



## Microeconomics III: Problem Set 9<sup>a</sup>

---

Thor Donsby Noe (thor.noe@econ.ku.dk) & Christopher Borberg (christopher.borberg@econ.ku.dk)  
November 20 2019

Department of Economics, University of Copenhagen

---

<sup>a</sup>Slides created for exercise class 3 and 4, with reservation for possible errors.

PS8, Ex. 1 (A): Mixed-Strategy NE and Pure-Strategy BNE

PS8, Ex. 2 (A): Mixed-Strategy BNE

The expected highest and second-highest draw from a uniform distribution

PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

PS8, Ex. 4: First-price sealed bid auctions with three bidders

PS8, Ex. 5: Winner's Curse

PS8, Ex. 3': Solving 3.b and 3.c through integration (optional)

PS8, Ex. 4': Solving 4.c through integration (optional)

**PS8, Ex. 1 (A): Mixed-Strategy NE  
and Pure-Strategy BNE**

---

## PS8, Ex. 1 (A): Mixed-Strategy NE and Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C	D
A	0, 2	2, 3
B	3, 1	$k$ , 8

- (a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).
- (b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

*[Hints on the next slide. Try to independently write down the approach/criteria for a mixed-strategy NE and a pure-strategy BNE respectively.]*

## PS8, Ex. 1 (A): Mixed-Strategy NE and Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C ( $q$ )	D ( $1-q$ )
A ( $p$ )	0, 2	2, 3
B ( $1-p$ )	3, 1	$k$ , 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

**Hint:** To find a mixed-strategy NE (MSNE):

Find the probabilities  $q$  for which Player 1 is indifferent, i.e.  $u_1(A, q) = u_1(B, q)$  and the probabilities  $p$  for which Player 2 is indifferent, i.e.  $u_2(C, p) = u_2(D, p)$ .

(b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

**Hint:** Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type,  $t_2$ ). For each possible strategy  $s_2(t_2)$ :

1. Given the different types  $t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$ , write up the best response of player 1:  $s_1^*(t_{1,k}) \equiv BR_1(s_2(t_2)|t_{1,k})$ .
2. If it also holds that  $s_2(t_2) = BR_2(s_1^*(t_{1,k})|t_2) \equiv s_2^*(t_2)$  then  $(s_1^*(t_{1,k}), s_2^*(t_2))$  is a BNE.

## PS8, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where  $k \in \mathbb{N}$ :

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

***First, find all pure-strategy NE given  $k$ .***

## PS8, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where  $k \in \mathbb{N}$ :

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

P2:  $C$  is strictly dominated by  $D$ , thus  $D$  is played in any NE, pure or mixed.

P1: For P2 playing  $D$  consider:

$k=1$ :

$k=2$ :

$k \geq 3$ :

## PS8, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where  $k \in \mathbb{N}$ :

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

P2:  $C$  is strictly dominated by  $D$ , thus  $D$  is played in any NE, pure or mixed.

P1: For P2 playing  $D$  consider:

$k=1$ : One PSNE:  $\{(A, D)\}$

$k=2$ : Two PSNE:  $\{(A, D); (B, D)\}$

$k \geq 3$ : One PSNE:  $\{(B, D)\}$

***Then find all mixed-strategy NE given  $k$ .***



## PS8, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where  $k \in \mathbb{N}$ :

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

P2:  $C$  is strictly dominated by  $D$ , thus  $D$  is played in any NE, pure or mixed.

P1: For P2 playing  $D$  consider:

$k = 1$  : One PSNE:  $\{(A, D)\}$

$k = 2$  : Two PSNE:  $\{(A, D); (B, D)\}$

$k \geq 3$  : One PSNE:  $\{(B, D)\}$

For P1 to mix, she has to be indifferent between  $A$  and  $B$ , thus we only need to look at:

$k = 2$  :

## PS8, Ex. 1.a (A): Mixed-Strategy NE

Consider this static game, where  $k \in \mathbb{N}$ :

	C (q)	D (1-q)
A (p)	0, 2	2, 3
B (1-p)	3, 1	k, 8

(a) For all possible values of  $k$ , find all Nash Equilibria (pure and mixed).

P2:  $C$  is strictly dominated by  $D$ , thus  $D$  is played in any NE, pure or mixed.

P1: For P2 playing  $D$  consider:

$k = 1$ : One PSNE:  $\{(A, D)\}$

$k = 2$ : Two PSNE:  $\{(A, D); (B, D)\}$

$k \geq 3$ : One PSNE:  $\{(B, D)\}$

For P1 to mix, she has to be indifferent between  $A$  and  $B$ , thus we only need to look at:

$k = 2$ : One MSNE:  $\{(p \in (0, 1), q = 0)\}$

## PS8, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C	D
A	0, 2	2, 3
B	3, 1	$k$ , 8

- (b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

*[Hint for BNE on next slide.]*

## PS8, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C	D
A	0, 2	2, 3
B	3, 1	$k$ , 8

- (b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

**Hint:** Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type,  $t_2$ ). For each possible strategy  $s_2(t_2)$ :

Step 1: Given the different types

$t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$ , write up the best response of player 1:  
 $s_1^*(t_{1,k}) \equiv BR_1(s_2(t_2)|t_{1,k})$ .

Step 2: If it also holds that

$s_2(t_2) = BR_2(s_1^*(t_{1,k})|t_2) \equiv s_2^*(t_2)$   
then  $(s_1^*(t_{1,k}), s_2^*(t_2))$  is a BNE.

## PS8, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C	D
A	0, 2	2, 3
B	3, 1	$k$ , 8

- (b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

**Hint:** Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type,  $t_2$ ). For each possible strategy  $s_2(t_2)$ :

Step 1: Given the different types

$t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$ , write up the best response of player 1:  
 $s_1^*(t_{1,k}) \equiv BR_1(s_2(t_2)|t_{1,k})$ .

Step 2: If it also holds that

$s_2(t_2) = BR_2(s_1^*(t_{1,k})|t_2) \equiv s_2^*(t_2)$   
then  $(s_1^*(t_{1,k}), s_2^*(t_2))$  is a BNE.

As  $C$  is strictly dominated, player 2 only has the viable strategy  $s_2(t_2) = D$ :

1. Best response of player 1,  $s_1^*(t_{1,k})$ :

$$BR_1(D|t_{1,k}) = (s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}) = (A, B)$$

## PS8, Ex. 1.b (A): Pure-Strategy BNE

Consider this static game, where  $k \in \mathbb{N}$ :

	C	D
A	0, 2	2, 3
B	3, 1	$k$ , 8

- (b) Now assume that player 1 knows  $k$ , but player 2 only knows that  $k = 1$  with probability  $\frac{1}{2}$  and  $k = 3$  with probability  $\frac{1}{2}$ . Find the Bayesian Nash Equilibrium of this game.

**Hint:** Find Bayesian Nash Equilibria (BNE) by going through the possible strategies for player 2 (the player with only one type,  $t_2$ ). For each possible strategy  $s_2(t_2)$ :

Step 1: Given the different types

$t_{1,k} \in T_1 = \{t_{1,k=1}, t_{1,k=3}\}$ , write up the best response of player 1:  
 $s_1^*(t_{1,k}) \equiv BR_1(s_2(t_2)|t_{1,k})$ .

Step 2: If it also holds that

$s_2(t_2) = BR_2(s_1^*(t_{1,k})|t_2) \equiv s_2^*(t_2)$   
then  $(s_1^*(t_{1,k}), s_2^*(t_2))$  is a BNE.

