Microeconomics III: Session 4

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October 9 2019

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Outline

PS4, Ex. 1 (A): MSNE and best-response functions

PS4, Ex. 3: The Focal Point (plotting BR functions)

Guide: Examine which equilibria are the most realistic in a static game

PS4, Ex. 4: Generalized Battle of the Sexes (plotting BR functions)

Take Home Assignment 1 (theorems and backwards induction)

PS4, Ex. 5: North-Atlantic, 1943 (MSNE)

PS4, Ex. 6: Stopping the bike thief (MSNE)

PS4, Ex. 7: To keep or split (backwards induction)

Problem Set 3, Ex. 5: Luxembourg as a rogue state (static game)

PS4, Ex. 8: Building a playground (Stackelberg game)

PS4, Ex. 1 (A): MSNE and best-response functions

PS4, Ex. 1 (A): MSNE and best-response functions

1. (A) Find all equilibria (pure and mixed) in the following games, first analytically and then through plotting the best-response functions.

(a)

		Player 2	
		L (q)	L (1-q)
Player 1	T (p)	3, 3	0, 0
	B (1-p)	0, 0	4, 4

(b)

		Player 2	
		L (q)	L (1-q)
Player 1	T (p)	1, 1	0, 0
	B (1-p)	1, 0	2, 1

Hint: Find the probabilities q for which Player 1 is indifferent, e.g. $u_1(T, q) = u_1(B, q)$. and the probabilities p for which Player 2 is indifferent, e.g. $u_2(L, p) = u_2(R, p)$.

PS4, Ex. 1.a (A): MSNE and best-response functions

(a) Find all equilibria (pure and mixed), first analytically and then through plotting the BR functions.

		Player 2	
		L (q)	L (1-q)
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Highlight the best responses in pure strategies.

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For which values of q is Player 1 indifferent?

Find q such that Player 1 expects to have equal payoffs from playing T and B:

$$E[u_1|T] = E[u_1|B]$$

$$=$$

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$$E[u_1|T] = E[u_1|B]$$

$$3q = 4(1-q) \Leftrightarrow q = \frac{4}{7}$$

Write up all NE (pure and mixed).

$$NE = (p^*, q^*) =$$

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The players have symmetric payoffs, thus:

$$NE = (p^*, q^*) = \{(0,0); (1,1); ...\}$$

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Write up Player 1's best-response (BR) function, p*(q)

$$BR_1(q) = \{$$

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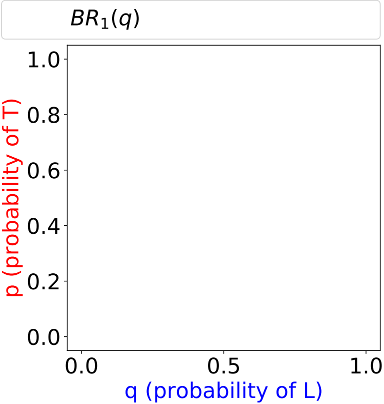
Plot Player 1's best-response (BR) function, p*(q)

Write up and plot the BR functions:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 4/7 \\ p \in [0,1] & \text{if} \quad q = 4/7 \\ p = 1 & \text{if} \quad q > 4/7 \end{cases}$$



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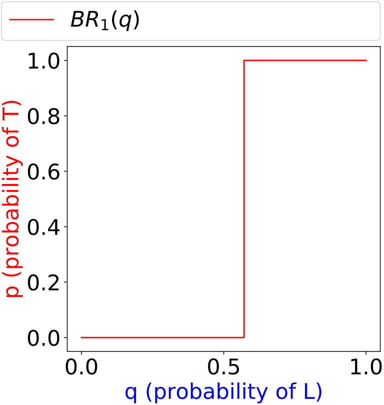

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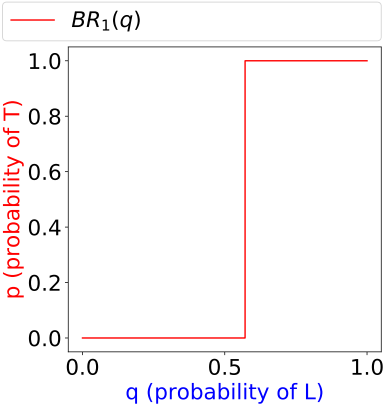

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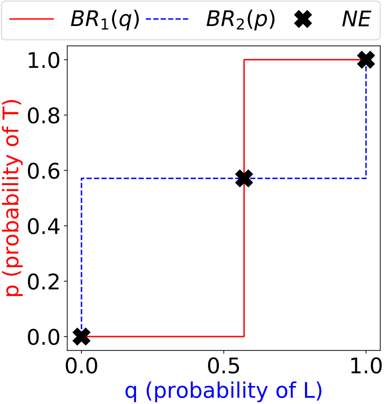

PS4, Ex. 1.b (A): MSNE and best-response functions

(b) Find all equilibria (pure and mixed), first analytically and then through plotting the BR functions.

		Player 2	
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Player 1	T (p)	1, 1	0, 0
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Highlight the best responses in pure strategies.

PS4, Ex. 1.b (A): MSNE and best-response functions

(b) Find all equilibria (pure and mixed), first analytically and then through plotting the BR functions.

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Player 1	L (q)	R (1-q)
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Find q such that Player 1 expects to have equal payoffs from playing T and B:

$$E[u_1|T] = E[u_1|B]$$

$$q = q + 2(1-q) \Leftrightarrow q = 1$$

For which values of p is Player 2 indifferent?

Find p such that Player 2 expect to have equal payoffs from playing L and R:

$$E[u_2|L] = E[u_2|R]$$

$$=$$

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$$E[u_2|L] = E[u_2|R]$$

$$p = 1 - p \Leftrightarrow p = \frac{1}{2}$$

and chooses $q = 1$ for $p > 1/2$.

Write up all NE (pure and mixed).

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(b) Find all NE, first analytically:

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The pure and mixed NE, (p^*, q^*), are:

$$\left\{(0,0); (1,1); \left(p \in \left[\frac{1}{2}, 1\right), q = 1\right)\right\}$$

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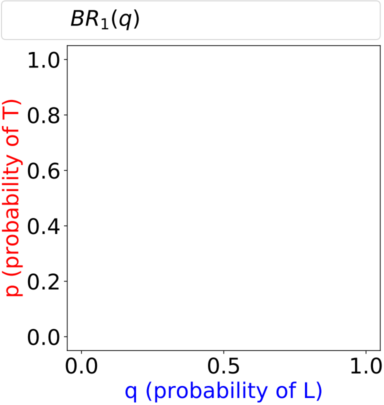
Plot Player 1's best-response (BR) function, $p^*(q)$

Then through plotting the BR functions:

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < 1 \\ p \in [0,1] & \text{if} \quad q = 1 \end{cases}$$



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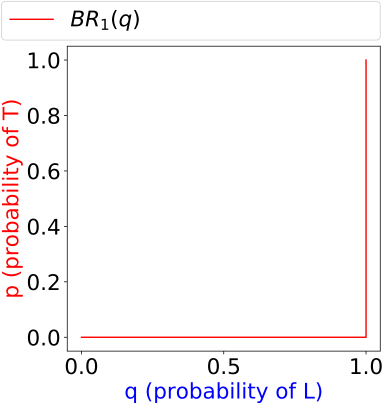

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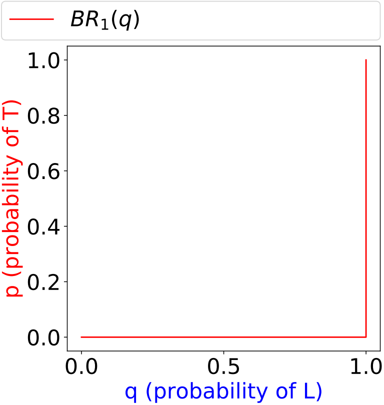

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Then through plotting the BR functions:

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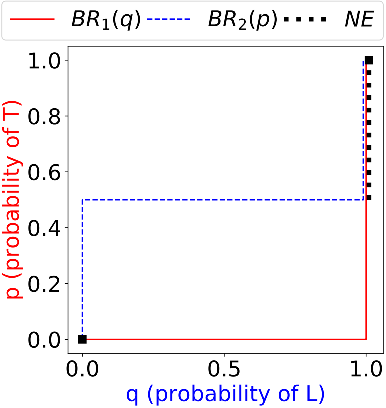

PS4, Ex. 3: The Focal Point (plotting BR functions)

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Thomas and Alice want to meet on a Friday night. There are two bars in their home town: “The Focal Point” and “The Other Place”. They have to decide independently where they go. If they meet in the same bar, they both get utility of 1. If they end up in different bars, they get utility of 0.

