. Use this to find a stationary solution, where $s_1 = s_3$.

Outcomes for Player 1:
Round 1 $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.
Round 3 s_3.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 2) Since the game is infinite, the players are playing the same game in Round 3 as in Round 1, thus, the outcome of Round 1 should be the same as in Round 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$s_1 = s_3 \Rightarrow$$
$$s^* = 1 - (1 - s^*\delta_1)\delta_2$$

Outcomes for Player 1:

Round 1 $s_1 = 1 - (1 - s_3\delta_1)\delta_2$.

Round 3 s_3.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 2) Since the game is infinite, the players are playing the same game in Round 3 as in Round 1, thus, the outcome of Round 1 should be the same as in Round 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$\begin{aligned} s_1 &= s_3 \Rightarrow \\ s^* &= 1-(1-s^*\delta_1)\delta_2 \Rightarrow \\ s^* &= 1-\delta_2+s^*\delta_1\delta_2 \Rightarrow \\ s^*(1-\delta_1\delta_2) &= 1-\delta_2 \Rightarrow \\ s^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

Outcomes for Player 1:
Round 1 $s_1 = 1-(1-s_3\delta_1)\delta_2$.
Round 3 s_3.

PS6, Ex. 5: Infinite-horizon bargaining with different discount factors

Consider Rubinstein's infinite-horizon bargaining game, but where each player has a different discount factor: δ_1, δ_2. Show that in the backwards induction outcome, player 1 offers the settlement $(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ which player 2 accepts.

(Step 2) Since the game is infinite, the players are playing the same game in Round 3 as in Round 1, thus, the outcome of Round 1 should be the same as in Round 3. Use this to find a stationary solution, where $s_1 = s_3$.

Stationary solution:

$$\begin{aligned} s_1 &= s_3 \Rightarrow \\ s^* &= 1-(1-s^*\delta_1)\delta_2 \Rightarrow \\ s^* &= 1-\delta_2+s^*\delta_1\delta_2 \Rightarrow \\ s^*(1-\delta_1\delta_2) &= 1-\delta_2 \Rightarrow \\ s^* &= \frac{1-\delta_2}{1-\delta_1\delta_2} \end{aligned}$$

Insert in $1-s^*$, juggle a bit, and get:

$$(s^*, 1-s^*) = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$$

Outcomes for Player 1:
Round 1 $s_1 = 1-(1-s_3\delta_1)\delta_2$.
Round 3 s_3.

PS6, Ex. 6: Cornout, colluding to every-ones benefit?

PS6, Ex. 6: Cornout, colluding to everyone's benefit?

(A two stage game with simultaneous moves) On July 12, 2001, the presidents of Toyota and PSA Group, Fujio Cho and Jean-Martin Folz, decided to jointly develop a small city car (...)

PS6, Ex. 6.a: Cornout, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Cost production: $c_{1q} = c_{2q} = 1 - x_1 - x_2$

3 Cost research: $c_i = x_i^2$

PS6, Ex. 6.a: Cornout, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step 1) Write up the payoff function, taking research as given.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Cost production:
$c_{1q} = c_{2q} = 1 - x_1 - x_2$

3 Cost research: $c_i = x_i^2$

PS6, Ex. 6.a: Cornout, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step 1) Write up the payoff function, taking research as given.

(Step 2) Write up the FOC and find the best-response function for q_i.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Cost production: $c_{1q} = c_{2q} = 1 - x_1 - x_2$

3 Cost research: $c_i = x_i^2$

4 *Payoff*$_i$: $\pi_i(q_i, q_j, x_i, x_j) = (2 - q_i - q_j)q_i - (1 - x_i - x_j)q_i - x_i^2$

PS6, Ex. 6.a: Cornout, colluding to everyone's benefit?

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step 1) Write up the payoff function, taking research as given.

(Step 2) Write up the FOC and find the best response function for q_i.

(Step 3) Use symmetry to find the NE by setting $q_i = q_j$.