As  $C$  is strictly dominated, player 2 only has the viable strategy  $s_2(t_2) = D$ :

1. Best response of player 1,  $s_1^*(t_{1,k})$ :

$$BR_1(D|t_{1,k}) = (s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}) = (A, B)$$

2. As  $D = BR_2((A, B)|t_2) \equiv s_2^*(t_2)$  we have a unique BNE:

$$((s_1^*|t_{1,k=1}, s_1^*|t_{1,k=3}), s_2^*(t_2)) = \{((A, B), D)\}$$

## **PS8, Ex. 2 (A): Mixed-Strategy BNE**

---

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Consider the same set-up as exercise 3.4 in Gibbons, but now with the following bi-matrices for Game 1 and Game 2 respectively:

G1:

	L	R
U	1, 1	0, 0
D	0, 0	2, 0

G2:

	L	R
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

*Exercise 3.4 in Gibbons (p. 169). Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:*

- Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.*
- Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.***
- Player 1 chooses either  $U$  or  $D$ ; player 2 simultaneously chooses either  $L$  or  $R$ .*
- Payoffs are given by the game drawn by nature.*



## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:		L	R
U		1, 1	0, 0
D		0, 0	2, 0

G2:		L	R
U		0, 0	0, 0
D		0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

Step 1: **Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.**

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:		L	R
U		1, 1	0, 0
D		0, 0	2, 0

G2:		L	R
U		0, 0	0, 0
D		0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

I.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:		L	R
U		1, 1	0, 0
D		0, 0	2, 0

G2:		L	R
U		0, 0	0, 0
D		0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

I.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

$BR_2((D, D)) = R$  but P2 is indifferent between  $L$  and  $R$  for  $s'_1 = (U, D)$ .

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:	L ( $q$ )	R ( $1-q$ )
U	1, 1	0, 0
D	0, 0	2, 0

G1:	L ( $q$ )	R ( $1-q$ )
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

- Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.
- Step 2: **Find the values of  $q$  (the probability that player 2 plays  $L$ ) such that player 1 will actually play  $s'_1$ .**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

I.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

$BR_2((D, D)) = R$  but P2 is indifferent between  $L$  and  $R$  for  $s'_1 = (U, D)$ .

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:	L ( $q$ )	R ( $1-q$ )
U	1, 1	0, 0
D	0, 0	2, 0

G1:	L ( $q$ )	R ( $1-q$ )
U	0, 0	0, 0
D	0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.

Step 2: **Find the values of  $q$  (the probability that player 2 plays  $L$ ) such that player 1 will actually play  $s'_1$ .**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

I.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

$BR_2((D, D)) = R$  but P2 is indifferent between  $L$  and  $R$  for  $s'_1 = (U, D)$ .

2. P1 wants to play  $U$  in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1-q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:		L ( $q$ )	R ( $1-q$ )
U		1, 1	0, 0
D		0, 0	2, 0

G2:		L ( $q$ )	R ( $1-q$ )
U		0, 0	0, 0
D		0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

- Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.
- Step 2: Find the values of  $q$  (the probability that player 2 plays  $L$ ) such that player 1 will actually play  $s'_1$ .
- Step 3: **Write up the mixed-strategy BNE where P1 plays a pure strategy and P2 randomizes between  $L$  and  $R$ .**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

i.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

$BR_2((D, D)) = R$  but P2 is indifferent between  $L$  and  $R$  for  $s'_1 = (U, D)$ .

2. P1 wants to play  $U$  in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1 - q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

## PS8, Ex. 2 (A): Mixed-Strategy BNE

Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not:

G1:		L ( $q$ )	R ( $1-q$ )
U		1, 1	0, 0
D		0, 0	2, 0

G2:		L ( $q$ )	R ( $1-q$ )
U		0, 0	0, 0
D		0, 0	1, 1

Find all mixed-strategy Bayesian Nash Equilibria of the following form: Player 1 plays a pure strategy, and Player 2 randomizes between  $L$  and  $R$ .

- Step 1: Find the player 1 strategies  $s'_1$  for which player 2 will want to mix.
- Step 2: Find the values of  $q$  (the probability that player 2 plays  $L$ ) such that player 1 will actually play  $s'_1$ .
- Step 3: **Write up the mixed-strategy BNE where P1 plays a pure strategy and P2 randomizes between  $L$  and  $R$ .**

1. As P1 learns whether nature has drawn G1 or G2 her strategy space is:

$$S_1 = \{(U, U), (U, D), (D, D), (D, U)\}$$

G2:  $U$  is weakly dominated by  $D$ , thus, P1 will play  $D$  in G2 as long as P2 plays  $R$  with positive probability.

i.e. P1 either plays  $(U, D)$  or  $(D, D)$ .

$BR_2((D, D)) = R$  but P2 is indifferent between  $L$  and  $R$  for  $s'_1 = (U, D)$ .

2. P1 wants to play  $U$  in G1 if:

$$\mathbb{E}[u_1|U] \geq \mathbb{E}[u_1|D] \Leftrightarrow$$

$$q \geq 2(1 - q) \Leftrightarrow 3q \geq 2 \Leftrightarrow q \geq \frac{2}{3}$$

3. For  $q$  being the probability that P2 plays  $L$ , the mixed-strategy BNE is:

$$BNE' = \left\{ \left( (U, D), q \geq \frac{2}{3} \right) \right\}$$

**The expected highest and  
second-highest draw from a uniform  
distribution**

---



## The expected highest and second-highest draw from a uniform distribution

To find *seller's expected revenue* from a sealed bid auction (e.g. bidders simultaneously submit their bids in sealed envelopes without knowing the bids of others) with symmetric bidders with valuation drawn from a uniform distribution, there are two different approaches:

1. One approach is to derive each *bidder's expected payment* as a function of her valuation and then integrate this expression using the PDF to get the *ex-ante expected payment* of each bidder which can then be added together to find seller's expected revenue.
2. However, a more simple approach is to for  $N$  number of bidders to calculate the expected highest value (first-price sealed bid auction) or the expected second-highest value (second-price sealed bid auction). Plugging the value into the bid-function gives the seller's expected revenue.

Deriving the bid-function (best-response function) is a prerequisite for both approaches.

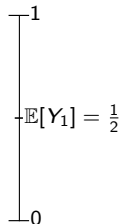
## The expected highest and second-highest draw from a uniform distribution

First, let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from the **standard uniform distribution**  $x \sim U(0, 1)$ . The highest draw  $Y_1$  and the second-highest draw  $Y_2$  of all  $N$  draws are expected to be:

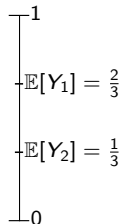
$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \text{ where } Y_1 = \max(X)$$

$$\mathbb{E}[Y_2] = \frac{N-1}{N+1}, \text{ where } Y_2 = \max(X \neq Y_1)$$

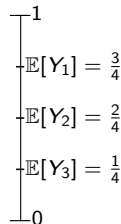
For  $N=1$ :



For  $N=2$ :



For  $N=3$ :



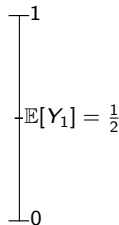
## The expected highest and second-highest draw from a uniform distribution

First, let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from the *standard* uniform distribution  $x \sim U(0, 1)$ . The highest draw  $Y_1$  and the second-highest draw  $Y_2$  of all  $N$  draws are expected to be:

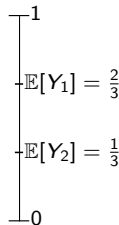
$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \text{ where } Y_1 = \max(X)$$

$$\mathbb{E}[Y_2] = \frac{N-1}{N+1}, \text{ where } Y_2 = \max(X \neq Y_1)$$

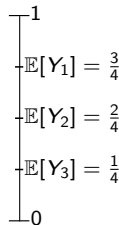
For  $N=1$ :