(a) Find all equilibria (pure and mixed). Which equilibrium do you consider the most realistic? Where would you go if you were one of them?

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Thomas gets a payoff of 1 if he meets Alice, and -1 otherwise. Alice gets a payoff of -1 for meeting Thomas, and 1 otherwise. Find all equilibria (pure and mixed).

(c) Now assume again that Thomas and Alice both want to meet (so that payoffs are as in part (a)), but now there are N bars in town, where N can be very large. Show that there are $2^N - 1$ equilibria (pure and mixed). Say that the bars have names: “The First Bar in Town”, “The Second Bar in Town”, and so on. Which equilibrium is the most realistic?

PS4, Ex. 3.a: The Focal Point (plotting BR functions)

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		Thomas	
		F (q)	O (1-q)
Alice	F (p)	1, 1	0, 0
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For which values of q is Alice indifferent?

$$\begin{aligned} E[u_A|Focal] &= E[u_A|Other] \\ &= \end{aligned}$$

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Alice is indifferent for:

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Which is the most realistic?

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Which is the most realistic?

$(\frac{1}{2}, \frac{1}{2})$ seems unlikely as expected payoffs are $\frac{1}{2}$ while being 1 for $(0,0)$ and $(1,1)$.

Where would you go?

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Taking advantage of symmetry:

$$NE = (p^*, q^*) = \left\{(0,0); (1,1); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

Which is the most realistic?

$(\frac{1}{2}, \frac{1}{2})$ seems unlikely as expected payoffs are $\frac{1}{2}$ while being 1 for $(0,0)$ and $(1,1)$.

Where would you go?

I would go to the ”The Focal Point” - it sounds like the place to meet.

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Thomas gets a payoff of 1 if he meets Alice, and -1 otherwise. Alice gets a payoff of -1 for meeting Thomas, and 1 otherwise. Find all equilibria (pure and mixed).

Write up the new matrix and highlight the best responses. What are the pure strategy NE?

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Thomas gets a payoff of 1 if he meets Alice, and -1 otherwise. Alice gets a payoff of -1 for meeting Thomas, and 1 otherwise. Find all equilibria (pure and mixed).

Thomas

Alice		F (q)	O (1-q)
	F (p)	-1, 1	1, -1
	O (1-p)	1, -1	-1, 1

There exist no NE in pure strategies.

For which values of q is Alice indifferent?

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		Thomas	
		F (q)	O (1-q)
Alice	F (p)	-1, 1	1, -1
	O (1-p)	1, -1	-1, 1

No PSNE. Alice is indifferent for:

$$E[u_A|Focal] = E[u_A|Other]$$

$$-q + (1-q) = q - (1-q) \Leftrightarrow q = \frac{1}{2}$$

For which values of p is Thomas indifferent?

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

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Thomas

Alice	F (q)	O (1-q)
F (p)	-1, 1	1, -1
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No PSNE. Alice is indifferent for:

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$$-q + (1-q) = q - (1-q) \Leftrightarrow q = \frac{1}{2}$$

Thomas is indifferent for:

$$E[u_T|Focal] = E[u_T|Other]$$

$$p - (1-p) = -p + (1-p) \Leftrightarrow p = \frac{1}{2}$$

Write up Alice's BR function, $p^*(q)$

$$BR_A(q) = \{$$

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Thomas is indifferent for:

$$E[u_T|Focal] = E[u_T|Other]$$

$$p - (1-p) = -p + (1-p) \Leftrightarrow p = \frac{1}{2}$$

The BR functions are:

$$BR_A(q) = \begin{cases} p = 1 & \text{if} \quad q < 1/2 \\ p \in [0,1] & \text{if} \quad q = 1/2 \\ p = 0 & \text{if} \quad q > 1/2 \end{cases}$$

$$BR_T(p) = \{$$

Write up Thomas' BR function, $q^*(p)$

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

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(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Find all NE.

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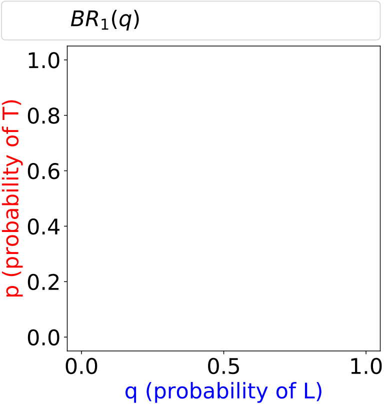

Plot Alice's BR function, $p^*(q)$

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Find all NE.

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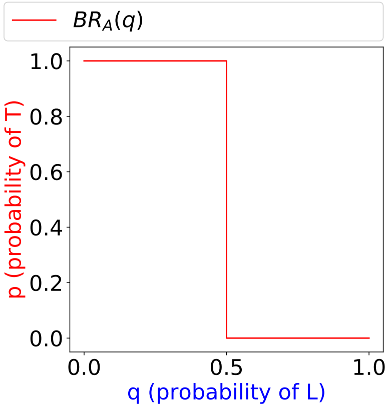

Plot Thomas' BR function, $q^*(p)$

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Find all NE.

		Thomas	
		F (q)	O (1-q)
Alice	F (p)	-1, 1	1, -1
	O (1-p)	1, -1	-1, 1

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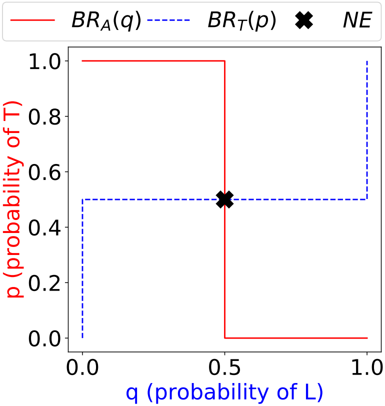

Write up all NE (pure and mixed).

$$NE = (p^*, q^*) =$$

PS4, Ex. 3.b: The Focal Point (plotting BR functions)

(b) Now assume that Thomas wants to meet Alice, but Alice does not want to meet Thomas. Find all NE.

		Thomas	
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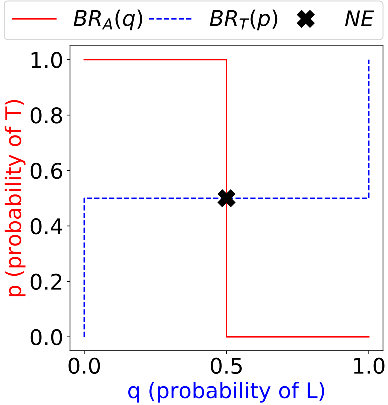

The only NE is the Mixed Strategy NE:

$$(p^*, q^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

(c) Now assume again that Thomas and Alice both want to meet (so that payoffs are as in part (a)), but now there are N bars in town, where N can be very large. Show that there are $2^N - 1$ equilibria (pure and mixed). Say that the bars have names: "The First Bar in Town", "The Second Bar in Town", and so on. Which equilibrium is the most realistic?

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For N=2: We have $3 = 2^N - 1$ equilibria:

$$(p^*, q^*) = \left\{(0,0); (1,1); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

		Thomas	
		Bar_1 (q)	Bar_2 (1-q)
Alice	Bar_1 (p)	1, 1	0, 0
	Bar_2 (1-p)	0, 0	1, 1

What about N=3?

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

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For N=2: We have found $3 = 2^N - 1$ equilibria:

$$(p^*, q^*) = \left\{(0,0); (1,1); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

Thomas

		Bar$_1$ (q)	*Bar*$_2$ (1-q)
Alice	*Bar*$_1$ (p)	1, 1	0, 0
	Bar$_2$ (1-p)	0, 0	1, 1

What about N=3?

Thomas

		Bar$_1$ (q_1)	*Bar*$_2$ (q_2)	*Bar*$_3$ (1-q_1-q_2)
Alice	*Bar*$_1$ (p_1)	1, 1	0, 0	0, 0
	Bar$_2$ (p_2)	0, 0	1, 1	0, 0
	Bar$_3$ (1-p_1-p_2)	0, 0	0, 0	1, 1

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

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Thomas

Alice

	Bar_1 (q)	Bar_2 (1-q)
Bar_1 (p)	1, 1	0, 0
Bar_2 (1-p)	0, 0	1, 1

For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

$$\left\{(0,0,0,0); (0,1,0,1); (1,0,1,0); \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right); \left(0, \frac{1}{2}, 0, \frac{1}{2}\right); \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$$

Thomas

Alice

	Bar_1 (q_1)	Bar_2 (q_2)	Bar_3 (1-q_1-q_2)
Bar_1 (p_1)	1, 1	0, 0	0, 0
Bar_2 (p_2)	0, 0	1, 1	0, 0
Bar_3 (1-p_1-p_2)	0, 0	0, 0	1, 1

What about any N?