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Cost production: $c_{1q} = c_{2q} = 1 - x_1 - x_2$

3 Cost research: $c_i = x_i^2$

4 *Payoff*$_i$: $\pi_i(q_i, q_j, x_i, x_j) = (2 - q_i - q_j)q_i - (1 - x_i - x_j)q_i - x_i^2$

6 $BR_i(q_j)$: $q_i = \frac{1}{2} - \frac{q_j}{2} + \frac{x_i + x_j}{2}$

(a) Given the levels of research x_1, x_2, find the resulting levels of output $(q_1(x_1, x_2))$ and $(q_2(x_1, x_2))$ in the second stage.

(Step 1) Write up the payoff function, taking research as given.

(Step 2) Write up the FOC and find the best response function for q_i.

(Step 3) Use symmetry to find the NE by setting $q_i = q_j$:

$$q_i = \frac{1}{2} - \frac{q_i}{2} + \frac{x_i + x_j}{2} \Rightarrow$$

$$q_i = \frac{1 + x_i + x_j}{3}$$

NE: $\left(\frac{1+x_i+x_j}{3}, \frac{1+x_i+x_j}{3}\right)$

Information so far:

1 Price: $P(q_1, q_2) = 2 - q_1 - q_2$

2 Cost production: $c_{1q} = c_{2q} = 1 - x_1 - x_2$

3 Cost research: $c_i = x_i^2$

4 *Payoff$_i$*: $\pi_i(q_i, q_j, x_i, x_j) = (2 - q_i - q_j)q_i - (1 - x_i - x_j)q_i - x_i^2$

6 $BR_i(q_j)$: $q_i = \frac{1}{2} - \frac{q_j}{2} + \frac{x_i+x_j}{2}$

PS6, Ex. 6.b: Cornout, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

Information so far:

1. $BR_i(x_i, x_j) : q_i(x_i, x_j) = \frac{1+x_i+x_j}{3}$
2. *Payoff*$_i$: $\pi_i(q_i, q_j, x_i, x_j) =$
   $[2 - q_i - q_j - (1 - x_i - x_j)]q_i - x_i^2$

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step 1) Write up the payoff function as a function of research

Information so far:

1. $BR_i(x_i, x_j) : q_i(x_i, x_j) = \frac{1+x_i+x_j}{3}$
2. $Payoff_i$: $\pi_i(q_i, q_j, x_i, x_j) = [2 - q_i - q_j - (1 - x_i - x_j)]q_i - x_i^2$

PS6, Ex. 6.b: Cornout, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step 1) Write up the payoff function as a function of research

(Step 2) Write up the FOC for x_i

Information so far:

1 $BR_i(x_i, x_j) : q_i(x_i, x_j) = \frac{1+x_i+x_j}{3}$

2 $Payoff_i$: $\pi_i(q_i, q_j, x_i, x_j) =$ $[2 - q_i - q_j - (1 - x_i - x_j)]q_i - x_i^2$

3 $Payoff_i(x_1, x_2)$:

$$
\begin{aligned}
\pi_i =& [2 - 2\frac{1 + x_i + x_j}{3} - (1 - x_i - x_j)] \\
& \frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow \\
=& \frac{1 + x_i + x_j}{3} \frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow \\
=& \frac{(1 + x_i + x_j)^2}{9} - x_i^2
\end{aligned}
$$

PS6, Ex. 6.b: Cornout, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step 1) Write up the payoff function as a function of research

(Step 2) Write up the FOC for x_i

(Step 3) Use symmetry to find the SPNE by setting $x_i = x_j$ isolating x_i and calculating q_i.

Information so far:

1 $BR_i(x_i, x_j) : q_i(x_i, x_j) = \frac{1+x_i+x_j}{3}$

2 $Payoff_i$: $\pi_i(q_i, q_j, x_i, x_j) = [2 - q_i - q_j - (1 - x_i - x_j)]q_i - x_i^2$

3 $Payoff_i(x_1, x_2)$:

$$\begin{aligned}\pi_i =&[2 - 2\frac{1 + x_i + x_j}{3} - (1 - x_i - x_j)]\\ &\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow\\ =&\frac{1 + x_i + x_j}{3}\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow\\ =&\frac{(1 + x_i + x_j)^2}{9} - x_i^2\end{aligned}$$

4 FOC_i: $\frac{2}{9}(1 + x_i + x_j) - 2x_i = 0$

PS6, Ex. 6.b: Cornout, colluding to everyone's benefit?