For  $N=2$ :



For  $N=3$ :



**Generalized**, let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from a uniform distribution  $x \sim U(a, b)$ . The highest draw  $Y_1$  and the second-highest draw  $Y_2$  of all  $N$  draws are expected to be:

$$\mathbb{E}[Y_1] = a + N \frac{b-a}{N+1},$$

where  $Y_1 = \max(X)$

$$\mathbb{E}[Y_2] = a + (N-1) \frac{b-a}{N+1},$$

where  $Y_2 = \max(X \neq Y_1)$

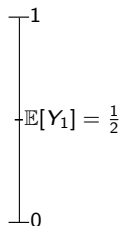
## The expected highest and second-highest draw from a uniform distribution

First, let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from the *standard* uniform distribution  $x \sim U(0, 1)$ . The highest draw  $Y_1$  and the second-highest draw  $Y_2$  of all  $N$  draws are expected to be:

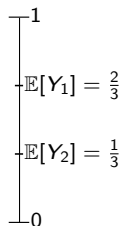
$$\mathbb{E}[Y_1] = \frac{N}{N+1}, \text{ where } Y_1 = \max(X)$$

$$\mathbb{E}[Y_2] = \frac{N-1}{N+1}, \text{ where } Y_2 = \max(X \neq Y_1)$$

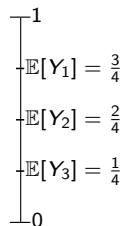
For  $N=1$ :



For  $N=2$ :



For  $N=3$ :



Generalized, let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from a uniform distribution  $x \sim U(a, b)$ . The highest draw  $Y_1$  and the second-highest draw  $Y_2$  of all  $N$  draws are expected to be:

$$\mathbb{E}[Y_1] = a + N \frac{b-a}{N+1}, \quad \text{where } Y_1 = \max(X)$$

$$\mathbb{E}[Y_2] = a + (N-1) \frac{b-a}{N+1}, \quad \text{where } Y_2 = \max(X \neq Y_1)$$

**E.g.** for  $N=1$ ,  $\mathbb{E}[Y_1]$  simply collapses to the expression for the expected mean,  $\mu$ :

$$\mathbb{E}[Y_1] = a + N \frac{b-a}{N+1} = a + 1 \frac{b-a}{1+1} = \frac{2a}{2} + \frac{b-a}{2} = \frac{a+b}{2} \equiv \mu$$

## The expected highest and second-highest draw from a uniform distribution

**Applied to auctions:** Consider  $N$  number of bidders where each bidder  $i$  has the value  $v_i$  that is independently drawn from the same uniform distribution  $v_i \sim U(a, b)$ .

1<sup>st</sup> step: The highest value  $Y_1$  and the second-highest value  $Y_2$  for all  $N$  bidders are expected to be:

$$\mathbb{E}[Y_1] = a + N \frac{b - a}{N + 1}, \quad \text{where } Y_1 = \max(V), \quad V = v_1, v_2, \dots, v_N$$

$$\mathbb{E}[Y_2] = a + (N - 1) \frac{b - a}{N + 1}, \quad \text{where } Y_2 = \max(V \neq Y_1)$$

2<sup>nd</sup> step: To calculate the seller's expected revenue, insert the expected highest value  $\mathbb{E}[Y_1]$  (first-price sealed bid auction) or the expected second-highest value  $\mathbb{E}[Y_2]$  (second-price sealed bid auction) in the derived bid-function.

## The expected highest and second-highest draw from a uniform distribution

**Proof:** [only for those interested] Let  $X = x_1, x_2, \dots, x_N$  be  $N$  independent and identically distributed (i.i.d.) draws from a uniform distribution  $x \sim U(a, b)$ . Denote the highest draw  $Y_1 = \max(X)$ . The cumulative distribution function (CDF) of  $Y_1$  is:  
 $G(x) = \mathbb{P}[Y_1 \leq x]$

$$\begin{aligned} &= \mathbb{P}[x_1 \leq x, x_1 \leq x, \dots, x_N \leq x], && \text{since } Y_1 \text{ is the max of } X \\ &= \mathbb{P}[x_1 \leq x] \times \mathbb{P}[x_1 \leq x] \times \dots \times \mathbb{P}[x_N \leq x], && \text{since draws are independent} \\ &= F(x) \times F(x) \times \dots \times F(x) = (F(x))^N, && F(x) \text{ is the CDF of } x : F(x) = \frac{x-a}{b-a} \quad (*) \end{aligned}$$

The first-derivative of the CDF gives the probability density function (PDF) of  $Y_1$ :

$$\begin{aligned} g(x) &= \frac{\delta G(x)}{\delta x} \\ &= F'(x)N(F(x))^{N-1} \\ &= f(x)N(F(x))^{N-1}, && f(x) \text{ is the PDF of } x : f(x) = \frac{1}{b-a} \quad (**) \end{aligned}$$

The expectation to  $Y_1$  is found by integrating  $x$  times the PDF of  $Y_1$ ,  $g(x)$ :

$$\begin{aligned} \mathbb{E}[Y_1] &= \int_a^b x \cdot g(x) \, dx \\ &= \int_a^b x \cdot f(x)N(F(x))^{N-1} \, dx \\ &= \int_a^b x \cdot \frac{1}{b-a} N \left( \frac{x-a}{b-a} \right)^{N-1} \, dx, && \text{using } (**) \text{ and } (*) \end{aligned}$$

While the general solution isn't too obvious, the integral solves easily for  $x \sim (0, 1)$ :

$$\mathbb{E}[Y_1] = \int_0^1 x \frac{1}{1-0} N \left( \frac{x-0}{1-0} \right)^{N-1} dx = \int_0^1 x N x^{N-1} dx = \int_0^1 N x^N dx = \left[ \frac{N}{N+1} x^{N+1} \right]_0^1 = \frac{N}{N+1} \quad 26$$

## **PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders**

---

## PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with

$$v_i \sim U(1, 3)$$

Thus, the values are *private*.

- (a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$ . Find  $c$  and  $d$ .
- (b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

*[Consider a uniform distribution  $x \sim U(a, b)$ . Try to write up the probability density function (PDF), cumulative distribution function (CDF), and the expectations to the highest value  $Y_1$  and to the second-highest value  $Y_2$  from  $N$  draws of  $x$ .]*



## PS8, Ex. 3: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with

$$v_i \sim U(1, 3)$$

Thus, the values are *private*.

- (a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .
- (b) Calculate the revenue to the seller.

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for  $N$  draws from a uniform distribution  $x \sim U(a, b)$  :

PDF: Probability density function:  $f(x) = \frac{1}{b-a}$

CDF: Cumulative distribution function:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$  where  $Y_1 = \max(X)$ ,  $X = x_1, x_2, \dots, x_N$

$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}$  where  $Y_2 = \max(X \neq Y_1)$

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

- (a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

1<sup>st</sup> step: **Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ .**

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

- (a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

1<sup>st</sup> step: **Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :**

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Results:

$$2^{\text{nd}}: \text{FOC: } \frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Results:

$$2^{\text{nd}}: \text{FOC: } \frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$$

$$\text{SOC: } -\frac{1}{c} = 0$$

i.e. expected utility is concave in  $b_i$ .

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned}\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ using CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i)\end{aligned}$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

3<sup>rd</sup> step: **To find  $c^*$  and  $d^*$ , compare the best-response function  $b_i(v_i)$  to (\*).**

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, Y_2 = \max(X \neq Y_1)$$

Results:

$$2^{\text{nd}}: \text{FOC: } \frac{1}{2c}[(v_i - 2b_i) + (d + c)] = 0$$

$$\text{SOC: } -\frac{1}{c} = 0$$

i.e. expected utility is concave in  $b_i$ .



## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

3<sup>rd</sup> step: **To find  $c^*$  and  $d^*$ , compare the best-response function  $b_i(v_i)$  to (\*).**

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, Y_2 = \max(X \neq Y_1)$$

Results:

$$\begin{aligned} 2^{\text{nd}}: \text{ FOC: } \frac{1}{2c}[(v_i - 2b_i) + (d + c)] &= 0 \\ \text{SOC: } -\frac{1}{c} &= 0 \end{aligned}$$

i.e. expected utility is concave in  $b_i$ .

3<sup>rd</sup>: From the FOC, the BR function is:

$$2b_i = v_i + d + c \Rightarrow$$

$$b_i(v_i) = \underbrace{\frac{1}{2}}_{c^*} v_1 + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

## PS8, Ex. 3.a: First- and second-price sealed bid auctions with two bidders

Consider a first-price sealed bid auction with two bidders, who have valuations  $v_1$  and  $v_2$ , respectively. These values are distributed independently uniformly with  $v_i \sim U(1, 3)$ , thus, the values are *private*.

(a) Show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:

$$b_i(v_i) = cv_i + d (*). \text{ Find } c \text{ and } d.$$

1<sup>st</sup> step: Assuming bidder  $j$  follows the proposed strategy  $b_j(v_j) = cv_j + d$ , calculate bidder  $i$ 's expected payoff from bidding  $b_i$ :

$$\begin{aligned} \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\ &= \mathbb{P}(b_i > b_j(v_j))(v_i - b_i) \\ &= \mathbb{P}(b_i > cv_j + d)(v_i - b_i) \\ &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\ &= \frac{\frac{b_i - d}{c} - 1}{3 - 1}(v_i - b_i), \text{ CDF} \\ &= \frac{b_i - d - c}{2c}(v_i - b_i) \end{aligned}$$

2<sup>nd</sup> step: Take the FOC and SOC wrt.  $b_i$ .

3<sup>rd</sup> step: To find  $c^*$  and  $d^*$ , compare the best-response function  $b_i(v_i)$  to (\*).

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, Y_2 = \max(X \neq Y_1)$$

$$\begin{aligned} 2^{\text{nd}}: \text{FOC: } \frac{1}{2c} [(v_i - 2b_i) + (d + c)] &= 0 \\ \text{SOC: } -\frac{1}{c} &= 0 \end{aligned}$$

i.e. expected utility is concave in  $b_i$ .

3<sup>rd</sup>: From the FOC, the BR function is:

$$b_i(v_i) = \underbrace{\frac{1}{2} v_1}_{c^*} + \underbrace{\frac{1}{2}(d + c)}_{d^*}$$

Inserting the first term in the second term,  $d^* = \frac{1}{2}(d^* + c^*) = \frac{1}{2}(d^* + \frac{1}{2})$ , which solves for  $c^* = d^* = \frac{1}{2}$ .

(b) Calculate the revenue to the seller. For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: **Calculate the expected highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ .**

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned}\mathbb{E}[Y_1] &= a + N \frac{b-a}{N+1} \\ &= 1 + 2 \frac{3-1}{2+1} \\ &= 1 + \frac{4}{3} = \frac{7}{3}\end{aligned}$$

For  $N$  draws from  $x \sim U(a, b)$ :

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, Y_2 = \max(X \neq Y_1)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$1^{\text{st}}: \mathbb{E}[Y_1] = \frac{7}{3}$$

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned}\mathbb{E}[Y_1] &= a + N \frac{b-a}{N+1} \\ &= 1 + 2 \frac{3-1}{2+1} \\ &= 1 + \frac{4}{3} = \frac{7}{3}\end{aligned}$$

2<sup>nd</sup> step: **Insert the expected highest value  $\mathbb{E}[Y_1]$  in the bid-function (\*) to find the seller's expected revenue.**

For  $N$  draws from  $x \sim U(a, b)$  :

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$ ,  $Y_1 = \max(X=x_1, \dots, x_N)$

$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}$ ,  $Y_2 = \max(X \neq Y_1)$

Results so far:

(\*)  $b_i(v_i) = cv_i + d$

(\*\*)  $\mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$

(3.a)  $c^* = d^* = \frac{1}{2}$

1<sup>st</sup>:  $\mathbb{E}[Y_1] = \frac{7}{3}$

## PS8, Ex. 3.b: First- and second-price sealed bid auctions with two bidders

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned}\mathbb{E}[Y_1] &= a + N \frac{b-a}{N+1} \\ &= 1 + 2 \frac{3-1}{2+1} \\ &= 1 + \frac{4}{3} = \frac{7}{3}\end{aligned}$$

2<sup>nd</sup> step: Insert the expected highest value  $\mathbb{E}[Y_1]$  in the bid-function (\*) to find the seller's expected revenue:

$$\begin{aligned}\mathbb{E}[\text{revenue}] &= b_i \left( \frac{7}{3} \right) \\ &= c^* \frac{7}{3} + d^* \\ &= \frac{1}{2} \cdot \frac{7}{3} + \frac{1}{2}, \quad \text{using (3.a)} \\ &= \frac{7}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3}\end{aligned}$$

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$1^{\text{st}}: \mathbb{E}[Y_1] = \frac{7}{3}$$

- (c) Suppose now that the object is sold by a *second-price sealed bid auction*.
- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
  - ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
  - iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$



(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

## PS8, Ex. 3.c.ii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of  $b_1$  unless  $b_1 < b_2$ , in which case P1 no longer wins, thus, gets zero payoff.

P1 loses: Payoff is independent of  $b_1$  unless  $b_1 > b_2$ , in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium.**  
Compare to the answer in (b).

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

For  $N$  draws from  $x \sim U(a, b)$ :

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$ ,  $Y_1 = \max(X=x_1, \dots, x_N)$

$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}$ ,  $Y_2 = \max(X \neq Y_1)$

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium.**  
Compare to the answer in (b).

For  $N$  draws from  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) **Calculate the expected second-highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ .**

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium.

**Compare to the answer in (b).**

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) Calculate the expected second-highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned} \mathbb{E}[Y_2] &= a + (N-1) \frac{b-a}{N+1} \\ &= 1 + 1 \frac{3-1}{2+1} \\ &= 1 + \frac{2}{3} = \frac{5}{3} \end{aligned}$$

## PS8, Ex. 3.c.iii: First- and second-price sealed bid auctions with two bidders

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

For  $N$  draws from  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

$$\mathbb{E}(Y_2) = a + (N-1) \frac{b-a}{N+1}, \quad Y_2 = \max(X \neq Y_1)$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) Calculate the expected second-highest value  $v_i$  for  $N = 2$  draws from the uniform distribution  $v_i \sim U(1, 3)$ :

$$\mathbb{E}[Y_2] = a + (N-1) \frac{b-a}{N+1}$$

$$= 1 + 1 \frac{3-1}{2+1}$$

$$= 1 + \frac{2}{3} = \frac{5}{3}$$

Knowing from (ii) that each bidder bids their valuation:

$$\text{Seller's revenue} = \mathbb{E}[Y_2] = \frac{5}{3}$$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

## **PS8, Ex. 4: First-price sealed bid auctions with three bidders**

---

## PS8, Ex. 4: First-price sealed bid auctions with three bidders

Consider the auction setting of the previous exercise. But now suppose that there are three identical bidders,  $i = 1, 2, 3$ , with values  $v_i$  where

$$v_i \sim U(1, 3)$$

and the values are independent, i.e. private. The auction is first-price sealed bid.