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

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For N=2: We have $3 = 2^N - 1$ equilibria:

$$(p^*, q^*) = \left\{(0,0); (1,1); \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

		Thomas	
		Bar_1 (q)	Bar_2 (1-q)
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For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

$$\left\{(0,0,0,0); (0,1,0,1); (1,0,1,0); \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right); \left(0, \frac{1}{2}, 0, \frac{1}{2}\right); \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$$

		Thomas		
		Bar_1 (q_1)	Bar_2 (q_2)	Bar_3 (1-q_1-q_2)
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	Bar_2 (p_2)	0, 0	1, 1	0, 0
	Bar_3 (1-p_1-p_2)	0, 0	0, 0	1, 1

For any N: It is plausible that the geometric continues for $N > 3$. Note that we're asked to "show" not "proof", thus, providing two examples is sufficient.

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

(c) Which equilibrium is the most realistic?

For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

$$\left\{(0,0,0,0);(0,1,0,1);(1,0,1,0);\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right);\left(\frac{1}{2},0,\frac{1}{2},0\right);\left(0,\frac{1}{2},0,\frac{1}{2}\right);\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\}$$

		Thomas		
		Bar_1 (q_1)	Bar_2 (q_2)	Bar_3 (1-q_1-q_2)
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	Bar_2 (p_2)	0, 0	1, 1	0, 0
	Bar_3 (1-p_1-p_2)	0, 0	0, 0	1, 1

Look at the expected payoffs from the pure and mixed equilibria when N=3...

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

(c) Which equilibrium is the most realistic?

For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

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		Thomas		
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In the three PSNE, the expected payoffs are: $\Big(E[u_A|q_1^*, q_2^*)], E[u_T|p_1^*, p_2^*)]\Big) =$

$$\{(1-q_1-q_2, 1-p_1-p_2);(q_2,p_2);(q_1,p_1)\} \sim \{(1,1);(1,1);(1,1)\}$$

What are the expected payoffs in the four MSNE?

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

(c) Which equilibrium is the most realistic?

For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

$$\left\{(0,0,0,0);(0,1,0,1);(1,0,1,0);\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right);\left(\frac{1}{2},0,\frac{1}{2},0\right);\left(0,\frac{1}{2},0,\frac{1}{2}\right);\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\}$$

		Thomas		
		Bar_1 (q_1)	Bar_2 (q_2)	Bar_3 (1-q_1-q_2)
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In the three PSNE, the expected payoffs are: $\left(E[u_A|q_1^*, q_2^*)], E[u_T|p_1^*, p_2^*)]\right) =$

$$\{(1-q_1-q_2, 1-p_1-p_2);(q_2,p_2);(q_1,p_1)\} \sim \{(1,1);(1,1);(1,1)\}$$

In the four MSNE, the expected payoffs are: $\left(E[u_A|q_1^*, q_2^*)], E[u_T|p_1^*, p_2^*)]\right) =$

$$\left\{\left(\frac{q_1+q_2}{2},\frac{p_1+p_2}{2}\right);\left(\frac{1-q_2}{2},\frac{1-p_2}{2}\right);\left(\frac{1-q_1}{2},\frac{1-p_1}{2}\right);\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$\sim\left\{\left(\frac{1}{2},\frac{1}{2}\right);\left(\frac{1}{2},\frac{1}{2}\right);\left(\frac{1}{2},\frac{1}{2}\right);\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

Which equilibria are the most realistic - and which is the least realistic?

PS4, Ex. 3.c: The Focal Point (plotting BR functions)

(c) Which equilibrium is the most realistic?

For N=3: We have $7 = 2^N - 1$ equilibria, $(p_1^*, p_2^*, q_1^*, q_2^*)$:

$$\left\{(0,0,0,0); (0,1,0,1); (1,0,1,0); \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right); \left(\frac{1}{2},0,\frac{1}{2},0\right); \left(0,\frac{1}{2},0,\frac{1}{2}\right); \left(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\}$$

		Thomas		
		Bar_1 (q_1)	Bar_2 (q_2)	Bar_3 (1-q_1-q_2)
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In the three PSNE, the expected payoffs are: $\left(E[u_A|q_1^*, q_2^*)], E[u_T|p_1^*, p_2^*)]\right) =$

$$\{(1 - q_1 - q_2, 1 - p_1 - p_2); (q_2, p_2); (q_1, p_1)\} \sim \{(1,1); (1,1); (1,1)\}$$

In the four MSNE, the expected payoffs are: $\left(E[u_A|q_1^*, q_2^*)], E[u_T|p_1^*, p_2^*)]\right) =$

$$\left\{\left(\frac{q_1+q_2}{2}, \frac{p_1+p_2}{2}\right); \left(\frac{1-q_2}{2}, \frac{1-p_2}{2}\right); \left(\frac{1-q_1}{2}, \frac{1-p_1}{2}\right); \left(\frac{1}{3}, \frac{1}{3}\right)\right\}$$

$$\sim \left\{\left(\frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{3}, \frac{1}{3}\right)\right\}$$

PSNE have higher expected payoffs but without communication it's not clear which one to go for. Due to coordination issues, the MSNE can be just as good, even though expected payoffs are reciprocal to the number of actions that a MSNE is split between.

Guide: Examine which equilibria are the most realistic in a static game

Guide: Examine which equilibria are the most realistic in a static game

1. Look at the pareto optimal solutions:
    a. If exactly one Pure Strategy Nash Equilibrium (PSNE) is pareto optimal, rational players should pick this solution.
    b. If there are multiple pareto optimal PSNE, there is a risk of miscoordination, as players can't tell which PSNE the other player is going for. Thus, playing a mix of these can be just as good as arbitrarily picking a pure strategy.
2. Look at the punishment in the case of miscoordination:
    - E.g. If Player 1 thinks they are going for a certain PSNE, but they miscoordinate and Player 2 plays something else, how hard will Player 1 be punished?
3. Use the two first points to talk about what rational players would do?
4. Finally, consider what would happen if one player could send a message? Or they had just played the game with mixed strategies, and by chance landed on a pareto optimal PSNE?

PS4, Ex. 4: Generalized Battle of the Sexes (plotting BR functions)

PS4, Ex. 4: Generalized Battle of the Sexes (plotting BR functions)

Consider the following Generalized Battle of the Sexes game, with $N > 1$:

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

(a) How can you interpret the parameter N?

(b) Solve for the mixed strategy Nash equilibrium (MSNE). When N becomes very large, what happens to the probability of successful coordination?

PS4, Ex. 4.a: Generalized Battle of the Sexes (plotting BR functions)

(a) How can you interpret $N > 1$?

Formally: N is the factor of additional utility for one's most preferred outcome.

Informally: N is a measure for the conflict of interests.

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

For which values of q is Player 1 indifferent?

PS4, Ex. 4.a: Generalized Battle of the Sexes (plotting BR functions)

(a) How can you interpret $N > 1$?

Formally: N is the factor of additional utility for one's most preferred outcome.

Informally: N is a measure for the conflict of interests.

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

$$E[u_1|C1] = E[u_1|C2]$$

$$Nq = 1 - q \Leftrightarrow q = \frac{1}{1+N}$$

For which values of p is Player 2 indifferent?

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

$$E[u_1|C1] = E[u_1|C2]$$

$$Nq = 1 - q \Leftrightarrow q = \frac{1}{1+N}$$

Player 2 is indifferent for:

$$E[u_2|C1] = E[u_2|C2]$$

$$p = N(1-p) \Leftrightarrow p = \frac{N}{1+N}$$

$$BR_1(q) = p^*(q) = \{$$

$$BR_2(p) = q^*(p) = \{$$

Find the best-response functions.

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

Write the mixed strategy NE, (p^*, q^*).