(b) Assume that the stage one decisions are made simultaneously and independently. That is, each firm i chooses x_i in order to maximize its own profit (foreseeing the outcome of stage two). Using your results from (a), find the levels of research and output in the SPNE: $x_1^*, x_2^*, q_1(x_1^*, x_2^*), q_2(x_1^*, x_2^*)$.

(Step 1) Write up the payoff function as a function of research

(Step 2) Write up the FOC for x_i

(Step 3) Use symmetry to find the SPNE by setting $x_i = x_j$ isolating x_i and calculating q_i:

$$\frac{2}{9}(1 + x_i + x_i) - 2x_i = 0 \Rightarrow x_i = \frac{1}{7}$$

$$q_i = \frac{1 + x_i + x_j}{3} = \frac{1}{3}\left(1 + \frac{1}{7} + \frac{1}{7}\right) = \frac{3}{7}$$

SPNE: $\left(\frac{1}{7}, \frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right)$

Information so far:

1 $BR_i(x_i, x_j) : q_i(x_i, x_j) = \frac{1+x_i+x_j}{3}$

2 $Payoff_i$: $\pi_i(q_i, q_j, x_i, x_j) = [2 - q_i - q_j - (1 - x_i - x_j)]q_i - x_i^2$

3 $Payoff_i(x_1, x_2)$:

$$\begin{aligned}\pi_i =&[2 - 2\frac{1 + x_i + x_j}{3} - (1 - x_i - x_j)] \\ &\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow \\ =&\frac{1 + x_i + x_j}{3}\frac{1 + x_i + x_j}{3} - x_i^2 \Rightarrow \\ =&\frac{(1 + x_i + x_j)^2}{9} - x_i^2\end{aligned}$$

4 FOC_i: $\frac{2}{9}(1 + x_i + x_j) - 2x_i = 0$

PS6, Ex. 6.c: Cornout, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^* *, x_2^* *, q_1(x_1^* *, x_2^* *)$ and $q_2(x_1^* *, x_2^* *)$.

Information so far:

1 $Payoff_i(x_1, x_2)$:
$\pi_i(x_i, x_j) = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

PS6, Ex. 6.c: Cornout, colluding to everyone's benefit?

(c) Assume now that the firms collude in the first stage. That is, they choose x_1 and x_2 to maximize their joint profit while taking into account that q_1 and q_2 will be chosen simultaneously and independently in stage two. Find the resulting levels of research and output: $x_1^* *, x_2^* *, q_1(x_1^* *, x_2^* *)$ and $q_2(x_1^* *, x_2^* *)$.

(Step 1) Using symmetry, write up the combined payoff function.

(Step 2) Write up the FOC for π_T wrt. x_T.

(Step 3) Find the outcome by isolating x_T and calculating q_T:

$$8\frac{(1+2x_T)}{9} - 4x_T = 0 \Rightarrow$$

$$\frac{8}{9} + \frac{16}{9}x_T - \frac{36}{9}x_T = 0 \Rightarrow$$

$$x_T = \frac{8}{9} \cdot \frac{9}{20} = \frac{2}{5}.$$

$$q_i = \frac{1+x_T}{3} = \frac{1}{3}\left(1+\frac{2}{5}\right) = \frac{3}{5}$$

Outcome: $\left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}\right)$

Information so far:

1 *Payoff*$_i(x_1, x_2)$:
$\pi_i(x_i, x_j) = \frac{(1+x_i+x_j)^2}{9} - x_i^2$

2 $\pi_T(x_i, x_j) = 2\frac{(1+x_i+x_j)^2}{9} - x_i^2 - x_j^2$
$\Rightarrow \pi_T(x_T) = 2\frac{(1+2x_T)^2}{9} - 2x_T^2$

3 *FOC*$_i$: $8\frac{(1+2x_T)}{9} - 4x_T = 0$

PS6, Ex. 6: Cornout, colluding to everyone's benefit?

(d) Based on your findings in (b) and (c), compare the outcomes in terms of consumer welfare [hint: it is enough to look at total output] and firms' profit [hint: no calculations are necessary]. Comment on the source of the difference

Since the quantity in c is higher, this also means that the price is lower. Higher quantity and lower price means there is a higher consumer welfare.