- (a) Again, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .
- (b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?
- (c) (More difficult). Calculate the revenue to the seller.



- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write bidder  $i$ 's expected payoff in the proposed equilibrium.

- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write bidder  $i$ 's expected payoff in the proposed equilibrium:

$$\mathbb{E}[u_i(b_i, v_i)] = \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i)$$

## PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write  $i$ 's expected payoff in eq.:

$$\begin{aligned}\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\&= \mathbb{P}(b_i > b_j(v_j), b_i > b_k(v_k))(v_i - b_i) \\&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using (*)} \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}\end{aligned}$$

## PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write  $i$ 's expected payoff in eq.:

$$\begin{aligned}\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\&= \mathbb{P}(b_i > b_j(v_j), b_i > b_k(v_k))(v_i - b_i) \\&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using (*)} \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}\end{aligned}$$

Take the FOC and isolate  $b_i^{**}(v_i)$ .

## PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write  $i$ 's expected payoff in eq.:

$$\begin{aligned}\mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\&= \mathbb{P}(b_i > b_j(v_j), b_i > b_k(v_k))(v_i - b_i) \\&= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using (*)} \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\&= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}\end{aligned}$$

$$FOC : \quad 0 = \frac{1}{2c} [2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2]$$

$$0 = 2(v_i - b_i) - (b_i - d - c), \quad \text{assuming } b_i - d - c \neq 0$$

$$b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3}(c + d)$$

## PS8, Ex. 4.a: First-price sealed bid auctions with three bidders

- (a) For three bidders, show that there is a symmetric Bayesian Nash Equilibrium in linear strategies:  $b_i(v_i) = cv_i + d$  (\*). Find  $c$  and  $d$ .

Hint: Use that  $v_j$  and  $v_k$  are independent (private) to write  $i$ 's expected payoff in eq.:

$$\begin{aligned}
 \mathbb{E}[u_i(b_i, v_i)] &= \mathbb{P}(i \text{ wins} | b_i)(v_i - b_i) \\
 &= \mathbb{P}(b_i > b_j(v_j), b_i > b_k(v_k))(v_i - b_i) \\
 &= \mathbb{P}(b_i > cv_j + d, b_i > cv_k + d)(v_i - b_i), && \text{using (*)} \\
 &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j, \frac{b_i - d}{c} > v_k\right)(v_i - b_i) \\
 &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
 &= \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right) \times \mathbb{P}\left(\frac{b_i - d}{c} > v_j\right)(v_i - b_i) \\
 &= \left(\frac{b_i - d - c}{2c}\right)^2 (v_i - b_i), && \text{using ex. (3.a)}
 \end{aligned}$$

$$FOC: \quad 0 = \frac{1}{2c} [2(b_i - d - c)(v_i - b_i) - (b_i - d - c)^2]$$

$$0 = 2(v_i - b_i) - (b_i - d - c), \quad \text{assuming } b_i - d - c \neq 0$$

$$\begin{aligned}
 b_i^{**}(v_i) &= \underbrace{\frac{2}{3}}_{c^* = \frac{2}{3}} v_i + \underbrace{\frac{1}{3}(c + d)}_{d^* = \frac{1}{3}(\frac{2}{3} + d^*) \Rightarrow d^* = \frac{1}{3}} = \underline{\underline{\frac{2}{3}v_i + \frac{1}{3}}} && Q.E.D.
 \end{aligned}$$

- (b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

BNE found:

$$(3.a) \quad b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2} \text{ for } i \in 1, 2, 3$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3} \text{ for } i \in 1, 2, 3$$

- (b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading ( $\frac{2}{3} > \frac{1}{2}$ ) and therefore a *higher* revenue for the seller.

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

BNE found:

$$(3.a) \quad b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2} \text{ for } i \in 1, 2, 3$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3} \text{ for } i \in 1, 2, 3$$



## PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading ( $\frac{2}{3} > \frac{1}{2}$ ) and therefore a *higher* revenue for the seller.

Looking at the bid strategies, we can confirm this, as each bidder submits higher bids with more opponents:

$$\begin{aligned}b_i^{**} &> b_i^* \Leftrightarrow \\ \frac{2}{3}v_i + \frac{1}{3} &> \frac{1}{2}v_i + \frac{1}{2} \Leftrightarrow \\ \frac{1}{6}v_i &> \frac{1}{6} \Leftrightarrow \\ v_i &> 1\end{aligned}$$

I.e. except for the rare case where all players have the valuation  $v = 1$ , the seller's revenue is strictly higher with three players than with two players.

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

BNE found:

$$(3.a) \quad b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2} \text{ for } i \in 1, 2, 3$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3} \text{ for } i \in 1, 2, 3$$

## PS8, Ex. 4.b: First-price sealed bid auctions with three bidders

(b) Do you expect seller to earn a higher or a lower revenue than in the previous auction? What is causing this effect?

Intuitively, more bidders decreases the chance of winning, which should lead to less bid shading ( $\frac{2}{3} > \frac{1}{2}$ ) and therefore a *higher* revenue for the seller.

Looking at the bid strategies, we can confirm this, as each bidder submits higher bids with more opponents:

$$\begin{aligned}b_i^{**} &> b_i^* \Leftrightarrow \\ \frac{2}{3}v_i + \frac{1}{3} &> \frac{1}{2}v_i + \frac{1}{2} \Leftrightarrow \\ \frac{1}{6}v_i &> \frac{1}{6} \Leftrightarrow \\ v_i &> 1\end{aligned}$$

I.e. except for the rare case where all players have the valuation  $v = 1$ , the seller's revenue is strictly higher with three players than with two players.

Furthermore, the expected highest value of  $N$  draws is increasing in  $N$ .

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

BNE found:

$$(3.a) \quad b_i^*(v_i) = \frac{1}{2}v_i + \frac{1}{2} \text{ for } i \in 1, 2, 3$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad b_i^{**}(v_i) = \frac{2}{3}v_i + \frac{1}{3} \text{ for } i \in 1, 2, 3$$

(c) (More difficult). Calculate the revenue to the seller.

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: **Calculate the expected highest value  $v_i$  for  $N = 3$  draws from the uniform distribution  $v_i \sim U(1, 3)$ .**

For  $N$  draws from  $x \sim U(a, b)$  :

$$\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}, \quad Y_1 = \max(X=x_1, \dots, x_N)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value For  $N$  draws from  $x \sim U(a, b)$  :  
 $v_i$  for  $N = 3$  draws from the uniform  $\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$ ,  $Y_1 = \max(X=x_1, \dots, x_N)$   
 distribution  $v_i \sim U(1, 3)$ : Results so far:

$$\mathbb{E}[Y_1] = a + N \frac{b-a}{N+1}$$

$$= 1 + 3 \frac{3-1}{3+1}$$

$$= 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}$$

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$1^{\text{st}}: \mathbb{E}[Y_1] = \frac{5}{2}$$

2<sup>nd</sup> step: **Insert the expected highest value  $\mathbb{E}[Y_1]$  in the bid-function (\*) to find the seller's expected revenue.**

## PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value For  $N$  draws from  $x \sim U(a, b)$  :

$v_i$  for  $N = 3$  draws from the uniform  $\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$ ,  $Y_1 = \max(X=x_1, \dots, x_N)$   
distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned}\mathbb{E}[Y_1] &= a + N \frac{b-a}{N+1} \\ &= 1 + 3 \frac{3-1}{3+1} \\ &= 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}\end{aligned}$$

2<sup>nd</sup> step: Insert the expected highest value

$\mathbb{E}[Y_1]$  in the bid-function (\*) to find the seller's expected revenue.