Player 2

Player 1

	C1 (q)	C2 (1-q)
C1 (p)	N, 1	0, 0
C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

$$E[u_1|C1] = E[u_1|C2]$$

$$Nq = 1 - q \Leftrightarrow q = \frac{1}{1+N}$$

Player 2 is indifferent for:

$$E[u_2|C1] = E[u_2|C2]$$

$$p = N(1-p) \Leftrightarrow p = \frac{N}{1+N}$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if} \quad q < \frac{1}{1+N} \\ p \in [0,1] & \text{if} \quad q = \frac{1}{1+N} \\ p = 1 & \text{if} \quad q > \frac{1}{1+N} \end{cases}$$

$$BR_2(p) = \begin{cases} q = 0 & \text{if} \quad p < \frac{N}{1+N} \\ q \in [0,1] & \text{if} \quad p = \frac{N}{1+N} \\ q = 1 & \text{if} \quad p > \frac{N}{1+N} \end{cases}$$

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
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Player 1 is indifferent for:

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$$NE = \left\{(0,0); (1,1); \left(\frac{1}{N+1}, \frac{N}{N+1}\right)\right\}$$

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

Player 2

Player 1

	C1 (q)	C2 (1-q)
C1 (p)	N, 1	0, 0
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Player 1 is indifferent for:

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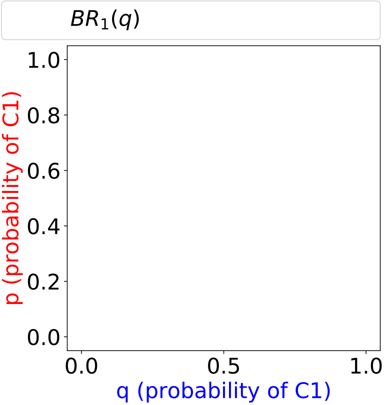
$$NE = \left\{(0,0); (1,1); \left(\frac{N}{N+1}, \frac{1}{N+1}\right)\right\}$$

When $N \to \infty$, what happens to the probability of successful coordination?



To illustrate it, plot Player 1's BR function, $p^*(q)$, e.g. for $N = 9$.

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

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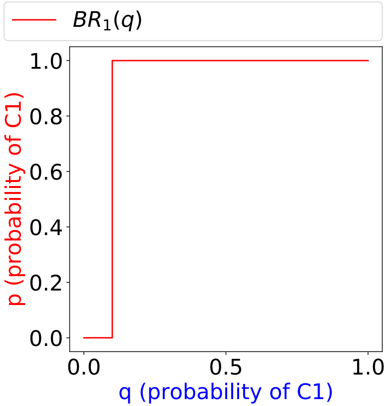

Plot Player 2's BR function, $q^*(p)$, for the same large value of N (e.g. $N = 9$).

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

$$Nq = 1 - q \Leftrightarrow q = \frac{1}{1+N}$$

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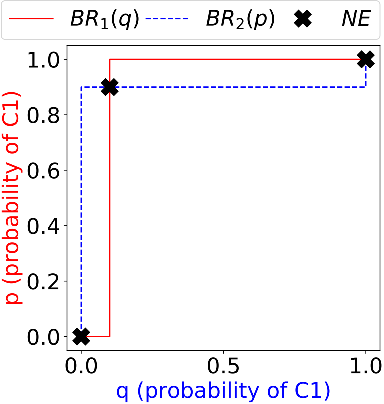
When $N \to \infty$, what happens to the probability of successful coordination?



In the MSNE, what happens to p^* and q^* when $N \to \infty$? What happens to the expected payoffs in the MSNE?

PS4, Ex. 4.b: Generalized Battle of the Sexes (plotting BR functions)

(b) Find the MSNE.

		Player 2	
		C1 (q)	C2 (1-q)
Player 1	C1 (p)	N, 1	0, 0
	C2 (1-p)	0, 0	1, N

Player 1 is indifferent for:

$$Nq = 1 - q \Leftrightarrow q = \frac{1}{1+N}$$

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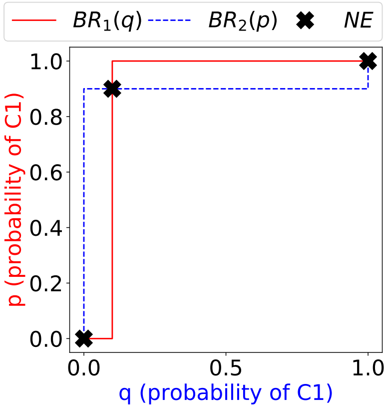
$$NE = \left\{ (0,0); (1,1); \left(\frac{N}{N+1}, \frac{1}{N+1} \right) \right\}$$

When $N \to \infty$, what happens to the probability of successful coordination?



$$MSNE : (p^*, q^*, u_1^*, u_2^*) \xrightarrow[N\to\infty]{} (1,0,0,0)$$

When N is large, coordination is difficult as Player 1 plays C1 most of the time and player 2 plays C2 most of the time.

Take Home Assignment 1 (theorems and backwards induction)

Take Home Assignment 1, Ex. 1-2: Theorems

(1) **Nash's theorem (John Nash, 1950):**

All finite games (finite number of players with finitely many strategies) have at least one Nash Equilibrium. Some of these game may only have an equilibrium in mixed strategies.

Take Home Assignment 1, Ex. 1-2: Theorems

(1) **Nash's theorem (John Nash, 1950):**

All finite games (finite number of players with finitely many strategies) have at least one Nash Equilibrium. Some of these game may only have an equilibrium in mixed strategies.

Refinement:

(2) **The Oddness Theorem (Robert Wilson, 1971; John Charles Harsanyi, 1973):**

Almost all finite games (finite number of players with finitely many strategies) have at a finite number of Nash Equilibria, and that number is also odd.

Take Home Assignment 1, Ex. 1-2: Theorems

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Refinement:

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Almost all finite games (finite number of players with finitely many strategies) have at a finite number of Nash Equilibria, and that number is also odd.

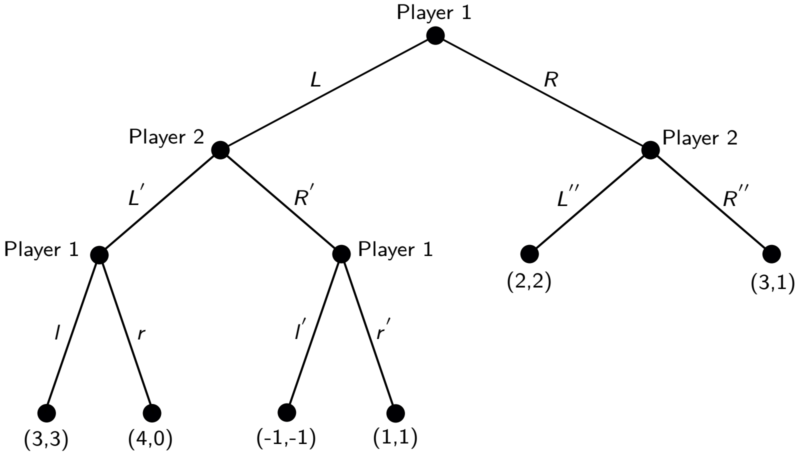
An exception is when one player is indifferent for a *pure* strategy of the other player, e.g. the games we have seen in

- Exercise 2 of the Take Home Assignment.
- Exercise 1.b of Problem Set 4.
- Exercise 7.b and 7.c of Problem Set 3.

In these cases we get an infinite set of equilibria, i.e. the real numbers in an interval.

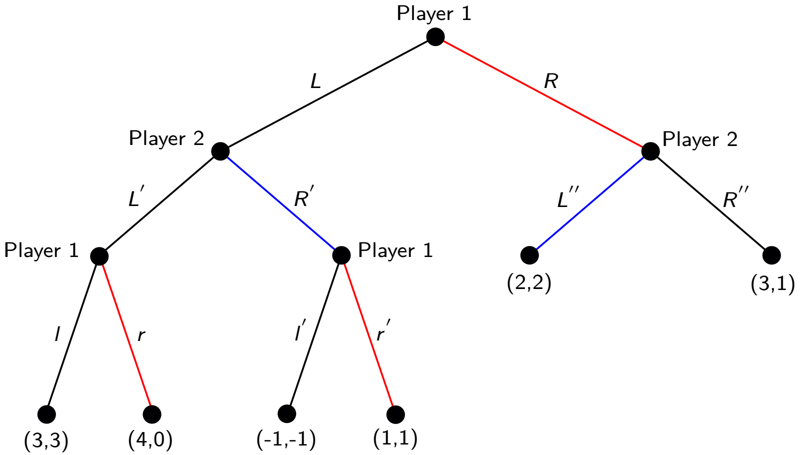
Take Home Assignment 1, Ex. 3: Backwards induction

3. A dynamic game.



Take Home Assignment 1, Ex. 3.a: Backwards induction

(3a) The Backwards Induction (BI) solution.