By definition the profit in c is higher.

- Write up the combined payoff function
- Write up the FOC for x_i
- The difference comes from the fact that the collusion in the first stage drives down the cost, the benefit of which is then distributed amongst companies and consumers

Total outcome:

(a) $(x_T^*, q_T^*) = \left(x_i + x_j, \frac{2}{3}(1 + x_i + x_j)\right)$

(b) $(x_T^*, q_T^*) = \left(\frac{2}{7}, \frac{6}{7}\right)$

(c) $(x_T^*, q_T^*) = \left(\frac{4}{5}, \frac{6}{5}\right)$

PS6, Ex. 7: To keep or split (imperfect information)

PS6, Ex. 7: To keep or split (imperfect information)

Consider the following 2×2 game where payoffs are monetary:

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

Before this game is played, Player 1 can choose whether, after the game is played, players should keep their own payoffs or split the aggregate payoff evenly between them. Player 2 observes this choice.

(a) Write down the game tree of this two-stage game: be careful to represent the simultaneous-move game in the second stage using information sets.

(b) Find the subgame perfect Nash Equilibria (SPNE).

(c) Now suppose that Player 2 cannot observe Player 1's choice in the first stage. Draw the game tree (again using information sets) and find the subgame perfect Nash Equilibria (SPNE).

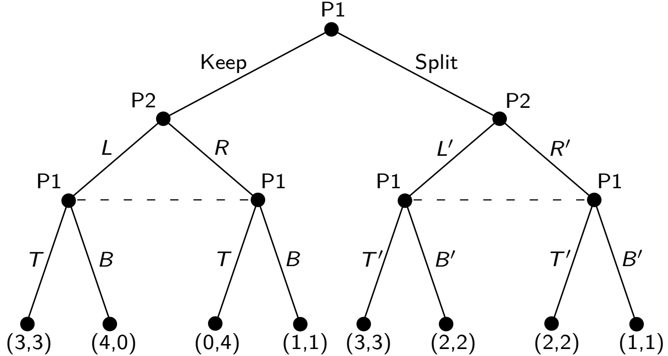
PS6, Ex. 7.a: To keep or split (imperfect information)

(a) Write down the game tree of this two-stage game: be careful to represent the simultaneous-move game in the second stage using information sets.

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

2nd stage: Player 2 chooses L or R (L' or R'). The action is private information.

3rd stage: Player 1 chooses T or B (T' or B') without knowing what Player 2 did.

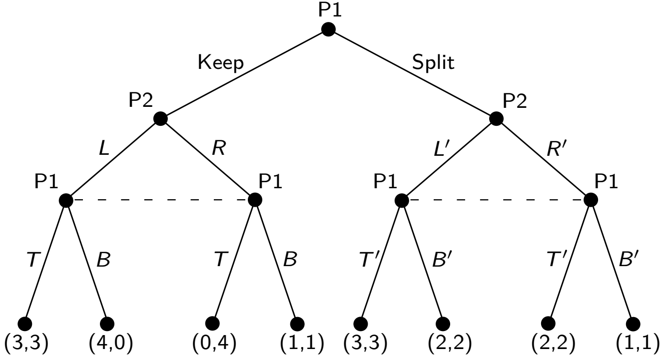


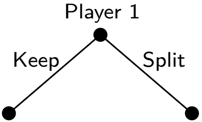
The order of stage 2 and 3 is arbitrary, but the 2nd stage must be private information.

(b) ***Find the subgame perfect Nash Equilibria (SPNE).***

PS6, Ex. 7.b: To keep or split (imperfect information)

(b) Find the subgame perfect Nash Equilibria (SPNE).