3<sup>rd</sup> step: **Does this live up to your expectation in ex. (4.b)?**

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$1^{\text{st}}: \mathbb{E}[Y_1] = \frac{5}{2}$$

2<sup>nd</sup>: The seller's expected revenue:

$$\mathbb{E}[\text{revenue}] = b_i \left( \frac{5}{2} \right)$$

$$= c^* \frac{5}{2} + d^*$$

$$= \frac{2}{3} \cdot \frac{5}{2} + \frac{1}{3},$$

$$= \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2 > \frac{5}{3}$$

using (4.a)

## PS8, Ex. 4.c: First-price sealed bid auctions with three bidders

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected highest value For  $N$  draws from  $x \sim U(a, b)$  :

$v_i$  for  $N = 3$  draws from the uniform  $\mathbb{E}(Y_1) = a + N \frac{b-a}{N+1}$ ,  $Y_1 = \max(X=x_1, \dots, x_N)$   
distribution  $v_i \sim U(1, 3)$ :

$$\begin{aligned}\mathbb{E}[Y_1] &= a + N \frac{b-a}{N+1} \\ &= 1 + 3 \frac{3-1}{3+1} \\ &= 1 + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}\end{aligned}$$

2<sup>nd</sup> step: Insert the expected highest value

$\mathbb{E}[Y_1]$  in the bid-function (\*) to find the seller's expected revenue.

3<sup>rd</sup> step: Does this live up to your expectation in ex. (4.b)?

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(3.b) \quad \mathbb{E}[\text{revenue}] = \frac{5}{3}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$1^{\text{st}}: \mathbb{E}[Y_1] = \frac{5}{2}$$

2<sup>nd</sup>: The seller's expected revenue:

$$\begin{aligned}\mathbb{E}[\text{revenue}] &= b_i \left( \frac{5}{2} \right) \\ &= c^* \frac{5}{2} + d^* \\ &= \frac{2}{3} \cdot \frac{5}{2} + \frac{1}{3}, && \text{using (4.a)} \\ &= \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2 > \frac{5}{3}\end{aligned}$$

3<sup>rd</sup>: The expected revenue is higher as more players increases competition and the expected highest valuation.

## PS8, Ex. 5: Winner's Curse

---



Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (a) Are the bidders' values private or common?
- (b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?
- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report says the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(a) Are the bidders' values private or common?

Though the reports investigating the values are private, the bidders' actual values are **common** since they are identical, i.e.  $v_1 = v_2 = v$ .

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report says the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Write up Bayes' rule.

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report says the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule: 
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: Use Bayes' rule and the given probabilities to write up the probability that the value of the oil-field is high after having received the report  $r_1 = H$ .

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report  $r_1 = H$ :

$$\begin{aligned}\mathbb{P}[H|r_1 = H] &= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H]} = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} \\ &= \frac{p^{\frac{1}{2}}}{p^{\frac{1}{2}} + (1-p)^{\frac{1}{2}}} = \frac{p^{\frac{1}{2}}}{\frac{1}{2}} = p \quad (\star)\end{aligned}$$

## PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report  $r_1 = H$ :

$$\begin{aligned}\mathbb{P}[H|r_1 = H] &= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H]} = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} \\ &= \frac{p \frac{1}{2}}{p \frac{1}{2} + (1-p) \frac{1}{2}} = \frac{p \frac{1}{2}}{\frac{1}{2}} = p \quad (*)\end{aligned}$$

Step 3: Use  $(*)$  to write up the expected value of the oil-field after receiving the report  $r_1 = H$  where the profits can be either high  $v_H$  or low  $v_L$ .

## PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report  $r_1 = H$ :

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p \frac{1}{2}}{p \frac{1}{2} + (1-p) \frac{1}{2}} = \frac{p \frac{1}{2}}{\frac{1}{2}} = p \quad (\star)$$

Step 3: Use  $(\star)$  to write up the expected value of the oil-field after receiving the report  $r_1 = H$  where the profits can be either high  $v_H$  or low  $v_L$ :

$$\mathbb{E}[v|r_1 = H] = \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L$$



## PS8, Ex. 5.b: Winner's Curse

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

(b) Assume that company 1 receives a report of  $H$ . Given this report, what is the expected value of the oil field to this company?

Step 1: Bayes' rule:  $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Step 2: The probability that the value of the oil-field is high given the report  $r_1 = H$ :

$$\mathbb{P}[H|r_1 = H] = \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[L]} = \frac{p \frac{1}{2}}{p \frac{1}{2} + (1-p) \frac{1}{2}} = \frac{p \frac{1}{2}}{\frac{1}{2}} = p \quad (*)$$

Step 3: Use  $(*)$  to write up the expected value of the oil-field after receiving the report  $r_1 = H$  where the profits can be either high  $v_H$  or low  $v_L$ :

$$\begin{aligned}\mathbb{E}[v|r_1 = H] &= \mathbb{P}[H|r_1 = H]v_H + \mathbb{P}[L|r_1 = H]v_L \\ &= \mathbb{P}[H|r_1 = H]v_H + (1 - \mathbb{P}[H|r_1 = H])v_L, \text{ using } (*) \\ &= pv_H + (1-p)v_L\end{aligned}$$

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report say the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: Write up the probability that the value of the oil-field is  $H$  given company 1 receives a report  $r_1 = H$  and wins the auction.

Two companies want to acquire the drilling rights to a North Sea oil field. However, the companies are unsure about the value of these rights. They know the drilling rights have an identical value for both companies, and this value is either high ( $H$ ) or low ( $L$ ) with equal probability.

The Danish government plans to hold an auction to sell off the rights, so each company sends a research team to the oil field to learn more about its value. The research team then sends a private report back to the company that sent it. Each report says the value is either  $H$  or  $L$ , and is correct with probability  $p$ , where  $\frac{1}{2} < p < 1$ . The probability of a mistake is independent across the two reports.

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\mathbb{P}[H|r_1 = H \wedge \text{win}] = \mathbb{P}[H|r_1 = H \wedge r_2 = L], \quad (\text{company 1 only wins if } r_2 = L)$$

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}
 \mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L) \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]} \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (***)
 \end{aligned}$$

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}
 \mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L) \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]} \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (***)
 \end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction.

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}
 \mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L\text{)} \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L]} \\
 &= \frac{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H \wedge r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H \wedge r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H]}{\mathbb{P}[r_1 = H|H] \times \mathbb{P}[r_2 = L|H] \times \mathbb{P}[H] + \mathbb{P}[r_1 = H|L] \times \mathbb{P}[r_2 = L|L] \times \mathbb{P}[L]} \\
 &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (***)
 \end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction:

$$\mathbb{E}[v|r_1 = H \wedge \text{win}] = \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + \mathbb{P}[L|r_1 = H \wedge \text{win}]v_L$$

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}\mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L\text{)} \\ &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (**)\end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + \mathbb{P}[L|r_1 = R \wedge \text{win}]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge \text{win}])v_L\end{aligned}$$

Now, insert  $(**)$  above and compare to the result in (3.b).



- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}\mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L\text{)} \\ &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (**)\end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + \mathbb{P}[L|r_1 = R \wedge \text{win}]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge \text{win}])v_L && \text{insert } (**) \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H], \text{ using (b)}} && \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c).