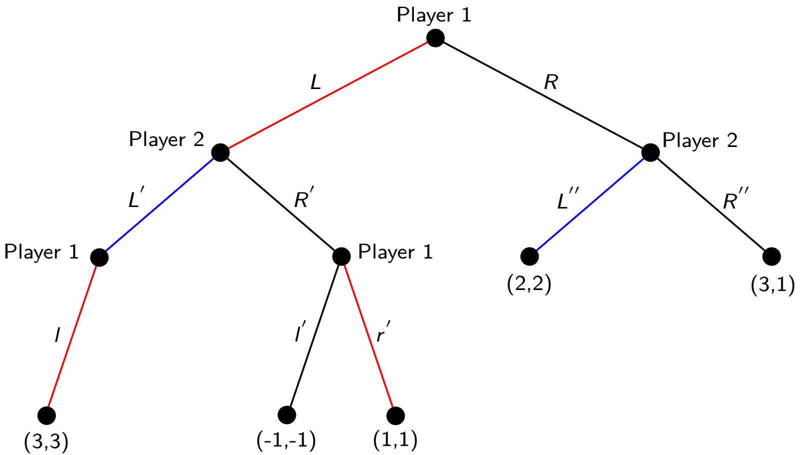


The BI solution is the complete strategy profile - on and off the equilibrium path:

$$(\textit{best responses for player } 1,\ \textit{best responses for player } 2) = (s_1, s_2) = (Rrr', R'L'')$$

Take Home Assignment 1, Ex. 3.b: Backwards induction

(3b) Player 1: Any improvement from deleting a strategy.

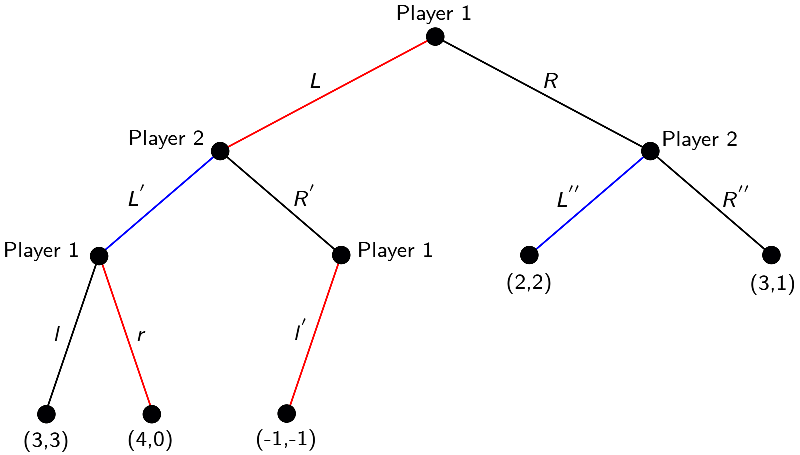


Most of you answered that Player 1 can improve his outcome by deleting r to get 3.

This is true, but it is suboptimal (for Player 1 at least).

Take Home Assignment 1, Ex. 3.b: Backwards induction

(3b) Player 1: The best improvement from deleting a strategy.

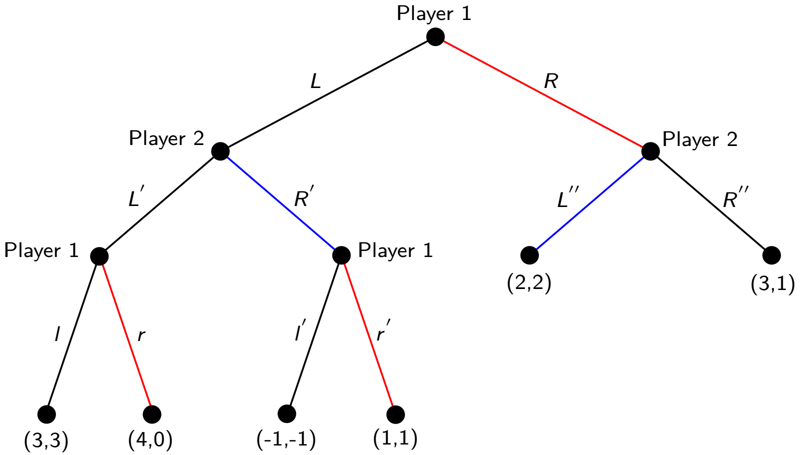


Player 1 can do better by deleting r'. Then he can use the ***first mover advantage*** to choose the left side of the tree and use the ***last mover advantage*** to pick r and get 4.

Player 2 has limited agency in the middle stage, but picks L' to avoid negative payoffs.

Take Home Assignment 1, Ex. 3.c: Backwards induction

(3c) Player 2: Show there can be no improvement from deleting a strategy,



The foolproof way: Delete L', R', L'', R'' one at the time and solve all 4 new games.

The smart way: Argue that the only possible improvement for Player 2 would be to end up in $(3,3)$, but Player 1 would never choose l over r.

PS4, Ex. 5: North-Atlantic, 1943 (MSNE)

PS4, Ex. 5: North-Atlantic, 1943 (MSNE)

North-Atlantic, 1943. An allied convoy, counting 100 ships, is heading east and it can choose between a northern route where icebergs are known to be numerous or a more southern route. The northern route is dangerous - because of the icebergs - and it is estimated that 6 ships will get lost due to icebergs. Below the surface, the wolf-pack lures. If the u-boats catch the convoy on the southern route, it is a field day, and 40 ships from the convoy are estimated to get lost. If the u-boats catch the convoy on the northern route, they do not have as much time hunting down the convoy - due to petrol shortages - and they are only expected to be able to sink 20 ships from the convoy. The wolf-pack does not have time to check both locations, north and south. Each headquarter (allied or nazi) has to decide whether to go north or south. Unfortunately, there is no radar etc, so one cannot observe the move of the enemy before taking a decision. Each headquarter has a simple payoff function. For the allied headquarter it equals the number of ships making it across the Atlantic. For the nazi headquarter payoff equals the number of ships lost by the allies.

(a) Write down this strategic situation in a bi-matrix.

(b) Find the Nash Equilibrium (equilibria?)

(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

PS4, Ex. 5.a: North-Atlantic, 1943 (MSNE)

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(a) Write down this strategic situation in a bi-matrix.

Let the chance of the nazis going north be noted by q, and the chance they go south be noted by 1-q. The chance the Allied go north is then noted by p, and the chance they go south is noted by 1-p.

PS4, Ex. 5.a: North-Atlantic, 1943 (MSNE)

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(a) Write down this strategic situation in a bi-matrix:

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

(b) Find the Nash Equilibrium (equilibria?)

PS4, Ex. 5.b: North-Atlantic, 1943 (MSNE)

(a) Write down this strategic situation in a bi-matrix:

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

(b) Find the Nash Equilibrium (equilibria?):

It's a zero-sum (100-sum) type game like matching-pennies or rock-paper-scissors. Thus, no pure strategy NE (PSNE), but a mixed strategy NE (MSNE) must exist.

Find q such that the Allied are indifferent.

PS4, Ex. 5.b: North-Atlantic, 1943 (MSNE)

(a) Write down this strategic situation in a bi-matrix:
It's a zero-sum (100-sum) type game:

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
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(b) Find the Nash Equilibrium (equilibria?): There are no PSNE, find the MSNE:

The Allied are indifferent for:

$$E[u_A|North] = E[u_A|South]$$

$$74q + 94(1-q) = 100q + 60(1-q) \Leftrightarrow \ldots \Leftrightarrow q = \frac{17}{30}$$

Find p such that the Nazis are indifferent.

PS4, Ex. 5.b: North-Atlantic, 1943 (MSNE)

(a) Write down this strategic situation in a bi-matrix:
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The Nazis are indifferent for:

$$E[u_N|North] = E[u_N|South]$$

$$26p = 6p + 40(1-p) \Leftrightarrow ... \Leftrightarrow p = \frac{2}{3}$$

Write up all Nash Equilibria, (p^*, q^*).

PS4, Ex. 5.b: North-Atlantic, 1943 (MSNE)

(a) Write down this strategic situation in a bi-matrix:
It's a zero-sum (100-sum) type game:

		Nazis	
		North (q)	South (1-q)
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	South (1-p)	100, 0	60, 40

(b) Find the Nash Equilibrium (equilibria?): There are no PSNE, find the MSNE:

The Allied are indifferent for:

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The Nazis are indifferent for:

$$E[u_N|North] = E[u_N|South]$$

$$26p = 6p + 40(1-p) \Leftrightarrow ... \Leftrightarrow p = \frac{2}{3}$$

The unique NE is:

$$NE = (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30}\right)$$

(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

PS4, Ex. 5.c: North-Atlantic, 1943 (MSNE)

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

$$NE = (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30}\right)$$

(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

First, write up the Allied's expected utility of playing North and South respectively.

PS4, Ex. 5.c: North-Atlantic, 1943 (MSNE)

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

$$NE = (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30}\right)$$

(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

In general, the Allied's expected utility of playing North and South respectively:

$$E[u_A|North] = 74q + 94(1-q) = 94 - 20q$$
$$E[u_A|South] = 100q + 60(1-q) = 60 + 40q$$

Then write up the Allied's expected utility in the equilibrium: $E[u_A|p^*, q^*]$.

PS4, Ex. 5.c: North-Atlantic, 1943 (MSNE)

		Nazis	
		North (q)	South (1-q)
Allied	North (p)	74, 26	94, 6
	South (1-p)	100, 0	60, 40

$$NE = (p^*, q^*) = \left(\frac{2}{3}, \frac{17}{30}\right)$$

(c) In equilibrium, what is the expected number of ships that make it across the Atlantic?