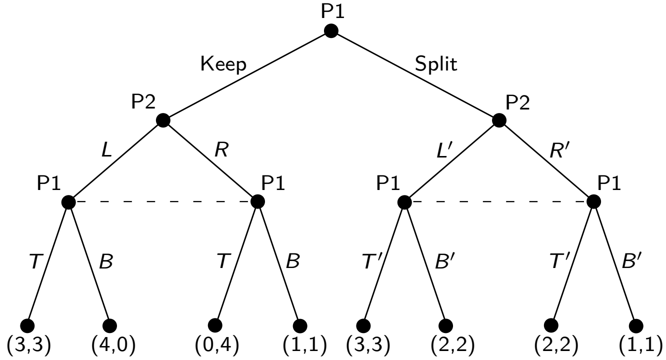


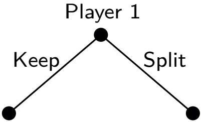
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L'	R'
T'	3, 3	2, 2
B'	2, 2	1, 1

PS6, Ex. 7.b: To keep or split (imperfect information)

(b) Find the subgame perfect Nash Equilibria (SPNE).



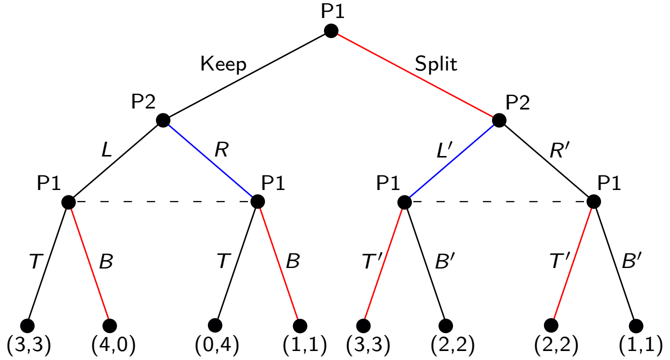


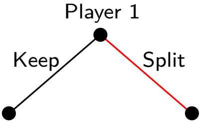
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L'	R'
T'	3, 3	2, 2
B'	2, 2	1, 1

PS6, Ex. 7.b: To keep or split (imperfect information)

(b) Find the subgame perfect Nash Equilibria (SPNE).





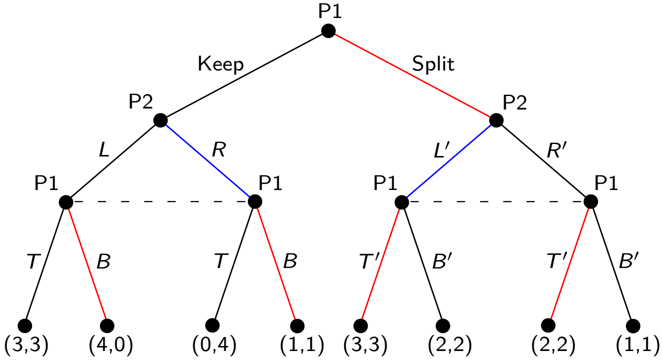
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

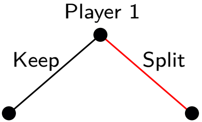
	L'	R'
T'	3, 3	2, 2
B'	2, 2	1, 1

Write up the full strategy profiles for the subgame perfect Nash Equilibria (SPNE).

PS6, Ex. 7.b: To keep or split (imperfect information)

(b) Find the subgame perfect Nash Equilibria (SPNE).





	L	R
T	3, 3	0, 4
B	4, 0	1, 1

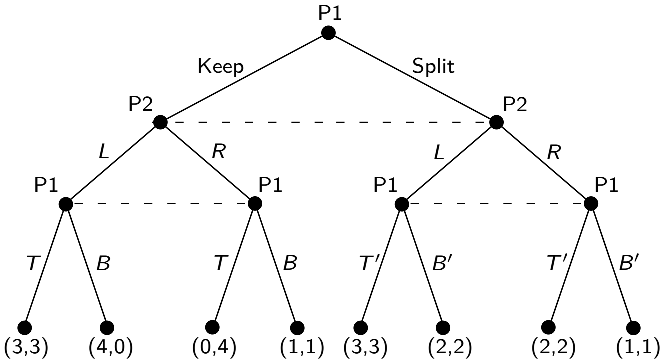
	L’	R’
T’	3, 3	2, 2
B’	2, 2	1, 1

$SPNE = \{(Split, B, T'), (R, L')\}$ with outcome (3,3).