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}\mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L\text{)} \\ &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (**)\end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + \mathbb{P}[L|r_1 = R \wedge \text{win}]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge \text{win}])v_L && \text{insert } (**)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H], \text{ using (b)}} && \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: Looking at the inequality above, explain the difference between (b) and (c).  
*In other words, why is company 1 less certain that the value is  $H$  after they actually win the auction?*

- (c) Continue to assume that company 1 receives a report of  $H$ , and suppose that this company bids  $b_H$  in the auction. Assume that company 2 will bid  $b_L < b_H$  if its own report is  $L$  and  $b_H$  if it is  $H$ . Suppose that company 2 wins the auction if it places the higher bid and also in the case of a tie. Use Bayes' to calculate the expected value of the oil field to company 1, conditional on it winning the auction. How does this value compare to your answer in (b)?

Step 1: The probability the value is  $H$  given company 1 receives a report  $r_1 = H$  and wins:

$$\begin{aligned}\mathbb{P}[H|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge r_2 = L], && \text{(company 1 only wins if } r_2 = L\text{)} \\ &= \frac{p(1-p)\frac{1}{2}}{p(1-p)\frac{1}{2} + (1-p)p\frac{1}{2}} = \frac{p(1-p)}{2(p(1-p))} = \frac{1}{2} && (**)\end{aligned}$$

Step 2: Use  $(**)$  to write up the expected value of the oil-field conditional on the report being  $r_1 = H$  and company 1 winning the auction:

$$\begin{aligned}\mathbb{E}[v|r_1 = H \wedge \text{win}] &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + \mathbb{P}[L|r_1 = R \wedge \text{win}]v_L \\ &= \mathbb{P}[H|r_1 = H \wedge \text{win}]v_H + (1 - \mathbb{P}[H|r_1 = H \wedge \text{win}])v_L && \text{insert } (**) \\ &= \frac{1}{2}v_H + \frac{1}{2}v_L < \underbrace{pv_H + (1-p)v_L}_{\mathbb{E}[v|r_1 = H], \text{ using (b)}} && \text{since } p > \frac{1}{2}\end{aligned}$$

Step 3: This is an example of The Winner's Curse: The equally trustworthy reports of the two companies cancel each other out. Since the valuations of the auctioned object are correlated, you are likely to win the object when you overestimate the value.

**PS8, Ex. 3': Solving 3.b and 3.c  
through integration (optional)**

---

(b) Calculate the revenue to the seller.

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

(b) Calculate the revenue to the seller.      Standard results for  $x \sim U(a, b)$  :

1<sup>st</sup> step: **Calculate the expected payment of bidder  $i$  with valuation  $v_i$ .**

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

Results so far:

(\*)  $b_i(v_i) = cv_i + d$

(\*\*)  $\mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$

(3.a)  $c^* = d^* = \frac{1}{2}$

(b) Calculate the revenue to the seller.

Standard results for  $x \sim U(a, b)$  :

1<sup>st</sup> step: **Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :**

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned}
 m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\
 &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\
 &= \frac{v_i - 1}{2} (cv_i + d) \\
 &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\
 &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\
 &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4}
 \end{aligned}$$

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$



## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned}
 m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\
 &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\
 &= \frac{v_i - 1}{2} (cv_i + d) \\
 &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\
 &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\
 &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4}
 \end{aligned}$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\ &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\ &= \frac{v_i - 1}{2} (cv_i + d) \\ &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\ &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\ &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\ &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\ &= \frac{v_i - 1}{2} (cv_i + d) \\ &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\ &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\ &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4} \end{aligned}$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\ &= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{8} \int_1^3 v_i^2 - 1 dv_i \\ &= \frac{1}{8} \left[ \frac{1}{3} v_i^3 - v_i \right]_1^3 \\ &= \frac{1}{8} \left( \frac{3^3}{3} - 3 - \frac{1^3}{3} + 1 \right) = \frac{5}{6} \end{aligned}$$

## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned}
 m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\
 &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\
 &= \frac{v_i - 1}{2} (cv_i + d) \\
 &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\
 &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\
 &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4}
 \end{aligned}$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

3<sup>rd</sup> step: **Write up the expected revenue to the seller.**

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\begin{aligned}
 \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\
 &= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\
 &= \frac{1}{8} \int_1^3 v_i^2 - 1 dv_i \\
 &= \frac{1}{8} \left[ \frac{1}{3} v_i^3 - v_i \right]_1^3 \\
 &= \frac{1}{8} \left( \frac{3^3}{3} - 3 - \frac{1^3}{3} + 1 \right) = \frac{5}{6}
 \end{aligned}$$

## PS8, Ex. 3.b': Solving 3.b through integration (optional)

(b) Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$\begin{aligned}
 m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) b_i(v_i) \\
 &= \frac{cv_i - c}{2c} (cv_i + d), \text{ using } (*), (**) \\
 &= \frac{v_i - 1}{2} (cv_i + d) \\
 &= \frac{v_i - 1}{2} \left( \frac{v_i}{2} + \frac{1}{2} \right), \text{ using (3.a)} \\
 &= \left( \frac{v_i}{2} - \frac{1}{2} \right) \left( \frac{v_i}{2} + \frac{1}{2} \right) \\
 &= \left( \frac{v_i}{2} \right)^2 - \left( \frac{1}{2} \right)^2 = \frac{v_i^2 - 1}{4}
 \end{aligned}$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

3<sup>rd</sup> step: The expected revenue to the seller is the ex-ante expected payment of both bidders:

$$\text{Seller's revenue} = \mathbb{E}[m_1(v_1)] + \mathbb{E}[m_2(v_2)] = \frac{5}{3}$$

Standard results for  $x \sim U(a, b)$ :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(**) \quad \mathbb{P}(i \text{ wins} | v_i) = \frac{b_i(v_i) - d - c}{2c} = \frac{cv_i - c}{2c}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\begin{aligned}
 \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\
 &= \int_1^3 \frac{v_i^2 - 1}{4} \cdot \frac{1}{3-1} dv_i \\
 &= \frac{1}{8} \int_1^3 v_i^2 - 1 dv_i \\
 &= \frac{1}{8} \left[ \frac{1}{3} v_i^3 - v_i \right]_1^3 \\
 &= \frac{1}{8} \left( \frac{3^3}{3} - 3 - \frac{1^3}{3} + 1 \right) = \frac{5}{6}
 \end{aligned}$$

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

## PS8, Ex. 3.c.i': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. **Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.**
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

Standard results for  $x \sim U(a, b)$  :

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

## PS8, Ex. 3.c.ii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) P1 wins: Payoff is independent of  $b_1$  unless  $b_1 < b_2$ , in which case P1 no longer wins, thus, gets zero payoff.

P1 loses: Payoff is independent of  $b_1$  unless  $b_1 > b_2$ , in which case P1 wins instead but bids more than her evaluation and gets negative payoff.

i.e. there is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .



## PS8, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) **Calculate player  $i$ 's expected payment in the BNE.**

Standard results for  $x \sim U(a, b)$  :

PDF:  $f(x) = \frac{1}{b-a}$

CDF:  $F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$

Mean:  $\mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$

## PS8, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) **Calculate player  $i$ 's expected payment in the BNE:**

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) \cdot \mathbb{E}[b_j^*(v_j) | b_j^*(v_j) < b_i^*(v_i)]$$

## PS8, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. **Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).**

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) Player  $i$ 's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) \cdot \mathbb{E}[b_j^*(v_j) | b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j | v_j < v_i] \\ &= \frac{v_i - 1}{3 - 1} \cdot \frac{1 + v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

## PS8, Ex. 3.c.iii': Solving 3.c through integration (optional)

(c) Suppose now that the object is sold by a *second-price sealed bid auction*.

- i. Suppose player 2 bids his valuation:  $b_2(v_2) = v_2$ . Write down the expected payoffs to player 1 from bidding  $b_1$ .
- ii. Using your previous answer, argue that there is a symmetric Bayesian Nash Equilibrium (BNE) in which both players bid their valuation.
- iii. Calculate the revenue to the seller from this equilibrium. Compare to the answer in (b).