In general, the Allied's expected utility of playing North and South respectively:

$$E[u_A|North] = 74q + 94(1-q) = 94 - 20q \tag{1}$$

$$E[u_A|South] = 100q + 60(1-q) = 60 + 40q \tag{2}$$

In equilibrium, the Allied's expected utility:

$$\begin{aligned} E[u_A|p^*, q^*] &= p^* E[u_A|North] + (1-p^*)E[u_A|South] \\ &= p^*(94 - 20q^*) + (1-p^*)(60 + 40q^*), \qquad \text{using eq. (1) and (2)} \\ &= \frac{2}{3}\left(94 - 20\frac{17}{30}\right) + \frac{1}{3}\left(60 + 40\frac{17}{30}\right) \\ &\approx 73.48 \end{aligned}$$

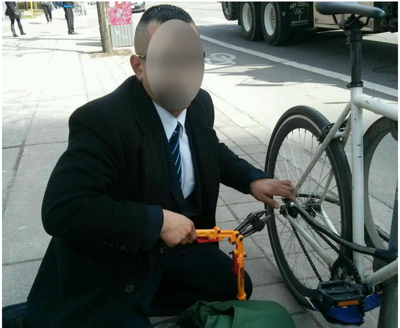
PS4, Ex. 6: Stopping the bike thief (MSNE)

PS4, Ex. 6: Stopping the bike thief (MSNE)

As in Problem Set 2, there are $N \geq 2$ people observing someone trying to steal a parked bike. Each of the witnesses would like the thief to be stopped, but prefers not to do it him/herself (because it is unpleasant and perhaps even dangerous). More precisely, if the thief is stopped by someone else, each of the witnesses gets a utility of $v > 0$. Every person who stops the thief gets a utility of $v - c > 0$, where c is the cost of interaction with the thief. Finally, if nobody stops the thief and the bike gets stolen, every witness gets a utility of 0. The witnesses decide whether or not to stop the thief simultaneously and independently.

a) Solve for a symmetric mixed strategy equilibrium of this game, where each witness stops the thief with probability $p \in (0, 1)$.

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?



PS4, Ex. 6.a: Stopping the bike thief (MSNE)

Payoffs for player $i \neq j$:

$$u_i(s_i, s_j) = \begin{cases} v > 0 & \text{if} \quad i \text{ does nothing and } j \text{ stops the thief} \\ v - c > 0 & \text{if} \quad i \text{ stops the thief} \\ 0 & \text{if} \quad \text{nobody stops the thief} \end{cases}$$

a) Solve for a symmetric mixed strategy equilibrium of this game, where each witness stops the thief with probability $p \in (0, 1)$.

Taking advantage of symmetry, find the probability p such that person i is indifferent between stopping the thief or not. That is, her expected payoff from stopping the thief equals her expected payoff from someone else stopping the thief.

PS4, Ex. 6.a: Stopping the bike thief (MSNE)

Payoffs for player $i \neq j$:

$$u_i(s_i, s_j) = \begin{cases} v > 0 & \text{if} \quad i \text{ does nothing and } j \text{ stops the thief} \\ v - c > 0 & \text{if} \quad i \text{ stops the thief} \\ 0 & \text{if} \quad \text{nobody stops the thief} \end{cases}$$

a) Solve for a symmetric mixed strategy equilibrium of this game, where each witness stops the thief with probability $p \in (0, 1)$.

Person i is indifferent between stopping the thief or not when her expected payoff from stopping the thief equals her expected payoff from someone else stopping the thief:

$$E[u_i | i \text{ stops thief}] = Prob[j \text{ stops thief}] \times E[u_i | j \text{ stops thief}], \quad i \neq j \in \mathcal{J} = 1, ..., 1 - N$$

$$E[u_i | i \text{ stops thief}] = (1 - Prob[\text{nobody in } \mathcal{J} \text{ stops thief}]) \times E[u_i | j \text{ stops thief}]$$

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$$v - c = \left(1 - (1 - p)^{N-1}\right) \times v$$

$$\frac{c}{v} = (1 - p)^{N-1}$$

$$p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}, \qquad 0 < \frac{c}{v} < 1$$

Is there a mixed strategy NE?

PS4, Ex. 6.a: Stopping the bike thief (MSNE)

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$$u_i(s_i, s_j) = \begin{cases} v > 0 & \text{if} \quad i \text{ does nothing and } j \text{ stops the thief} \\ v - c > 0 & \text{if} \quad i \text{ stops the thief} \\ 0 & \text{if} \quad \text{nobody stops the thief} \end{cases}$$

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$$\begin{aligned} v - c &= \left(1 - (1 - p)^{N-1}\right) \times v \\ \frac{c}{v} &= (1 - p)^{N-1} \\ p^* &= 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}, \qquad 0 < \frac{c}{v} < 1 \end{aligned} \tag{3}$$

The MSNE is where each persons stops the thief with probability $p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}$

PS4, Ex. 6.b: Stopping the bike thief (MSNE)

The MSNE is where each persons stops the thief with probability $p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}$

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?

Calculate the probability that the thief is stopped.

PS4, Ex. 6.b: Stopping the bike thief (MSNE)

The MSNE is where each persons stops the thief with probability $p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}$

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?

Calculate the probability that the thief is stopped.

$$Prob[\text{the thief is stopped}] = 1 - Prob[\text{nobody stops the thief}] \qquad \vdots$$

PS4, Ex. 6.b: Stopping the bike thief (MSNE)

The MSNE is where each persons stops the thief with probability $p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}$

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?

$$\begin{aligned}
\mathit{Prob}[\text{the thief is stopped}] &= 1 - \mathit{Prob}[\text{nobody stops the thief}] \\
&= 1 - (1-p^*)^N \\
&= 1 - (1-p^*)^{N-1}(1-p^*) \\
&= 1 - \frac{c}{v}(1-p^*), && \text{inserting eq. (3) from 6.a} \\
&= 1 - \frac{c}{v}\left(1 - 1 + \left(\frac{c}{v}\right)^{\frac{1}{N-1}}\right), && \text{inserting the MSNE} \\
&= 1 - \frac{c}{v}^{\left(1+\frac{1}{N-1}\right)}
\end{aligned}$$

When $N \to \infty$, what happens to the probability of the thief being caught?

PS4, Ex. 6.b: Stopping the bike thief (MSNE)

The MSNE is where each persons stops the thief with probability $p^* = 1 - \left(\frac{c}{v}\right)^{\frac{1}{N-1}}$

b) Discuss what happens to p as the number of witness becomes very large. What happens then to the probability that the thief will get stopped? What is the intuition for this result?

$$\begin{aligned} Prob[\text{the thief is stopped}] &= 1 - Prob[\text{nobody stops the thief}] \\ &= 1 - (1-p^*)^N \\ &= 1 - (1-p^*)^{N-1}(1-p^*) \\ &= 1 - \frac{c}{v}(1-p^*), && \text{inserting eq. (3) from 6.a} \\ &= 1 - \frac{c}{v}\left(1 - 1 + \left(\frac{c}{v}\right)^{\frac{1}{N-1}}\right), && \text{inserting the MSNE} \\ &= 1 - \frac{c}{v}^{\left(1+\frac{1}{N-1}\right)} \xrightarrow[N\to\infty]{} 1 - \frac{c}{v} \end{aligned}$$

When $N \to \infty$, the probability of the thief being caught decreases due to an increase in the incentive to freeride. Remember: there's a cost to stopping the thief yourself:

$$\begin{aligned} E[u_i|p^*] &= v \cdot Prob[\text{the thief is stopped}] - c \cdot p^* \\ &= v \cdot \left(1 - (1-p^*)^N\right) - c \cdot p^* \\ &= v - c^{\left(1+\frac{1}{N-1}\right)} - c + \left(\frac{c^2}{v}\right)^{\frac{1}{N-1}} \xrightarrow[N\to\infty]{} v - 2c + 1 \end{aligned}$$

PS4, Ex. 7: To keep or split (backwards induction)

PS4, Ex. 7: To keep or split (backwards induction)

Consider the following 2 × 2 game where payoffs are monetary:

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

Before this game is played, Player 1 can choose whether, after the game is played, players should keep their own payoffs or split the aggregate payoff evenly between them.

(a) Draw the game tree of this two-stage game (assuming that Players 1's choice of whether to split payoffs is revealed to Player 2 before the second stage).

(b) Solve by backwards induction.