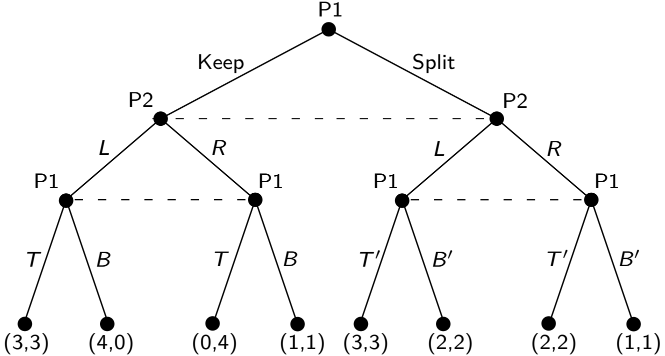
(c) Now suppose that Player 2 cannot observe Player 1’s choice in the first stage. ***Draw the game tree (again using information sets)*** and find the subgame perfect Nash Equilibria (SPNE).

PS6, Ex. 7.c: To keep or split (imperfect information)

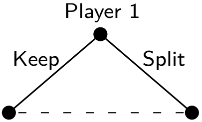
(c) Now suppose that Player 2 cannot observe Player 1's choice in the first stage. Draw the game tree (again using information sets) ***and find the subgame perfect Nash Equilibria (SPNE).***



(c) Find the subgame perfect Nash Equilibria (SPNE).



With 2nd and 3rd stage in normal form (Player 1 knows her own action in 1st stage):



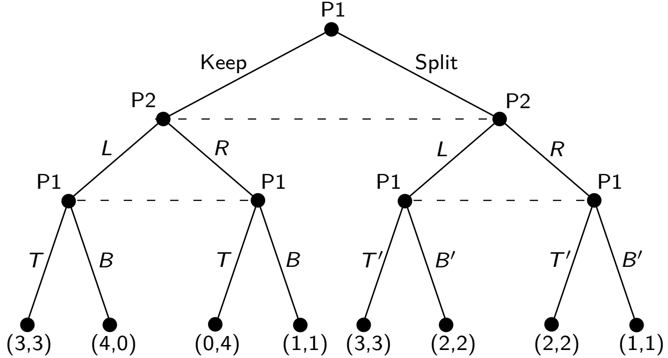
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L	R
T'	3, 3	2, 2
B'	2, 2	1, 1

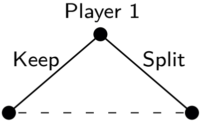
How many subgames are there?

PS6, Ex. 7.c: To keep or split (imperfect information)

(c) Find the subgame perfect Nash Equilibria (SPNE).



With 2nd and 3rd stage in normal form:



	L	R
T	3, 3	0, 4
B	4, 0	1, 1

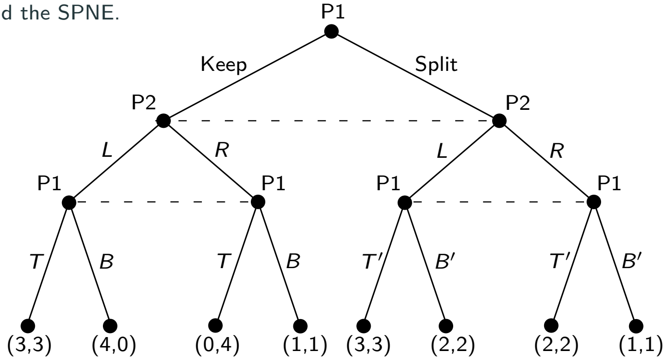
	L	R
T'	3, 3	2, 2
B'	2, 2	1, 1

There is only one subgame; the full game itself.

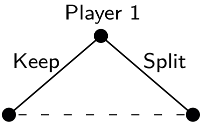
Write up the game in normal form and solve it.

PS6, Ex. 7.c: To keep or split (imperfect information)

(c) Find the SPNE.