Standard results for  $x \sim U(a, b)$  :

$$\text{PDF: } f(x) = \frac{1}{b-a}$$

$$\text{CDF: } F(x) = \frac{x-a}{b-a} \Rightarrow \mathbb{P}(c > x) = \frac{c-a}{b-a}$$

$$\text{Mean: } \mu = \frac{a+b}{2} \Rightarrow \mathbb{E}(c < x) = \frac{a+x}{2}$$

(i) The expected payoffs of P1 given  $b_2$ :

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 > b_2 \\ (v_1 - b_2)/2 & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

(ii) There is no incentive to deviate from  $BNE = (b_1^*, b_2^*) = \{(v_1, v_2)\}$ .

(iii) Player  $i$ 's expected payment in BNE:

$$\begin{aligned} m_i(v_i) &= \mathbb{P}(i \text{ wins} | v_i) \cdot \mathbb{E}[b_j^*(v_j) | b_j^*(v_j) < b_i^*(v_i)] \\ &= \mathbb{P}(v_i > v_j) \cdot \mathbb{E}[v_j | v_j < v_i] \\ &= \frac{v_i - 1}{3 - 1} \cdot \frac{1 + v_i}{2}, \text{ using CDF and Mean} \\ &= \frac{v_i + v_i^2 - 1^2 - v_i}{2^2} = \frac{v_i^2 - 1}{4} \end{aligned}$$

As this is the same as in (3.b), we know:

$$\text{Ex-ante expected payment} = \mathbb{E}[m_i(v_i)] = \frac{5}{6}$$

$$\text{Seller's revenue} = 2 \cdot \mathbb{E}[m_i(v_i)] = \frac{5}{3}$$

Thus, the outcome is the exact same as for the *first-price sealed bid auction*.

**PS8, Ex. 4': Solving 4.c through  
integration (optional)**

---

(c) (More difficult). Calculate the revenue to the seller.

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: **Calculate the expected payment of bidder  $i$  with valuation  $v_i$ .**

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: **Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :**

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$



## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger)$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*)$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3} v_i + \frac{1}{3} \right), \text{ using (4.a)}$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of

bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3}v_i + \frac{1}{3} \right), \text{ using } (4.a)$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from } (4.a)$$

2<sup>nd</sup> step: **Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.**

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of

bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3} v_i + \frac{1}{3} \right), \text{ using } (4.a)$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from } (4.a)$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

2<sup>nd</sup> step: **Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.**

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3}v_i + \frac{1}{3} \right), \text{ using } (4.a) \quad \mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

**Why is the ex-ante expected payment lower than in exercise 3.b?**

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\begin{aligned} \mathbb{E}[m_i(v_i)] &= \int_1^3 m_i(v_i) f_i(v_i) dv_i \\ &= \int_1^3 \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \cdot \frac{1}{3-1} dv_i \\ &= \frac{1}{24} \left[ \frac{2}{4} v_i^4 - \frac{3}{3} v_i^3 + v_i \right]_1^3 \\ &= \frac{1}{24} \left( \frac{33}{2} - \frac{1}{2} \right) = \frac{2}{3} < \frac{5}{6} \end{aligned}$$

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of

bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3}v_i + \frac{1}{3} \right), \text{ using } (4.a) \quad \mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

2<sup>nd</sup> step: Find the ex-ante expected payment  
by integrating  $m_i(v_i)$  using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from } (4.a)$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$= \int_1^3 \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \cdot \frac{1}{3-1} dv_i$$

$$= \frac{1}{24} \left[ \frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i \right]_1^3$$

$$= \frac{1}{24} \left( \frac{33}{2} - \frac{1}{2} \right) = \frac{2}{3} < \frac{5}{6}$$

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of

bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3}v_i + \frac{1}{3} \right), \text{ using } (4.a) \quad \mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3<sup>rd</sup> step: **Calculate the seller's revenue and compare to exercise (3.b).**

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$= \int_1^3 \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \cdot \frac{1}{3-1} dv_i$$

$$= \frac{1}{24} \left[ \frac{2}{4}v_i^4 - \frac{3}{3}v_i^3 + v_i \right]_1^3$$

$$= \frac{1}{24} \left( \frac{33}{2} - \frac{1}{2} \right) = \frac{2}{3} < \frac{5}{6}$$

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of

bidder  $i$  with valuation  $v_i$ :  
 $m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger)$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*)$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3} v_i + \frac{1}{3} \right), \text{ using (4.a)}$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right)$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3<sup>rd</sup> step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from (4.a)}$$

$$(3.a) \quad c^* = d^* = \frac{1}{2}$$

$$(4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$\mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \int_1^3 \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \cdot \frac{1}{3-1} dv_i$$

$$= \frac{1}{24} \left[ \frac{2}{4} v_i^4 - \frac{3}{3} v_i^3 + v_i \right]_1^3$$

$$= \frac{1}{24} \left( \frac{33}{2} - \frac{1}{2} \right) = \frac{2}{3} < \frac{5}{6}$$

$$3^{\text{rd}}: \text{Revenue} = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$$

**Why is seller's revenue higher than in exercise 3.b?**

## PS8, Ex. 4.c': Solving 4.c through integration (optional)

(c) (More difficult). Calculate the revenue to the seller.

1<sup>st</sup> step: Calculate the expected payment of bidder  $i$  with valuation  $v_i$ :

$$m_i(v_i) = \mathbb{P}(i \text{ wins} | v_i) b_i(v_i)$$

$$= \left( \frac{b_i - d - c}{2c} \right)^2 b_i(v_i), \text{ using } (\dagger) \quad (3.a) \quad c^* = d^* = \frac{1}{2}$$

$$= \left( \frac{cv_i - c}{2c} \right)^2 (cv_i + d), \text{ using } (*) \quad (4.a) \quad c^* = \frac{2}{3}, \quad d^* = \frac{1}{2}$$

$$= \left( \frac{v_i - 1}{2} \right)^2 \left( \frac{2}{3} v_i + \frac{1}{3} \right), \text{ using } (4.a) \quad \mathbb{E}[m_i(v_i)] = \int_1^3 m_i(v_i) f_i(v_i) dv_i$$

$$= \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \quad = \int_1^3 \left( \frac{2v_i^3 - 3v_i^2 + 1}{12} \right) \cdot \frac{1}{3-1} dv_i$$

2<sup>nd</sup> step: Find the ex-ante expected payment by integrating  $m_i(v_i)$  using the PDF.

Though the bids are higher, the expected payment from each bidder is lower due to a lower probability of winning.

3<sup>rd</sup> step: Calculate the seller's revenue and compare to exercise (3.b).

Results so far:

$$(*) \quad b_i(v_i) = cv_i + d$$

$$(\dagger) \quad \mathbb{P}(i \text{ wins} | v_i) = \left( \frac{b_i - d - c}{2c} \right)^2 \text{ from } (4.a)$$

2<sup>nd</sup>: Ex-ante payment of bidder  $i$ :

$$= \frac{1}{24} \left[ \frac{2}{4} v_i^4 - \frac{3}{3} v_i^3 + v_i \right]_1^3$$

$$= \frac{1}{24} \left( \frac{33}{2} - \frac{1}{2} \right) = \frac{2}{3} < \frac{5}{6}$$

$$3^{\text{rd}}: \text{Revenue} = 3 \cdot \mathbb{E}[m_i(v_i)] = 2 > \frac{5}{3}$$

The seller can expect higher revenue as more players increases competition and the chance of one having high valuation.