PS4, Ex. 7.a: To keep or split (backwards induction)

(a) Draw the game tree:

	L	R
T	3, 3	0, 4
B	4, 0	1, 1

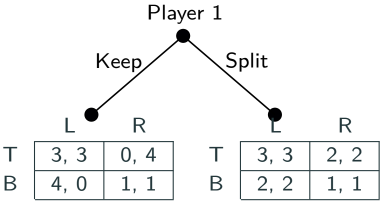
2nd stage: The above is the static game for a keep game, find the static game for a split game and draw the full game tree.

PS4, Ex. 7.a: To keep or split (backwards induction)

(a) Draw the game tree:

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

2nd stage: They play the static game and payoffs are realized.



	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L	R
T	3, 3	2, 2
B	2, 2	1, 1

(b) Solve by backwards induction:

PS4, Ex. 7.b: To keep or split (backwards induction)

(a) Draw the game tree:

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

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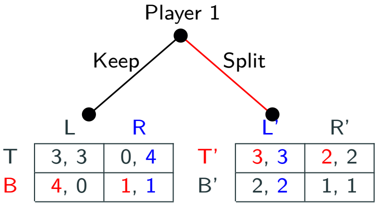


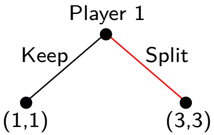
	L	R
T	3, 3	0, 4
B	4, 0	1, 1

	L'	R'
T'	3, 3	2, 2
B'	2, 2	1, 1

(b) Solve by backwards induction:

2nd stage: Each bi-matrix has a unique NE that can be founds using IESDS.

1st stage: Player 1's choice can be reduced to choosing between the subgame NE in each bi-matrix:



BI gives the subgame perfect NE:

$$SPNE = (Split\ B\ T',\ R\ L')$$

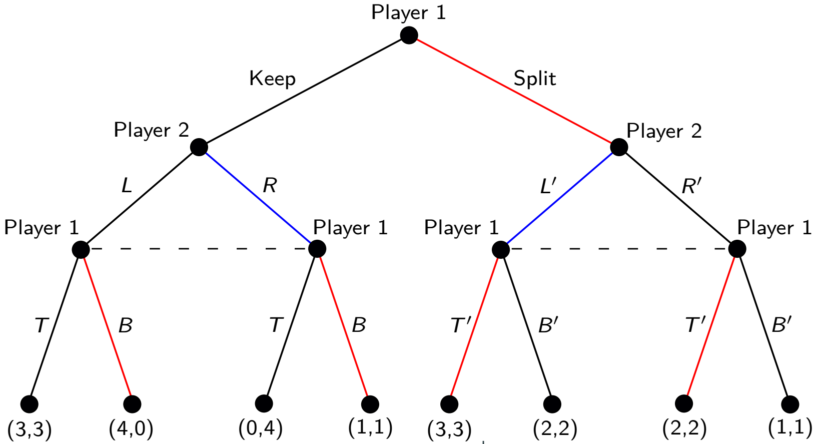
PS4, Ex. 7: To keep or split (backwards induction)

Alternatively, draw the game tree in extensive form to find $SPNE : (Split\ B\ T',\ R\ L')$

1st stage: Player 1 chooses Keep or Split. Player 2 observes the choice.

2nd stage: Player 2 chooses L or R (L' or R'). The action is private information.

3rd stage: Player 1 chooses T or B (T' or B') without knowing what Player 2 did.



The order of stage 2 and 3 is arbitrary, but the 2nd stage must be private information.

Problem Set 3, Ex. 5: Luxembourg as a rogue state (static game)

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Assume that Luxembourg has turned into a rogue state. It is close to acquiring nuclear weapons, which would threaten the stability in the whole region. The Vatican (V) and Denmark (D) are preparing an attack on Luxembourg's nuclear research facilities to stop or slow down its nuclear program. The probability that the attack will be a success is

$$p(s_V, s_D) = s_V + s_D - s_v s_D,$$

where $s_i \in [0, 1]$ is the share of its military capacity that country i ($i \in \{V, D\}$) uses in the attack. If the attack is successful then each country receives a payoff of 1. The cost of participating in the attack for country i is

$$c_i(s_i) = s_i^2$$

The objective of each country is to maximize its expected payoff from the attack minus the cost.

(a) Suppose that the Vatican and Denmark choose the shares of military capacity to use in the attack simultaneously and independently. Find the Nash equilibrium (NE) of this game.

(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

Problem Set 3, Ex. 5.a: Luxembourg as a rogue state (static game)

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Write expected payoff for player $i \neq j$.

Problem Set 3, Ex. 5.a: Luxembourg as a rogue state (static game)

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Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i.

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Find the best-response function for i:

$$FOC: \ \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

What is the NE?

(Hint: is the game symmetric?)

Problem Set 3, Ex. 5.a: Luxembourg as a rogue state (static game)

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Find the best-response function for i:

$$FOC: \ \frac{\delta u_i}{\delta s_i} = 1 + 0 - s_j - 2s_i = 0$$

$$s_i = \frac{1 - s_j}{2}$$

Taking advantage of symmetry $s_i^* = s_j^*$:

$$s_i^* = \frac{1 - s_i^*}{2}$$

$$2s_i^* + s_i^* = 1$$

$$s_i^* = \frac{1}{3} \equiv s^{NE}$$

i.e. $NE = \left\{(s_D^*, s_V^*) = (\frac{1}{3}, \frac{1}{3})\right\}$

Problem Set 3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

Find the best-response function for i:

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(b) Find the social optimum (SO) under the condition that the two countries use the same share of their military capacity. I.e., find the $\bar{s}_V = \bar{s}_D = \bar{s}$ that maximizes aggregate payoff from the attack minus costs. Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

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Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Find the social planner target function.

Problem Set 3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

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$$= 2\bar{s} - 2\bar{s}^2$$

The social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO).

Problem Set 3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

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(b) Find the SO given shares are equal:

Expected payoff for i, $\bar{s}_D = \bar{s}_V = \bar{s}$:

$$u_i(\bar{s}) = \underbrace{\bar{s} + \bar{s} - \bar{s}\bar{s}}_{\text{Probability of success}} - \underbrace{\bar{s}^2}_{\text{Cost}}$$

$$= 2\bar{s} - 2\bar{s}^2$$

Social planner target function:

$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$FOC: \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{S} = 0$$

$$\bar{S} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

Compare with the equilibrium from question (a) and give an intuitive explanation of your findings.

Problem Set 3, Ex. 5.b: Luxembourg as a rogue state (static game)

(a) Find the NE in the static game:

Expected payoff for player $i \neq j$:

$$u_i(s_i, s_j) = \underbrace{s_i + s_j - s_i s_j}_{\text{Probability of success}} - \underbrace{s_i^2}_{\text{Cost}}$$

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$$\pi^S(\bar{s}) = \underbrace{2}_{\text{Countries}} (2\bar{s} - 2\bar{s}^2) = 4\bar{s} - 4\bar{s}^2$$

Find the social optimum (SO):

$$FOC: \ \frac{\delta \pi^S}{\delta s_i} = 4 - 8\bar{S} = 0$$

$$\bar{S} = \frac{4}{8} = \frac{1}{2} > \frac{1}{3}$$

The SO is higher than the NE as the positive externality is not rewarded, which leads to an incentive to free ride.

PS4, Ex. 8: Building a playground (Stackelberg game)

PS4, Ex. 8: Building a playground (Stackelberg game)

Two neighbors are building a common playground for their children. The time spent on the project by neighbor *i* is $x_i \geq 0$, $i = 1, 2$. The resulting quality of the playground is

$$q(x_1, x_2) = x_1 + x_2 - x_1 x_2$$

Spending time on the project is costly. More precisely, the cost function of the neighbors are:

$$C_i(x_i) = x_i^2, \quad i = 1, 2$$

The payoff of neighbor *i*, U_i, is equal to the quality of the playground minus his cost.

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

PS4, Ex. 8.a: Building a playground (Stackelberg game)

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(Step 1) Write up the payoff function

Information so far

1 Quality: $q(x_1, x_2) = x_1 + x_2 - x_1 x_2$
2 Cost: $C_i(x_i) = x_i^2, \quad i = 1, 2$
3 Payoff: Quality-Cost

PS4, Ex. 8.a: Building a playground (Stackelberg game)

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(Step 1) Write up the payoff function

(Step 2) Write up the FOC and find the best response function

Information so far

1 *Quality* : $q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 *Cost* : $C_i(x_i) = x_i^2, \quad i = 1, 2$

3 *Payoff* : $U_i =$
$x_1 + x_2 - x_1 x_2 - x_i^2 \quad i = 1, 2$

PS4, Ex. 8.a: Building a playground (Stackelberg game)

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(Step 1) Write up the payoff function

(Step 2) Write up the FOC and find the best response function

(Step 3) This is a symmetric game, so the BR are the same for both players, use this to find the NE by substituting (6) into (5) and isolating x_1

Information so far

1 *Quality* : $q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 *Cost* : $C_i(x_i) = x_i^2, \quad i = 1, 2$

3 *Payoff* : $U_i = x_1 + x_2 - x_1 x_2 - x_i^2 \quad i = 1, 2$

4 *FOC* : $1 - x_j - 2x_i = 0$

5 $BR_1 : x_1 = (1 - x_2)/2$

6 $BR_2 : x_2 = (1 - x_1)/2$

PS4, Ex. 8.a: Building a playground (Stackelberg game)

(a) Suppose the neighbors decide how much time to spend on the project simultaneously and independently. Derive the best response functions. Find the Nash equilibrium of this game.