With 2nd and 3rd stage in normal form:



	L	R
T	3, 3	0, 4
B	4, 0	1, 1

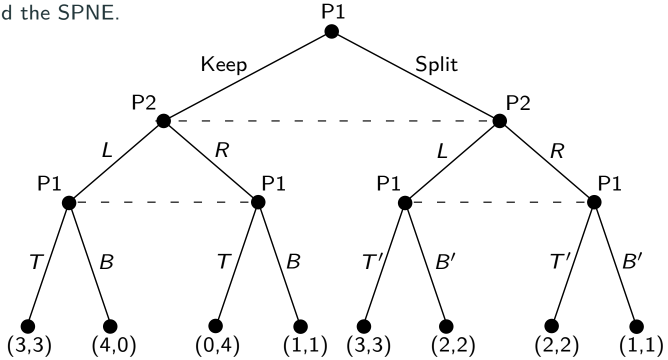
	L	R
T'	3, 3	2, 2
B'	2, 2	1, 1

Full game:

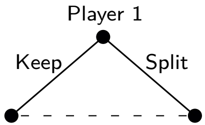
	L	R
Keep, T, T'	3, 3	0, 4
Keep, T, B'	3, 3	0, 4
Keep, B, T'	4, 0	1, 1
Keep, B, B'	4, 0	1, 1
Split, T, T'	3, 3	2, 2
Split, B, T'	3, 3	2, 2
Split, T, B'	2, 2	1, 1
Split, B, B'	2, 2	1, 1

PS6, Ex. 7.c: To keep or split (imperfect information)

(c) Find the SPNE.



With 2nd and 3rd stage in normal form:



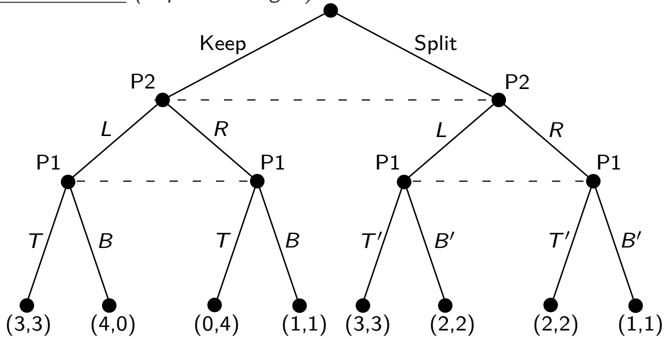
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L	R
T'	3, 3	2, 2
B'	2, 2	1, 1

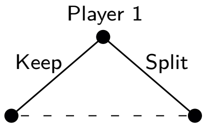
Full game:

	L	R
Keep, T, T'	3, 3	0, 4
Keep, T, B'	3, 3	0, 4
Keep, B, T'	4, 0	1, 1
Keep, B, B'	4, 0	1, 1
Split, T, T'	3, 3	2, 2
Split, B, T'	3, 3	2, 2
Split, T, B'	2, 2	1, 1
Split, B, B'	2, 2	1, 1

(c) ***No SPNE exists*** *(in pure strategies)*



With 2nd and 3rd stage in normal form:



	L	R
T	3, 3	0, 4
B	4, 0	1, 1

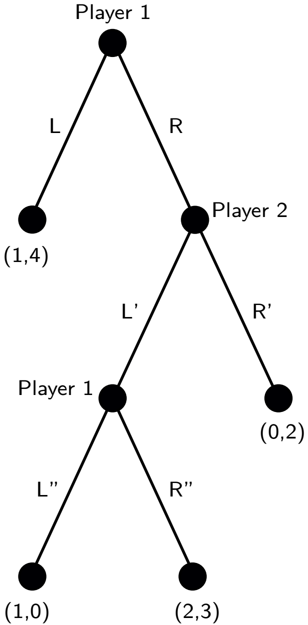
	L	R
T'	3, 3	2, 2
B'	2, 2	1, 1

Full game:

	L	R
Keep, T, T'	3, 3	0, 4
Keep, T, B'	3, 3	0, 4
Keep, B, T'	4, 0	1, 1
Keep, B, B'	4, 0	1, 1
Split, T, T'	3, 3	2, 2
Split, B, T'	3, 3	2, 2
Split, T, B'	2, 2	1, 1
Split, B, B'	2, 2	1, 1

Code examples

Code examples



Matrix, no player names:

	L (q)	R (1-q)
T (p)		
B (1-p)		

Matrix, no colors:

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)		
	B (1-p)		

Matrix, with colors:

		Player 2	
		L (q)	R (1-q)
Player 1	T (p)	1, 1	
	B (1-p)		