(Step 1) Write up the payoff function

(Step 2) Write up the FOC and find the best response function

(Step 3) This is a symmetric game, so the BR are the same for both players, use this to find the NE by substituting (6) into (5) and isolating x_1

(NE) $x_1 = (1-(1-x_1)/2)/2 \Rightarrow x_1 = \frac{1}{3}$
$x_2 = (1-(1-x_2)/2)/2 \Rightarrow x_2 = \frac{1}{3}$
$NE : (\frac{1}{3}, \frac{1}{3})$

Information so far

1 $Quality : q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 $Cost : C_i(x_i) = x_i^2, \quad i = 1, 2$

3 $Payoff : U_i = x_1 + x_2 - x_1 x_2 - x_i^2 \quad i = 1, 2$

4 $FOC : 1 - x_j - 2x_i = 0$

5 $BR_1 : x_1 = (1 - x_2)/2$

6 $BR_2 : x_2 = (1 - x_1)/2$

PS4, Ex. 8.b: Building a playground (Stackelberg game)

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(Step 1) Write up the new payoff function for player one, where he takes player 2s best response as given. In order words, write his payoff as a function of x_1 and $BR_2(x_1)$

Information so far

1 $Quality : q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 $Cost : C_i(x_i) = x_i^2, \quad i = 1, 2$

3 $U_1(x_1, x_2) = x_1 + x_2 - x_1 x_2 - x_1^2$

4 $BR_2 : x_2 = (1 - x_1)/2$

PS4, Ex. 8.b: Building a playground (Stackelberg game)

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(Step 1) Write up the new payoff function for player one, where he takes player 2s best response as given. In order words, write his payoff as a function of x_1 and $BR_2(x_1)$

(Step 2) Write up the FOC and find the best response function for player 1, as a function of x_1 and $BR_2(x_1)$

Information so far

1 $Quality : q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 $Cost : C_i(x_i) = x_i^2, \quad i = 1, 2$

3 $U_1(x_1, x_2) = x_1 + x_2 - x_1 x_2 - x_1^2$

4 $BR_2 : x_2 = (1 - x_1)/2$

5 $U_1(x_1, BR_2(x_1)) :$
$x_1 + (1 - x_1)/2 - x_1(1 - x_1)/2 - x_1^2$

PS4, Ex. 8.b: Building a playground (Stackelberg game)

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(Step 1) Write up the new payoff function for player one, where he takes player 2s best response as given. In order words, write his payoff as a function of x_1 and $BR_2(x_1)$

(Step 2) Write up the FOC and find the best response function for player 1, as a function of x_1 and $BR_2(x_1)$

(Step 3) Use the value for x_1 to find x_2 and write up the SPNE

Information so far

1 $Quality : q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 $Cost : C_i(x_i) = x_i^2, \quad i = 1, 2$

3 $U_1(x_1, x_2) = x_1 + x_2 - x_1 x_2 - x_1^2$

4 $BR_2 : x_2 = (1 - x_1)/2$

5 $U_1(x_1, BR_2(x_1)) :$
$x_1 + (1 - x_1)/2 - x_1(1 - x_1)/2 - x_1^2$

6 $FOC_1 : 1 - \frac{1}{2} - \frac{1}{2} - x_1 = 0$

7 $BR_1 : x_1 = 0$

PS4, Ex. 8.b: Building a playground (Stackelberg game)

(b) Suppose now that the game is played in two stages. First, neighbor 1 decides how much time to spend on the project. Neighbor 2 observes this and then chooses how much time to put in himself. Find the backwards induction outcome of this game.

(Step 1) Write up the new payoff function for player one, where he takes player 2s best response as given. In order words, write his payoff as a function of x_1 and $BR_2(x_1)$

(Step 2) Write up the FOC and find the best response function for player 1, as a function of x_1 and $BR_2(x_1)$

(Step 3) Use the value for x_1 to find x_2 and write up the SPNE

(SPNE) $x_1 = 0 \quad x_2 = (1-0)/2 \Rightarrow x_2 = \frac{1}{2}$
$SPNE : (0, \frac{1}{2})$

Information so far

1 $Quality : q(x_1, x_2) = x_1 + x_2 - x_1 x_2$

2 $Cost : C_i(x_i) = x_i^2, \quad i = 1, 2$

3 $U_1(x_1, x_2) = x_1 + x_2 - x_1 x_2 - x_1^2$

4 $BR_2 : x_2 = (1 - x_1)/2$

5 $U_1(x_1, BR_2(x_1)) :$
$x_1 + (1 - x_1)/2 - x_1(1 - x_1)/2 - x_1^2$

6 $FOC_1 : 1 - \frac{1}{2} - \frac{1}{2} - x_1 = 0$

7 $BR_1 : x_1 = 0$

PS4, Ex. 8.c: Building a playground (Stackelberg game)

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

(Step 1) What are the payoffs for each player in the two games? What is the total utility?

Information so far

G1 $NE = \left(\frac{1}{3}, \frac{1}{3}\right)$

G2 $SPNE = \left(0, \frac{1}{2}\right)$

Utility $U_i = x_1 + x_2 - x_1 x_2 - x_i^2, \quad i = 1, 2$

PS4, Ex. 8.c: Building a playground (Stackelberg game)

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

(Step 1) What are the payoffs for each player in the two games? What is the total utility?

(Step 2) Compare and explain.

(Bonus) If bargaining is possible, does a pareto improvement exist to the outcome in the 2nd game?

Information so far

G1 $NE = \left(\frac{1}{3}, \frac{1}{3}\right)$

G2 $SPNE = \left(0, \frac{1}{2}\right)$

Utility $U_i = x_1 + x_2 - x_1 x_2 - x_i^2, \quad i = 1, 2$

G1 $U_1 = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}^2 - \frac{1}{3}^2 = \frac{4}{9}$

G1 $U_2 = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}^2 - \frac{1}{3}^2 = \frac{4}{9}$

G1 $U_T = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$

G2 $U_1' = 0 + \frac{1}{2} - 0 \cdot \frac{1}{2} - 0^2 = \frac{1}{2}$

G2 $U_2' = 0 + \frac{1}{3} - 0 \cdot \frac{1}{2} - \frac{1}{2}^2 = \frac{1}{4}$

G2 $U_T' = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

PS4, Ex. 8.c: Building a playground (Stackelberg game)

(c) Compare the games from (a) and (b) with respect to the payoff that each neighbor obtains. Give an intuitive explanation of your results.

(Player 1) Gets a higher payoff when he gets to choose first (first mover advantage). He chooses to freeride, relying on Player 2 to pick up the slack.

(Player 2) Gets a lower payoff when she chooses second. Even though Player 1 freerides, it is still optimal for her to pick up some of the slack.

(Total U) Overall utility is lower in the 2nd game. With bargaining, P2 could offer P1 compensation in order to remove the freeride opportunity. E.g. In game 2, if P2 could offer P1 1/9 in order for them to choose at the same time instead. P1 would accept the offer and get the payoff $5/9 > 1/2$ and P2 would transfer 1/9 and be left with $3/9 > 1/4$.

Information so far

G1 $NE = \left(\frac{1}{3}, \frac{1}{3}\right)$

G2 $SPNE = \left(0, \frac{1}{2}\right)$

Utility $U_i = x_1 + x_2 - x_1 x_2 - x_i^2, \quad i = 1, 2$

G1 $U_1 = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}^2 - \frac{1}{3}^2 = \frac{4}{9}$

G1 $U_2 = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}^2 - \frac{1}{3}^2 = \frac{4}{9}$

G1 $U_T = \frac{4}{9} + \frac{4}{9} = \frac{8}{9}$

G2 $U_1' = 0 + \frac{1}{2} - 0 \cdot \frac{1}{2} - 0^2 = \frac{1}{2}$

G2 $U_2' = 0 + \frac{1}{3} - 0 \cdot \frac{1}{2} - \frac{1}{2}^2 = \frac{1}{4}$

G2 $U_T' = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

→ The Coase Theorem (Coase, 1960).